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Abstract

Let F be a non-discrete non-Archimedean locally compact field and $\mathcal{O}_F$ the ring of integers in F. The main results of this paper are the classification of ergodic probability measures on the space $\mathrm{Mat}(\mathbb{N}, F)$ of infinite matrices with entries in F with respect to the natural action of the group $\mathrm{GL}(\infty, \mathcal{O}_F) \times \mathrm{GL}(\infty, \mathcal{O}_F)$ and the classification, for non-dyadic F, of ergodic probability measures on the space $\mathrm{Sym}(\mathbb{N}, F)$ of infinite symmetric matrices with respect to the natural action of the group $\mathrm{GL}(\infty, \mathcal{O}_F)$.

1. Introduction

Given a group action on a topological space, it is natural to try to describe the corresponding space of ergodic invariant probability measures. For some very classical actions, such as, for example, that of the shift on the space of infinite binary sequences, the space of ergodic measures is huge and does not seem to admit a reasonable description. On the other hand, for a number of natural actions of infinite-dimensional groups, a complete classification is possible. For example, De Finetti's Theorem (1937) claims that for the action of the infinite symmetric group on the space of infinite binary sequences, all ergodic probability measures are Bernoulli, and Schoenberg's Theorem (1951) claims that for the action of the infinite orthogonal group on the space of infinite $\mathbb{R}$-valued sequences, all ergodic probability measures are Gaussian. In both these examples, the space of ergodic probability measures is one dimensional. Pickrell [Pic87, Pic90, Pic91] and, by a different method, Olshanski and Vershik [OV96], classified all ergodic unitarily invariant measures on the space of infinite Hermitian matrices. In this case, an ergodic measure is determined by infinitely many parameters.

In this paper, we study classification of ergodic measures for actions related to the following inductive limit group

$$\mathrm{GL}(\infty, \mathcal{O}_F) := \varinjlim \mathrm{GL}(n, \mathcal{O}_F), \tag{1}$$

where $\mathcal{O}_F$ is the ring of integers in a non-discrete locally compact non-Archimedean field F and $\mathrm{GL}(n, \mathcal{O}_F)$ is the compact group of invertible $n \times n$ matrices over $\mathcal{O}_F$. Denote by $\mathrm{Mat}(\mathbb{N}, F)$ (respectively $\mathrm{Sym}(\mathbb{N}, F)$) the space of infinite matrices (respectively infinite symmetric matrices) over F. Our main results are:

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(i) the classification of the ergodic probability measures for the group action of $\mathrm{GL}(\infty, \mathcal{O}_F) \times \mathrm{GL}(\infty, \mathcal{O}_F)$ on $\mathrm{Mat}(\mathbb{N}, F)$ defined by

$$((g_1, g_2), M) \mapsto g_1 M g_2^{-1}, \quad g_1, g_2 \in \mathrm{GL}(\infty, \mathcal{O}_F), \quad M \in \mathrm{Mat}(\mathbb{N}, F);$$

(ii) the classification of the ergodic probability measures for the group action of $\mathrm{GL}(\infty, \mathcal{O}_F)$ on $\mathrm{Sym}(\mathbb{N}, F)$ defined by

$$(g, M) \mapsto g M g^t, \quad g \in \mathrm{GL}(\infty, \mathcal{O}_F), \quad S \in \mathrm{Sym}(\mathbb{N}, F),$$

where g^t is the transposition of g.

We proceed to the precise formulation. Let F be a non-discrete locally compact non-Archimedean field (for example, the field of p-adic numbers). Let $|\cdot|$ be the absolute value on F. The ring $\mathcal{O}_F$ of integers in F is given by $\{x \in F : |x| \leqslant 1\}$. The unique maximal and principal ideal of $\mathcal{O}_F$ is given by $\{x \in F : |x| < 1\}$. Throughout the paper, we fix any generator ϖ of $\{x \in F : |x| < 1\}$, that is, $\{x \in F : |x| < 1\} = \varpi \mathcal{O}_F$. The quotient $\mathcal{O}_F / \varpi \mathcal{O}_F$ is a finite field with q elements.

Define the inductively compact group $\mathrm{GL}(\infty, \mathcal{O}_F)$ by (1). Set

$$\mathrm{Mat}(\mathbb{N}, F) := \{X = (X_{ij})_{i,j \in \mathbb{N}} \mid X_{ij} \in F\}.$$

Let $\mathrm{Mat}(\infty, F)$ denote the subspace of $\mathrm{Mat}(\mathbb{N}, F)$ consisting of matrices whose all but a finite number of coefficients are zero. Define also

$$\mathrm{Sym}(\mathbb{N}, F) := \{X \in \mathrm{Mat}(\mathbb{N}, F) \mid X_{ij} = X_{ji}, \forall i, j \in \mathbb{N}\},$$

and let $\mathrm{Sym}(\infty, F) := \mathrm{Sym}(\mathbb{N}, F) \cap \mathrm{Mat}(\infty, F)$.

Throughout the paper, by the usual identification $\mathrm{Mat}(\mathbb{N}, F) \simeq F^{\mathbb{N} \times \mathbb{N}}$, we equip $\mathrm{Mat}(\mathbb{N}, F)$ with the Tychonoff's product topology. The set $\mathrm{Sym}(\mathbb{N}, F)$ is equipped with the induced topology by the natural embedding $\mathrm{Sym}(\mathbb{N}, F) \subset \mathrm{Mat}(\mathbb{N}, F)$.

1.1 Classification of ergodic measures on $\mathrm{Mat}(\mathbb{N}, F)$

Let Δ be the set of non-increasing sequences in $\mathbb{Z} \cup \{-\infty\}$, that is,

$$\Delta := \{\Bbbk = (k_j)_{j=1}^{\infty} \mid k_j \in \mathbb{Z} \cup \{-\infty\}; k_1 \geqslant k_2 \geqslant \cdots\}. \tag{2}$$

By the inclusion $\Delta \subset (\mathbb{Z} \cup \{-\infty\})^{\mathbb{N}}$, we equip Δ with the induced topology of the Tychonoff's product topology on $(\mathbb{Z} \cup \{-\infty\})^{\mathbb{N}}$.

To each sequence $\Bbbk \in \Delta$, we now assign an ergodic $\mathrm{GL}(\infty, \mathcal{O}_F) \times \mathrm{GL}(\infty, \mathcal{O}_F)$-invariant probability measure on $\mathrm{Mat}(\mathbb{N}, F)$. Let

$$X_i^{(n)}, \quad Y_i^{(n)}, \quad Z_{ij}, \quad i, j, n = 1, 2, \ldots$$

be independent random variables, each sampled with respect to the normalized Haar measure on the compact additive group $\mathcal{O}_F$. In what follows, we use the convention $\varpi^{\infty} = 0$.

Definition 1.1. Given an element $\Bbbk \in \Delta$, let

$$\mu_{\Bbbk} := \mathcal{L}(M_{\Bbbk})$$

be the probability distribution of the infinite random matrix $M_{\Bbbk}$ defined as follows. Write $k := \lim_{n \to \infty} k_n \in \mathbb{Z} \cup \{-\infty\}$ and set

$$M_{\Bbbk} := \left[\sum_{n:\, k_n > k} \varpi^{-k_n} X_i^{(n)} Y_j^{(n)} + \varpi^{-k} Z_{ij} \right]_{i,j \in \mathbb{N}}.$$

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Remark 1.2. For the constant sequence $\Bbbk = (-\infty, \dots, -\infty, \dots) \in \Delta$, using the convention $\varpi^{\infty} = 0$, the corresponding matrix $M_{\Bbbk}$ defined in Definition 1.1 is the zero matrix $O \in \mathrm{Mat}(\mathbb{N}, F)$ and hence the measure $\mu_{\Bbbk}$ is the Dirac measure at the point $O \in \mathrm{Mat}(\mathbb{N}, F)$.

Let $\mathcal{P}_{\mathrm{erg}}(\mathrm{Mat}(\mathbb{N}, F))$ be the space of ergodic $\mathrm{GL}(\infty, \mathcal{O}_F) \times \mathrm{GL}(\infty, \mathcal{O}_F)$-invariant probability measures on $\mathrm{Mat}(\mathbb{N}, F)$, endowed with the induced weak topology. The classification of $\mathcal{P}_{\mathrm{erg}}(\mathrm{Mat}(\mathbb{N}, F))$ is given by the following.

Theorem 1.3. *The map $\Bbbk \mapsto \mu_{\Bbbk}$ is a homeomorphism between Δ and $\mathcal{P}_{\mathrm{erg}}(\mathrm{Mat}(\mathbb{N}, F))$.*

Remark 1.4. By Theorem 1.3, the space $\mathcal{P}_{\mathrm{erg}}(\mathrm{Mat}(\mathbb{N}, F))$ is weakly closed in the space of all Borel measures on $\mathrm{Mat}(\mathbb{N}, F)$ and is σ-compact; moreover, any measure $\mu_{\Bbbk} \in \mathcal{P}_{\mathrm{erg}}(\mathrm{Mat}(\mathbb{N}, F))$ is compactly supported.

Let us explain Theorem 1.3 in more detail. We have the following elementary ergodic measures.

- (Haar type measures) For any $k \in \mathbb{Z}$, the normalized Haar measure on $\mathrm{Mat}(\mathbb{N}, \varpi^{-k}\mathcal{O}_F)$ is $\mathrm{GL}(\infty, \mathcal{O}_F) \times \mathrm{GL}(\infty, \mathcal{O}_F)$-ergodic.
- (Non-symmetric Wishart type measures) Let $X_1, Y_1, X_2, Y_2, \dots$ be independent and uniformly distributed on $\mathcal{O}_F$. For any $k \in \mathbb{Z}$, the probability distribution of the random matrix

$$[\varpi^{-k} X_i Y_j]_{i,j \in \mathbb{N}}$$

  is $\mathrm{GL}(\infty, \mathcal{O}_F) \times \mathrm{GL}(\infty, \mathcal{O}_F)$-ergodic.

Theorem 1.3 implies that any ergodic $\mathrm{GL}(\infty, \mathcal{O}_F) \times \mathrm{GL}(\infty, \mathcal{O}_F)$-invariant probability measure on $\mathrm{Mat}(\mathbb{N}, F)$ can be obtained as a possibly infinite convolution of the above two types of elementary ones.

1.2 Classification of ergodic measures on $\mathrm{Sym}(\mathbb{N}, F)$

In what follows, when dealing with the ergodic measures on $\mathrm{Sym}(\mathbb{N}, F)$, we always assume that the field F is non-dyadic, that is, the cardinality of the field of residue class $\mathcal{O}_F/\varpi\mathcal{O}_F$ is not a power of 2.

The group of units of $\mathcal{O}_F$ is given by $\mathcal{O}_F^{\times} := \{x \in F : |x| = 1\}$. Denote by $(\mathcal{O}_F^{\times})^2$ the subgroup of $\mathcal{O}_F^{\times}$ defined by

$$(\mathcal{O}_F^{\times})^2 := \{x \in \mathcal{O}_F^{\times} : \text{there exists } a \in F \text{ such that } x = a^2\}.$$

If F is non-dyadic, then the quotient $\mathcal{O}_F^{\times}/(\mathcal{O}_F^{\times})^2$ has two elements. Throughout the paper, we fix a non-square unit $\varepsilon \in \mathcal{O}_F^{\times} \setminus (\mathcal{O}_F^{\times})^2$.

We now explicitly describe the parametrization of ergodic $\mathrm{GL}(\infty, \mathcal{O}_F)$-invariant probability measures on $\mathrm{Sym}(\mathbb{N}, F)$.

Recall the definition (2) of the set Δ of non-increasing sequences in $\mathbb{Z} \cup \{-\infty\}$. A sequence $(k_j)_{j \in \mathbb{N}} \in \Delta$ is called finite if $k_j = -\infty$ for all sufficiently large j. In this case, either $k_1 = -\infty$, and then we identity the sequence with an empty sequence, or $j_0 := \max\{j \mid k_j \in \mathbb{Z}\} \in \mathbb{N}$, and then we identify $(k_j)_{j \in \mathbb{N}}$ with $(k_j)_{j=1}^{j_0}$ and j_0 is called the *length* of the sequence. Conversely, for any finite non-increasing sequence $(k_j)_{j=1}^{n}$ in $\mathbb{Z}$, we identify it with the element in Δ by adding infinitely many $-\infty$ at the end of $(k_j)_{j=1}^{n}$.

For any $k \in \mathbb{Z}$, let $\mathbb{Z}_{>k}$ denote the set of integers strictly larger than k. We introduce the following four subsets of Δ:

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– $\Delta[k]$, the set of *non-increasing* sequences of *finite length* in $\mathbb{Z}_{>k}$;
– $\Delta^\sharp[k]$, the set of *strictly decreasing* sequences in $\mathbb{Z}_{>k}$ (which are automatically of finite length);
– $\Delta[-\infty]$, the set of *non-increasing* sequences in $\mathbb{Z}$ *of finite length* or *of infinite length tending to* $-\infty$;
– $\Delta^\sharp[-\infty]$, the set of *strictly decreasing* sequences of *finite or infinite length* in $\mathbb{Z}$.

Note that for any $k \in \mathbb{Z} \cup \{-\infty\}$, the following relations hold:

$$\Delta^\sharp[k] \subset \Delta[k], \quad \Delta^\sharp[k] \subset \Delta^\sharp[-\infty] \quad \text{and} \quad \Delta^\sharp[k] \subset \Delta[-\infty].$$

We introduce the parameter space

$$\Omega := \bigsqcup_{k \in \mathbb{Z} \cup \{-\infty\}} \{k\} \times \Delta[k] \times \Delta^\sharp[k], \tag{3}$$

where $\{k\}$ is the singleton with a single element k. The space Ω is equipped with the topology induced by the inclusion:

$$\Omega \subset (\mathbb{Z} \cup \{-\infty\}) \times (\mathbb{Z} \cup \{-\infty\})^{\mathbb{N}} \times (\mathbb{Z} \cup \{-\infty\})^{\mathbb{N}}.$$

To each element $\mathbb{h} \in \Omega$, we assign an ergodic $\mathrm{GL}(\infty, \mathcal{O}_F)$-invariant probability measure on $\mathrm{Sym}(\mathbb{N}, F)$ as follows. Let

$$X_i^{(n)}, \quad Y_i^{(n)}, \quad H_{ij}, \quad i \leqslant j, n = 1, 2, \dots$$

be independent random variables uniformly distributed on $\mathcal{O}_F$. In particular, define

$$H = [H_{ij}]_{i,j \in \mathbb{N}} \tag{4}$$

by setting $H_{ij} = H_{ji}$ if $i > j$, then H is an infinite symmetric random matrix sampled uniformly from $\mathrm{Sym}(\mathbb{N}, \mathcal{O}_F)$.

Definition 1.5. For any $\mathbb{h} \in \Omega$ given by

$$\mathbb{h} = (k; \mathbb{k}, \mathbb{k}'), \quad \text{with } k \in \mathbb{Z} \cup \{-\infty\},\ \mathbb{k} \in \Delta[k],\ \mathbb{k}' \in \Delta^\sharp[k],$$

we define

$$\nu_{\mathbb{h}} := \mathcal{L}(S_{\mathbb{h}}),$$

as the probability distribution of the infinite symmetric random matrix $S_{\mathbb{h}}$ defined as follows. First let

$$W_{\mathbb{k}} := \left[\sum_{n=1}^{\infty} \varpi^{-k_n} X_i^{(n)} X_j^{(n)} \right]_{i,j \in \mathbb{N}}, \quad W_{\mathbb{k}'} := \left[\sum_{n=1}^{\infty} \varpi^{-k'_n} Y_i^{(n)} Y_j^{(n)} \right]_{i,j \in \mathbb{N}},$$

and then set

$$S_{\mathbb{h}} := W_{\mathbb{k}} + \varepsilon W_{\mathbb{k}'} + \varpi^{-k} H.$$

Remark 1.6. For the element $\mathbb{h} = (-\infty; \emptyset, \emptyset)$, where $\emptyset$ means the empty sequence in $\mathbb{Z}$, the corresponding matrix $S_{\mathbb{h}}$ defined in Definition 1.5 is the zero matrix $O \in \mathrm{Sym}(\mathbb{N}, F)$ and hence the measure $\nu_{\mathbb{h}}$ is the Dirac measure at the point $O \in \mathrm{Sym}(\mathbb{N}, F)$.

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Remark 1.7. The strictly decreasing assumption on the sequences $\Bbbk' \in \Delta^{\sharp}[k]$ is imposed for the uniqueness of parametrization. The reason is the following:

$$\mathcal{L}\left(\left[\sum_{n=1}^{2} X_i^{(n)} X_j^{(n)}\right]_{i,j\in\mathbb{N}}\right) = \mathcal{L}\left(\varepsilon\left[\sum_{n=1}^{2} X_i^{(n)} X_j^{(n)}\right]_{i,j\in\mathbb{N}}\right). \tag{5}$$

For the detail, see Remark 4.8 and the proof of Proposition 5.3 below.

Let $\mathcal{P}_{\mathrm{erg}}(\mathrm{Sym}(\mathbb{N}, F))$ be the space of ergodic $\mathrm{GL}(\infty, \mathcal{O}_F)$-invariant probability measures on $\mathrm{Sym}(\mathbb{N}, F)$, endowed with the induced weak topology. The classification of $\mathcal{P}_{\mathrm{erg}}(\mathrm{Sym}(\mathbb{N}, F))$ is given by the following.

Theorem 1.8. *Assume that F is non-dyadic. Then the map $\mathbb{h} \mapsto \nu_{\mathbb{h}}$ is a homeomorphism between Ω and $\mathcal{P}_{\mathrm{erg}}(\mathrm{Sym}(\mathbb{N}, F))$.*

Remark 1.9. By Theorem 1.8, the space $\mathcal{P}_{\mathrm{erg}}(\mathrm{Sym}(\mathbb{N}, F))$ is weakly closed in the space of all Borel measures on $\mathrm{Sym}(\mathbb{N}, F)$ and is σ-compact. Moreover, any measure $\nu_{\mathbb{h}} \in \mathcal{P}_{\mathrm{erg}}(\mathrm{Sym}(\mathbb{N}, F))$ is compactly supported.

Theorem 1.8 can be explained as follows. We have the following elementary ergodic measures.

- (Haar type measures) For any $k \in \mathbb{Z}$, the normalized Haar measure on $\mathrm{Sym}(\mathbb{N}, \varpi^{-k}\mathcal{O}_F)$ is $\mathrm{GL}(\infty, \mathcal{O}_F)$-ergodic.
- (Symmetric Wishart type measures) Let $X_1, X_2, \ldots$ be independent copies of F-valued random variables, all of which are uniformly distributed on $\mathcal{O}_F$. For any $k \in \mathbb{Z}$, the distributions of the infinite rank-one random matrices

$$[\varpi^{-k} X_i X_j]_{i,j\in\mathbb{N}} \quad \text{and} \quad [\varepsilon\varpi^{-k} X_i X_j]_{i,j\in\mathbb{N}}$$

  are $\mathrm{GL}(\infty, \mathcal{O}_F)$-ergodic.

Theorem 1.8 implies that any ergodic $\mathrm{GL}(\infty, \mathcal{O}_F)$-invariant probability measure on $\mathrm{Sym}(\mathbb{N}, F)$ can be obtained as a possibly infinite convolution of the above two types of elementary ones.

1.3 Characteristic functions of ergodic measures

Let $\chi \in \widehat{F}$ be a *fixed character* of F such that

$$\chi|_{\mathcal{O}_F} \equiv 1 \quad \text{and} \quad \chi \text{ is not constant on } \varpi^{-1}\mathcal{O}_F. \tag{6}$$

Given a Borel probability measure μ on $\mathrm{Mat}(\mathbb{N}, F)$, its characteristic function, or Fourier transform, $\widehat{\mu}$ is defined on $\mathrm{Mat}(\infty, F)$ by the formula

$$\widehat{\mu}(A) := \int_{\mathrm{Mat}(\mathbb{N},F)} \chi(\mathrm{tr}(AM))\, \mu(dM), \quad A \in \mathrm{Mat}(\infty, F).$$

Similarly, given a Borel probability measure ν on $\mathrm{Sym}(\mathbb{N}, F)$, its characteristic function $\widehat{\nu}$ is defined on $\mathrm{Sym}(\infty, F)$ by the formula

$$\widehat{\nu}(A) := \int_{\mathrm{Sym}(\mathbb{N},F)} \chi(\mathrm{tr}(AS))\, \nu(dS), \quad A \in \mathrm{Sym}(\infty, F).$$

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Let $\mathrm{Mat}(n,F)$ be the space of $n\times n$ matrices with entries in F. Every $A\in\mathrm{Mat}(n,F)$ can be written (see Lemma 2.4 below) in the form

$$A=a\cdot\mathrm{diag}(\varpi^{-k_1},\dots,\varpi^{-k_n})\cdot b,\quad a,b\in\mathrm{GL}(n,\mathcal{O}_F),k_i\in\mathbb{Z}\cup\{-\infty\}.$$

These numbers $k_1,k_2,\dots,k_n$, taken with multiplicities, are uniquely determined by A and are called the *singular numbers* of A. The collection of the singular numbers of the matrix A is denoted by $\mathrm{Sing}(A)$.

After the computation of characteristic functions for the probability measures in the list of measures defined in Definition 1.1 (see Proposition 4.1 below), Theorem 1.3 can be reformulated in the following form.

Theorem 1.10. *The characteristic functions of ergodic* $\mathrm{GL}(\infty,\mathcal{O}_F)\times\mathrm{GL}(\infty,\mathcal{O}_F)$*-invariant probability measures on* $\mathrm{Mat}(\mathbb{N},F)$ *are exactly of the form*

$$\varphi(A)=\prod_{\ell\in\mathrm{Sing}(A)}\exp\Bigg(-\log q\cdot\sum_{j=1}^{\infty}(k_j+\ell)\mathbb{1}_{\{k_j+\ell\geqslant 1\}}\Bigg),\quad A\in\mathrm{Mat}(\infty,F),$$

where $\Bbbk=(k_j)_{j\in\mathbb{N}}\in\Delta$ *is the parameter sequence.*

For formulating a similar statement in the symmetric case, we need to introduce a function $\theta:F\to\mathbb{C}$ by

$$\theta(x):=\int_{\mathcal{O}_F}\chi(z^2\cdot x)\,dz. \tag{7}$$

Properties of the function θ are summarized in Proposition 4.5 below.

Let $\mathrm{Sym}(n,F)$ be the space of $n\times n$ symmetric matrices with entries in F. Note that if the field F is non-dyadic, then any $A\in\mathrm{Sym}(n,F)$ can be written (see Lemma 2.7 below) in the form

$$A=g\cdot\mathrm{diag}(x_1,\dots,x_n)\cdot g^t,\quad g\in\mathrm{GL}(n,\mathcal{O}_F).$$

After the computation of characteristic functions for the probability measures in the list of measures defined in Definition 1.5 (see Proposition 4.4), Theorem 1.8 can be reformulated in the following form.

Theorem 1.11. *Assume that* F *is non-dyadic. Then the characteristic functions of the ergodic* $\mathrm{GL}(\infty,\mathcal{O}_F)$*-invariant probability measures on* $\mathrm{Sym}(\mathbb{N},F)$ *are exactly given by*

$$\Phi(\mathrm{diag}(x_1,\dots,x_m,0,\dots))=\prod_{i=1}^{m}\Bigg[\mathbb{1}_{\mathcal{O}_F}(\varpi^{-k}x_i)\prod_{j=1}^{\infty}\theta(\varpi^{-k_j}x_i)\prod_{j=1}^{\infty}\theta(\varepsilon\varpi^{-k'_j}x_i)\Bigg],$$

where $\Bbbh=(k;(k_j)_{j\in\mathbb{N}},(k'_j)_{j\in\mathbb{N}})$ *is a parameter in* Ω *introduced in (3).*

1.4 Spherical representations

Our classification theorems, Theorems 1.3 and 1.8, can equivalently be formulated as a classification of *spherical representations* of analogues, in our context, of the infinite dimensional Cartan motion groups

$$\mathrm{Mat}(\infty,F)\rtimes(\mathrm{GL}(\infty,\mathcal{O}_F)\times\mathrm{GL}(\infty,\mathcal{O}_F))\quad\text{and}\quad\mathrm{Sym}(\infty,F)\rtimes\mathrm{GL}(\infty,\mathcal{O}_F)$$

respectively. We explain this in more detail for $\mathrm{Sym}(\infty,F)\rtimes\mathrm{GL}(\infty,\mathcal{O}_F)$.

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Recall that $\mathrm{Sym}(n,F) \rtimes \mathrm{GL}(n,\mathcal{O}_F)$ is the semi-direct product of the additive group $\mathrm{Sym}(n,F)$ and the general linear group $\mathrm{GL}(n,\mathcal{O}_F)$. Elements of $\mathrm{Sym}(n,F) \rtimes \mathrm{GL}(n,\mathcal{O}_F)$ are pairs $(A,g), A \in \mathrm{Sym}(n,F), g \in \mathrm{GL}(n,\mathcal{O}_F)$ and the rule of multiplication is given by

$$(A,g)\cdot(B,h) = (A + gBg^t, gh).$$

The group $\mathrm{Sym}(\infty,F) \rtimes \mathrm{GL}(\infty,\mathcal{O}_F)$ is defined in a similar way and is of course the inductive limit of the sequence $\mathrm{Sym}(n,F) \rtimes \mathrm{GL}(n,\mathcal{O}_F)$. The groups $\mathrm{Sym}(\infty,F)$ and $\mathrm{GL}(\infty,\mathcal{O}_F)$ are canonically identified with subgroups of $\mathrm{Sym}(\infty,F) \rtimes \mathrm{GL}(\infty,\mathcal{O}_F)$ by the following embeddings

$$A \mapsto (A,1) \quad \text{and} \quad g \mapsto (0,g),$$

where $A \in \mathrm{Sym}(\infty,F)$ and $g \in \mathrm{GL}(\infty,\mathcal{O}_F)$.

A unitary representation ρ of $\mathrm{Sym}(\infty,F) \rtimes \mathrm{GL}(\infty,\mathcal{O}_F)$ in a Hilbert space $H(\rho)$ is called spherical if it is irreducible and the subspace $H(\rho)^{\mathrm{GL}(\infty,\mathcal{O}_F)}$ of $\mathrm{GL}(\infty,\mathcal{O}_F)$-invariant vectors in $H(\rho)$ is non-trivial; in which case, by irreducibility, the subspace $H(\rho)^{\mathrm{GL}(\infty,\mathcal{O}_F)}$ has dimension one. A vector $h \in H(\rho)^{\mathrm{GL}(\infty,\mathcal{O}_F)}$ of norm 1 is called a *spherical vector* of ρ and the function

$$\varphi_\rho(g) := (\rho(g)h,h), \quad g \in \mathrm{GL}(\infty,\mathcal{O}_F)$$

is called the *spherical function* of ρ. The spherical function φ_ρ is an invariant of the spherical representation ρ and it uniquely determines ρ. As a bi-invariant function with respect to the subgroup $\mathrm{GL}(\infty,\mathcal{O}_F) \subset \mathrm{Sym}(\infty,F) \rtimes \mathrm{GL}(\infty,\mathcal{O}_F)$, the function φ_ρ is uniquely determined by its restriction $\varphi_\rho|_{\mathrm{Sym}(\infty,F)}$.

Given an ergodic $\mathrm{GL}(\infty,\mathcal{O}_F)$-invariant probability measure ν on $\mathrm{Sym}(\mathbb{N},F)$, one may define a spherical representation ρ_ν in the Hilbert space $L^2(\mathrm{Sym}(\mathbb{N},F),\nu)$ as follows:

$$(\rho(g)\xi)(S) = \xi(g^{-1}S), \quad g \in \mathrm{GL}(\infty,\mathcal{O}_F),$$
$$(\rho(A)\xi)(S) = \chi(\mathrm{tr}(AS))\xi(S), \quad A \in \mathrm{Sym}(\infty,F),$$

where $\xi \in L^2(\mathrm{Sym}(\mathbb{N},F),\nu)$ and $S \in \mathrm{Sym}(\mathbb{N},F)$. The spherical vector can be chosen as the constant function $\xi_0(S) \equiv 1$.

Proposition 1.12. *The map $\nu \mapsto \rho_\nu$ defines a bijection between the set of ergodic $\mathrm{GL}(\infty,\mathcal{O}_F)$-invariant probability measures on $\mathrm{Sym}(\mathbb{N},F)$ and the set of spherical representations of the group $\mathrm{Sym}(\infty,F) \rtimes \mathrm{GL}(\infty,\mathcal{O}_F)$.*

The proof of Proposition 1.12 is the same as that of Olshanski and Vershik [OV96, Proposition 1.5].

1.5 An outline of the argument

Our argument relies on the Vershik–Kerov ergodic method in the spirit of Olshanski and Vershik [OV96]. The implementation of individual steps is, however, quite different. In the case of measures on $\mathrm{Mat}(\mathbb{N},F)$ and the case of measures on $\mathrm{Sym}(\mathbb{N},F)$, the main steps are:

- *explicit construction of ergodic measures, see Definitions 1.1 and 1.5;*
- *the asymptotic formulae for the analogues of Harish-Chandra–Izykson–Zuber orbital integrals, see Theorem 7.1, Theorem 7.4 in the case of measures on* $\mathrm{Mat}(\mathbb{N},F)$ *and Theorem 7.5, Theorem 7.6 in the case of measures on* $\mathrm{Sym}(\mathbb{N},F)$ *;*
- *proof of completeness of the lists of ergodic measures, see Theorems 8.8 and 8.14 respectively.*

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We now explain our method in greater detail in the case of measures on $\mathrm{Sym}(\mathbb{N}, F)$.

(1) *The Vershik–Kerov method: approximation of ergodic measures by orbital measures.* While we follow the general scheme of Vershik and Kerov, a number of details are different.

Given $x \in \mathrm{Sym}(\mathbb{N}, F)$ and $n \in \mathbb{N}$, let $m_{\mathrm{GL}(n,\mathcal{O}_F)}(x)$ denote the unique $\mathrm{GL}(n, \mathcal{O}_F)$-invariant probability measure on $\mathrm{Sym}(\mathbb{N}, F)$ supported on the orbit $\mathrm{GL}(n, \mathcal{O}_F) \cdot x \subset \mathrm{Sym}(\mathbb{N}, F)$. Let $\mathscr{ORB}_\infty(\mathrm{Sym}(\mathbb{N}, F))$ be the class of probability measures ν on $\mathrm{Sym}(\mathbb{N}, F)$ verifying the condition: there exists a sequence of positive integers $n_1 < n_2 < \cdots$ and a sequence $(\nu_{n_k})_{k\in\mathbb{N}}$ of probability measures with ν_{n_k} being a $\mathrm{GL}(n_k, \mathcal{O}_F)$-orbital measure supported on $\mathrm{Sym}(n_k, F) \subset \mathrm{Sym}(\mathbb{N}, F)$, such that ν_{n_k} converges weakly to ν.

As a variant of Vershik's Theorem (see Theorem 6.3 below), we obtain the following inclusion:

$$\mathcal{P}_{\mathrm{erg}}(\mathrm{Sym}(\mathbb{N}, F)) \subset \mathscr{ORB}_\infty(\mathrm{Sym}(\mathbb{N}, F)).$$

Note that, *a priori*, we do not know whether the inverse inclusion holds.

(2) *Main ingredients: Classification of* $\mathscr{ORB}_\infty(\mathrm{Sym}(\mathbb{N}, F))$.

(i) *Computation of orbital integrals.* To describe $\mathscr{ORB}_\infty(\mathrm{Sym}(\mathbb{N}, F))$, we need to understand the asymptotic behaviour of the characteristic functions of orbital measures of the compact groups $\mathrm{GL}(n, \mathcal{O}_F)$. Recalling the assumption on the character $\chi \in \widehat{F}$ in (6), we obtain an asymptotic formula for the following orbital integral:

$$\int_{\mathrm{GL}(n,\mathcal{O}_F)} \chi(\mathrm{tr}(g \cdot \mathrm{diag}(x_1, \ldots, x_n) \cdot g^t \cdot \mathrm{diag}(a_1, \ldots, a_r, 0, \ldots))) \, dg, \tag{8}$$

where dg is the normalized Haar measure of $\mathrm{GL}(n, \mathcal{O}_F)$. The formula we obtain for the integral (8) is *uniformly asymptotically multiplicative*, that is, the orbital integral (8) has the same asymptotic behaviour, uniformly on the choices of $x_1, \ldots, x_n$, as the following product of much simpler orbital integrals:

$$\prod_{j=1}^{r} \int_{\mathrm{GL}(n,\mathcal{O}_F)} \chi(\mathrm{tr}(g \cdot \mathrm{diag}(x_1, \ldots, x_n) \cdot g^t \cdot \mathrm{diag}(a_j, 0, 0, \ldots))) \, dg.$$

See Theorem 7.6 for the details.

Explicit computation of the above orbital integral requires some Fourier analysis on the field F and quite a few combinatorial arguments in which we compute the cardinality of various sets of matrices over the finite field $\mathbf{F}_q$.

(ii) *Multiplicativity of characteristic functions for limits of orbital measures.* An immediate consequence of the uniform asymptotic multiplicativity for the orbital integral (8) is that for any $\nu \in \mathscr{ORB}_\infty(\mathrm{Sym}(\mathbb{N}, F))$, its characteristic function $\widehat{\nu}$ possesses an exact multiplicativity property, that is, for any $r \in \mathbb{N}$ and any $x_1, \ldots, x_r \in F$,

$$\widehat{\nu}(\mathrm{diag}(x_1, \ldots, x_r, 0, 0, \ldots)) = \prod_{j=1}^{r} \widehat{\nu}(x_j e_{11}), \tag{9}$$

where e_{11} is the elementary matrix whose $(1, 1)$-coefficient is 1. This multiplicativity result implies in particular that the classification of the class $\mathscr{ORB}_\infty(\mathrm{Sym}(\mathbb{N}, F))$ is reduced to the classification of the class of functions on F defined by $x \mapsto \widehat{\nu}(xe_{11})$.

(3) *Ergodicity: Proof of the inclusion:*

$$\mathscr{ORB}_\infty(\mathrm{Sym}(\mathbb{N}, F)) \subset \mathcal{P}_{\mathrm{erg}}(\mathrm{Sym}(\mathbb{N}, F)).$$

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Our direct proof of ergodicity for all measures in $\mathscr{ORB}_\infty(\mathrm{Sym}(\mathbb{N}, F))$ uses an argument of Okounkov and Olshanski [OO98]: ergodicity for measures in $\mathscr{ORB}_\infty(\mathrm{Sym}(\mathbb{N}, F))$ is derived from the De Finetti Theorem, see Theorem 3.10 below for the details. This approach of proving ergodicity can be also applied to different situations, such as that of Olshanski and Vershik in [OV96].

(4) *Proof of the equality*

$$\mathcal{P}_{\mathrm{erg}}(\mathrm{Sym}(\mathbb{N}, F)) = \{\nu_{\mathbb{h}} : h \in \Omega\}. \tag{10}$$

By comparing the characteristic functions $\widehat{\nu_{\mathbb{h}}}$ for all the measures $\nu_{\mathbb{h}}$ with $\mathbb{h} \in \Omega$ and that of measures in $\mathscr{ORB}_\infty(\mathrm{Sym}(\mathbb{N}, F))$, we obtain the equality

$$\mathscr{ORB}_\infty(\mathrm{Sym}(\mathbb{N}, F)) = \{\nu_{\mathbb{h}} : \mathbb{h} \in \Omega\}.$$

Combining this equality with the results obtained in the previous steps, we finally get the desired equality (10).

1.6 Organization of the paper

The exposition, which we tried to make essentially self-contained, is organized as follows.

In § 2, we recall the definition of ergodic measures and the necessary definitions related to non-discrete locally compact non-Archimedean fields, linear groups over them and Fourier transforms in this setting.

In § 3, we prove that all the measures on $\mathrm{Mat}(\mathbb{N}, F)$ from the family $\{\mu_{\mathbb{k}} \mid \mathbb{k} \in \Delta\}$ introduced in Definition 1.1 are $\mathrm{GL}(\infty, \mathcal{O}_F) \times \mathrm{GL}(\infty, \mathcal{O}_F)$-invariant and ergodic and that all the measures on $\mathrm{Sym}(\mathbb{N}, F)$ from the family $\{\nu_{\mathbb{h}} \mid \mathbb{h} \in \Omega\}$ introduced in Definition 1.5 are $\mathrm{GL}(\infty, \mathcal{O}_F)$-invariant and ergodic.

In § 4, we give explicit formulae for characteristic functions of measures from the two families $\{\mu_{\mathbb{k}} : \mathbb{k} \in \Delta\}$ and $\{\nu_{\mathbb{h}} : \mathbb{h} \in \Omega\}$.

In § 5, we prove that the parametrization maps $\mathbb{k} \mapsto \mu_{\mathbb{k}}$ from Δ to $\mathcal{P}_{\mathrm{erg}}(\mathrm{Mat}(\mathbb{N}, F))$ and $\mathbb{h} \mapsto \nu_{\mathbb{h}}$ from Ω to $\mathcal{P}_{\mathrm{erg}}(\mathrm{Sym}(\mathbb{N}, F))$ are injective.

In § 6, we introduce orbital measures and recall the Vershik–Kerov ergodic method for dealing with ergodic measures for inductively compact groups.

In § 7, we obtain the asymptotic formula for orbital integrals of the type (8).

In § 8, we complete the classifications by proving that the parametrization maps $\mathbb{k} \mapsto \mu_{\mathbb{k}}$ and $\mathbb{h} \mapsto \nu_{\mathbb{h}}$ are surjective.

In § 9, we show that the parametrization maps $\mathbb{k} \mapsto \mu_{\mathbb{k}}$ and $\mathbb{h} \mapsto \nu_{\mathbb{h}}$ are homeomorphisms between corresponding topological spaces.

Proofs of some routine technical lemmata are given in Appendix A.

2. Preliminaries

2.1 Ergodic measures

2.1.1 *Ergodicity.* Let $\mathcal{X}$ be a Polish space, that is, it is homeomorphic to a complete metric space that has a countable dense subset. Denote by $\mathcal{P}(\mathcal{X})$ the set of Borel probability measures on $\mathcal{X}$. Denote by $C_b(\mathcal{X})$ the space of bounded continuous complex-valued functions on $\mathcal{X}$. Recall that a sequence $(\mu_n)_{n\in\mathbb{N}}$ in $\mathcal{P}(\mathcal{X})$ is said to converge weakly to $\mu \in \mathcal{P}(\mathcal{X})$ and is denoted by $\mu_n \Longrightarrow \mu$ if for any $f \in C_b(\mathcal{X})$, we have

$$\lim_{n\to\infty} \int_{\mathcal{X}} f(x)\, \mu_n(dx) = \int_{\mathcal{X}} f(x)\, \mu(dx).$$

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Given a group action of a group G on $\mathcal{X}$, we denote by $\mathcal{P}^G_{\mathrm{inv}}(\mathcal{X})$ the set of G-invariant Borel probability measures on $\mathcal{X}$. By definition, a G-invariant Borel probability measure $\mu \in \mathcal{P}^G_{\mathrm{inv}}(\mathcal{X})$ is ergodic, if for any G-invariant Borel subset $\mathcal{A} \subset \mathcal{X}$, either $\mu(\mathcal{A}) = 0$ or $\mu(\mathcal{X} \setminus \mathcal{A}) = 0$. The totality of ergodic G-invariant probability measures on $\mathcal{X}$ is denoted by $\mathcal{P}^G_{\mathrm{erg}}(\mathcal{X})$. If the group action is clear from the context, we denote $\mathcal{P}^G_{\mathrm{inv}}(\mathcal{X})$ and $\mathcal{P}^G_{\mathrm{erg}}(\mathcal{X})$ simply by $\mathcal{P}_{\mathrm{inv}}(\mathcal{X})$ and $\mathcal{P}_{\mathrm{erg}}(\mathcal{X})$ respectively.

2.1.2 *Indecomposability and ergodicity.* Using the notation as before, a measure $\mu \in \mathcal{P}^G_{\mathrm{inv}}(\mathcal{X})$ is called *indecomposable* in $\mathcal{P}^G_{\mathrm{inv}}(\mathcal{X})$ if the equality $\mu = \alpha\mu_1 + (1-\alpha)\mu_2$ with $\alpha \in (0,1)$, μ_1, $\mu_2 \in \mathcal{P}^G_{\mathrm{inv}}(\mathcal{X})$, implies $\mu = \mu_1 = \mu_2$. Recall that a Borel subset $\mathcal{A} \subset \mathcal{X}$ is called *almost G-invariant* with respect to a measure $\mu \in \mathcal{P}^G_{\mathrm{inv}}(\mathcal{X})$ if for every $g \in G$, we have $\mu(\mathcal{A}\Delta g.\mathcal{A}) = 0$, where $g.\mathcal{A} = \{g.x : x \in \mathcal{A}\}$ and $\mathcal{A}\Delta g.\mathcal{A} = (\mathcal{A} \setminus g.\mathcal{A}) \cup (g.\mathcal{A} \setminus \mathcal{A})$.

A Borel probability measure $\mu \in \mathcal{P}^G_{\mathrm{inv}}(\mathcal{X})$ is indecomposable in $\mathcal{P}^G_{\mathrm{inv}}(\mathcal{X})$ if and only if any Borel subset $\mathcal{A} \subset \mathcal{X}$ which is almost G-invariant with respect to the measure μ satisfies either $\mu(\mathcal{A}) = 0$ or $\mu(\mathcal{X} \setminus \mathcal{A}) = 0$ (see [Bog69, Fom50] and [Buf14, Proposition 1]).

Indecomposable Borel probability measures in $\mathcal{P}^G_{\mathrm{inv}}(\mathcal{X})$ are *a fortiori* ergodic. For actions of general groups, ergodic probability measures may fail to be indecomposable, see [Buf14, § 5] for a counterexample of Kolmogorov. Nevertheless, it is proved in [Buf14, Proposition 2] that for action of inductively compact groups, and thus including the actions in our main Theorems 1.3 and 1.8, etc, the notions of indecomposability and ergodicity coincide.

Note that for the action of countable groups, the notions of indecomposability and ergodicity coincide. Indeed, if G is a countable group and $\mu \in \mathcal{P}^G_{\mathrm{erg}}(\mathcal{X})$, then for any G-almost invariant Borel subset $\mathcal{A} \subset \mathcal{X}$ with respect to the measure μ, the Borel subset $\bigcap_{g\in G} g.\mathcal{A}$ is G-invariant and hence, by the ergodicity of μ, we have $\mu(\bigcap_{g\in G} g.\mathcal{A}) \in \{0,1\}$. Since $\mathcal{A}$ is G-almost invariant with respect to μ and since G is countable, we also have $\mu(\mathcal{A}\Delta \bigcap_{g\in G} g.\mathcal{A}) = 0$. Therefore, $\mu(\mathcal{A}) \in \{0,1\}$. This proves that μ is indecomposable.

2.2 Fields and integers

Let F be a non-discrete locally compact non-Archimedean field. The classification of local fields (see, e.g., Ramakrishnan and Valenza's book [RV99, Theorem 4-12]) implies that F is isomorphic to one of the following fields:

- a finite extension of the field $\mathbb{Q}_p$ of p-adic numbers for some prime p;
- the field of formal Laurent series over a finite field.

Let $|\cdot|$ be the absolute value on F and denote by $d(\cdot,\cdot)$ the ultrametric on F defined by $d(x,y) = |x-y|$. The ring of integers in F is given by

$$\mathcal{O}_F := \{x \in F : |x| \leqslant 1\}.$$

The subset $\mathfrak{m} := \{x \in F : |x| < 1\}$ is the unique maximal ideal of $\mathcal{O}_F$. The ideal $\mathfrak{m}$ is principal. Any generator of $\mathfrak{m}$ is called a *uniformizer* of F. Throughout the paper, we fix any uniformizer ϖ of F, that is, $\mathfrak{m} = \varpi\mathcal{O}_F$. The field $\mathcal{O}_F/\varpi\mathcal{O}_F$ is finite with $q = p^f$ elements for a prime number p and a positive integer $f \in \mathbb{N}$. If $q = 2^f$, then we say that F is *dyadic*, otherwise, we say that F is *non-dyadic*.

We write $\mathbf{F}_q := \mathcal{O}_F/\varpi\mathcal{O}_F$. The quotient map is denoted by

$$\pi : \mathcal{O}_F \to \mathbf{F}_q = \mathcal{O}_F/\varpi\mathcal{O}_F.$$

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Fix a *complete set of representatives* $\mathcal{C}_q \subset \mathcal{O}_F$ of cosets of $\varpi\mathcal{O}_F$ in $\mathcal{O}_F$ and assume that $0 \in \mathcal{C}_q$. The restriction of the quotient map π on the finite set $\mathcal{C}_q$ is a bijection:

$$\pi : \mathcal{C}_q \xrightarrow{\text{bijection}} \mathbf{F}_q. \tag{11}$$

Any element of F is uniquely expanded as a convergent series in F:

$$x = \sum_{n=v}^{\infty} a_n \varpi^n \quad (v \in \mathbf{Z}, a_n \in \mathcal{C}_q, a_v \neq 0). \tag{12}$$

If $x \in F$ is given by the series (12), then we define the *F-valuation* of x by $\mathrm{ord}_F(x) := v$. By convention, we set $\mathrm{ord}_F(0) = \infty$. The absolute value and the F-valuation of any element $x \in F$ are related by the formula $|x| = q^{-\mathrm{ord}_F(x)}$.

2.3 Group actions

Let $\mathrm{GL}(n, F)$ and $\mathrm{GL}(n, \mathcal{O}_F)$ denote the groups of invertible $n \times n$ matrices over F and $\mathcal{O}_F$ respectively. The group $\mathrm{GL}(n, \mathcal{O}_F)$ is embedded naturally into $\mathrm{GL}(n+1, \mathcal{O}_F)$ by

$$a \in \mathrm{GL}(n, \mathcal{O}_F) \mapsto \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}(n+1, \mathcal{O}_F). \tag{13}$$

Define an inductive limit group

$$\mathrm{GL}(\infty, \mathcal{O}_F) := \varinjlim \mathrm{GL}(n, \mathcal{O}_F).$$

Equivalently, $\mathrm{GL}(\infty, \mathcal{O}_F)$ is the group of infinite invertible matrices $g = (g_{ij})_{i,j \in \mathbb{N}}$ over $\mathcal{O}_F$ such that $g_{ij} = \delta_{ij}$ if $i + j$ is large enough.

Let $\mathrm{Mat}(n, F)$ and $\mathrm{Mat}(n, \mathcal{O}_F)$ denote the spaces of all $n \times n$ matrices over F and $\mathcal{O}_F$ respectively. Define

$$\mathrm{Mat}(\mathbb{N}, F) := \{X = (X_{ij})_{i,j \in \mathbb{N}} \mid X_{ij} \in F\}.$$

Let $\mathrm{Mat}(\infty, F)$ denote the subspace of $\mathrm{Mat}(\mathbb{N}, F)$ consisting of matrices whose all but a finite number of coefficients are zeros. Define also

$$\mathrm{Sym}(n, F) := \{X \in \mathrm{Mat}(n, F) \mid X_{ij} = X_{ji}, \forall 1 \leqslant i, j \leqslant n\},$$
$$\mathrm{Sym}(\mathbb{N}, F) := \{X \in \mathrm{Mat}(\mathbb{N}, F) \mid X_{ij} = X_{ji}, \forall i, j \in \mathbb{N}\}.$$

Set $\mathrm{Sym}(\infty, F) := \mathrm{Sym}(\mathbb{N}, F) \cap \mathrm{Mat}(\infty, F)$.

Two natural group actions under consideration in this paper are:

– the group action of $\mathrm{GL}(\infty, \mathcal{O}_F) \times \mathrm{GL}(\infty, \mathcal{O}_F)$ on $\mathrm{Mat}(\mathbb{N}, F)$ defined by

$$((g_1, g_2), M) \mapsto g_1 M g_2^{-1}, \quad g_1, g_2 \in \mathrm{GL}(\infty, \mathcal{O}_F), \quad M \in \mathrm{Mat}(\mathbb{N}, F);$$

– the group action of $\mathrm{GL}(\infty, \mathcal{O}_F)$ on $\mathrm{Sym}(\mathbb{N}, F)$ defined by

$$(g, M) \mapsto g M g^t, \quad g \in \mathrm{GL}(\infty, \mathcal{O}_F), \quad S \in \mathrm{Sym}(\mathbb{N}, F),$$

where g^t is the transposition of g.

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2.4 Conventions

Given a finite set B, we denote by $\#B$ its cardinality.

Let $(\Sigma, \mathcal{B}, m)$ be a measurable space equipped with a positive measure m and let f be a real or complex valued integrable function defined on Σ. If $A \subset \Sigma$ is measurable and $0 < m(A) < \infty$, then we write

$$\fint_A f(x)\, dm(x) := \frac{1}{m(A)} \int_A f(x)\, dm(x). \tag{14}$$

For any random variable Y, we denote its distribution by $\mathcal{L}(Y)$.

We establish conventions concerning the empty set $\emptyset$. Let $(r_i)_{i\in I}$ be a family of real numbers (or complex numbers for the last two formulae). We set

$$\inf_{i\in\emptyset} r_i = +\infty, \quad \sup_{i\in\emptyset} r_i = -\infty, \quad \sum_{i\in\emptyset} r_i = 0, \quad \prod_{i\in\emptyset} r_i = 1.$$

The following conventions will also be used.

- As elements in F, $\varpi^{\infty} = \varpi^{+\infty} = 0 \in F$.
- As elements in $\mathbb{R} \cup \{+\infty\}$, $q^{\infty} = q^{+\infty} = +\infty$ and $q^{-\infty} = 0 \in \mathbb{R}$.

2.5 Haar measure on $\mathrm{GL}(n, \mathcal{O}_F)$

For any $n \in \mathbb{N}$, denote by $d\mathrm{vol}_n$ the Haar measure on F^n normalized by the condition $\mathrm{vol}_n(\mathcal{O}_F^n) = 1$. If there is no confusion, we will use the simplified notation $\mathrm{vol}(\cdot)$ for $\mathrm{vol}_n(\cdot)$.

Remark 2.1. The Haar measure vol_n on F^n is preserved by any linear map represented by a matrix from the group $\mathrm{GL}(n, \mathcal{O}_F)$.

For any n, we fix a Haar measure $\mathrm{vol}(\cdot)$ on $\mathrm{Mat}(n, F)$ normalized by $\mathrm{vol}(\mathrm{Mat}(n, \mathcal{O}_F)) = 1$. Up to a multiplicative constant, the Haar measure on the locally compact group $\mathrm{GL}(n, F)$ is uniquely given (see, e.g., Neretin [Ner13]) by

$$|\det(M)|^{-n} \cdot \mathrm{vol}(dM). \tag{15}$$

Let $\mathrm{GL}(n, \mathbf{F}_q)$ be the group of invertible $n \times n$ matrices over $\mathbf{F}_q$. Set

$$\mathrm{GL}(n, \mathcal{C}_q) := \{t = (t_{ij})_{1\leqslant i,j\leqslant n} \in \mathrm{GL}(n, \mathcal{O}_F) : t_{ij} \in \mathcal{C}_q\}.$$

Proposition 2.2. *A standard partition of* $\mathrm{GL}(n, \mathcal{O}_F)$ *is given by*

$$\mathrm{GL}(n, \mathcal{O}_F) = \bigsqcup_{t\in \mathrm{GL}(n, \mathcal{C}_q)} (t + \mathrm{Mat}(n, \varpi\mathcal{O}_F)). \tag{16}$$

In particular, we have

$$\mathrm{vol}(\mathrm{GL}(n, \mathcal{O}_F)) = \prod_{j=1}^{n} (1 - q^{-j}). \tag{17}$$

Proof. By definition, $a = (a_{ij})_{1\leqslant i,j\leqslant n} \in \mathrm{GL}(n, \mathcal{O}_F)$ implies that $a_{ij} \in \mathcal{O}_F$ and $|\det(a)| = 1$. Now take any $x \in \mathrm{Mat}(n, \varpi\mathcal{O}_F)$. First, we have $a + x \in \mathrm{Mat}(n, \mathcal{O}_F)$. Second, write $x = \varpi y$ with $y \in \mathrm{Mat}(n, \mathcal{O}_F)$. By definition, there exists $z \in \mathcal{O}_F$, such that $\det(a + x) = \det(a) + \varpi z$. Since $|\varpi z| \leqslant q^{-1}$, by ultrametricity, we obtain $|\det(a + x)| = |\det(a) + \varpi z| = 1$, whence $a + x \in \mathrm{GL}(n, \mathcal{O}_F)$ and the set on the right-hand side of (16) is contained in $\mathrm{GL}(n, \mathcal{O}_F)$. Conversely, since $\mathcal{C}_q$

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is a complete set of representatives of the cosets of $\varpi\mathcal{O}_F$ in $\mathcal{O}_F$, for any $A \in \mathrm{GL}(n, \mathcal{O}_F)$, there exists a unique $t \in \mathrm{GL}(n, \mathcal{C}_q)$, such that $A \equiv t (\mathrm{mod}\ \varpi\mathcal{O}_F)$. This completes the proof of (16).

Recalling that

$$\#\mathrm{GL}(n, \mathcal{C}_q) = \#\mathrm{GL}(n, \mathbf{F}_q) = \prod_{j=0}^{n-1} (q^n - q^j),$$

we arrive at (17). □

By Proposition 2.2, $\mathrm{GL}(n, \mathcal{O}_F)$ is an open subgroup of $\mathrm{GL}(n, F)$. It follows that the restriction on $\mathrm{GL}(n, \mathcal{O}_F)$ of the Haar measure (15) on $\mathrm{GL}(n, F)$ is a Haar measure on $\mathrm{GL}(n, \mathcal{O}_F)$. Consequently, the normalized Haar measure on $\mathrm{GL}(n, \mathcal{O}_F)$ is given by

$$\frac{\mathrm{vol}(\cdot)}{\prod_{j=1}^{n}(1 - q^{-j})}. \tag{18}$$

Let $T(n)$ be sampled uniformly from the finite set $\mathrm{GL}(n, \mathcal{C}_q)$; let $V(n)$ be sampled with respect to the normalized Haar measure on $\mathrm{Mat}(n, \varpi\mathcal{O}_F)$ and independent of $T(n)$.

Proposition 2.3. *The random matrix $T(n) + V(n)$ is a Haar random matrix on $\mathrm{GL}(n, \mathcal{O}_F)$; that is, the distribution law $\mathcal{L}(T(n) + V(n))$ coincides with the normalized Haar measure on $\mathrm{GL}(n, \mathcal{O}_F)$.*

Proof. The proof follows immediately from Proposition 2.2 and the formula (18) for the normalized Haar measure on $\mathrm{GL}(n, \mathcal{O}_F)$. □

2.6 Diagonalization in Mat(n, F)

Lemma 2.4 (See, e.g., Neretin [Ner13, § 1.3]). *Every matrix $A \in \mathrm{Mat}(n, F)$ can be written in the form*

$$A = a \cdot \mathrm{diag}(\varpi^{-k_1}, \varpi^{-k_2}, \ldots, \varpi^{-k_n}) \cdot b, \quad (a, b \in \mathrm{GL}(n, \mathcal{O}_F)), \tag{19}$$

where $k_1 \geqslant k_2 \geqslant \cdots \geqslant k_n \geqslant -\infty$ and $\mathrm{diag}(\varpi^{-k_1}, \varpi^{-k_2}, \ldots, \varpi^{-k_n})$ is the diagonal matrix with diagonal coefficients $\varpi^{-k_1}, \varpi^{-k_2}, \ldots, \varpi^{-k_n}$. Moreover, the n-tuple $(k_1, k_2, \ldots, k_n)$ is uniquely determined by the matrix A.

2.7 Square-units in F

The group of units of the ring $\mathcal{O}_F$ is given by $\mathcal{O}_F^\times := \{x \in F : |x| = 1\}$. Let $(\mathcal{O}_F^\times)^2$ be the subgroup of $\mathcal{O}_F^\times$ defined by

$$(\mathcal{O}_F^\times)^2 := \{x \in \mathcal{O}_F^\times : \text{there exists } a \in F \text{ such that } x = a^2\}.$$

Let $\mathcal{C}_q^\times$ denote the set $\mathcal{C}_q \setminus \{0\}$ and define

$$(\mathcal{C}_q^\times)^2 := \{a \in \mathcal{C}_q^\times \mid \text{there exists } y \in F \text{ such that } a = y^2\}.$$

Write

$$(\mathbf{F}_q^\times)^2 := \{a \in \mathbf{F}_q^\times \mid \text{there exists } c \in \mathbf{F}_q^\times \text{ such that } a = c^2\}.$$

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Lemma 2.5. *Assume that F is non-dyadic. Then*

$$(\mathcal{O}_F^\times)^2 = \bigsqcup_{a\in(\mathcal{C}_q^\times)^2} (a+\varpi\mathcal{O}_F). \tag{20}$$

Consequently, the map π in (11) induces a bijection:

$$\pi : (\mathcal{C}_q^\times)^2 \xrightarrow{\text{bijection}} (\mathbf{F}_q^\times)^2. \tag{21}$$

Proof. Since F is non-dyadic, we have $|2| = 1$. For any $x = \alpha^2 \in (\mathcal{O}_F^\times)^2$, since $\mathcal{C}_q$ is a complete set of representatives for $\mathcal{O}_F/\varpi\mathcal{O}_F$, there exists $a \in \mathcal{C}_q^\times$, such that $x \equiv a (\mathrm{mod}\ \varpi\mathcal{O}_F)$, that is,

$$|x-a| < 1. \tag{22}$$

Take any $b \in \mathcal{O}_F^\times$ such that $|b-x| < 1 = |2\alpha|^2$. Then the polynomial $P_b(X) = X^2 - b \in \mathcal{O}_F[X]$ satisfies

$$|P_b(\alpha)| < |P_b'(\alpha)|^2. \tag{23}$$

By Hensel's lemma (see Cassels [Cas86, pp. 49–51]), the inequality (23) implies that there exists $\beta \in F$ such that

$$P_b(\beta) = \beta^2 - b = 0 \quad \text{and} \quad |\beta-\alpha| \leqslant \frac{|P_b(\alpha)|}{|P_b'(\alpha)|} < |P_b'(\alpha)| = 1. \tag{24}$$

In particular, we have

$$\{b \in \mathcal{O}_F^\times : |b-x| < 1\} = x + \varpi\mathcal{O}_F \subset (\mathcal{O}_F^\times)^2. \tag{25}$$

Combining (22) and (25), we get $a \in (\mathcal{O}_F^\times)^2$. Hence

$$(\mathcal{O}_F^\times)^2 \subset \bigsqcup_{a\in(\mathcal{C}_q^\times)^2} (a+\varpi\mathcal{O}_F).$$

Conversely, since $(\mathcal{C}_q^\times)^2 \subset (\mathcal{O}_F^\times)^2$, for any $a \in (\mathcal{C}_q^\times)^2$, replacing x by a in the above argument, the inclusion (25) implies $a + \varpi\mathcal{O}_F \subset (\mathcal{O}_F^\times)^2$. Hence

$$\bigsqcup_{a\in(\mathcal{C}_q^\times)^2} (a+\varpi\mathcal{O}_F) \subset (\mathcal{O}_F^\times)^2. \qquad \square$$

Remark 2.6. Recall that if F is non-dyadic, then the quotient group $\mathcal{O}_F^\times/(\mathcal{O}_F^\times)^2$ has two elements.

2.8 Diagonalization in $\mathrm{Sym}(n, F)$

In what follows, we fix a non-square unit $\varepsilon \in \mathcal{O}_F^\times \setminus (\mathcal{O}_F^\times)^2$. By Remark 2.6, the following set

$$\mathcal{T} := \{\varpi^{-k} \mid k \in \mathbb{Z}\} \sqcup \{\varpi^{-k}\varepsilon \mid k \in \mathbb{Z}\} \sqcup \{0\} \tag{26}$$

is a complete set of representatives for the quotient $F/(\mathcal{O}_F^\times)^2$.

Lemma 2.7. *Assume that F is non-dyadic. Then any symmetric matrix $A \in \mathrm{Sym}(n, F)$ can be written in the form*

$$A = g \cdot \mathrm{diag}(x_1, \dots, x_n) \cdot g^t, \quad x_1, \dots, x_n \in \mathcal{T}, g \in \mathrm{GL}(n, \mathcal{O}_F).$$

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Proof. Since $\mathcal{T}$ is a complete set of representatives for the quotient $F/(\mathcal{O}_F^\times)^2$, it suffices to show that any symmetric matrix $A \in \mathrm{Sym}(n, F)$ is diagonalizable. Assume that A is not a zero matrix. We claim that, up to passing A to gAg^t for some $g \in \mathrm{GL}(n, \mathcal{O}_F)$, we may assume that the $(1,1)$-coefficient of A has maximal absolute value.

Case 1: Assume first that there exists $1 \leqslant i_0 \leqslant n$, such that $|A_{i_0 i_0}| = \max_{1 \leqslant i,j \leqslant n} |A_{ij}|$. If $i_0 = 1$, then there is nothing to prove. Otherwise, take $g = M_{(1 i_0)}$, where $M_{(1 i_0)}$ is the permutation matrix associated with the transposition $(1 i_0)$. Then the matrix gAg^t has $A_{i_0 i_0}$ as its $(1,1)$-coefficient.

Case 2: Now assume that there exists $i_0 < j_0$ such that

$$|A_{i_0 j_0}| = \max_{1 \leqslant i,j \leqslant n} |A_{ij}| > \max_{1 \leqslant i \leqslant n} |A_{ii}|. \tag{27}$$

Let us do operations on the submatrix indexed by $\{i_0, j_0\} \times \{i_0, j_0\}$ as follows:

$$\begin{bmatrix} 2A_{i_0 j_0} + A_{i_0 i_0} + A_{j_0 j_0} & A_{i_0 j_0} + A_{j_0 j_0} \\ A_{i_0 j_0} + A_{j_0 j_0} & A_{j_0 j_0} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} A_{i_0 i_0} & A_{i_0 j_0} \\ A_{i_0 j_0} & A_{j_0 j_0} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

Since F is non-dyadic and non-Archimedean, (27) implies

$$|2A_{i_0 j_0} + A_{i_0 i_0} + A_{j_0 j_0}| = |A_{i_0 j_0} + A_{j_0 j_0}| = |A_{i_0 j_0}|.$$

This shows that we can reduce the second case to the first case where a diagonal coefficient has maximal absolute value.

Now assume that

$$A = \begin{bmatrix} x & c^t \\ c & A_1 \end{bmatrix}, \quad c \in F^{n-1},$$

such that x attains the maximal absolute value of all coefficients of A. Then $x^{-1}c \in \mathcal{O}_F^{n-1}$ and we have

$$\begin{bmatrix} 1 & 0 \\ -x^{-1}c & 1 \end{bmatrix} \begin{bmatrix} x & c^t \\ c & A_1 \end{bmatrix} \begin{bmatrix} 1 & -x^{-1}c^t \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} x & 0 \\ 0 & A_1 - x^{-1}cc^t \end{bmatrix}.$$

By continuing the above procedure on the submatrix $A_1 - x^{-1}cc^t$, we prove finally that A is diagonalizable. □

Remark 2.8. The assumption that F is non-dyadic is necessary in Lemma 2.7. Indeed, if

$$\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix}, \quad \begin{bmatrix} a & c \\ b & d \end{bmatrix} \in \mathrm{GL}(2, \mathcal{O}_F),$$

then

$$bc + ad = 0, \quad ad - bc \in \mathcal{O}_F^\times.$$

It follows that $2ad \in \mathcal{O}_F^\times$ and hence $2 \in \mathcal{O}_F^\times$. This implies that F is non-dyadic.

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2.9 Characteristic functions

Denote by $\widehat{F}$ the Pontryagin dual of the additive group F. Elements in $\widehat{F}$ are called characters of F. Throughout the paper, we fix a non-trivial character $\chi \in \widehat{F}$ such that

$$\chi|_{\mathcal{O}_F} \equiv 1 \quad \text{and} \quad \chi \text{ is not constant on } \varpi^{-1}\mathcal{O}_F. \tag{28}$$

For any $y \in F$, define a character $\chi_y \in \widehat{F}$ by $\chi_y(x) = \chi(yx)$. The map $y \mapsto \chi_y$ from F to $\widehat{F}$ defines a group isomorphism.

We write explicitly characteristic functions of probability measures in the following situations.

(i) If μ is a Borel probability measure on F^m, then $\widehat{\mu}$ is defined on F^m by

$$\widehat{\mu}(y) := \int_{F^m} \chi(x \cdot y)\mu\,(dx),$$

where $x \cdot y := \sum_{j=1}^m x_j y_j$.

(ii) If μ is a Borel probability measure on $\mathrm{Mat}(n, F)$, then $\widehat{\mu}$ is defined on $\mathrm{Mat}(n, F)$ by

$$\widehat{\mu}(A) := \int_{\mathrm{Mat}(n,F)} \chi(\mathrm{tr}(AM))\mu\,(dM).$$

(iii) If μ is a Borel probability measure on $\mathrm{Mat}(\mathbb{N}, F)$, then $\widehat{\mu}$ is defined on $\mathrm{Mat}(\infty, F)$ by

$$\widehat{\mu}(A) := \int_{\mathrm{Mat}(\mathbb{N},F)} \chi(\mathrm{tr}(AM))\mu\,(dM). \tag{29}$$

(iv) If ν is a Borel probability measure on $\mathrm{Sym}(n, F)$, then $\widehat{\nu}$ is defined on $\mathrm{Sym}(n, F)$ by

$$\widehat{\nu}(A) := \int_{\mathrm{Sym}(n,F)} \chi(\mathrm{tr}(AS))\nu(dS).$$

(v) If ν is a Borel probability measure on $\mathrm{Sym}(\mathbb{N}, F)$, then $\widehat{\nu}$ is defined on $\mathrm{Sym}(\infty, F)$ by

$$\widehat{\nu}(A) := \int_{\mathrm{Sym}(\mathbb{N},F)} \chi(\mathrm{tr}(AS))\nu(dS).$$

Since the corresponding groups are locally compact, Theorem 31.5 in Hewitt and Ross [HR70, p. 212] implies that in cases (i), (ii) and (iv), the characteristic function $\widehat{\mu}$ determines μ uniquely. The same statement holds for cases (iii) and (v). Indeed, although the additive groups $\mathrm{Mat}(\mathbb{N}, F)$ and $\mathrm{Mat}(\infty, F)$ are not locally compact and we can not apply the result on locally compact groups directly, we may use the fact that any Borel probability measure μ on $\mathrm{Mat}(\mathbb{N}, F)$ is uniquely determined by its finite dimensional projections $(\mathrm{Cut}_n^\infty)_*(\mu)$ and (29) contains all information for $\widehat{(\mathrm{Cut}_n^\infty)_*(\mu)}$, $n = 1, 2, \ldots$. Case (v) is treated similarly.

Remark 2.9. If μ is a probability measure on $\mathrm{Mat}(n, F)$ which is invariant under the action of the group $\mathrm{GL}(n, \mathcal{O}_F) \times \mathrm{GL}(n, \mathcal{O}_F)$, then for any $a, b \in \mathrm{GL}(n, \mathcal{O}_F)$, we have

$$\widehat{\mu}(a \cdot \mathrm{diag}(\varpi^{-k_1}, \varpi^{-k_2}, \ldots, \varpi^{-k_n}) \cdot b) = \widehat{\mu}(\mathrm{diag}(\varpi^{-k_1}, \varpi^{-k_2}, \ldots, \varpi^{-k_n})). \tag{30}$$

Similarly, if ν is a $\mathrm{GL}(n, \mathcal{O}_F)$-invariant probability measure on $\mathrm{Sym}(n, F)$, then for any $g \in \mathrm{GL}(n, \mathcal{O}_F)$, we have

$$\widehat{\nu}(g \cdot \mathrm{diag}(x_1, \ldots, x_n) \cdot g^t) = \widehat{\nu}(\mathrm{diag}(x_1, \ldots, x_n)). \tag{31}$$

Similar statements hold for $\mathrm{GL}(\infty, \mathcal{O}_F) \times \mathrm{GL}(\infty, \mathcal{O}_F)$-invariant probability measures on $\mathrm{Mat}(\mathbb{N}, F)$ and for $\mathrm{GL}(\infty, \mathcal{O}_F)$-invariant probability measures on $\mathrm{Sym}(\mathbb{N}, F)$.

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Let $m \in \mathbb{N}$. Given any Borel probability measures $\mu_1, \dots, \mu_m$ on $\mathrm{Mat}(\mathbb{N}, F)$ (respectively $\mathrm{Sym}(\mathbb{N}, F)$), their convolution $\mu_1 * \cdots * \mu_m$ is defined as follows: let $M_1, \dots, M_m$ be independent random matrices such that $\mathcal{L}(M_i) = \mu_i, i = 1, \dots, m$ and set

$$\mu_1 * \cdots * \mu_m := \mathcal{L}(M_1 + \cdots + M_m).$$

The characteristic function of $\mu_1 * \cdots * \mu_m$ is given by the formula

$$(\mu_1 * \cdots * \mu_m)^\wedge = \prod_{i=1}^{m} \widehat{\mu_i}. \tag{32}$$

3. Invariance and ergodicity

In this section, we prove that all the measures on $\mathrm{Mat}(\mathbb{N}, F)$ from the family $\{\mu_{\Bbbk} = \mathcal{L}(M_{\Bbbk}) \mid \Bbbk \in \Delta\}$ introduced in Definition 1.1 are $\mathrm{GL}(\infty, \mathcal{O}_F) \times \mathrm{GL}(\infty, \mathcal{O}_F)$-invariant and ergodic and that all the measures on $\mathrm{Sym}(\mathbb{N}, F)$ from the family $\{\nu_{\Bbbh} = \mathcal{L}(S_{\Bbbh}) \mid \Bbbh \in \Omega\}$ introduced in Definition 1.5 are $\mathrm{GL}(\infty, \mathcal{O}_F)$-invariant and ergodic.

3.1 $\mathrm{GL}(\infty, \mathcal{O}_F) \times \mathrm{GL}(\infty, \mathcal{O}_F)$-invariance for probability measures $\mu_{\Bbbk}$

Proposition 3.1. *For any $\Bbbk \in \Delta$, the probability measure $\mu_{\Bbbk}$ on $\mathrm{Mat}(\mathbb{N}, F)$ is $\mathrm{GL}(\infty, \mathcal{O}_F) \times \mathrm{GL}(\infty, \mathcal{O}_F)$-invariant.*

Recall that the normalized integral $⨍$ is introduced in (14).

Remark 3.2. For any $n \geqslant 1$, we have

$$⨍_{\varpi^{-n}\mathcal{O}_F} \chi(x)\, dx = 0. \tag{33}$$

Indeed, for any fixed $n \geqslant 1$, the character χ defines a non-trivial character $\widetilde{\chi}$ of the finite group $\Gamma_n := \varpi^{-n}\mathcal{O}_F/\mathcal{O}_F$ by

$$\widetilde{\chi}(\gamma) := \chi(x) \quad (\text{if } \gamma = x + \mathcal{O}_F, x \in \varpi^{-n}\mathcal{O}_F).$$

By orthogonality of the character $\widetilde{\chi}$ and the trivial character, we have

$$⨍_{\varpi^{-n}\mathcal{O}_F} \chi(x)\, dx = \frac{1}{\#\Gamma_n} \sum_{\gamma \in \Gamma_n} \widetilde{\chi}(\gamma) = 0,$$

and (33) is proved.

Lemma 3.3. *For any $y \in F$ and any $l \in \mathbb{Z}$, we have*

$$⨍_{\varpi^{l}\mathcal{O}_F} \chi(xy)\, dx = \mathbb{1}_{\varpi^{-l}\mathcal{O}_F}(y). \tag{34}$$

Proof. First assume that $y \in \varpi^{-l}\mathcal{O}_F$. Then for any $x \in \varpi^{l}\mathcal{O}_F$, we have $xy \in \mathcal{O}_F$. Consequently, by (28), we have $⨍_{\varpi^{l}\mathcal{O}_F} \chi(xy)\, dx = 1$. Now assume that $y \notin \varpi^{-l}\mathcal{O}_F$. Since the Haar measure $d\mathrm{vol}$ on F is invariant under the multiplication action by any element $u \in \mathcal{O}_F^\times$, without loss of generality, we may assume that $y = \varpi^k$ with $k \leqslant -l - 1$. By (33), we have

$$⨍_{\varpi^{l}\mathcal{O}_F} \chi(xy)\, dx = ⨍_{\varpi^{l}\mathcal{O}_F} \chi(\varpi^k x)\, dx = ⨍_{\mathcal{O}_F} \chi(\varpi^{k+l} z)\, dz = 0.$$

This completes the proof of (34). □

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LEMMA 3.4. *For any $m \in \mathbb{N}$, the distribution of the random vector $(X_i^{(1)})_{i=1}^m$ is $\mathrm{GL}(m, \mathcal{O}_F)$-invariant.*

Proof. Write $X = (X_i^{(1)})_{i=1}^m$. It suffices to prove that for any $A \in \mathrm{GL}(m, \mathcal{O}_F)$ and any $y = (y_1, \ldots, y_m) \in F^m$, we have

$$\mathbb{E}[\chi(X \cdot y)] = \mathbb{E}[\chi((AX) \cdot y)]. \tag{35}$$

By the independence between $X_i^{(1)} : i = 1, \ldots, m$ and Lemma 3.3, we have

$$\mathbb{E}[\chi(X \cdot y)] = \prod_{j=1}^{n} \mathbb{E}[\chi(X_j y_j)] = \prod_{j=1}^{n} \mathbb{1}_{\mathcal{O}_F}(y_j) = \mathbb{1}_{\mathcal{O}_F^m}(y).$$

Similarly,

$$\mathbb{E}[\chi((AX) \cdot y)] = \mathbb{E}[\chi(X \cdot (A^t y))] = \mathbb{1}_{\mathcal{O}_F^m}(A^t y) = \mathbb{1}_{\mathcal{O}_F^m}(y).$$

Hence we get (35). The proof of Lemma 3.4 is completed. □

Proof of Proposition 3.1. It suffices to prove that the following probability measures

$$\mathcal{L}[X_i^{(1)} Y_j^{(1)}]_{i,j \in \mathbb{N}} \quad \text{and} \quad \mathcal{L}([Z_{ij}]_{i,j \in \mathbb{N}})$$

are $\mathrm{GL}(\infty, \mathcal{O}_F) \times \mathrm{GL}(\infty, \mathcal{O}_F)$-invariant. The invariance of both measures follows immediately from Lemma 3.4. □

3.2 $\mathrm{GL}(\infty, \mathcal{O}_F) \times \mathrm{GL}(\infty, \mathcal{O}_F)$-ergodicity for probability measures $\mu_{\mathbb{k}}$

THEOREM 3.5. *For any $\mathbb{k} \in \Delta$, the probability measure $\mu_{\mathbb{k}}$ on $\mathrm{Mat}(\mathbb{N}, F)$ is $\mathrm{GL}(\infty, \mathcal{O}_F) \times \mathrm{GL}(\infty, \mathcal{O}_F)$-ergodic.*

The map $M \mapsto (M_{ii})_{i \in \mathbb{N}}$ from $\mathrm{Mat}(\mathbb{N}, F)$ to $F^{\mathbb{N}}$ induces an affine map

$$\Psi : \mathcal{P}(\mathrm{Mat}(\mathbb{N}, F)) \to \mathcal{P}(F^{\mathbb{N}}).$$

Let $S(n)$ denote the group of permutations of the set $\{1, 2, \ldots, n\}$ and set $S(\infty) := \bigcup_{n \in \mathbb{N}} S(n)$. The group $S(\infty)$ acts naturally on $F^{\mathbb{N}}$ by permutations of coordinates.

LEMMA 3.6. *For any $\mu \in \mathcal{P}_{\mathrm{inv}}(\mathrm{Mat}(\mathbb{N}, F))$, we have $\Psi(\mu) \in \mathcal{P}_{\mathrm{inv}}^{S(\infty)}(F^{\mathbb{N}})$. Moreover, the restriction map*

$$\Psi : \mathcal{P}_{\mathrm{inv}}(\mathrm{Mat}(\mathbb{N}, F)) \to \mathcal{P}_{\mathrm{inv}}^{S(\infty)}(F^{\mathbb{N}}) \tag{36}$$

is an affine embedding.

Proof. Let $\mu \in \mathcal{P}_{\mathrm{inv}}(\mathrm{Mat}(\mathbb{N}, F))$. For any $\sigma \in S(\infty)$, the associated permutation matrix M_σ, defined by

$$M_\sigma(i, j) := 1_{\sigma(i)=j},$$

is an element in $\mathrm{GL}(\infty, \mathcal{O}_F)$. By the invariance of μ under the multiplication by all permutation matrices $M_\sigma, \sigma \in S(\infty)$ on left and on right, it is easy to see that $\Psi(\mu) \in \mathcal{P}_{\mathrm{inv}}^{S(\infty)}(F^{\mathbb{N}})$.

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Now we show that the map (36) is injective. By the definition of pushforward map, the Fourier transform of $\Psi(\mu)$ is given as follows: for any $x_1, \dots, x_r \in F$,

$$\widehat{\Psi(\mu)}((x_1, \dots, x_r, 0, 0, \dots)) = \widehat{\mu}(\mathrm{diag}(x_1, \dots, x_r, 0, 0, \dots)). \tag{37}$$

By Remark 2.9, $\widehat{\mu}$ is determined by

$$\widehat{\mu}(\mathrm{diag}(x_1, \dots, x_r, 0, 0, \dots)), \quad x_1, \dots, x_r \in F.$$

Now, the equality (37) implies that $\widehat{\mu}$ and hence μ itself is determined uniquely by $\Psi(\mu)$. The injectivity of the map (36) is proved. □

In what follows, for any convex set C, we denote by $\mathrm{Ext}(C)$ the set of extreme points of C.

Remark 3.7. Recall here the definition of indecomposability in § 2.1.2. Since $S(\infty)$ is countable, we have

$$\mathrm{Ext}(\mathcal{P}_{\mathrm{inv}}^{S(\infty)}(\mathcal{X}^{\mathbb{N}})) = \mathcal{P}_{\mathrm{erg}}^{S(\infty)}(\mathcal{X}^{\mathbb{N}}).$$

Proof of Theorem 3.5. By construction, for any $\Bbbk \in \Delta$, the measure $\Psi(\mu_\Bbbk)$ is Bernoulli. Indeed, the definition

$$\mu_\Bbbk = \mathcal{L}\left(\left[\sum_{n:k_n>k} \varpi^{-k_n} X_i^{(n)} Y_j^{(n)} + \varpi^{-k} Z_{ij}\right]_{i,j\in\mathbb{N}}\right)$$

implies that the diagonal marginal measure $\Psi(\mu_\Bbbk)$ is

$$\Psi(\mu_\Bbbk) = \mathcal{L}\left(\left(\sum_{n:k_n>k} \varpi^{-k_n} X_i^{(n)} Y_i^{(n)} + \varpi^{-k} Z_{ii}\right)_{i\in\mathbb{N}}\right),$$

and our original assumption, that all $X_i^{(n)}, Y_i^{(n)}, Z_{ii}$ are independent sampled with respect to the normalized Haar measure on the compact additive group $\mathcal{O}_F$, implies that the measure $\Psi(\mu_\Bbbk)$ is Bernoulli. Consequently, by the De Finetti Theorem, $\Psi(\mu_\Bbbk)$ is $S(\infty)$-ergodic. Therefore, by Remark 3.7,

$$\Psi(\mu_\Bbbk) \in \mathcal{P}_{\mathrm{erg}}^{S(\infty)}(F^{\mathbb{N}}) = \mathrm{Ext}(\mathcal{P}_{\mathrm{inv}}^{S(\infty)}(F^{\mathbb{N}})).$$

It is clear that for any convex subset C of $\mathrm{Ext}(\mathcal{P}_{\mathrm{inv}}^{S(\infty)}(F^{\mathbb{N}}))$, we have

$$C \cap \mathrm{Ext}(\mathcal{P}_{\mathrm{inv}}^{S(\infty)}(F^{\mathbb{N}})) \subset \mathrm{Ext}(C).$$

Taking $C = \Psi(\mathcal{P}_{\mathrm{inv}}(\mathrm{Mat}(\mathbb{N}, F)))$, we see that

$$\Psi(\mu_\Bbbk) \in \mathrm{Ext}(\Psi(\mathcal{P}_{\mathrm{inv}}(\mathrm{Mat}(\mathbb{N}, F))))). \tag{38}$$

By Lemma 3.6, Ψ is an affine embedding. Hence

$$\mathrm{Ext}(\Psi(\mathcal{P}_{\mathrm{inv}}(\mathrm{Mat}(\mathbb{N}, F)))) = \Psi(\mathrm{Ext}(\mathcal{P}_{\mathrm{inv}}(\mathrm{Mat}(\mathbb{N}, F))))).$$

The relation (38) implies $\mu_\Bbbk \in \mathrm{Ext}(\mathcal{P}_{\mathrm{inv}}(\mathrm{Mat}(\mathbb{N}, F)))$. Since indecomposability implies ergodicity, we get the desired relation $\mu_\Bbbk \in \mathcal{P}_{\mathrm{erg}}(\mathrm{Mat}(\mathbb{N}, F))$. □

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3.3 $\mathrm{GL}(\infty, \mathcal{O}_F)$-invariance for probability measures $\nu_{\mathbb{h}}$

Proposition 3.8. *For any $\mathbb{h} \in \Omega$, the probability measure $\nu_{\mathbb{h}}$ on $\mathrm{Sym}(\mathbb{N}, F)$ is $\mathrm{GL}(\infty, \mathcal{O}_F)$-invariant.*

Lemma 3.9. *The normalized Haar measure on $\mathrm{Sym}(n, \mathcal{O}_F)$ is invariant under the natural action of the group $\mathrm{GL}(n, O_F)$.*

Proof. For any $g \in \mathrm{GL}(n, \mathcal{O}_F)$, the linear map

$$\begin{array}{ccc} \mathrm{Sym}(n, \mathcal{O}_F) & \longrightarrow & \mathrm{Sym}(n, \mathcal{O}_F) \\ S & \mapsto & gSg^t \end{array} \tag{39}$$

is invertible. Clearly, if we use the group identification

$$\mathrm{Sym}(n, \mathcal{O}_F) \simeq \mathcal{O}_F^{(n^2+n)/2},$$

then the linear map (39) is represented by an invertible matrix from $\mathrm{GL}((n^2+n)/2, \mathcal{O}_F)$. Hence by Remark 2.1, it preserves the normalized Haar measure. □

Proof of Proposition 3.8. It suffices to prove the $\mathrm{GL}(\infty, \mathcal{O}_F)$-invariance of the following probability measures on $\mathrm{Sym}(\mathbb{N}, F)$:

$$\mathcal{L}([X_i^{(1)} X_j^{(1)}]_{i,j\in\mathbb{N}}) \quad \text{and} \quad \mathcal{L}([H_{ij}]_{i,j\in\mathbb{N}}).$$

The $\mathrm{GL}(\infty, \mathcal{O}_F)$-invariance of $\mathcal{L}([X_i^{(1)} X_j^{(1)}]_{i,j\in\mathbb{N}})$ follows immediately from Lemma 3.4, while the $\mathrm{GL}(\infty, \mathcal{O}_F)$-invariance of $\mathcal{L}([H_{ij}]_{i,j\in\mathbb{N}})$ follows from Lemma 3.9. □

3.4 $\mathrm{GL}(\infty, \mathcal{O}_F)$-ergodicity for probability measures $\nu_{\mathbb{h}}$

Theorem 3.10. *For any $\mathbb{h} \in \Omega$, the probability measure $\nu_{\mathbb{h}}$ on $\mathrm{Sym}(\mathbb{N}, F)$ is $\mathrm{GL}(\infty, \mathcal{O}_F)$-ergodic.*

The map $S \mapsto (S_{ii})_{i\in\mathbb{N}}$ from $\mathrm{Sym}(\mathbb{N}, F)$ to $F^{\mathbb{N}}$ induces an affine map

$$\Phi : \mathcal{P}(\mathrm{Sym}(\mathbb{N}, F)) \to \mathcal{P}(F^{\mathbb{N}}).$$

Lemma 3.11. *For any $\nu \in \mathcal{P}_{\mathrm{inv}}(\mathrm{Sym}(\mathbb{N}, F))$, we have $\Phi(\nu) \in \mathcal{P}_{\mathrm{inv}}^{S(\infty)}(F^{\mathbb{N}})$. Moreover, the restriction map*

$$\Phi : \mathcal{P}_{\mathrm{inv}}(\mathrm{Sym}(\mathbb{N}, F)) \to \mathcal{P}_{\mathrm{inv}}^{S(\infty)}(F^{\mathbb{N}})$$

is an affine embedding.

Proof. The proof is similar to that of Lemma 3.6. □

Proof of Theorem 3.10. The proof is similar to that of Theorem 3.5 by using Lemma 3.11 instead of Lemma 3.6. □

4. Explicit computation of characteristic functions

In this section, we give explicit formulae for characteristic functions of measures from the two families $\{\mu_{\mathbb{k}} : \mathbb{k} \in \Delta\}$ and $\{\nu_{\mathbb{h}} : \mathbb{h} \in \Omega\}$.

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4.1 Measures on $\mathrm{Mat}(\mathbb{N}, F)$

By the elementary observation (30), for studying the characteristic functions of $\mu_{\mathbb{k}}$, it suffices to compute

$$\widehat{\mu_{\mathbb{k}}}(\mathrm{diag}(\varpi^{-\ell_1}, \dots, \varpi^{-\ell_r}, 0, 0, \dots))$$

for any $r \in \mathbb{N}$ and any $\ell_1, \dots, \ell_r \in \mathbb{Z}$.

Proposition 4.1. *For any $\mathbb{k} = (k_n)_{n\in\mathbb{N}} \in \Delta$ and any $\ell \in \mathbb{Z}$, we have*

$$\widehat{\mu_{\mathbb{k}}}(\varpi^{-\ell} e_{11}) = \exp\left(-\log q \cdot \sum_{j=1}^{\infty} (k_j + \ell) \mathbb{1}_{\{k_j+\ell\geqslant 1\}}\right). \tag{40}$$

More generally, for any $\ell_1, \dots, \ell_r \in \mathbb{Z}$, we have

$$\widehat{\mu_{\mathbb{k}}}(\mathrm{diag}(\varpi^{-\ell_1}, \dots, \varpi^{-\ell_r}, 0, 0, \dots)) = \prod_{i=1}^{r} \widehat{\mu_{\mathbb{k}}}(\varpi^{-\ell_i} e_{11}). \tag{41}$$

Introduce a function $\Theta : F \to \mathbb{C}$ by

$$\Theta(x) = \int_{\mathcal{O}_F \times \mathcal{O}_F} \chi(z_1 z_2 \cdot x)\, dz_1\, dz_2. \tag{42}$$

Lemma 4.2. *The function Θ is given by*

$$\Theta(x) = q^{-\ell \mathbb{1}_{\{\ell\geqslant 1\}}}, \quad |x| = q^{\ell}.$$

Proof. Let $|x| = q^{\ell}$. Then there exists $u \in \mathcal{O}_F^{\times}$, such that $x = \varpi^{-\ell} u$. By rotation invariance, $\Theta(x) = \Theta(\varpi^{-\ell})$. Now by Lemma 3.3,

$$\begin{aligned}\Theta(\varpi^{-\ell}) &= \mathbb{E}[\chi(\varpi^{-\ell} X_1^{(1)} Y_1^{(1)})] = \mathbb{E}(\mathbb{E}[\chi(\varpi^{-\ell} X_1^{(1)} Y_1^{(1)}) \mid Y_1^{(1)}]) \\ &= \mathbb{E}(\mathbb{1}_{\mathcal{O}_F}(\varpi^{-\ell} Y_1^{(1)})) = \mathbb{P}(|Y_1^{(1)}| \leqslant q^{-\ell}) = q^{-\ell \mathbb{1}_{\{\ell\geqslant 1\}}}.\end{aligned}$$

□

Remark 4.3. In the formulae below, for graphical convenience, we write $\exp(-\log q \cdot \ell \mathbb{1}_{\{\ell\geqslant 1\}})$ instead of $q^{-\ell \mathbb{1}_{\{\ell\geqslant 1\}}}$.

Proof of Proposition 4.1. The identity (41) follows from the independence between all diagonal coefficient of $M_{\mathbb{k}}$. So we only need to prove the identity (40).

First assume that $\mathbb{k} = (k_n)_{n\in\mathbb{N}} \in \Delta$ is such that $\lim_{n\to\infty} k_n = -\infty$. By the independence between all $X_1^{(n)}$ and $Y_1^{(n)}, n \in \mathbb{N}$, we have

$$\begin{aligned}\widehat{\mu_{\mathbb{k}}}(\varpi^{-\ell} e_{11}) &= \mathbb{E}\left[\chi\left(\mathrm{tr}\left(\left[\sum_{n=1}^{\infty} \varpi^{-k_n} X_i^{(n)} Y_j^{(n)}\right]_{i,j\in\mathbb{N}} \varpi^{-\ell} e_{11}\right)\right)\right] \\ &= \mathbb{E}\left[\chi\left(\sum_{n=1}^{\infty} \varpi^{-k_n-\ell} X_1^{(n)} Y_1^{(n)}\right)\right] = \prod_{n=1}^{\infty} \mathbb{E}[\chi(\varpi^{-k_n-\ell} X_1^{(n)} Y_1^{(n)})].\end{aligned}$$

By Lemma 4.2, we get

$$\begin{aligned}\widehat{\mu_{\mathbb{k}}}(\varpi^{-\ell} e_{11}) &= \prod_{n=1}^{\infty} \exp(-\log q \cdot (k_n + \ell) \mathbb{1}_{\{k_n+\ell\geqslant 1\}}) \\ &= \exp\left(-\log q \cdot \sum_{n=1}^{\infty} (k_n + \ell) \mathbb{1}_{\{k_n+\ell\geqslant 1\}}\right).\end{aligned}$$

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Now assume that there exists $m \in \mathbb{N} \cup \{0\}$ and $k \in \mathbb{Z}$, such that

$$k_1 \geqslant \cdots \geqslant k_m > k \quad \text{and} \quad k_n = k \text{ for any } n \geqslant m+1.$$

By previous computation and the formula (32) for the characteristic functions of convolutions of probability measures, we only need to consider the case when $\Bbbk = (k_n)_{n\in\mathbb{N}}$ is such that

$$k_n = k \in \mathbb{Z} \quad \text{for any } n \in \mathbb{N}.$$

In this case, $\mu_\Bbbk = \mathcal{L}(\varpi^{-k}Z)$ with Z an infinite random matrix sampled uniformly from $\mathrm{Mat}(\mathbb{N}, \mathcal{O}_F)$. Hence by Lemma 3.3, we obtain

$$\widehat{\mu_\Bbbk}(\varpi^{-\ell}e_{11}) = \mathbb{E}[\chi(\varpi^{-\ell-k}Z_{11})] = \mathbb{1}_{\mathcal{O}_F}(\varpi^{-\ell-k}) = \mathbb{1}_{\{k+\ell\leqslant 0\}}.$$

But if $k_n = k$ for any $n \in \mathbb{N}$, we have

$$\mathbb{1}_{\{k+\ell\leqslant 0\}} = \exp\left(-\log q \cdot \sum_{n=1}^{\infty}(k_n+\ell)\mathbb{1}_{\{k_n+\ell\geqslant 1\}}\right). \tag{43}$$

This proves the identity (40) in the second case and we complete the proof of Proposition 4.1. □

4.2 Measures on Sym($\mathbb{N}, F$)

By the elementary observation (31), for studying the characteristic function of $\nu_\Bbbh$, it suffices to compute

$$\widehat{\nu_\Bbbh}(\mathrm{diag}(x_1, \ldots, x_r, 0, 0, \ldots))$$

for any $r \in \mathbb{N}$ and any $x_1, \ldots, x_r \in F$.

Recall the definition (7) for the function θ:

$$\theta(x) = \int_{\mathcal{O}_F} \chi(z^2 \cdot x)\, dz.$$

Proposition 4.4. *Let $\Bbbh = (k; \Bbbk, \Bbbk') \in \Omega$. Then for any $x \in F$, we have*

$$\widehat{\nu_\Bbbh}(xe_{11}) = \mathbb{1}_{\mathcal{O}_F}(\varpi^{-k}x) \cdot \prod_{n=1}^{\infty}\theta(\varpi^{-k_n}x)\prod_{n=1}^{\infty}\theta(\varepsilon\varpi^{-k'_n}x). \tag{44}$$

More generally, for any $r \in \mathbb{N}$ and any $x_1, \ldots, x_r \in F$, we have

$$\widehat{\nu_\Bbbh}(\mathrm{diag}(x_1, \ldots, x_r, 0, 0, \ldots)) = \prod_{i=1}^{r}\widehat{\nu_\Bbbh}(x_i e_{11}). \tag{45}$$

Define a function $L_2 : \mathcal{O}_F^\times \to \{-1, 1\}$ by setting $L_2(u) = 1$ if u is a square element in $\mathcal{O}_F^\times$ and setting $L_2(u) = -1$ if u is a non-square element in $\mathcal{O}_F^\times$. Write

$$\varrho_q := \begin{cases} 1 & \text{if } q \equiv 1 (\mathrm{mod}\ 4), \\ i & \text{if } q \equiv 3 (\mathrm{mod}\ 4). \end{cases} \tag{46}$$

Recall that for any $x \in F$, we have $|x| = q^{-\mathrm{ord}_F(x)}$.

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Proposition 4.5. *The function $\theta : F \to \mathbb{C}$ is continuous and satisfies the following properties.*

(i) *If $|x| \leqslant 1$, then $\theta(x) = 1$.*
(ii) *If $|x| > 1$ and $\mathrm{ord}_F(x) \equiv 0 (\mathrm{mod}\ 2)$, then $\theta(x) = |x|^{-1/2}$.*
(iii) *If $|x| > 1$ and $\mathrm{ord}_F(x) \equiv 1 (\mathrm{mod}\ 2)$, then by writing $x = \varpi^{-\ell} u$ with $\ell \in \mathbb{N}$ and $u \in \mathcal{O}_F^\times$, we have*

$$\theta(x) = s_\chi \varrho_q \cdot \frac{L_2(u)}{|x|^{1/2}}, \tag{47}$$

where $s_\chi \in \{-1, 1\}$ depends on the choice of χ.

In particular, θ satisfies the following property:

$$\theta(x)^2 = \theta(\varepsilon x)^2 \neq 0 \quad \text{for all } x \in F. \tag{48}$$

Moreover, if $x = \varpi^{-\ell} u$ with $\ell \in \mathbb{Z} \cup \{-\infty\}$ and $u \in \mathcal{O}_F^\times$

$$|\theta(x)|^2 = \exp(-\log|x| \cdot \mathbb{1}_{|x|>1}) = \exp(-\ell \log q \cdot \mathbb{1}_{\{\ell \geqslant 1\}}). \tag{49}$$

Let us postpone the proof of Proposition 4.5 to § 10.

Remark 4.6. Proposition 4.5 is related to the calculations of the Gaussian integrals in Neretin [Ner11, § 11.1] and Vladimirov *et al.* [VVZ94, ch. V] in the particular case when the field F is the field $\mathbb{Q}_p$ of p-adic numbers.

Lemma 4.7. *Fix an element $a \in F$. Then for any $\Bbbk = (k_j)_{j\in\mathbb{N}} \in \Delta$, the infinite product*

$$\prod_{j=1}^{\infty} \theta(a \cdot \varpi^{-k_j}) \tag{50}$$

converges. Moreover, if $\lim_{j\to\infty} k_j = k \in \mathbb{Z}$, then

$$\prod_{j=1}^{\infty} \theta(a \cdot \varpi^{-k_j}) = \mathbb{1}_{\mathcal{O}_F}(a \cdot \varpi^{-k}) \cdot \prod_{j \in \{n | k_n > k\}} \theta(a \cdot \varpi^{-k_j}). \tag{51}$$

Proof. Let $\Bbbk = (k_j)_{j\in\mathbb{N}} \in \Delta$. Since $\Bbbk$ is a non-increasing sequence, we have

$$|a \cdot \varpi^{-k_1}| \geqslant |a \cdot \varpi^{-k_2}| \geqslant \cdots \geqslant |a \cdot \varpi^{-k_j}| \cdots .$$

Then either there exists $j_0 \in \mathbb{N}$, such that $|a \cdot \varpi^{-k_j}| \leqslant 1$ for all $j \geqslant j_0$ or $|a \cdot \varpi^{-k_j}| > 1$ for all $j \in \mathbb{N}$. Consequently, the infinite product (50) either is a finite product or equals 0. The identity (51) follows immediately form (49). □

Proof of Proposition 4.4. The identity (45) follows from the independence between all diagonal coefficient of $S_{\mathbb{h}}$. So we only need to prove the identity (44).

Case 1: $\mathbb{h} = (-\infty; \Bbbk, \Bbbk'), \Bbbk \in \Delta[-\infty], \Bbbk' \in \Delta^\sharp[-\infty]$.

In this case, we have

$$\nu_{\mathbb{h}} = \mathcal{L}(W_\Bbbk + \varepsilon W_{\Bbbk'}) = \mathcal{L}(W_\Bbbk) * \mathcal{L}(\varepsilon W_{\Bbbk'}).$$

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Thus for proving (45), it suffices to prove it for the probability measures $\mathcal{L}(W_{\mathbb{k}})$ and $\mathcal{L}(\varepsilon W_{\mathbb{k}'})$. For instance, we have

$$\begin{aligned}\widehat{\mathcal{L}(W_{\mathbb{k}})}(xe_{11}) &= \mathbb{E}\left[\chi\left(\mathrm{tr}\left(\left[\sum_{n=1}^{\infty}\varpi^{-k_n}X_i^{(n)}X_j^{(n)}\right]_{i,j\in\mathbb{N}}xe_{11}\right)\right)\right]\\ &= \mathbb{E}\left[\chi\left(\sum_{n=1}^{\infty}\varpi^{-k_n}(X_1^{(n)})^2x\right)\right] = \mathbb{E}\left[\prod_{n=1}^{\infty}\chi(\varpi^{-k_n}(X_1^{(n)})^2x)\right].\end{aligned}$$

Then by dominated convergence theorem and the independence between all $X_1^{(n)}, n\in\mathbb{N}$, we get

$$\widehat{\mathcal{L}(W_{\mathbb{k}})}(xe_{11}) = \prod_{n=1}^{\infty}\mathbb{E}[\chi(\varpi^{-k_n}(X_1^{(n)})^2x)] = \prod_{n=1}^{\infty}\theta(\varpi^{-k_n}x).$$

Similar computation works for $\mathcal{L}(\varepsilon W_{\mathbb{k}'})$.

Case 2: There exists $k\in\mathbb{Z}$ and $\mathbb{h} = (k;\mathbb{k},\mathbb{k}'), \mathbb{k}\in\Delta[k], \mathbb{k}'\in\Delta^\sharp[k]$.

In this case, we have

$$\nu_{\mathbb{h}} = \mathcal{L}(W_{\mathbb{k}} + \varepsilon W_{\mathbb{k}'} + \varpi^{-k}H) = \mathcal{L}(W_{\mathbb{k}}) * \mathcal{L}(\varepsilon W_{\mathbb{k}'}) * \mathcal{L}(\varpi^{-k}H).$$

By (32), for proving (45), it suffices to prove it for the probability measures

$$\mathcal{L}(W_{\mathbb{k}}), \quad \mathcal{L}(\varepsilon W_{\mathbb{k}'}) \quad \text{and} \quad \mathcal{L}(\varpi^{-k}H).$$

By the computation in Case 1, we only need to verify (45) for $\mathcal{L}(\varpi^{-k}H)$. A simple computation yields the desired identity

$$\widehat{\mathcal{L}(\varpi^{-k}H)}(xe_{11}) = \mathbb{E}[\chi(\mathrm{tr}(\varpi^{-k}Hxe_{11}))] = \mathbb{E}[\chi(\varpi^{-k}H_{11}x)] = \mathbb{1}_{\mathcal{O}_F}(\varpi^{-k}x). \qquad \square$$

Remark 4.8. Let us prove the identity (5) mentioned in Remark 1.7. Write

$$\sigma_1 := \mathcal{L}\left(\left[\sum_{n=1}^{2}X_i^{(n)}X_j^{(n)}\right]_{i,j\in\mathbb{N}}\right), \quad \sigma_2 := \mathcal{L}\left(\varepsilon\left[\sum_{n=1}^{2}X_i^{(n)}X_j^{(n)}\right]_{i,j\in\mathbb{N}}\right).$$

Note that σ_1,σ_2 are both $\mathrm{GL}(\infty,\mathcal{O}_F)$-invariant. Since for any $x\in F$, we have $\theta(x)^2 = \theta(\varepsilon x)^2$. Consequently

$$\widehat{\sigma_1}(\mathrm{diag}(x_1,\ldots,x_r,0,\ldots)) = \prod_{i=1}^{r}\theta(x_i)^2 = \prod_{i=1}^{r}\theta(\varepsilon x_i)^2 = \widehat{\sigma_2}(\mathrm{diag}(x_1,\ldots,x_r,0,\ldots)).$$

It follows that $\sigma_1 = \sigma_2$.

5. Uniqueness of parametrization

In this section, we will prove two uniqueness results, Propositions 5.1 and 5.3.

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5.1 Measures on $\mathrm{Mat}(\mathbb{N}, F)$

Proposition 5.1. *Let $\Bbbk, \widetilde{\Bbbk} \in \Delta$. Then $\mu_{\Bbbk} = \mu_{\widetilde{\Bbbk}}$ if and only if $\Bbbk = \widetilde{\Bbbk}$.*

Lemma 5.2. *The map*

$$\Bbbk = (k_j)_{j\in\mathbb{N}} \mapsto \left(\sum_{j=1}^{\infty}(k_j+\ell)\mathbb{1}_{\{k_j+\ell\geqslant 1\}}\right)_{\ell\in\mathbb{Z}}$$

from Δ to $(\mathbb{Z}\cup\{+\infty\})^{\mathbb{Z}}$ is injective.

Proof. We need to show that if $\Bbbk = (k_j)_{j\in\mathbb{N}}$ and $\widetilde{\Bbbk} = (\widetilde{k}_j)_{j\in\mathbb{N}}$ are two distinct elements in Δ, then there exists $\ell \in \mathbb{Z}$, such that

$$\sum_{j=1}^{\infty}(k_j+\ell)\mathbb{1}_{\{k_j+\ell\geqslant 1\}} \neq \sum_{j=1}^{\infty}(\widetilde{k}_j+\ell)\mathbb{1}_{\{\widetilde{k}_j+\ell\geqslant 1\}}. \tag{52}$$

By assumption, there exists $j_0 \in \mathbb{N}$, such that

$$k_j = \widetilde{k}_j \quad \text{for any } 1 \leqslant j < j_0 \quad \text{and} \quad k_{j_0} \neq \widetilde{k}_{j_0}. \tag{53}$$

By symmetry, let us assume that $k_{j_0} > \widetilde{k}_{j_0}$. Under this assumption (whether $\widetilde{k}_{j_0}$ equals to $-\infty$ or not), we will have $k_{j_0} \in \mathbb{Z}$. Now by taking $\ell = 1 - k_{j_0} \in \mathbb{Z}$, we have

$$\begin{cases} k_j + \ell = \widetilde{k}_j + \ell \geqslant 1 & \text{for any } 1 \leqslant j < j_0, \\ k_{j_0} + \ell = 1 \quad \text{and} \quad \widetilde{k}_{j_0} + \ell \leqslant 0. \end{cases}$$

Consequently,

$$\sum_{j=1}^{\infty}(k_j+\ell)\mathbb{1}_{\{k_j+\ell\geqslant 1\}} \geqslant \sum_{j=1}^{j_0}(k_j+\ell) = 1 + \sum_{j=1}^{j_0-1}(k_j+\ell),$$

while

$$\sum_{j=1}^{\infty}(\widetilde{k}_j+\ell)\mathbb{1}_{\{\widetilde{k}_j+\ell\geqslant 1\}} = \sum_{j=1}^{j_0-1}(\widetilde{k}_j+\ell) = \sum_{j=1}^{j_0-1}(k_j+\ell).$$

Thus we prove that the inequality (52) holds for $\ell = 1 - k_{j_0}$. □

Proof of Proposition 5.1. Proposition 5.1 follows from Proposition 4.1, Lemma 5.2 and the fact any probability measure on $\mathrm{Mat}(\mathbb{N}, F)$ is uniquely determined by its characteristic function. □

5.2 Measures on $\mathrm{Sym}(\mathbb{N}, F)$

Proposition 5.3. *Let $\mathbb{h}, \widetilde{\mathbb{h}} \in \Omega$. Then $\nu_{\mathbb{h}} = \nu_{\widetilde{\mathbb{h}}}$ if and only if $\mathbb{h} = \widetilde{\mathbb{h}}$.*

Remark 5.4. Any element $\Bbbk = (k_j)_{j\in\mathbb{N}} \in \Delta$ is uniquely determined by the bi-infinite sequence in $\mathbb{N}\cup\{\infty\}$:

$$(\#\{j \in \mathbb{N} \mid k_j = \ell\})_{\ell\in\mathbb{Z}}.$$

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Proof of Proposition 5.3. Let $\mathbb{h} = (k; \mathbb{k}, \mathbb{k}')$ and $\widetilde{\mathbb{h}} = (\widetilde{k}; \widetilde{\mathbb{k}}, \widetilde{\mathbb{k}}')$ be two elements in Ω such that $\nu_{\mathbb{h}} = \nu_{\widetilde{\mathbb{h}}}$. By Proposition 4.4, this is *equivalent* to the following identity: for any $x \in F$,

$$\mathbb{1}_{\mathcal{O}_F}(\varpi^{-k}x) \cdot \prod_{n=1}^{\infty} \theta(\varpi^{-k_n}x) \prod_{n=1}^{\infty} \theta(\varepsilon\varpi^{-k'_n}x) = \mathbb{1}_{\mathcal{O}_F}(\varpi^{-\widetilde{k}}x) \cdot \prod_{n=1}^{\infty} \theta(\varpi^{-\widetilde{k}_n}x) \prod_{n=1}^{\infty} \theta(\varepsilon\varpi^{-\widetilde{k}'_n}x). \quad (54)$$

Recall the identity (49). By taking the modulus and square of both sides of (54) and substituting $x = \varpi^{-\ell}u$ with $\ell \in \mathbb{Z}$ and $u \in \mathcal{O}_F^{\times}$, we obtain

$$\begin{aligned} &\mathbb{1}_{\{k+\ell\leqslant 0\}} \cdot q^{-\sum_{n=1}^{\infty}(k_n+\ell)\mathbb{1}_{\{k_n+\ell\geqslant 1\}} - \sum_{n=1}^{\infty}(k'_n+\ell)\mathbb{1}_{\{k'_n+\ell\geqslant 1\}}} \\ &\quad = \mathbb{1}_{\{\widetilde{k}+\ell\leqslant 0\}} \cdot q^{-\sum_{n=1}^{\infty}(\widetilde{k}_n+\ell)\mathbb{1}_{\{\widetilde{k}_n+\ell\geqslant 1\}} - \sum_{n=1}^{\infty}(\widetilde{k}'_n+\ell)\mathbb{1}_{\{\widetilde{k}'_n+\ell\geqslant 1\}}}. \end{aligned} \quad (55)$$

Claim 1: $k = \widetilde{k}$.

Indeed, if $k = -\infty$, then the left-hand side of the identity (55) never vanishes. Consequently, so does the right-hand side. It follows that $\widetilde{k} = -\infty$. If $k \in \mathbb{Z}$. Then the left-hand side of the identity (55) vanishes at $\ell = 1 - k$. Consequently, so does the right-hand side of (55) vanishes both at $\ell = 1 - k$. It follows that $\widetilde{k} + 1 - k > 0$ or equivalently $\widetilde{k} \geqslant k$. By symmetry, we have $k = \widetilde{k}$.

Claim 2: $(\mathbb{k}, \mathbb{k}') = (\widetilde{\mathbb{k}}, \widetilde{\mathbb{k}}')$.

For simplifying notation, let us define $\mathbb{k}^*, \widetilde{\mathbb{k}}^* \in \Delta$ as follows: if $k = \widetilde{k} = -\infty$, then set $\mathbb{k}^* := \mathbb{k}$, $\widetilde{\mathbb{k}}^* := \widetilde{\mathbb{k}}$; if $k = \widetilde{k} \in \mathbb{Z}$, then both $\mathbb{k}$ and $\widetilde{\mathbb{k}}$ are finite sequences in $\mathbb{Z}_{>k}$, set $\mathbb{k}^*$ and $\widetilde{\mathbb{k}}^*$ by adding infinitely many k. Clearly, for proving $(\mathbb{k}, \mathbb{k}') = (\widetilde{\mathbb{k}}, \widetilde{\mathbb{k}}')$, it suffices to prove that $(\mathbb{k}^*, \mathbb{k}') = (\widetilde{\mathbb{k}}^*, \widetilde{\mathbb{k}}')$. By Remark 5.4, it suffices to prove that for any $l \in \mathbb{Z}$,

$$\begin{aligned} \#\{n \in \mathbb{N} \mid k_n^* = l\} &= \#\{n \in \mathbb{N} \mid \widetilde{k}_n^* = l\}, \\ \#\{n \in \mathbb{N} \mid k'_n = l\} &= \#\{n \in \mathbb{N} \mid \widetilde{k}'_n = l\}. \end{aligned} \quad (56)$$

Applying (43) to $\mathbb{1}_{\{k+\ell\leqslant 0\}}$ and $\mathbb{1}_{\{\widetilde{k}+\ell\leqslant 0\}}$, we may write (54) as

$$q^{-\sum_{n=1}^{\infty}(k_n^*+\ell)\mathbb{1}_{\{k_n^*+\ell\geqslant 1\}} - \sum_{n=1}^{\infty}(k'_n+\ell)\mathbb{1}_{\{k'_n+\ell\geqslant 1\}}} = q^{-\sum_{n=1}^{\infty}(\widetilde{k}_n^*+\ell)\mathbb{1}_{\{\widetilde{k}_n^*+\ell\geqslant 1\}} - \sum_{n=1}^{\infty}(\widetilde{k}'_n+\ell)\mathbb{1}_{\{\widetilde{k}'_n+\ell\geqslant 1\}}}. \quad (57)$$

By Lemma 5.2 and Remark 5.4, the equality (57) implies that for any $l \in \mathbb{Z}$, we have

$$\#\{n \in \mathbb{N} \mid k_n^* = l\} + \#\{n \in \mathbb{N} \mid k'_n = l\} = \#\{n \in \mathbb{N} \mid \widetilde{k}_n^* = l\} + \#\{n \in \mathbb{N} \mid \widetilde{k}'_n = l\}. \quad (58)$$

The identity (58) implies in particular that the two identities in (56) hold or are violated simultaneously. Now assume by contradiction that there exists $l_0 \in \mathbb{Z}$, such that the identities in (56) are violated. Obviously, such l_0 verifies

$$k < l_0 \leqslant \max\{k_n, \widetilde{k}_n, k'_n, \widetilde{k}'_n\} < +\infty.$$

Now let $l_{\max} \in \mathbb{Z}$ be the largest l_0 such that the identities in (56) are violated. Substituting $x = \varpi^{l_{\max}-1}u$ with $u \in \mathcal{O}_F^{\times}$ into the identity (54), we obtain

$$\begin{aligned} &\prod_{n_1=1}^{\infty} \theta(\varpi^{-k_{n_1}}\varpi^{l_{\max}-1}u) \prod_{n_2=1}^{\infty} \theta(\varepsilon\varpi^{-k'_{n_2}}\varpi^{l_{\max}-1}u) \\ &\quad = \prod_{m_1=1}^{\infty} \theta(\varpi^{-\widetilde{k}_{m_1}}\varpi^{l_{\max}-1}u) \prod_{m_2=1}^{\infty} \theta(\varepsilon\varpi^{-\widetilde{k}'_{m_2}}\varpi^{l_{\max}-1}u). \end{aligned} \quad (59)$$

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By the assumption of $l_{\max}$, we know that for any $l > l_{\max}$,

$$\begin{aligned} \#\{n \in \mathbb{N} \mid k_n^* = l\} &= \#\{n \in \mathbb{N} \mid \widetilde{k}_n^* = l\}, \\ \#\{n \in \mathbb{N} \mid k'_n = l\} &= \#\{n \in \mathbb{N} \mid \widetilde{k}'_n = l\}. \end{aligned} \tag{60}$$

Since $l_{\max} > k$, by definition of $\mathbb{k}^*$ and $\widetilde{\mathbb{k}}^*$ we also have for any $l > l_{\max}$,

$$\begin{aligned} \#\{n \in \mathbb{N} \mid k_n^* = l\} &= \#\{n \in \mathbb{N} \mid k_n = l\}, \\ \#\{n \in \mathbb{N} \mid \widetilde{k}_n^* = l\} &= \#\{n \in \mathbb{N} \mid \widetilde{k}_n = l\}. \end{aligned} \tag{61}$$

By Proposition 4.5, the function θ never vanishes. Hence by the identities (60) and (61), we can remove simultaneously all those terms concerning $k_{n_1} > l_{\max}, k'_{n_2} > l_{\max}$ and $\widetilde{k}_{m_1} > l_{\max}$, $\widetilde{k}'_{m_2} > l_{\max}$ from both sides of identity (59). Again by Proposition 4.5, for any $k_{n_1} < l_{\max}$, $k'_{n_2} < l_{\max}$ and $\widetilde{k}_{m_1} < l_{\max}, \widetilde{k}'_{m_2} < l_{\max}$, we have

$$\theta(\varpi^{-k_{n_1}}\varpi^{l_{\max}-1}u) = \theta(\varepsilon\varpi^{-k'_{n_2}}\varpi^{l_{\max}-1}u) = \theta(\varpi^{-\widetilde{k}_{m_1}}\varpi^{l_{\max}-1}u) = \theta(\varepsilon\varpi^{-\widetilde{k}'_{m_2}}\varpi^{l_{\max}-1}u) = 1.$$

Consequently, we may remove simultaneously all those terms concerning $k_{n_1} < l_{\max}, k'_{n_2} < l_{\max}$ and $\widetilde{k}_{m_1} < l_{\max}, \widetilde{k}'_{m_2} < l_{\max}$ from both sides of identity (59) as well. Then we arrive at the identity

$$\theta(\varpi^{-1}u)^D \cdot \theta(\varepsilon\varpi^{-1}u)^{D'} = \theta(\varpi^{-1}u)^{\widetilde{D}} \cdot \theta(\varepsilon\varpi^{-1}u)^{\widetilde{D'}}, \tag{62}$$

where

$$\begin{aligned} D &:= \#\{n \in \mathbb{N} \mid k_n = l_{\max}\} \quad \text{and} \quad D' := \#\{n \in \mathbb{N} \mid k'_n = l_{\max}\}, \\ \widetilde{D} &:= \#\{n \in \mathbb{N} \mid \widetilde{k}_n = l_{\max}\} \quad \text{and} \quad \widetilde{D}' := \#\{n \in \mathbb{N} \mid \widetilde{k}'_n = l_{\max}\}. \end{aligned}$$

By definitions for $\Delta[k]$ and $\Delta^\sharp[k]$, we must have

$$D, \quad \widetilde{D} \in \mathbb{N} \cup \{0\} \quad \text{and} \quad D', \widetilde{D'} \in \{0, 1\}. \tag{63}$$

The identity (58) now implies that

$$D + D' = \widetilde{D} + \widetilde{D'}. \tag{64}$$

By definition of $l_{\max}$, we have $D \neq D'$. Without loss of generality, we may assume that $D' = 0$ and $\widetilde{D'} = 1$. Then the identity (62) becomes

$$\theta(\varpi^{-1}u)^D = \theta(\varpi^{-1}u)^{\widetilde{D}} \cdot \theta(\varepsilon\varpi^{-1}u). \tag{65}$$

But now $D = \widetilde{D} + 1$ and since θ never vanishes, the identity (65) is equivalent to

$$\theta(\varpi^{-1}u) = \theta(\varepsilon\varpi^{-1}u).$$

Since $|\varpi^{-1}u| > 1, \mathrm{ord}_F(\varpi^{-1}u) = -1 \equiv 1 (\mathrm{mod}\ 2)$ and $L_2(\varepsilon) = -1$, by Proposition 4.5, we have $\theta(\varepsilon\varpi^{-1}u) = -\theta(\varpi^{-1}u)$. Consequently, we would have $\theta(\varpi^{-1}u) = \theta(\varepsilon\varpi^{-1}u) = 0$. This contradicts to the non-vanishing property of θ. Hence we complete the proof of Claim 2.

Combining Claims 1 and 2, we complete the proof of Proposition 5.3. □

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6. Ergodic measures as limits of orbital measures: the Vershik–Kerov ergodic method

In this section, we recall the Vershik–Kerov ergodic method for dealing with ergodic measures for inductively compact groups. The general setting is as follows. Let

$$K(1) \subset K(2) \subset \cdots \subset K(n) \subset \cdots \subset K(\infty),$$

be an increasing chain of topological groups such that for any $n \in \mathbb{N}$, the group $K(n)$ is compact and $K(\infty)$ is the inductive limit:

$$K(\infty) = \varinjlim K(n).$$

For any $n \in \mathbb{N}$, let $m_{K(n)}$ denote the normalized Haar measure of $K(n)$. Fix a group action of $K(\infty)$ on a Polish space $\mathcal{X}$.

Definition 6.1 (Orbital measures). For any $x \in \mathcal{X}$ and any $n \in \mathbb{N}$, we define the $K(n)$-orbital measure generated by x, denoted by $m_{K(n)}(x)$, as the unique $K(n)$-invariant probability measure on $\mathcal{X}$ supported on the $K(n)$-orbit $K(n) \cdot x := \{g \cdot x \mid g \in K(n)\}$. In other words, $m_{K(n)}(x)$ is the image of $m_{K(n)}$ under the map $g \mapsto g \cdot x$ from $K(n)$ to $\mathcal{X}$.

Definition 6.2. Let $\mathscr{L}^{K(\infty)}(\mathcal{X}) \subset \mathcal{P}(\mathcal{X})$ be the set of probability measures μ on $\mathcal{X}$ such that there exists $x \in \mathcal{X}$ verifying $m_{K(n)}(x) \Longrightarrow \mu$.

Theorem 6.3 (Vershik [Ver74, Theorem 1]). *The following inclusion holds:*

$$\mathcal{P}_{\mathrm{erg}}^{K(\infty)}(\mathcal{X}) \subset \mathscr{L}^{K(\infty)}(\mathcal{X}). \tag{66}$$

More precisely, if μ is an ergodic $K(\infty)$-invariant Borel probability measure on $\mathcal{X}$, then for μ-almost every point $x \in \mathcal{X}$, the weak convergence $m_{K(n)}(x) \Longrightarrow \mu$ holds.

Vershik's method in [Ver74] was further developed in a series of papers [VK81a, VK81b, KV86, OV96] by Kerov *et al.*

Remark 6.4. In general, the converse inclusion $\mathscr{L}^{K(\infty)}(\mathcal{X}) \subset \mathcal{P}_{\mathrm{erg}}^{K(\infty)}(\mathcal{X})$ does not hold. There is however a simple situation, see the note [Qiu17], where $\mathcal{P}_{\mathrm{erg}}^{K(\infty)}(\mathcal{X})$ always coincides with $\mathscr{L}^{K(\infty)}(\mathcal{X})$.

For simplifying notation, in what follows, we write

$$\mathscr{L}(\mathrm{Mat}(\mathbb{N}, F)) := \mathscr{L}^{\mathrm{GL}(\infty,\mathcal{O}_F)\times\mathrm{GL}(\infty,\mathcal{O}_F)}(\mathrm{Mat}(\mathbb{N}, F)),$$
$$\mathscr{L}(\mathrm{Sym}(\mathbb{N}, F)) := \mathscr{L}^{\mathrm{GL}(\infty,\mathcal{O}_F)}(\mathrm{Sym}(\mathbb{N}, F)).$$

For any $n \in \mathbb{N}$, we set

$$\mathscr{ORB}_n(\mathrm{Mat}(n, F)) := \{m_{\mathrm{GL}(n,\mathcal{O}_F)\times\mathrm{GL}(n,\mathcal{O}_F)}(M) \mid M \in \mathrm{Mat}(n, F)\}.$$

By identifying $\mathrm{Mat}(n, F)$ in a natural way with the subset of $\mathrm{Mat}(\mathbb{N}, F)$, we have $\mathcal{P}(\mathrm{Mat}(n, F)) \subset \mathcal{P}(\mathrm{Mat}(\mathbb{N}, F))$. In particular,

$$\mathscr{ORB}_n(\mathrm{Mat}(n, F)) \subset \mathcal{P}(\mathrm{Mat}(\mathbb{N}, F)).$$

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Definition 6.5. Let $\mathscr{ORB}_\infty(\mathrm{Mat}(\mathbb{N},F))$ denote the set of probability measures μ on $\mathrm{Mat}(\mathbb{N},F)$ such that there exists a subsequence of positive integers $n_1<n_2<\cdots$ and a sequence $(\mu_{n_k})_{k\in\mathbb{N}}$ of orbital measures with $\mu_{n_k}\in\mathscr{ORB}_{n_k}(\mathrm{Mat}(n_k,F))$, so that

$$\mu_{n_k}\Longrightarrow\mu.$$

Similarly, in the symmetric case, for any $n\in\mathbb{N}$, we set

$$\mathscr{ORB}_n(\mathrm{Sym}(n,F)):=\{m_{\mathrm{GL}(n,\mathcal{O}_F)}(S)\mid S\in\mathrm{Sym}(n,F)\}.$$

By identifying $\mathrm{Sym}(n,F)$ in a natural way with a subspace of $\mathrm{Sym}(\mathbb{N},F)$, we have

$$\mathscr{ORB}_n(\mathrm{Sym}(n,F))\subset\mathcal{P}(\mathrm{Sym}(\mathbb{N},F)).$$

Definition 6.6. Let $\mathscr{ORB}_\infty(\mathrm{Sym}(\mathbb{N},F))$ denote the set of probability measures ν on $\mathrm{Sym}(\mathbb{N},F)$ such that there exists a subsequence of positive integers $n_1<n_2<\cdots$ and a sequence $(\nu_{n_k})_{k\in\mathbb{N}}$ of orbital measures with $\nu_{n_k}\in\mathscr{ORB}_{n_k}(\mathrm{Sym}(n_k,F))$, so that

$$\nu_{n_k}\Longrightarrow\nu.$$

Remark 6.7. It is easy to see that we have

$$\mathscr{ORB}_\infty(\mathrm{Mat}(\mathbb{N},F))\subset\mathcal{P}_{\mathrm{inv}}(\mathrm{Mat}(\mathbb{N},F)),$$
$$\mathscr{ORB}_\infty(\mathrm{Sym}(\mathbb{N},F))\subset\mathcal{P}_{\mathrm{inv}}(\mathrm{Sym}(\mathbb{N},F)).$$

Proposition 6.8. *The following two inclusions hold:*

$$\mathscr{L}(\mathrm{Mat}(\mathbb{N},F))\subset\mathscr{ORB}_\infty(\mathrm{Mat}(\mathbb{N},F)),\tag{67}$$
$$\mathscr{L}(\mathrm{Sym}(\mathbb{N},F))\subset\mathscr{ORB}_\infty(\mathrm{Sym}(\mathbb{N},F)).\tag{68}$$

Proof. For any $n,m\in\mathbb{N}$ such that $m\geqslant n$, let

$$\mathrm{Cut}_n^\infty:\mathrm{Mat}(\mathbb{N},F)\to\mathrm{Mat}(n,F),$$
$$\mathrm{Cut}_n^m:\mathrm{Mat}(m,F)\to\mathrm{Mat}(n,F)$$

be the maps of cutting the $n\times n$ left-upper corner.

For simplifying notation, write

$$\mathcal{K}(n):=\mathrm{GL}(n,\mathcal{O}_F)\times\mathrm{GL}(n,\mathcal{O}_F).\tag{69}$$

Let $\mu\in\mathscr{L}(\mathrm{Mat}(\mathbb{N},F))$. By definition, there exists an infinite matrix $X_0\in\mathrm{Mat}(\mathbb{N},F)$ and a subsequence $(n_k)_{k\in\mathbb{N}}$ of positive integers such that

$$m_{\mathcal{K}(n_k)}(X_0)\Longrightarrow\mu.$$

This is equivalent to saying that for any $N\in\mathbb{N}$, we have

$$(\mathrm{Cut}_N^\infty)_*[m_{\mathcal{K}(n_k)}(X_0)]\Longrightarrow(\mathrm{Cut}_N^\infty)_*(\mu).\tag{70}$$

Take

$$X_k=\mathrm{Cut}_{n_k}^\infty(X_0)\in\mathrm{Mat}(n_k,F).$$

Then we have

$$m_{\mathcal{K}(n_k)}(X_k)\Longrightarrow\mu.$$

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Indeed, it suffices to prove that for any $N \in \mathbb{N}$, we have

$$(\mathrm{Cut}_N^\infty)_*[m_{\mathcal{K}(n_k)}(X_k)] \Longrightarrow (\mathrm{Cut}_N^\infty)_*(\mu). \tag{71}$$

For any $k \in \mathbb{N}$, we clearly have

$$(\mathrm{Cut}_{n_k}^\infty)_*[m_{\mathcal{K}(n_k)}(X_k)] = (\mathrm{Cut}_{n_k}^\infty)_*[m_{\mathcal{K}(n_k)}(X_0)]. \tag{72}$$

But if $n_k \geqslant N$, we have $\mathrm{Cut}_N^\infty \circ \mathrm{Cut}_{n_k}^\infty = \mathrm{Cut}_N^\infty$. Combining with (72), we see that, once $n_k \geqslant N$, we have

$$(\mathrm{Cut}_N^\infty)_*[m_{\mathcal{K}(n_k)}(X_k)] = (\mathrm{Cut}_N^\infty)_*[m_{\mathcal{K}(n_k)}(X_0)].$$

Now it is clear that (70) implies (71). The first inclusion (67) is proved. The proof of the second inclusion (68) is the same. □

7. Asymptotic multiplicativity for orbital integrals

7.1 $\mathrm{GL}(n, \mathcal{O}_F) \times \mathrm{GL}(n, \mathcal{O}_F)$-orbital integrals

Recall Definition 42 and Lemma 4.2:

$$\Theta(x) = \int_{\mathcal{O}_F \times \mathcal{O}_F} \chi(z_1 z_2 \cdot x)\, dz_1\, dz_2 = q^{-\ell \mathbb{1}_{\{\ell \geqslant 1\}}}, \quad |x| = q^\ell.$$

In what follows, we use the notation (69).

Theorem 7.1. *Let $n, r \in \mathbb{N}$ be such that $r \leqslant n$. Suppose that D and A are two diagonal matrices given by*

$$D = \mathrm{diag}(x_1, \ldots, x_n), \quad A = \mathrm{diag}(a_1, \ldots, a_r, 0, \ldots, 0)$$

where $x_1, \ldots, x_n, a_1, \ldots, a_r \in F$. Then

$$\left| \int_{\mathcal{K}(n)} \chi(\mathrm{tr}(g_1 D g_2 A))\, dg_1\, dg_2 - \prod_{i=1}^{r} \prod_{j=1}^{n} \Theta(a_i x_j) \right| \leqslant 2 \left(1 - \prod_{w=0}^{r-1} (1 - q^{w-n}) \right)^2, \tag{73}$$

where $dg_1\, dg_2$ is the normalized Haar measure on $\mathcal{K}(n)$.

In particular, for any $a \in F$, we have

$$\left| \int_{\mathcal{K}(n)} \chi(a \cdot \mathrm{tr}(g_1 D g_2 e_{11}))\, dg_1\, dg_2 - \prod_{j=1}^{n} \Theta(a x_j) \right| \leqslant 2 q^{-2n}. \tag{74}$$

Remark 7.2. Obviously, we have

$$\int_{\mathcal{K}(n)} \chi(\mathrm{tr}(g_1 D g_2 A))\, dg_1\, dg_2 = \int_{\mathcal{K}(n)} \chi(\mathrm{tr}(g_1 D g_2^{-1} A))\, dg_1\, dg_2.$$

The following elementary lemma will be useful.

Lemma 7.3. *Let $r \leqslant n$ be two positive integers. Define*

$$S(r \times n) := \{M \in \mathrm{Mat}(r \times n, \mathcal{C}_q) \mid \mathrm{rank}_{\mathbf{F}_q}(M (\mathrm{mod}\ \varpi \mathcal{O}_F)) = r\}.$$

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Then we have

$$\#S(r\times n)=\prod_{w=0}^{r-1}(q^n-q^w). \tag{75}$$

For any rectangular matrix $M\in S(r\times n)$, we have

$$\#\{t\in \mathrm{GL}(n,\mathcal{C}_q)\mid t_{ij}=M_{ij},\forall 1\leqslant i\leqslant r,1\leqslant j\leqslant n\}=\prod_{w=0}^{n-w-1}(q^n-q^{r+w}).$$

In particular, the above cardinality does not depend on the choice of $M\in S(r\times n)$.

Proof of Theorem 7.1. Fix $n,r\in\mathbb{N}$ and fix the two diagonal matrices D and A. Let $T=T(n)$, $T'=T'(n)$ be two independent copies of random matrices sampled uniformly from the finite set $\mathrm{GL}(n,\mathcal{C}_q)$, and let $V=V(n),V'=V'(n)$ be two independent random matrices sampled uniformly from $\mathrm{Mat}(n,\varpi\mathcal{O}_F)$ and independent of T,T'. By Proposition 2.3, we have

$$\int_{\mathcal{K}(n)}\chi(\mathrm{tr}(g_1Dg_2A))\,dg_1\,dg_2=\mathbb{E}[\chi(\mathrm{tr}((T+V)D(T'+V')A))].$$

Since the transposed random matrix $(T'+V')^t$ and the original random matrix $T'+V'$ have the same distribution, we have

$$\begin{aligned}&\mathbb{E}[\chi(\mathrm{tr}((T+V)D(T'+V')A))]\\&=\mathbb{E}[\chi(\mathrm{tr}((T+V)D(T'+V')^tA))]\\&=\mathbb{E}\left[\chi\left(\sum_{i=1}^{r}\sum_{j=1}^{n}(T_{ij}+V_{ij})(T'_{ij}+V'_{ij})a_ix_j\right)\right]\\&=\frac{1}{[\#\mathrm{GL}(n,\mathcal{C}_q)]^2}\sum_{t,t'\in\mathrm{GL}(n,\mathcal{C}_q)}\mathbb{E}\left[\chi\left(\sum_{i=1}^{r}\sum_{j=1}^{n}(t_{ij}+V_{ij})(t'_{ij}+V'_{ij})a_ix_j\right)\right].\end{aligned}$$

By Lemma 7.3, we obtain

$$\begin{aligned}&\mathbb{E}[\chi(\mathrm{tr}((T+V)D(T'+V')A))]\\&=\frac{1}{[\#S(r\times n)]^2}\sum_{M,M'\in S(r\times n)}\mathbb{E}\left[\chi\left(\sum_{i=1}^{r}\sum_{j=1}^{n}(M_{ij}+V_{ij})(M'_{ij}+V'_{ji})a_ix_j\right)\right].\end{aligned}$$

For simplifying notation, for any pair of rectangular matrices $M,M'\in\mathrm{Mat}(r\times n,\mathcal{C}_q)$, we write

$$F(M,M'):=\mathbb{E}\left[\chi\left(\sum_{i=1}^{r}\sum_{j=1}^{n}(M_{ij}+V_{ij})(M'_{ij}+V'_{ji})a_ix_j\right)\right].$$

Then we may write

$$\mathbb{E}[\chi(\mathrm{tr}((T+V)D(T'+V')A))]=\frac{1}{[\#S(r\times n)]^2}\underbrace{\left[\sum_{M,M'\in\mathrm{Mat}(r\times n,\mathcal{C}_q)}F(M,M')+\mathcal{E}_1\right]}_{\text{denoted by }I},$$

where

$$\mathcal{E}_1:=-\sum_{M,M'\in\mathrm{Mat}(r\times n,\mathcal{C}_q)\setminus S(r\times n)}F(M,M').$$

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Note that if M_{ij} and V_{ij} are sampled independently and uniformly from the finite set $\mathcal{C}_q$ and from the compact additive group $\varpi\mathcal{O}_F$ respectively, then $M_{ij}+V_{ij}$ is uniformly distributed on $\mathcal{O}_F$. It follows that

$$\prod_{i=1}^{r}\prod_{j=1}^{n}\Theta(a_ix_j)=\frac{1}{[\#\mathrm{Mat}(r\times n,\mathcal{C}_q)]^2}\sum_{M,M'\in\mathrm{Mat}(r\times n,\mathcal{C}_q)}F(M,M').$$

Consequently,

$$\begin{aligned}\mathbb{E}[\chi(\mathrm{tr}((T+V)D(T'+V')A))]&=\frac{I}{[\#\mathrm{Mat}(r\times n,\mathcal{C}_q)]^2}+\mathcal{E}_2\\&=\prod_{i=1}^{r}\prod_{j=1}^{n}\Theta(a_ix_j)+\frac{\mathcal{E}_1}{[\#\mathrm{Mat}(r\times n,\mathcal{C}_q)]^2}+\mathcal{E}_2,\end{aligned}$$

where

$$\mathcal{E}_2:=\frac{I}{[\#S(r\times n)]^2}-\frac{I}{[\#\mathrm{Mat}(r\times n,\mathcal{C}_q)]^2}.$$

Now let us estimate these error terms. By the obvious estimate $|F(M,M')|\leqslant 1$, we have

$$|\mathcal{E}_1|\leqslant[\#(\mathrm{Mat}(r\times n,\mathcal{C}_q)\setminus S(r\times n))]^2.$$

Note that $|I|\leqslant[\#S(r\times n)]^2$. Hence

$$\begin{aligned}|\mathcal{E}_2|&=|I|\biggl(\frac{1}{[\#S(r\times n)]^2}-\frac{1}{[\#\mathrm{Mat}(r\times n,\mathcal{C}_q)]^2}\biggr)\\&\leqslant\biggl(\frac{\#\mathrm{Mat}(r\times n,\mathcal{C}_q)-\#S(r\times n)}{\#\mathrm{Mat}(r\times n,\mathcal{C}_q)}\biggr)^2.\end{aligned}$$

Taking (75) into account, we get

$$\begin{aligned}&\biggl|\mathbb{E}[\chi(\mathrm{tr}((T+V)D(T'+V')A))]-\prod_{i=1}^{r}\prod_{j=1}^{n}\Theta(a_ix_j)\biggr|\\&\quad\leqslant\frac{|\mathcal{E}_1|}{[\#\mathrm{Mat}(r\times n,\mathcal{C}_q)]^2}+|\mathcal{E}_2|\\&\quad\leqslant 2\biggl(\frac{\#\mathrm{Mat}(r\times n,\mathcal{C}_q)-\#S(r\times n)}{\#\mathrm{Mat}(r\times n,\mathcal{C}_q)}\biggr)^2\\&\quad\leqslant 2\biggl(1-\prod_{w=0}^{r-1}(1-q^{w-n})\biggr)^2.\end{aligned}$$

□

Theorem 7.4 (Uniform asymptotic multiplicativity). *Let $n,r\in\mathbb{N}$ be such that $r\leqslant n$. Suppose that D and A are two diagonal matrices given by*

$$D=\mathrm{diag}(x_1,\dots,x_n),\quad A=\mathrm{diag}(a_1,\dots,a_r,0,\dots,0),$$

where $x_1,\dots,x_n,a_1,\dots,a_r\in F$. Then

$$\begin{aligned}&\biggl|\int_{\mathcal{K}(n)}\chi(\mathrm{tr}(g_1Dg_2A))\,dg_1\,dg_2-\prod_{i=1}^{r}\int_{\mathcal{K}(n)}\chi(a_i\cdot\mathrm{tr}(g_1Dg_2e_{11}))\,dg_1\,dg_2\biggr|\\&\quad\leqslant 2\biggl(1-\prod_{w=0}^{r-1}(1-q^{w-n})\biggr)^2+2rq^{-2n}.\end{aligned}\tag{76}$$

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Proof. By inequalities (73) and (74), we have

$$\int_{\mathcal{K}(n)} \chi(\mathrm{tr}(g_1 D g_2 A))\, dg_1\, dg_2 = \prod_{i=1}^{r}\prod_{j=1}^{n} \Theta(a_i x_j) + \varepsilon_0$$
$$= \prod_{i=1}^{r} \Bigg(\underbrace{\int_{\mathcal{K}(n)} \chi(a_i \cdot \mathrm{tr}(g_1 D g_2 e_{11}))\, dg_1\, dg_2 + \varepsilon_i}_{=\prod_{j=1}^{n} \Theta(a_i x_j)} \Bigg) + \varepsilon_0,$$

with the errors $\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_r$ controlled by

$$|\varepsilon_0| \leqslant 2\Bigg(1 - \prod_{w=0}^{r-1}(1 - q^{w-n})\Bigg)^2 \quad \text{and} \quad |\varepsilon_i| \leqslant 2q^{-2n}, \quad i = 1, \ldots, r.$$

Using the elementary inequalities $|\prod_{j=1}^{n} \Theta(a_i x_j)| \leqslant 1$ and by a simple computation, we get

$$\begin{aligned}
&\Bigg| \int_{\mathcal{K}(n)} \chi(\mathrm{tr}(g_1 D g_2 A))\, dg_1\, dg_2 - \prod_{i=1}^{r} \int_{\mathcal{K}(n)} \chi(a_i \cdot \mathrm{tr}(g_1 D g_2 e_{11}))\, dg_1\, dg_2 \Bigg| \\
&\leqslant \varepsilon_0 + \varepsilon_1 + \cdots + \varepsilon_r \\
&\leqslant 2\Bigg(1 - \prod_{w=0}^{r-1}(1 - q^{w-n})\Bigg)^2 + 2rq^{-2n}.
\end{aligned}$$

□

7.2 $\mathrm{GL}(n, \mathcal{O}_F)$-orbital integrals

Recall the definition (7) for the function $\theta : F \to \mathbb{C}$.

Theorem 7.5. *Let $n, r \in \mathbb{N}$ be such that $r \leqslant n$. Given two diagonal matrices D and A,*

$$D = \mathrm{diag}(x_1, \ldots, x_n), \quad A = \mathrm{diag}(a_1, \ldots, a_r, 0, \ldots, 0),$$

where $x_1, \ldots, x_n, a_1, \ldots, a_r \in F$, we have

$$\Bigg| \int_{\mathrm{GL}(n, \mathcal{O}_F)} \chi(\mathrm{tr}(g D g^t A))\, dg - \prod_{i=1}^{r}\prod_{j=1}^{n} \theta(a_i x_j) \Bigg| \leqslant 2 \cdot \Bigg(1 - \prod_{w=0}^{r-1}(1 - q^{w-n})\Bigg). \tag{77}$$

In particular, for any $a \in F$, we have

$$\Bigg| \int_{\mathrm{GL}(n, \mathcal{O}_F)} \chi(a \cdot \mathrm{tr}(g D g^t e_{11}))\, dg - \prod_{j=1}^{n} \theta(a \cdot x_j) \Bigg| \leqslant 2q^{-n}. \tag{78}$$

Proof. Fix $n, r \in \mathbb{N}$ and fix the two diagonal matrices D and A. Let $T = T(n)$ be a random matrix uniformly distributed on the finite set $\mathrm{GL}(n, \mathbf{F}_q)$, and let $V = V(n)$ be a random matrix uniformly distributed on $\mathrm{Mat}(n, \varpi\mathcal{O}_F)$ and independent of T. By Proposition 2.3, we have

$$\int_{\mathrm{GL}(n, \mathcal{O}_F)} \chi(\mathrm{tr}(g D g^t A))\, dg = \mathbb{E}\Bigg[\chi\Bigg(\sum_{i=1}^{r}\sum_{j=1}^{n} a_i x_j (T_{ij} + V_{ij})^2\Bigg)\Bigg].$$

By similar arguments as in the proof of Theorem 7.1, we may get the desired inequality (77). The second inequality (78) follows immediately by taking $e = 1$ and $a_1 = a$. □

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Theorem 7.6 (Uniform asymptotic multiplicativity). *Let $n, r \in \mathbb{N}$ be such that $r \leqslant n$. Given two diagonal matrices D and A,*

$$D = \operatorname{diag}(x_1, \ldots, x_n), \quad A = \operatorname{diag}(a_1, \ldots, a_r, 0, \ldots, 0),$$

where $x_1, \ldots, x_n, a_1, \ldots, a_e \in F$, we have

$$\begin{aligned}&\left| \int_{\mathrm{GL}(n,\mathcal{O}_F)} \chi(\operatorname{tr}(gDg^tA))\, dg - \prod_{i=1}^{r} \int_{\mathrm{GL}(n,\mathcal{O}_F)} \chi(a_i \operatorname{tr}(gDg^te_{11}))\, dg \right| \\ &\quad \leqslant 2\left(1 - \prod_{w=0}^{r-1}(1 - q^{w-n})\right) + 2rq^{-n}.\end{aligned}$$

Proof. The proof is similar to that of Theorem 7.4. □

8. The completion of the classification of ergodic measures

8.1 The case of $\mathcal{P}_{\mathrm{erg}}(\mathrm{Mat}(\mathbb{N}, F))$

Theorem 8.1 (Multiplicativity theorem for orbital limit measures). *Assume that μ is an element in $\mathscr{ORB}_\infty(\mathrm{Mat}(\mathbb{N}, F))$. Then for any $r \in \mathbb{N}$ and for any finite sequence $x_1, \ldots, x_r$ in F, we have*

$$\widehat{\mu}(\operatorname{diag}(x_1, \ldots, x_r, 0, 0, \ldots)) = \prod_{j=1}^{r} \widehat{\mu}(x_j e_{11}). \tag{79}$$

In particular, we have

$$\mathscr{ORB}_\infty(\mathrm{Mat}(\mathbb{N}, F)) = \mathcal{P}_{\mathrm{erg}}(\mathrm{Mat}(\mathbb{N}, F)).$$

Proof. Let $\mu \in \mathscr{ORB}_\infty(\mathrm{Mat}(\mathbb{N}, F))$. Then by definition, there exists an increasing sequence $(n_k)_{k\in\mathbb{N}}$ in $\mathbb{N}$ and a sequence $(\mu_{n_k})_{k\in\mathbb{N}}$ of orbital measures with $\mu_{n_k} \in \mathscr{ORB}_{n_k}(\mathrm{Mat}(n_k, F))$, such that

$$\mu_{n_k} \Longrightarrow \mu \quad \text{as } k \to \infty. \tag{80}$$

Take any $x_1, \ldots, x_r \in F$. By the inequality (76), we have

$$\lim_{k\to\infty} \left| \widehat{\mu_{n_k}}(\operatorname{diag}(x_1, \ldots, x_r, 0, \ldots)) - \prod_{j=1}^{r} \widehat{\mu_{n_k}}(x_j e_{11}) \right| = 0. \tag{81}$$

Combining (80) and (81), we get the desired identity (79).

By Vershik's Theorem 6.3 and Proposition 6.8, to obtain (81), we only need to prove the inclusion

$$\mathscr{ORB}_\infty(\mathrm{Mat}(\mathbb{N}, F)) \subset \mathcal{P}_{\mathrm{erg}}(\mathrm{Mat}(\mathbb{N}, F)). \tag{82}$$

Recall the definition (36) of the affine map Ψ:

$$\Psi : \mathcal{P}_{\mathrm{inv}}(\mathrm{Mat}(\mathbb{N}, F)) \to \mathcal{P}_{\mathrm{inv}}^{S(\infty)}(F^{\mathbb{N}}).$$

The identity (79) implies that for any $\mu \in \mathscr{ORB}_\infty(\mathrm{Mat}(\mathbb{N}, F))$, the marginal measure on the diagonal matrices $\Psi(\mu)$ is a Bernoulli measure on $F^{\mathbb{N}}$. Consequently, by exactly the same argument as in the proof of Theorem 3.5, we can prove the desired inclusion (82). □

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An immediate consequence of Theorem 8.1 and the argument as in the proof of Theorem 8.1 is the following Ismagilov–Olshanski multiplicativity in our setting.

Corollary 8.2 (Ismagilov–Olshanski multiplicativity). *An invariant probability measure $\mu \in \mathcal{P}_{\mathrm{inv}}(\mathrm{Mat}(\mathbb{N}, F))$ is ergodic if and only if for any $r \in \mathbb{N}$ and for any finite sequence $x_1, \ldots, x_r$ in F, we have*

$$\widehat{\mu}(\mathrm{diag}(x_1, \ldots, x_r, 0, 0, \ldots)) = \prod_{j=1}^{r} \widehat{\mu}(x_j e_{11}).$$

Remark 8.3. The reader may compare our method with the methods used in, for instance, Olshansk and Vershik [OV96]. The Olshanski–Vershik argument relies on the Ismagilov–Olshanski multiplicativity: the multiplicativity of the characteristic function is equivalent to the ergodicity of the corresponding probability measure. In different contexts, this multiplicativity is established, e.g., by Ismagilov [Ism69, Ism70], Nessonov [Nes86], Voiculescu [Voi76], Olshanski [Ol'78], Stratila and Voiculescu [SV82], Pickrell [Pic90], Vershik and Kerov [VK90], Olshanski [Ol'90].

In our situation, the Ismagilov–Olshanski multiplicativity follows as a corollary.

Now we may concentrate on the classification of $\mathscr{ORB}_\infty(\mathrm{Mat}(\mathbb{N}, F))$. For this purpose, we need to study the weak convergence of probability measures on $\mathrm{Mat}(\mathbb{N}, F)$. The following standard proposition implies that the weak convergence of probability measures on $\mathrm{Mat}(\mathbb{N}, F)$ is equivalent to the locally uniform convergence of corresponding characteristic functions. For completeness, we include its proof in Appendix A.

Proposition 8.4. *A sequence of invariant probability measures $(\mu_n)_{n\in\mathbb{N}}$ in $\mathcal{P}_{\mathrm{inv}}(\mathrm{Mat}(\mathbb{N}, F))$ converges weakly to an invariant probability measure $\mu \in \mathcal{P}_{\mathrm{inv}}(\mathrm{Mat}(\mathbb{N}, F))$ if and only if for any $r \in \mathbb{N}$ and any $\ell_1, \ldots, \ell_r \in \mathbb{Z}$, we have*

$$\widehat{\mu}(\mathrm{diag}(\varpi^{-\ell_1}, \ldots, \varpi^{-\ell_r}, 0, \ldots)) = \lim_{n\to\infty} \widehat{\mu}_n(\mathrm{diag}(\varpi^{-\ell_1}, \ldots, \varpi^{-\ell_r}, 0, \ldots))$$

and the convergence is uniform on any subset of type

$$\{(\ell_1, \ldots, \ell_r) \mid \ell_1, \ldots, \ell_r \in \mathbb{Z}_{\leqslant C}\}.$$

Lemma 8.5. *Let μ be a Borel probability measure on $\mathrm{Mat}(\mathbb{N}, F)$. The function $x \in F \mapsto \widehat{\mu}(xe_{11}) \in \mathbb{C}$ is uniformly continuous. In particular, we have*

$$\lim_{\ell\to-\infty} \widehat{\mu}(\varpi^{-\ell} e_{11}) = 1.$$

Proof. Note that the function $x \mapsto \widehat{\mu}(xe_{11})$ is the characteristic function of the marginal probability measure $(\mathrm{Cut}_1^\infty)_*\mu$ on F. The uniform continuity of this function then follows immediately, see, e.g., Hewitt and Ross [HR70, Theorem 31.5, p. 212]. □

Lemma 8.6. *Assume that we are given a sequence of probability measures $(\mu_n)_{n\in\mathbb{N}}$, such that $\mu_n \in \mathscr{ORB}_n(\mathrm{Mat}(n, F))$. A necessary and sufficient condition for this sequence to be tight is the following.*

(C_1) *There exists $\gamma \in \mathbb{Z}$, such that the supports $\mathrm{supp}(\mu_n)$ are all contained in the following compact subset of $\mathrm{Mat}(\mathbb{N}, F)$:*

$$\{X \in \mathrm{Mat}(\mathbb{N}, F) \mid |X_{ij}| \leqslant q^\gamma, \forall i, j \in \mathbb{N}\}.$$

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Proof. The above condition (C_1) is clearly sufficient for the sequence to be tight. Now suppose that the sequence $(\mu_n)_{n\in\mathbb{N}}$ is tight. By assumption, suppose that μ_n is the $\mathrm{GL}(n,\mathcal{O}_F)\times\mathrm{GL}(n,\mathcal{O}_F)$-orbital measure supported on an orbit generated by $X_n\in\mathrm{Mat}(n,\mathcal{O}_F)$. By Lemma 2.4, we may assume that

$$X_n=\mathrm{diag}(x_1^{(n)},\dots,x_n^{(n)}),\quad |x_1^{(n)}|\geqslant\cdots\geqslant|x_n^{(n)}|.$$

Assume by contradiction that the condition (C_1) is not satisfied. Then there exists a subsequence $(n_k)_{k\in\mathbb{N}}$ of positive integers, such that

$$\lim_{k\to\infty}|x_1^{(n_k)}|=\infty. \tag{83}$$

Passing to a subsequence if necessary, we may assume that there exists a probability measure μ on $\mathrm{Mat}(\mathbb{N},F)$, such that $\mu_{n_k}\Longrightarrow\mu$. By Lemma 8.5, for any $a\in F$, we have

$$\lim_{|a|\to0}\lim_{k\to\infty}\widehat{\mu}_{n_k}(ae_{11})=\lim_{|a|\to0}\widehat{\mu}(ae_{11})=1.$$

That is,

$$\lim_{|a|\to0}\lim_{k\to\infty}\int_{\mathcal{K}(n_k)}\chi(a\cdot\mathrm{tr}(g_1X_{n_k}g_2e_{11}))\,dg_1\,dg_2=1. \tag{84}$$

By (74), the relation (84) implies that

$$\lim_{|a|\to0}\lim_{k\to\infty}\prod_{j=1}^{n_k}\Theta(a\cdot x_j^{(n_k)})=1. \tag{85}$$

Since $|\Theta(x)|\leqslant1$, for any $a\in F^\times$, we have

$$\lim_{k\to\infty}\left|\prod_{j=1}^{n_k}\Theta(a\cdot x_j^{(n_k)})\right|\leqslant\lim_{k\to\infty}|\Theta(a\cdot x_1^{(n_k)})|=0.$$

This contradicts to (85). Thus the condition (C_1) is necessary for the sequence $(\mu_n)_{n\in\mathbb{N}}$ to be tight. □

Recall that

$$\Delta=\{\Bbbk=(k_j)_{j=1}^\infty\mid k_j\in\mathbb{Z}\cup\{-\infty\};k_1\geqslant k_2\geqslant\cdots\},$$

as a subset of $(\mathbb{Z}\cup\{-\infty\})^{\mathbb{N}}$, is assumed to be equipped with the subspace topology of Tychonoff's product topology on $(\mathbb{Z}\cup\{-\infty\})^{\mathbb{N}}$.

LEMMA 8.7. *Let $\ell\in\mathbb{Z}$. Then*

$$\Bbbk\mapsto f(\Bbbk):=\sum_{j=1}^\infty(k_j+\ell)\mathbb{1}_{\{k_j+\ell\geqslant1\}}$$

defines a continuous map from Δ to $\mathbb{Z}\cup\{+\infty\}$.

Proof. It suffices to prove Lemma 8.7 for $\ell=0$. We want to prove that $\Bbbk\mapsto f(\Bbbk)$ is continuous at some point $\Bbbk^{(0)}=(k_j^{(0)})_{j\in\mathbb{N}}\in\Delta$.

Case 1: $f(\Bbbk^{(0)})=+\infty$. This means that $k_j^{(0)}\geqslant1$ for any $j\in\mathbb{N}$. Consequently, for any $A\in\mathbb{R}$, we may take n large enough so that $\sum_{j=1}^n k_j^{(0)}>A$. Then for any $\Bbbk\in\Delta$ sufficiently close to $\Bbbk^{(0)}$, we have $k_j=k_j^{(0)}$ for $j=1,\dots,n$. For such $\Bbbk$, we have $f(\Bbbk)\geqslant\sum_{j=1}^n k_j>A$.

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Case 2: $f(\Bbbk^{(0)}) < +\infty$. Then choose n so that

$$k_1^{(0)} \geqslant \cdots \geqslant k_{n-1}^{(0)} > 0 \geqslant k_n^{(0)} \geqslant \cdots .$$

For any $\Bbbk \in \Delta$ sufficiently close to $\Bbbk^{(0)}$, we have $k_j = k_j^{(0)}$ for $j = 1, \ldots, n-1$ and $k_n \leqslant 0$. For such $\Bbbk$, we have $f(\Bbbk) = \sum_{j=1}^{n-1} k_j = f(\Bbbk^{(0)})$.

The proof of Lemma 8.7 is completed. □

Theorem 8.8. *The map* $\Bbbk \mapsto \mu_\Bbbk$ *induces a bijection between* Δ *and* $\mathcal{P}_{\mathrm{erg}}(\mathrm{Mat}(\mathbb{N}, F))$.

Proof. The injectivity of the map $\Bbbk \mapsto \mu_\Bbbk$ from Δ to $\mathcal{P}_{\mathrm{erg}}(\mathrm{Mat}(\mathbb{N}, F))$ has already been proved in Proposition 5.1. We only need to prove that the map is also surjective.

Let $\mu \in \mathcal{P}_{\mathrm{erg}}(\mathrm{Mat}(\mathbb{N}, F))$. By Theorem 8.1, $\mu \in \mathscr{ORB}_\infty(\mathrm{Mat}(\mathbb{N}, F))$. Consequently, there exists a sequence $(n_l)_{l\in\mathbb{N}}$ of positive integers and a sequence $(\mu_{n_l})_{l\in\mathbb{N}}$ of orbital measures such that $\mu_{n_l} \in \mathscr{ORB}_{n_l}(\mathrm{Mat}(n_l, F))$ and

$$\mu_{n_l} \Longrightarrow \mu \quad \text{as } l \to \infty. \tag{86}$$

By Lemma 2.4, we may assume that μ_{n_l} is the $\mathcal{K}(n_l)$-orbital measure supported on the orbit $\mathcal{K}(n_l) \cdot X_{n_l}$ with

$$X_{n_l} = \mathrm{diag}(\varpi^{-k_1^{(n_l)}}, \ldots, \varpi^{-k_{n_l}^{(n_l)}}), \quad k_1^{(n_l)} \geqslant \cdots \geqslant k_{n_l}^{(n_l)} \geqslant -\infty.$$

By Lemma 8.6, the convergence (86) implies that $\sup_{l\in\mathbb{N}} k_1^{(n_l)} < \infty$. Consequently, passing to a subsequence of $(n_l)_{l\in\mathbb{N}}$ if necessary, we may assume that for any $j \in \mathbb{N}$, there exists $k_j \in \mathbb{Z} \cup \{-\infty\}$ such that

$$\lim_{l\to\infty} k_j^{(n_l)} = k_j. \tag{87}$$

The convergence (86) and the relation (74) now imply that, for any $\ell \in \mathbb{Z}$, we have

$$\begin{aligned} \widehat{\mu}(\varpi^{-\ell} e_{11}) &= \lim_{l\to\infty} \prod_{j=1}^{\infty} \Theta(\varpi^{-\ell} \varpi^{-k_j^{(n_l)}}) \\ &= \lim_{l\to\infty} \exp\Big(-\log q \cdot \sum_{j=1}^{\infty} (k_j^{(n_l)} + \ell) \mathbb{1}_{\{k_j^{(n_l)} + \ell \geqslant 1\}} \Big). \end{aligned}$$

By Lemma 8.7 and (87), we get

$$\widehat{\mu}(\varpi^{-\ell} e_{11}) = \exp\Big(-\log q \cdot \sum_{j=1}^{\infty} (k_j + \ell) \mathbb{1}_{\{k_j + \ell \geqslant 1\}} \Big). \tag{88}$$

Let us define $\Bbbk := (k_j)_{j\in\mathbb{N}} \in \Delta$. Comparing (99) with the formula (40) in Proposition 4.1, we get

$$\widehat{\mu}(\varpi^{-\ell} e_{11}) = \widehat{\mu_\Bbbk}(\varpi^{-\ell} e_{11}) \quad \text{for all } \ell \in \mathbb{Z}. \tag{89}$$

But by the multiplicativity of $\widehat{\mu}$ established in Theorem 8.9 and the multiplicativity of $\widehat{\mu}_\Bbbk$ established in Proposition 4.1, the above identity (89) implies $\mu = \mu_\Bbbk$.

The proof of Theorem 8.8 is completed. □

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8.2 The case of $\mathcal{P}_{\mathbf{erg}}(\mathbf{Sym}(\mathbb{N}, F))$

Theorem 8.9 (Multiplicativity theorem for orbital limit measures). *Let ν be an element in $\mathscr{ORB}_\infty(\mathrm{Sym}(\mathbb{N}, F))$. Then for any $r \in \mathbb{N}$ and for any finite sequence $(x_1, \ldots, x_r)$ in F, we have*

$$\widehat{\nu}(\mathrm{diag}(x_1, \ldots, x_r, 0, 0, \ldots)) = \prod_{j=1}^{r} \widehat{\nu}(x_i e_{11}).$$

In particular, we have

$$\mathscr{ORB}_\infty(\mathrm{Sym}(\mathbb{N}, F)) = \mathcal{P}_{\mathrm{erg}}(\mathrm{Sym}(\mathbb{N}, F)).$$

Proof. The proof is similar to that of Theorem 8.1. □

Corollary 8.10 (Ismagilov–Olshanski multiplicativity). *An invariant probability measure $\mu \in \mathcal{P}_{\mathrm{inv}}(\mathrm{Sym}(\mathbb{N}, F))$ is ergodic if and only if for any $r \in \mathbb{N}$ and for any finite sequence $x_1, \ldots, x_r$ in F, we have*

$$\widehat{\nu}(\mathrm{diag}(x_1, \ldots, x_r, 0, 0, \ldots)) = \prod_{j=1}^{r} \widehat{\nu}(x_j e_{11}).$$

Lemma 8.11. *Let ν be a Borel probability measure on $\mathrm{Sym}(\mathbb{N}, F)$. Then we have*

$$\lim_{|x|\to 0} \widehat{\nu}(x e_{11}) = 1.$$

Proof. The proof is similar to that of Lemma 8.5. □

Lemma 8.12. *Assume that we are given a sequence of probability measures $(\nu_n)_{n\in\mathbb{N}}$, such that $\nu_n \in \mathscr{ORB}_n(\mathrm{Sym}(\mathbb{N}, F))$. The necessary and sufficient condition for this sequence to be tight is the following.*

(C_2) *There exists $\gamma \in \mathbb{Z}$, such that the supports $\mathrm{supp}(\nu_n)$ are all contained in the following compact subset of $\mathrm{Sym}(\mathbb{N}, F)$:*

$$\{X \in \mathrm{Sym}(\mathbb{N}, F) \mid |X_{ij}| \leqslant q^{\gamma}, \forall i, j \in \mathbb{N}\}.$$

Proof. The above condition (C_2) is clearly sufficient for the sequence to be tight. Now suppose that the sequence is tight, we shall prove that (C_2) is satisfied. By assumption, suppose that ν_n is the $\mathrm{GL}(n, \mathcal{O}_F)$-orbital measure supported on the orbit $\mathrm{GL}(n, \mathcal{O}_F) \cdot X_n$. By Lemma 2.7, we may assume that

$$X_n = \mathrm{diag}(x_1^{(n)}, \ldots, x_n^{(n)}), \quad |x_1^{(n)}| \geqslant \cdots \geqslant |x_n^{(n)}|.$$

Now we argue by contradiction. If the condition is not satisfied, then there exists a subsequence $(n_k)_{k\in\mathbb{N}}$ of positive integers such that

$$\lim_{k\to\infty} |x_1^{(n_k)}| = \infty. \tag{90}$$

Passing to a subsequence if necessary, we may assume that there exists a probability measure ν on $\mathrm{Sym}(\mathbb{N}, F)$, such that $\nu_{n_k} \Longrightarrow \nu$. By Lemma 8.11, for any $a \in F$, we have

$$\lim_{|a|\to 0} \lim_{k\to\infty} \widehat{\nu}_{n_k}(a e_{11}) = \lim_{|a|\to 0} \widehat{\nu}(a e_{11}) = 1.$$

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That is,

$$\lim_{|a|\to 0}\lim_{k\to\infty}\int_{\mathrm{GL}(n_k,\mathcal{O}_F)}\chi(a\cdot\mathrm{tr}(gX_{n_k}g^te_{11}))\,dg=1. \tag{91}$$

By (78) in Theorem 7.5, the relation (91) implies that

$$\lim_{|a|\to 0}\lim_{k\to\infty}\prod_{j=1}^{n_k}\theta(a\cdot x_j^{(n_k)})=1. \tag{92}$$

Using the elementary inequality $|\theta(x)|\leqslant 1$ and (90) and Proposition 4.5, for any $a\in F^\times$, we have

$$\lim_{k\to\infty}\left|\prod_{j=1}^{n_k}\theta(a\cdot x_j^{(n_k)})\right|\leqslant\lim_{k\to\infty}|\theta(a\cdot x_1^{(n_k)})|=0.$$

This contradicts to (92). Thus the condition (C_2) is necessary for the sequence $(\nu_n)_{n\in\mathbb{N}}$ to be tight. □

8.2.1 *Classification of* $\mathcal{P}_{\mathrm{erg}}(\mathrm{Sym}(\mathbb{N},F))$. Recall that by Lemma 4.7, for any $a\in F$ and any $\Bbbk=(k_j)_{j\in\mathbb{N}}\in\Delta$, we may define an infinite product $\prod_{j=1}^{\infty}\theta(a\cdot\varpi^{-k_j})$.

Lemma 8.13. *Let $a\in F$ be a fixed element. Then*

$$\Bbbk=(k_j)_{j\in\mathbb{N}}\mapsto\prod_{j=1}^{\infty}\theta(a\cdot\varpi^{-k_j})$$

defines a continuous map from Δ to $\mathbb{C}$.

Proof. If $a=0$, the assertion is obvious. Now assume that $|a|=q^\gamma$ with $\gamma\in\mathbb{Z}$. Suppose that $\Bbbk^{(n)}=(k_j^{(n)})_{j\in\mathbb{N}}$ converges to $\Bbbk=(k_j)_{j\in\mathbb{N}}$. That is, for any $j\in N$,

$$\lim_{n\to\infty}k_j^{(n)}=k_j. \tag{93}$$

We need to show that

$$\lim_{n\to\infty}\prod_{j=1}^{\infty}\theta(a\cdot\varpi^{-k_j^{(n)}})=\prod_{j=1}^{\infty}\theta(a\cdot\varpi^{-k_j}). \tag{94}$$

First write $k:=\lim_{j\to\infty}k_j\in\mathbb{Z}\cup\{-\infty\}$.

Case 1: $k+\gamma\leqslant 0$.

In this case, there exists $j_0\in\mathbb{N}$, such that $k_{j_0}+\gamma\leqslant 0$. Since $\mathbb{Z}\cup\{-\infty\}$ is a discrete space, by (93), there exists $n_0\in\mathbb{N}$, such that for any $n\geqslant n_0$,

$$\sup_{j\geqslant j_0}(k_j^{(n)}+\gamma)=k_{j_0}^{(n)}+\gamma\leqslant 0. \tag{95}$$

Hence by property (i) in Proposition 4.5, for any $j\geqslant j_0$ and $n\geqslant n_0$, we have

$$\theta(a\cdot\varpi^{-k_j^{(n)}})=\theta(a\cdot\varpi^{-k_j})=1.$$

Consequently, we have

$$\lim_{n\to\infty}\prod_{j=1}^{\infty}\theta(a\cdot\varpi^{-k_j^{(n)}})=\lim_{n\to\infty}\prod_{j=1}^{j_0-1}\theta(a\cdot\varpi^{-k_j^{(n)}})=\prod_{j=1}^{j_0-1}\theta(a\cdot\varpi^{-k_j})=\prod_{j=1}^{\infty}\theta(a\cdot\varpi^{-k_j}).$$

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Case 2: $k+\gamma>0$.

In this case, by the previous argument, we have

$$\prod_{j=1}^{\infty}\theta(a\cdot\varpi^{-k_j})=0.$$

By (93), for any $N\in\mathbb{N}$, there exists $n_0\in\mathbb{N}$, such that for any $n\geqslant n_0$ and any $1\leqslant j\leqslant j_0+N$, we have $k_j^{(n)}=k_j$. This implies

$$\begin{aligned}\limsup_{n\to\infty}\left|\prod_{j=1}^{\infty}\theta(a\cdot\varpi^{-k_j^{(n)}})\right|&\leqslant\limsup_{N\to\infty}\left(\limsup_{n\to\infty}\left|\prod_{j=1}^{j_0+N}\theta(a\cdot\varpi^{-k_j^{(n)}})\right|\right)\\&\leqslant\limsup_{N\to\infty}\left|\prod_{j=1}^{j_0+N}\theta(a\cdot\varpi^{-k_j})\right|=\left|\prod_{j=1}^{\infty}\theta(a\cdot\varpi^{-k_j})\right|=0.\end{aligned}$$

Hence the desired relation (94) holds. □

Theorem 8.14. *Assume that F is non-dyadic. Then the map $\mathbb{h}\mapsto\nu_{\mathbb{h}}$ induces a bijection between Ω and $\mathcal{P}_{\mathrm{erg}}(\mathrm{Sym}(\mathbb{N},F))$.*

Proof. The injectivity of the map $\mathbb{h}\mapsto\nu_{\mathbb{h}}$ from Ω to $\mathcal{P}_{\mathrm{erg}}(\mathrm{Sym}(\mathbb{N},F))$ has already been proved in Proposition 5.3. We only need to prove that the map is also surjective.

Assume that $\nu\in\mathcal{P}_{\mathrm{erg}}(\mathrm{Sym}(\mathbb{N},F))$. Since

$$\mathcal{P}_{\mathrm{erg}}(\mathrm{Sym}(\mathbb{N},F))=\mathscr{ORB}_{\infty}(\mathrm{Sym}(\mathbb{N},F)),$$

there exists a sequence $(\nu_{n_l})_{l\in\mathbb{N}}$ of orbital measures satisfying $\nu_{n_l}\in\mathscr{ORB}_{n_l}(\mathrm{Sym}(n_l,F))$ such that

$$\nu_{n_l}\Longrightarrow\nu\quad\text{as } l\to\infty.\tag{96}$$

By Lemma 2.7, we may assume that ν_{n_l} is the $\mathrm{GL}(n_l,\mathcal{O}_F)$-orbital measure supported on the orbit $\mathrm{GL}(n_l,\mathcal{O}_F)\cdot X_{n_l}$ with

$$X_{n_l}=\mathrm{diag}(x_1^{(n_l)},\dots,x_{n_l}^{(n_l)}),\quad x_1^{(n_l)},\dots,x_{n_l}^{(n_l)}\in\mathcal{T},$$

where $\mathcal{T}$ is given in (26). If for any multi-set[1] B with elements in F, we denote by B^* the multi-set of non-zero elements of B. Then there exist

$$\Bbbk^{(n_l)}=(k_j^{(n_l)})_{j\in\mathbb{N}},\quad \Bbbk'^{(n_l)}=(k_j'^{(n_l)})_{j\in\mathbb{N}}\in\Delta$$

such that the following two multi-sets coincide:

$$\{x_1^{(n_l)},\dots,x_{n_l}^{(n_l)}\}^*=(\{\varpi^{-k_j^{(n_l)}}\mid j\in\mathbb{N}\}\cup\{\varepsilon\varpi^{-k_j'^{(n_l)}}\mid j\in\mathbb{N}\})^*.$$

By Lemma 8.12, the weak convergence (96) implies that

$$\sup_{l\in\mathbb{N}}k_1^{(n_l)}<\infty\quad\text{and}\quad\sup_{l\in\mathbb{N}}k_1'^{(n_l)}<\infty.$$

[1] By multi-set, we mean that the multiplicities of elements are respected. In particular, if B is a multi-set, then $B\cup B$ will be a multi-set, such that the multiplicities for each element is multiplied by 2.

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Consequently, passing to a subsequence if necessary, we may assume that for any $j \in \mathbb{N}$, there exist $k_j, k'_j \in \mathbb{Z} \cup \{-\infty\}$ and we have

$$\lim_{l\to\infty} k_j^{(n_l)} = k_j, \quad \lim_{l\to\infty} k_j'^{(n_l)} = k'_j. \tag{97}$$

Now by the weak convergence (96) and the relation (78), for any $x \in F$, we have

$$\begin{aligned}\widehat{\nu}(xe_{11}) &= \lim_{n\to\infty} \prod_{j=1}^{\infty} \theta(x\varpi^{-k_j^{(n)}}) \prod_{j=1}^{\infty} \theta(x\varepsilon\varpi^{-k_j'^{(n)}}) \\ &= \lim_{l\to\infty} \prod_{j=1}^{\infty} \theta(x\varpi^{-k_j^{(n_l)}}) \prod_{j=1}^{\infty} \theta(x\varepsilon\varpi^{-k_j'^{(n_l)}}).\end{aligned}$$

By the continuity of the map in Lemma 8.13 and (97), we get

$$\widehat{\nu}(xe_{11}) = \prod_{j=1}^{\infty} \theta(x\varpi^{-k_j}) \prod_{j=1}^{\infty} \theta(x\varepsilon\varpi^{-k'_j}). \tag{98}$$

Now by using the identity $\theta(x\varpi^{-k_j})^2 = \theta(x\varepsilon\varpi^{-k'_j})^2$ for any $x \in F$ and $j \in \mathbb{N}$ and by moving certain elements (in $\mathbb{Z}$ and with multiplicities larger than 1) from $(k'_j)_{j\in\mathbb{N}}$ to the sequence $(k_j)_{j\in\mathbb{N}}$, we get a new non-increasing sequence $(\widetilde{k}_j)_{j\in\mathbb{N}}$ and a new strictly decreasing sequence $(\widetilde{k}'_j)_{j\in\mathbb{N}}$ of finite or infinite lengths in $\mathbb{Z} \cup \{-\infty\}$ such that the identity (98) is transformed to

$$\widehat{\nu}(xe_{11}) = \prod_{j=1}^{\infty} \theta(x\varpi^{-\widetilde{k}_j}) \prod_{j=1}^{\infty} \theta(x\varepsilon\varpi^{-\widetilde{k}'_j}). \tag{99}$$

Assume first that $\lim_{j\to\infty} \widetilde{k}_j = -\infty$. Then

$$\mathbb{h} = (-\infty; (\widetilde{k}_j)_{j\in\mathbb{N}}, (\widetilde{k}'_j)_{j\in\mathbb{N}})$$

is an element in Ω. Comparing (99) with the formula in Proposition 4.4, we get

$$\widehat{\nu}(xe_{11}) = \widehat{\nu_{\mathbb{h}}}(xe_{11}) \quad \text{for all } x \in F.$$

But by the multiplicativity of $\widehat{\nu}$ established in Theorem 8.9 and the multiplicativity of $\widehat{\nu}_{\mathbb{h}}$ established in Proposition 4.4, the above identity implies $\nu = \nu_{\mathbb{h}}$.

Assume now that $\lim_{j\to\infty} \widetilde{k}_j = k \in \mathbb{Z}$. Then using the relation (51), we have

$$\prod_{j=1}^{\infty} \theta(x\varpi^{-\widetilde{k}_j}) = \mathbb{1}_{\mathcal{O}_F}(x\varpi^{-k}) \prod_{j\in\{n|\widetilde{k}_n>k\}} \theta(x\varpi^{-\widetilde{k}_j}). \tag{100}$$

Moreover, we also have

$$\mathbb{1}_{\mathcal{O}_F}(x\varpi^{-k}) \prod_{j=1}^{\infty} \theta(x\varepsilon\varpi^{-\widetilde{k}'_j}) = \mathbb{1}_{\mathcal{O}_F}(x\varpi^{-k}) \prod_{j\in\{n|\widetilde{k}'_n>k\}} \theta(x\varepsilon\varpi^{-\widetilde{k}'_j}). \tag{101}$$

It suffices to check for $x \in F$ such that $|x\varpi^{-k}| \leqslant 1$. For $\widetilde{k}'_j$ such that $\widetilde{k}'_j \leqslant k$ (if they exist), we have $|x\varepsilon\varpi^{-\widetilde{k}'_j}| = |x\varpi^{-k}\varepsilon\varpi^{k-\widetilde{k}'_j}| \leqslant 1$. Hence $\theta(x\varepsilon\varpi^{-\widetilde{k}'_j}) = 1$ by property (i) in Proposition 4.5. The identity (101) is proved. Write

$$\widehat{k}_j := \begin{cases} \widetilde{k}_j & \text{if } \widetilde{k}_j > k, \\ -\infty & \text{if } \widetilde{k}_j = k \end{cases} \quad \text{and} \quad \widehat{k}'_j := \begin{cases} \widetilde{k}'_j & \text{if } \widetilde{k}'_j > k, \\ -\infty & \text{if } \widetilde{k}'_j \leqslant k. \end{cases}$$

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Now, $\mathbb{h} = (k; (\widehat{k}_j)_{j\in\mathbb{N}}, (\widehat{k}'_j)_{j\in\mathbb{N}})$, being an element in $\{k\} \times \Delta[k] \times \Delta^\sharp[k]$, is an element of Ω. Combining (99)–(101), we obtain

$$\begin{aligned}\widehat{\nu}(xe_{11}) &= \prod_{j=1}^{\infty} \theta(x\varpi^{-\widetilde{k}_j}) \prod_{j=1}^{\infty} \theta(x\varepsilon\varpi^{-\widetilde{k}'_j}) \\ &= \mathbb{1}_{\mathcal{O}_F}(x\varpi^{-k}) \Bigg[\prod_{j\in\{n|\widetilde{k}_n>k\}} \theta(x\varpi^{-\widetilde{k}_j})\Bigg] \Bigg[\prod_{j\in\{n|\widetilde{k}'_n>k\}} \theta(x\varepsilon\varpi^{-\widetilde{k}'_j})\Bigg].\end{aligned}$$

In this case, we also have

$$\widehat{\nu}(xe_{11}) = \widehat{\nu_{\mathbb{h}}}(xe_{11}) \quad \text{for all } x \in F.$$

By the same argument as above in using the multiplicativities of $\widehat{\nu}$ and $\widehat{\nu}_{\mathbb{h}}$, we get $\nu = \nu_{\mathbb{h}}$.

The proof of Theorem 8.14 is completed. □

9. Properties of the parametrization

9.1 The parametrizations are homeomorphisms

Proof of Theorem 1.3. By Theorem 8.8, we only need to prove that the map $\mathbb{k} \mapsto \mu_{\mathbb{k}}$ from Δ to $\mathcal{P}_{\mathrm{erg}}(\mathrm{Mat}(\mathbb{N}, F))$ and its inverse are both continuous. Note that since Δ and $\mathcal{P}_{\mathrm{erg}}(\mathrm{Mat}(\mathbb{N}, F))$ are metrizable, their topologies are determined by convergence of sequences.

If a sequence $(\mathbb{k}^{(n)})_{n\in\mathbb{N}}$ converges in Δ to a point $\mathbb{k}^{(0)} \in \Delta$, then

$$\sup_{n,j} k_j^{(n)} < \infty. \tag{102}$$

Consequently, the family of the measures $\mu_{\mathbb{k}^{(n)}}$, all being supported on a common compact subset of $\mathrm{Mat}(\mathbb{N}, F)$, is tight. Thus to prove that $\mu_{\mathbb{k}^{(n)}}$ converges weakly to $\mu_{\mathbb{k}^{(0)}}$, it suffices to prove that the latter one is the unique accumulation point of the former family of measures. Now let μ be an accumulation point of the sequence $(\mu_{\mathbb{k}^{(n)}})_{n\in\mathbb{N}}$. By definition, there exists a subsequence $(n_j)_{j\in\mathbb{N}}$ such that

$$\mu = \lim_{j\to\infty} \mu_{\mathbb{k}^{(n_j)}}.$$

Since $\mathbb{k}^{(n)} \longrightarrow \mathbb{k}^{(0)}$, by explicit formula (40) in Proposition 4.1 and Lemma 8.7, the characteristic function of μ is given by

$$\begin{aligned}&\widehat{\mu}(\mathrm{diag}(\varpi^{-\ell_1}, \ldots, \varpi^{-\ell_r}, 0, 0, \ldots)) \\ &\quad = \lim_{j\to\infty} \widehat{\mu}_{\mathbb{k}^{(n_j)}}(\mathrm{diag}(\varpi^{-\ell_1}, \ldots, \varpi^{-\ell_r}, 0, 0, \ldots)) \\ &\quad = \widehat{\mu}_{\mathbb{k}^{(0)}}(\mathrm{diag}(\varpi^{-\ell_1}, \ldots, \varpi^{-\ell_r}, 0, 0, \ldots)), \quad (\ell_j \in \mathbb{Z}).\end{aligned}$$

This implies that we have $\mu = \mu_{\mathbb{k}^{(0)}}$.

Conversely, if $\mu_{\mathbb{k}^{(n)}}$ converges to $\mu_{\mathbb{k}^{(0)}}$. By using the same argument as in the proof of Lemma 8.6, we can still get the relation (102). Again by compactness argument, it suffices to show that $\mathbb{k}^{(0)}$ is the unique accumulation point for the sequence $\mathbb{k}^{(n)}$. But if $\mathbb{k}$ is an accumulation point of $\mathbb{k}^{(n)}$, then $\mu_{\mathbb{k}}$ is an accumulation point of $\mu_{\mathbb{k}^{(n)}}$, whence $\mu_{\mathbb{k}} = \mu_{\mathbb{k}^{(0)}}$. Combining with Proposition 5.1 we have $\mathbb{k} = \mathbb{k}^{(0)}$.

The proof of Theorem 1.3 is completed. □

Proof of Theorem 1.8. By Theorem 8.14, we only need to prove that the map $\mathbb{h} \mapsto \nu_{\mathbb{h}}$ from Ω to $\mathcal{P}_{\mathrm{erg}}(\mathrm{Sym}(\mathbb{N}, F))$ and its inverse are both continuous. The proof of this part is similar to that of Theorem 1.3 as above. □

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9.2 The parametrizations are semi-group homomorphisms

On Δ is equipped with an Abelian semi-group structure. Given any two points $\Bbbk = (k_n)_{n\in\mathbb{N}}$ and $\Bbbk' = (k'_n)_{n\in\mathbb{N}}$ in Δ, we define $\Bbbk \oplus_\Delta \Bbbk'$ as follows.

(i) If $\inf k_n = \inf k'_n = -\infty$, then we define $\Bbbk\oplus_\Delta\Bbbk' \in \Delta$ to be the non-increasing rearrangement of the sequence $(\widetilde{k_n})_{n\in\mathbb{N}}$, where

$$\widetilde{k_{2n-1}} := k_n \quad \text{and} \quad \widetilde{k_{2n}} := k'_n \quad (n = 1, 2, \ldots).$$

(ii) If $k = \max\{\inf k_n, \inf k'_n\} \in \mathbb{Z}$, then we define $\Bbbk \oplus \Bbbk' \in \Delta$ to be the non-increasing rearrangement of the sequence $(k^*_n)_{n\in\mathbb{N}}$, that is any sequence exhausting the integers larger than k and from $\Bbbk$ and $\Bbbk'$, repeated with corresponding multiplicity. For instance, if $\Bbbk = (6, 2, 2, -3, -3, -3, -3, \ldots), \Bbbk' = (4, 3, 0, -1, -\infty, \cdots)$, then we define

$$\Bbbk \oplus_\Delta \Bbbk' = (6, 4, 3, 2, 2, 0, -1, -3, -3, -3, -3, \ldots).$$

Clearly, we have the following.

Proposition 9.1. *The map $\Bbbk \mapsto \mu_\Bbbk$ defines a semi-group isomorphism between $(\Delta, \oplus_\Delta)$ and $(\mathcal{P}_{\mathrm{erg}}(\mathrm{Mat}(\mathbb{N}, F)), *)$. More precisely, we have*

$$\mu_\Bbbk * \mu_{\Bbbk'} = \mu_{\Bbbk\oplus_\Delta\Bbbk'}.$$

An Abelian semigroup structure $\oplus_\Omega$ on Ω such that $\mathbb{h} \to \nu_\mathbb{h}$ defines semi-groups isomorphism between $(\Omega, \oplus_\Omega)$ and $(\mathcal{P}_{\mathrm{erg}}(\mathrm{Sym}(\mathbb{N}, F)), *)$ is introduced in the same way.

10. Proof of Proposition 4.5

In this section, we always assume that F is non-dyadic. We will use the following change of variables in the integration over a local field. To introduce the formula for change of variables, we need the notion of F-analytic functions. A function $\varphi : U \to V$, with U, V open subsets of F, is called F-analytic, if in some neighbourhood of any point in U it is given by a convergent power series, it is called F-bi-analytic, if φ is invertible such that both $\varphi : U \to V$ and $\varphi^{-1} : V \to U$ are F-analytic.

Theorem 10.1 (Change of variables, see Schoissengeier [Sch92]). *Let $\varphi : U \to V$ be a F-bi-analytic function. Then for any integrable function $f : U \to \mathbb{C}$, we have*

$$\int_U f(\varphi(x))|\varphi'(x)|\,dx = \int_V f(y)\,dy,$$

where φ' is the formal derivative of φ.

We will also need the following classical result from number theory concerning Gauss sums for finite field $\mathbf{F}_q$. For the reader's convenience, we include its standard proof in Appendix A.

Denote by λ_2 the unique multiplicative character for $\mathbf{F}_q^\times$ of order 2, that is,

$$\lambda_2(a) = \begin{cases} 1 & \text{if } a \in (\mathbf{F}_q^\times)^2, \\ -1 & \text{if } a \notin (\mathbf{F}_q^\times)^2. \end{cases}$$

By convention, we extend the definition of λ_2 to the whole finite field $\mathbf{F}_q$ by setting $\lambda_2(0) = 0$.

Denote the set of additive characters of $\mathbf{F}_q$ by $\widehat{\mathbf{F}}_q$. Given any $\tau \in \widehat{\mathbf{F}}_q\backslash\{1\}$ (that is, τ is a non-trivial character of $\mathbf{F}_q$) and any $a \in \mathbf{F}_q$, denote by τ_a the character of $\mathbf{F}_q$ defined by $\tau_a(x) = \tau(a \cdot x)$. It is a standard fact that the map $a \mapsto \tau_a$ is an group isomorphism between $\mathbf{F}_q$ and $\widehat{\mathbf{F}}_q$.

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LEMMA 10.2 (Gauss sums). *Fix an element $\tau \in \widehat{\mathbf{F}}_q \backslash \{1\}$. Then for any $a \in \mathbf{F}_q^\times$, we have*

$$\sum_{x \in \mathbf{F}_q} \tau_a(x^2) = \lambda_2(a) \cdot \sum_{x \in \mathbf{F}_q} \tau(x^2). \tag{103}$$

Moreover,

$$\left\{ \sum_{x \in \mathbf{F}_q} \tau_a(x^2) \;\middle|\; a \in \mathbf{F}_q^\times \right\} = \{\varrho_q \sqrt{q}, -\varrho_q \sqrt{q}\}.$$

Denote by $\mathcal{O}_F^*$ the set of non-zero elements in $\mathcal{O}_F$ and denote by $(\mathcal{O}_F^*)^2$ the square elements in $\mathcal{O}_F^*$, that is,

$$(\mathcal{O}_F^*)^2 := \{a \in \mathcal{O}_F^* \text{: there exists } b \in F \text{ such that } a = b^2\}.$$

Denote the square function $x \mapsto x^2$ by $\psi(x) = x^2$.

PROPOSITION 10.3. *There exists a partition*

$$\mathcal{O}_F^* = U_1 \sqcup U_2,$$

such that the square function $\psi(x) = x^2$ induces two F-bi-analytic functions:

$$\psi : U_i \to (\mathcal{O}_F^*)^2, \quad i = 1, 2.$$

Recall that by Lemma 2.5, the group $(\mathcal{O}_F^\times)^2$ is a disjoint union of $(q-1)/2$ balls of radius q^{-1}:

$$(\mathcal{O}_F^\times)^2 = \bigsqcup_{a \in (\mathcal{C}_q^\times)^2} (a + \varpi \mathcal{O}_F).$$

LEMMA 10.4. *Any element $a \in (\mathcal{O}_F^\times)^2$ has two square roots $\alpha_1, \alpha_2 \in \mathcal{O}_F^\times$ such that $(\alpha_1 + \varpi\mathcal{O}_F) \cap (\alpha_2 + \varpi\mathcal{O}_F) = \emptyset$ and we have*

$$\psi(\alpha_i + \varpi \mathcal{O}_F) \subset a + \varpi \mathcal{O}_F, \quad i = 1, 2. \tag{104}$$

Moreover, we have two bijective maps:

$$\alpha_i + \varpi \mathcal{O}_F \xrightarrow{x \mapsto \psi(x) = x^2} a + \varpi \mathcal{O}_F, \quad i = 1, 2. \tag{105}$$

Proof. For any $a \in (\mathcal{O}_F^\times)^2$, there exist exactly two elements $\alpha_1, \alpha_2 \in \mathcal{O}_F^\times$, such that

$$\alpha_1 = -\alpha_2 \quad \text{and} \quad \alpha_1^2 = \alpha_2^2 = a.$$

Hence $|\alpha_1 - \alpha_2| = |2\alpha| = |\alpha| = 1$ and

$$(\alpha_1 + \varpi \mathcal{O}_F) \cap (\alpha_2 + \varpi \mathcal{O}_F) = \emptyset.$$

For any $z \in \mathcal{O}_F$ and $i = 1, 2$, we have

$$(\alpha_i + \varpi z)^2 = a + \varpi(2z\alpha_i + \varpi z^2) \in a + \varpi \mathcal{O}_F.$$

This proves (104).

By Hensel's lemma (24), for $i = 1, 2$, if $a' \in a + \varpi \mathcal{O}_F$, then there exists $\alpha_i' \in \alpha_i + \varpi \mathcal{O}_F$ such that $(\alpha_i')^2 = a'$. Hence the maps (105) for $i = 1, 2$ are both surjective. Now fix $i \in \{1, 2\}$.

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If $\delta_1, \delta_2 \in \alpha_i + \varpi\mathcal{O}_F$ are such that $\delta_1^2 = \delta_2^2$, then either $\delta_1 = \delta_2$ or $\delta_1 = -\delta_2$. However, if $\delta_1 = -\delta_2$, then

$$|\delta_1 - \delta_2| = |2\delta_1| = |2||\delta_1| = 1.$$

This contradicts to the following estimate:

$$|\delta_1 - \delta_2| = |(\delta_1 - \alpha_i) - (\delta_2 - \alpha_i)| \leqslant \max(|\delta_1 - \alpha_i|, |\delta_2 - \alpha_i|) \leqslant q^{-1}.$$

Hence we must have $\beta_1 = \beta_2$. This proves the injectivity of the map (105). □

Proof of Proposition 10.3. Clearly, we have

$$\mathcal{O}_F^* = \bigsqcup_{k=0}^{\infty} \varpi^k \mathcal{O}_F^\times \quad \text{and} \quad (\mathcal{O}_F^*)^2 = \bigsqcup_{k=0}^{\infty} \varpi^{2k} (\mathcal{O}_F^\times)^2.$$

For any $k = 0, 1, \ldots$, the square map φ maps $\varpi^k \mathcal{O}_F^\times$ surjectively into $\varpi^{2k}(\mathcal{O}_F^\times)^2$. For proving Proposition 10.3, it suffices to prove that for any $k = 0, 1, \ldots$, the set $\varpi^k \mathcal{O}_F^\times$ can be divided into two parts, such that the square map ψ maps each part surjectively into $\varpi^{2k}(\mathcal{O}_F^\times)^2$ and the restriction of ψ on each part is F-bi-analytic.

We only need to prove this assertion for $k = 0$, since the other $k \geqslant 1$ can be reduced to the case $k = 0$ by a suitable dilation. By Lemma 10.4, $\mathcal{O}_F^\times$ can be divided into two parts $\mathcal{O}_F^\times = V_1 \sqcup V_2$, such that the two maps $\psi : V_i \to (\mathcal{O}_F^\times)^2, i = 1, 2$ are both bijective. The analyticity of the inverse maps $(\psi|_{V_i})^{-1}$ follows from the Inverse Mapping Theorem in non-Archimedean setting, see, e.g., Abhyankar [Abh64, p. 87]. □

Corollary 10.5. *For any integrable function $f : \mathcal{O}_F \to \mathbb{C}$, we have*

$$\int_{\mathcal{O}_F} f(z^2)|z|\, dz = 2 \int_{(\mathcal{O}_F^*)^2} f(y)\, dy. \tag{106}$$

Proof. Note that since F is non-dyadic, $2 \in \mathcal{O}_F^\times$. Using the notation in Proposition 10.3 and Theorem 10.1

$$\int_{\mathcal{O}_F} f(z^2)|z|\, dz = \int_{\mathcal{O}_F^*} f(\psi(z))|\psi'(z)|\, dz = \sum_{i=1}^{2} \int_{U_i} f(\psi(z))|\psi'(z)|\, dz = 2 \int_{(\mathcal{O}_F^*)^2} f(y)\, dy. \quad \square$$

Proof of Proposition 4.5. Property (i) in Proposition 4.5 is trivial. We proceed with the proof of properties (ii) and (iii).

For any $x \in F^*$, define

$$f(z) = \frac{1}{|z|^{1/2}} \chi(xz).$$

Then by substituting f into the identity (106), we get

$$\begin{aligned} \theta(x) &= \int_{\mathcal{O}_F} \chi(xz^2)\, dz = \int_{\mathcal{O}_F} f(z^2)|2z|\, dz \\ &= 2 \int_{(\mathcal{O}_F^*)^2} f(y)\, dy = 2 \int_{(\mathcal{O}_F^*)^2} \frac{1}{|y|^{1/2}} \chi(xy)\, dy. \end{aligned} \tag{107}$$

Define

$$g(y) := \frac{2}{|y|^{1/2}} \mathbb{1}_{(\mathcal{O}_F^*)^2}(y). \tag{108}$$

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It is a standard fact that $g \in L^1(F, dx)$. The identity (107) can now be rewritten as

$$\theta(x) = \widehat{g}(x).$$

Since we have

$$(\mathcal{O}_F^*)^2 = \bigsqcup_{k=0}^{\infty} \varpi^{2k}(\mathcal{O}_F^\times)^2 = \bigsqcup_{k=0}^{\infty} \bigsqcup_{a\in(\mathcal{C}_q^\times)^2} (\varpi^{2k}a + \varpi^{2k+1}\mathcal{O}_F),$$

the function g defined by formula (108) can be written in the form

$$g(y) = 2\sum_{k=0}^{\infty} q^k \sum_{a\in(\mathcal{C}_q^\times)^2} \mathbb{1}_{\varpi^{2k}a+\varpi^{2k+1}\mathcal{O}_F}(y).$$

Consequently, we have

$$\theta(x) = \widehat{g}(x) = 2\sum_{k=0}^{\infty} q^{-k-1}\mathbb{1}_{\varpi^{-2k-1}\mathcal{O}_F}(x) \sum_{a\in(\mathcal{C}_q^\times)^2} \chi(\varpi^{2k}ax). \tag{109}$$

Property (ii) in Proposition 4.5. If $\mathrm{ord}_F(x) = -2k_0$ with $k_0 \geqslant 1$, then $x = \varpi^{-2k_0}u$ with $u \in \mathcal{O}_F^\times$. Substituting $x = \varpi^{-2k_0}u$ into (109) and using the assumption (28) on χ, we obtain

$$\begin{aligned}\theta(x) &= 2\sum_{k=k_0}^{\infty} q^{-k-1} \sum_{a\in(\mathcal{C}_q^\times)^2} \chi(a\varpi^{2k-2k_0}u)\\ &= 2\sum_{k=k_0}^{\infty} q^{-k-1}\cdot\frac{q-1}{2} = q^{-k_0} = |x|^{-1/2}.\end{aligned}$$

We thus complete the proof of property (ii) in Proposition 4.5.

Property (iii) in Proposition 4.5. If $\mathrm{ord}_F(x) = -2k_0 - 1$ with $k_0 \geqslant 0$, then $x = \varpi^{-2k_0-1}u$ with $u \in \mathcal{O}_F^\times$. By substituting $x = \varpi^{-2k_0-1}u$ into (109) and using the fact that $a\varpi^{2k-2k_0-1}u \in \mathcal{O}_F$ for any $a \in (\mathcal{C}_q^\times)^2, k \geqslant k_0+1$ and the assumption (28) on the choice of χ, we obtain

$$\begin{aligned}\theta(x) &= 2\sum_{k=k_0}^{\infty} q^{-k-1} \sum_{a\in(\mathcal{C}_q^\times)^2} \chi(a\varpi^{2k-2k_0-1}u)\\ &= 2q^{-k_0-1} \sum_{a\in(\mathcal{C}_q^\times)^2} \chi(a\varpi^{-1}u) + 2\sum_{k=k_0+1}^{\infty} q^{-k-1}\cdot\frac{q-1}{2}\\ &= q^{-k_0-1}\Bigg(2\sum_{a\in(\mathcal{C}_q^\times)^2} \chi(a\varpi^{-1}u) + 1\Bigg).\end{aligned} \tag{110}$$

Now define a function $z \mapsto \chi(z\varpi^{-1}u)$ for $z \in \mathcal{O}_F$. Since this function takes same value on every coset of $\varpi\mathcal{O}_F$ in $\mathcal{O}_F$, we define a non-trivial additive character τ on $\mathbf{F}_q = \mathcal{O}_F/\varpi\mathcal{O}_F$ by the following formula:

$$\tau(z + \varpi\mathcal{O}_F) := \chi(z\varpi^{-1}u).$$

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Thus we have

$$\theta(x) = q^{-k_0-1}\left(2\sum_{a\in(\mathcal{C}_q^\times)^2}\tau(a+\varpi\mathcal{O}_F)+1\right)$$
$$= q^{-k_0-1}\sum_{z+\varpi\mathcal{O}_F\in\mathbf{F}_q}\tau(z^2+\varpi\mathcal{O}_F). \tag{111}$$

Indeed, the second equality in (111) follows from the fact that for every element in $a\in(\mathcal{C}_q^\times)^2$, there exist exactly two distinct square roots: $z+\varpi\mathcal{O}_F$ and $-z+\varpi\mathcal{O}_F$, that is,

$$(z+\varpi\mathcal{O}_F)^2\equiv(-z+\varpi\mathcal{O}_F)^2\equiv z^2+\varpi\mathcal{O}_F\equiv a+\varpi\mathcal{O}_F(\mathrm{mod}\ \varpi\mathcal{O}_F).$$

Now by applying Theorem 10.2, we have

$$|\theta(x)| = q^{-k_0-(1/2)} = |x|^{-1/2}.$$

Now assume that $v\in\mathcal{O}_F^\times\backslash(\mathcal{O}_F^\times)^2$. Then by changing x to vx (equivalently, changing u to $v\cdot u$) in (110) and applying (103), we obtain

$$\theta(vx) = q^{-k_0-1}\left(2\sum_{a\in(\mathcal{C}_q^\times)^2}\chi(a\varpi^{-1}vu)+1\right)$$
$$= q^{-k_0-1}\left(2\sum_{a\in(\mathcal{C}_q^\times)^2}\tau(v\cdot a+\varpi\mathcal{O}_F)+1\right)$$
$$= q^{-k_0-1}\sum_{z+\varpi\mathcal{O}_F\in\mathbf{F}_q}\tau(v\cdot z^2+\varpi\mathcal{O}_F) = -\theta(x).$$

We thus complete the proof of property (iii) in Proposition 4.5. □

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Appendix A

A.1 Proof of Proposition 8.4

Lemma A.1. *Let $m\in\mathbb{N}$ be a positive integer. Suppose that $(\sigma_n)_{n\in\mathbb{N}}$ is a sequence of probability measures on F^m, such that*

$$\lim_{n\to\infty}\widehat{\sigma}_n(x) = \phi(x) \quad \textit{for all } x\in F^m.$$

Assume that the function ϕ is continuous at the origin $0\in F^m$. Then there exists a probability measure σ on F^m, such that $\widehat{\sigma}(x)=\phi(x)$ and

$$\sigma_n \Longrightarrow \sigma \quad \textit{as } n\to\infty. \tag{A.1}$$

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Proof. First we show that under the hypothesis of Lemma A.1, the sequence of probability measures $(\sigma_n)_{n\in\mathbb{N}}$ is tight. By using Fubini's theorem and Lemma 3.3, for any $k \in \mathbb{N}$, we have

$$\begin{aligned}
&⨍_{(\varpi^k\mathcal{O}_F)^m} (1-\widehat{\sigma}_n(x))\,d\mathrm{vol}(x)\\
&\quad= ⨍_{(\varpi^k\mathcal{O}_F)^m} d\mathrm{vol}(x)\int_{F^m}(1-\chi(x\cdot z))\,d\sigma_n(z)\\
&\quad= \int_{F^m} d\sigma_n(z)\,⨍_{(\varpi^k\mathcal{O}_F)^m}(1-\chi(x\cdot z))\,d\mathrm{vol}(x)\\
&\quad= \int_{F^m}(1-\mathbb{1}_{(\varpi^{-k}\mathcal{O}_F)^m}(z))\,d\sigma_n(z)\\
&\quad= \sigma_n(\{z\in F^m : \|z\|\geqslant q^k\}).
\end{aligned}$$

By bounded convergence theorem, we have

$$⨍_{(\varpi^k\mathcal{O}_F)^m}(1-\widehat{\sigma}_n(x))\,d\mathrm{vol}(x)\xrightarrow{n\to\infty}⨍_{(\varpi^k\mathcal{O}_F)^m}(1-\phi(x))\,d\mathrm{vol}(x).$$

By assumption, ϕ is continuous at 0. Since $\phi(0)=1$, for any $\varepsilon>0$ there exists k large enough such that

$$⨍_{(\varpi^k\mathcal{O}_F)^m}(1-\phi(x))\,d\mathrm{vol}(x)\leqslant\varepsilon/2.$$

Fix such an integer $k\in\mathbb{N}$, and choose $n_0\in\mathbb{N}$ such that for any $n\geqslant n_0$, we have

$$⨍_{(\varpi^k\mathcal{O}_F)^m}(1-\widehat{\sigma}_n(x))\,d\mathrm{vol}(x)\leqslant\varepsilon.$$

That is, for any $n\geqslant n_0$, we have $\sigma_n(\{z\in F^m:\|z\|\geqslant q^k\})\leqslant\varepsilon$. Note that we may choose k' large enough such that

$$\sup_{1\leqslant n\leqslant n_0}\sigma_n(\{z\in F^m:\|z\|\geqslant q^{k'}\})\leqslant\varepsilon.$$

Hence by taking $K=\max(k,k')$, we get

$$\sup_{n\in\mathbb{N}}\sigma_n(\{z\in F^m:\|z\|\geqslant q^K\})\leqslant\varepsilon.$$

This proves the tightness of the sequence $(\sigma_n)_{n\in\mathbb{N}}$.

Now, for proving (A.1), we only need to show that any weakly convergent subsequence $(\sigma_{n_k})_{k\in\mathbb{N}}$ has the same limit point σ. Indeed, assume that $\sigma_{n_k}\Longrightarrow\sigma$. Then

$$\widehat{\sigma}(x)=\lim_{k\to\infty}\widehat{\sigma}_{n_k}(x)=\phi(x),$$

which does not depend on the choice of the subsequence. We now complete the proof by using the fact that the measure is uniquely determined by its characteristic function. □

LEMMA A.2. *Let $m\in\mathbb{N}$ be a positive integer. Suppose that $(\sigma_n)_{n\in\mathbb{N}}$ and σ are probability measures on F^m. Then $\sigma_n\Longrightarrow\sigma$ as $n\to\infty$ if and only if $\widehat{\sigma}_n(x)$ converges uniformly to $\widehat{\sigma}(x)$ for x in any compact subset of F^m.*

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Proof. Assume first that $\widehat{\sigma}_n(x)$ converges uniformly to $\widehat{\sigma}(x)$ on any compact subset of F^m. Then, by Lemma A.1, $\sigma_n \Longrightarrow \sigma$ as $n \to \infty$.

Conversely, assume that $\sigma_n \Longrightarrow \sigma$ as $n \to \infty$. For any $x \in F^m$, the function $z \mapsto \chi(x \cdot z)$ is bounded and continuous, hence $\widehat{\sigma}_n(x)$ converges to $\widehat{\sigma}(x)$. By the Arzelà–Ascoli theorem, for proving the uniform convergence of $\widehat{\sigma}_n$ on any compact subset, it suffices to prove the sequence of characteristic functions $\widehat{\sigma}_n$ is equicontinuous on any compact subset of F^m. In fact, let us prove that these characteristic functions are equicontinuous on the whole space F^m. For any $\varepsilon > 0$, since the sequence $(\sigma_n)_{n\in\mathbb{N}}$ of probability measures is tight, there exists $k \in \mathbb{N}$ large enough, such that

$$\sup_{n\in\mathbb{N}} \sigma_n(\{z \in F^m : \|z\| \geqslant q^k\}) \leqslant \varepsilon/2.$$

Now if $y \in F^m$ is such that $\|y\| \leqslant q^{-k}$, then for any $x \in F^m$, we have

$$\begin{aligned} |\widehat{\sigma}_n(x+y) - \widehat{\sigma}_n(x)| &\leqslant \int_{F^m} |\chi(y\cdot z) - 1|\, d\sigma_n(z) \\ &\leqslant 2\sigma_n(\{z \in F^m : \|z\| \geqslant q^k\}) \leqslant \varepsilon. \end{aligned}$$

This proves the equicontinuity of the sequence $(\widehat{\sigma}_n)_{n\in\mathbb{N}}$ and completes the proof of Lemma A.2. □

Now we may prove Proposition 8.4 by using the following two points.

- Characteristic functions $\widehat{\mu}_n$ and $\widehat{\mu}$ are all invariant under the action of the group $\mathcal{K}(\infty) = \mathrm{GL}(\infty, \mathcal{O}_F) \times \mathrm{GL}(\infty, \mathcal{O}_F)$.
- Checking the convergence $\mu_n \Longrightarrow \mu$ is equivalent to checking, for all $r \in \mathbb{N}$, the convergence $(\mathrm{Cut}_r^\infty)_*\mu_n \Longrightarrow (\mathrm{Cut}_r^\infty)_*\mu$.

A.2 Proof of Lemma 10.2

Let $a \in \mathbf{F}_q^\times$. First we claim that

$$\sum_{x\in\mathbf{F}_q} \tau_a(x^2) = \sum_{x\in\mathbf{F}_q} \lambda_2(x)\tau_a(x). \tag{A.2}$$

Indeed,

$$\begin{aligned} \sum_{x\in\mathbf{F}_q} \lambda_2(x)\tau_a(x) &= \sum_{x\in\mathbf{F}_q} \left(\sum_{y\in\mathbf{F}_q} \mathbb{1}_{\{y^2=x\}}(y) - 1 \right) \tau_a(x) \\ &= \sum_{y\in\mathbf{F}_q} \sum_{x\in\mathbf{F}_q} \mathbb{1}_{\{y^2=x\}}(y)\tau_a(x) - \sum_{y\in\mathbf{F}_q} \tau_a(x) = \sum_{y\in\mathbf{F}_q} \tau_a(y^2). \end{aligned}$$

Consequently, by using the fact that λ_2 is a multiplicative character of $\mathbf{F}_q^\times$, we have

$$\begin{aligned} \sum_{x\in\mathbf{F}_q} \tau_a(x^2) &= \sum_{x\in\mathbf{F}_q} \lambda_2(x)\tau(ax) = \sum_{x\in\mathbf{F}_q} \lambda_2(a^{-1}y)\tau(y) \\ &= \lambda_2(a^{-1}) \sum_{x\in\mathbf{F}_q} \lambda_2(y)\tau(y) = \lambda_2(a) \cdot \sum_{x\in\mathbf{F}_q} \tau(x^2). \end{aligned}$$

Hence the identity (103) is proved.

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Let us now show that $|\sum_{x\in\mathbf{F}_q}\tau_a(x^2)|=\sqrt{q}$. Indeed, by (A.2), we have

$$\begin{aligned}\Big|\sum_{x\in\mathbf{F}_q}\tau_a(x^2)\Big|^2 &= \sum_{x,y\in\mathbf{F}_q}\lambda_2(x)\lambda_2(y)\overline{\tau_a(x)}\tau_a(y)\\ &= \sum_{x\in\mathbf{F}_q^\times}\sum_{y\in\mathbf{F}_q}\lambda_2(x)\lambda_2(y)\tau_a(y-x).\end{aligned}$$

For fixed $x\in\mathbf{F}_q^\times$, by change of variables $y=xz$, we get

$$\begin{aligned}\Big|\sum_{x\in\mathbf{F}_q}\tau_a(x^2)\Big|^2 &= \sum_{x\in\mathbf{F}_q^\times}\sum_{z\in\mathbf{F}_q}\lambda_2(x)\lambda_2(xz)\tau_a(xz-x)\\ &= \sum_{z\in\mathbf{F}_q}\sum_{x\in\mathbf{F}_q^\times}\lambda_2(z)\tau_a(xz-x)\\ &= (q-1)+\sum_{z\in\mathbf{F}_q\setminus\{1\}}\lambda_2(z)\sum_{x\in\mathbf{F}_q^\times}\tau_a(x(z-1))\\ &= (q-1)-\sum_{z\in\mathbf{F}_q\setminus\{1\}}\lambda_2(z)=q.\end{aligned}$$

For concluding the proof, we need to show that

$$\sum_{x\in\mathbf{F}_q}\tau(x^2)\in\varrho_q\cdot\mathbb{R}. \tag{A.3}$$

However, if $-1\notin(\mathbf{F}_q^\times)^2$, then $\lambda_2(a)=-1$ and we have

$$\overline{\sum_{x\in\mathbf{F}_q}\tau(x^2)}=\sum_{x\in\mathbf{F}_q}\tau(-x^2)=\sum_{x\in\mathbf{F}_q}\tau_{-1}(x^2)=-\sum_{x\in\mathbf{F}_q}\tau(x^2).$$

It follows that $\sum_{x\in\mathbf{F}_q}\tau(x^2)\in i\mathbb{R}$. By similar argument, we can show that if $-1\in(\mathbf{F}_q^\times)^2$, then $\sum_{x\in\mathbf{F}_q}\tau(x^2)\in\mathbb{R}$. By definition of ϱ_q and the cyclic structure of the group $\mathbf{F}_q^\times$, we get the desired relation (A.3).

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