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Mirror symmetry and Hitchin system on Deligne–Mumford curves: Strominger–Yau–Zaslow duality

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Abstract. We systematically study the moduli stacks of Higgs bundles, spectral curves, and Norm maps on Deligne–Mumford curves. As an application, under some mild conditions, we prove the Strominger–Yau–Zaslow duality for the moduli spaces of Higgs bundles over a hyperbolic stacky curve.

1 Introduction

1.1 Strominger–Yau–Zaslow

Mirror symmetry stemmed from the study of superstring compactification in the late 1980s. Its first precise formulation was given by Candelas, dela Ossa, Green, and Parkes. They conjectured a formula for the number of rational curves of given degree on a quintic Calabi–Yau in terms of the periods of the holomorphic three form on another "mirror" Calabi–Yau manifold, which is related to the theory of closed strings in physics (see [COPL91]). In the mid-1990s, two developments emerged, inspired by the open string theory: Kontsevich's proposal of homological mirror symmetry (see [Kon95]) and the proposal of Strominger–Yau–Zaslow (SYZ; see [SYZ96]).

Let us focus on the SYZ's proposal. Consider a pair of compact Calabi–Yau threefolds M and $\check{M}$ related by mirror symmetry in the sense: the set of BPS A-branes on M is isomorphic to the set of BPS B-branes on $\check{M}$, while the set of BPS B-branes on M is isomorphic to the set of BPS A-branes on $\check{M}$. The simplest BPS B-branes on M are points, and their moduli space is M itself. For every point in M, the corresponding BPS A-brane on $\check{M}$ is a pair (T, L), where T is a special Lagrangian submanifold of $\check{M}$ and L is a flat $U(1)$-bundle on T. Then, there is a family of special Lagrangian submanifolds on $\check{M}$ parametrized by points of M. According to McLean's theorem [Mcl98], the deformation space of a special Lagrangian submanifold T is unobstructed and has real dimension $b_1(T)$ (the first Betti number of T). On the other hand, the moduli space of flat $U(1)$-bundles on T is $H^1(T, \mathbb{R}/\mathbb{Z})$ (a torus of real dimension $b_1(T)$). Thus, the

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total dimension of the moduli space of (T,L) is $2b_1(T)$. Since the moduli space is M, we must have $b_1(T)=3$. Then, M is fibered by tori of dim 3. Exchanging the roles of $\check{M}$ and M, we conclude that $\check{M}$ is also fibered by three-dimensional tori. Motivated by those, SYZ made a conjecture called SYZ conjecture: every n-dimensional Calabi–Yau manifold M admits a mirror $\check{M}$ (which is also a Calabi–Yau manifold of dim n). And, there exists a real manifold N of dimension n together with two smooth fibrations h, $\check{h}$

$$\begin{tikzcd} M \arrow[dr, "h"'] & & \check{M} \arrow[dl, "\check{h}"] \\ & N & \end{tikzcd}$$

where the generic fiber is a special Lagrangian n-torus. Moreover, h and $\check{h}$ are dual in the sense that for a common regular point $b\in N$ of h and $\check{h}$, we have

$$h^{-1}(b)=H^1(\check{h}^{-1}(b),\mathbb{R}/\mathbb{Z}),\quad \check{h}^{-1}(b)=H^1(h^{-1}(b),\mathbb{R}/\mathbb{Z}).$$

Hitchin [Hit01] extended the formulation of SYZ conjecture to Calabi–Yau manifolds with B-fields, where B-fields are flat unitary gerbes in mathematics. Suppose that $\boldsymbol{B}$ is a flat unitary gerbe on a Calabi–Yau X such that the restriction of $\boldsymbol{B}$ to every special Lagrangian torus fiber T is trivial. Since the set of isomorphism classes of flat unitary gerbes on T is $H^2(T,\mathbb{R}/\mathbb{Z})$, a trivialization of $\boldsymbol{B}$ on T is a 1-cochain whose coboundary is $\boldsymbol{B}$ and two trivializations are equivalent if they differ by an exact cocycle. Then, the set $\mathrm{Triv}^{U(1)}(T,\boldsymbol{B})$ of equivalence classes of trivializations of $\boldsymbol{B}$ on T is an $H^1(T,\mathbb{R}/\mathbb{Z})$-torsor. The SYZ mirror of Calabi–Yau X with a B-field $\boldsymbol{B}$ is defined to be the moduli space of pairs (T,t) where T is a special Lagrangian torus and t is a flat trivialization of $\boldsymbol{B}$ on T. Note that if $\boldsymbol{B}$ is a trivial flat unitary gerbe, we obtain the original SYZ mirror. More precisely, two n-dimensional Calabi–Yau orbifolds X and $\check{X}$, equipped with B-fields $\boldsymbol{B}$ and $\check{\boldsymbol{B}}$, respectively, are said to be mirror partners, if there is an n-dimensional real orbifold Y and two smooth surjections μ, $\check{\mu}$

$$\begin{tikzcd} X \arrow[dr, "\mu"'] & & \check{X} \arrow[dl, "\check{\mu}"] \\ & Y & \end{tikzcd}$$

such that for every regular value $x\in Y$ of μ and $\check{\mu}$, the fibers $\mu^{-1}(x)$ and $\check{\mu}^{-1}(x)$ are special Lagrangian tori and dual to each other in the sense that there are smooth identifications

$$\mu^{-1}(x)=\mathrm{Triv}^{U(1)}(\check{\mu}^{-1}(x),\check{\boldsymbol{B}})\quad\text{and}\quad \check{\mu}^{-1}(x)=\mathrm{Triv}^{U(1)}(\mu^{-1}(x),\boldsymbol{B}).$$

In [HT03], Hausel and Thaddeus showed that the moduli spaces of flat connections on a curve with structure groups $\mathbf{SL}_r$ and $\mathbf{PGL}_r$ are mirror partners in the above sense. Their work has been extended to the G_2 case by Hitchin [Hit07] and to all semisimple algebraic groups by Donagi and Pantev [DP12]. For the case of parabolic Higgs bundles, Biswas and Dey [BD12] proved the SYZ conjecture for full flags parabolic Higgs bundles with structure groups $\mathbf{SL}_r$ and $\mathbf{PGL}_r$.

1.2 Moduli spaces of Higgs bundles, Hitchin morphisms, and Norm maps

In [Nir08], Nironi constructed the moduli stacks (spaces) of coherent sheaves on projective Deligne–Mumford stacks. We use his construction to study the moduli stacks (spaces) of Higgs bundles on Deligne–Mumford curves. In fact, Simpson used coverings by smooth projective varieties to give description of the moduli stacks of Higgs bundles with vanishing Chern classes on Deligne–Mumford curves (see [Sim11]). For the stacky curves (or orbifold curves), Biswas–Majumder–Wong [BMW13], Borne [Bor07], Nasatyr–Steer [NS95], and others had considered the problem.

Let $\mathcal{X}$ be a complex hyperbolic Deligne–Mumford curve with coarse moduli space $\pi : \mathcal{X} \to X$. We show that the moduli stack $\mathcal{M}_{\mathrm{Dol}}(\mathbf{GL}_r)$ of rank r Higgs bundles on $\mathcal{X}$ is locally of finite type over $\mathbb{C}$. Fix a polarization $(\mathcal{E}, \mathcal{O}_X(1))$ on $\mathcal{X}$, where $\mathcal{E}$ is a generating sheaf (see Section 2.2) and $\mathcal{O}_X(1)$ is an ample line bundle on X. We introduce the notion of modified slope for Higgs bundles on $\mathcal{X}$. Using the modified slope, we define semistable(stable) Higgs bundles. As usual, we can represent the moduli stack $\mathcal{M}^{ss}_{\mathrm{Dol},P}(\mathbf{GL}_r)$ of semistable Higgs bundles with modified Hilbert polynomial P as a quotient stack. Moreover, we show that $\mathcal{M}^{ss}_{\mathrm{Dol},P}(\mathbf{GL}_r)$ admits a good moduli space $M^{ss}_{\mathrm{Dol},P}(\mathbf{GL}_r)$.

Fix a line bundle L on $\mathcal{X}$. The $\mathbf{SL}_r$-Higgs bundles is a Higgs bundle (E, ϕ) with $\det(E) = L$ and $\mathrm{tr}(\phi) = 0$. We also prove that the moduli stack $\mathcal{M}_{\mathrm{Dol}}(\mathbf{SL}_r)$ of $\mathbf{SL}_r$-Higgs bundles is locally of finite type over $\mathbb{C}$. And, we show that the moduli stack $\mathcal{M}^{ss}_{\mathrm{Dol},P}(\mathbf{SL}_r)$ of semistable Higgs bundles with modified Hilbert polynomial P is a quotient stack and admits a good moduli space $M^{ss}_{\mathrm{Dol},P}(\mathbf{SL}_r)$ which is a closed subscheme of $M^{ss}_{\mathrm{Dol},P}(\mathbf{GL}_r)$.

Recall that for a principal $\mathbf{PGL}_r$-bundle $\mathcal{P}$, there is an associated cohomology class $\alpha \in H^2(\mathcal{X}, \mu_r)$, which is the obstruction of lifting $\mathcal{P}$ to a principal $\mathbf{SL}_r$-bundle. We call $\mathcal{P}$ with topological type α. A $\mathbf{PGL}_r$-Higgs bundle is said to be with topological type α if the principal $\mathbf{PGL}_r$-bundle is with topological type α. In order to show the algebraicity of moduli stack $\mathcal{M}^{\alpha}_{\mathrm{Dol}}(\mathbf{PGL}_r)$ of $\mathbf{PGL}_r$-Higgs bundles with topological type α, we divide two cases: **Case I**. Assume that the image of α in $H^2(\mathcal{X}, \mathbb{G}_m)$ is zero. Therefore, there is a line bundle L on $\mathcal{X}$ such that $\delta(L) = -\alpha$ in the Kummer exact sequence (23). We prove that $\mathcal{M}_{\mathrm{Dol}}(\mathbf{SL}_r)$ is a $\mathcal{J}_r$-torsor over $\mathcal{M}^{\alpha}_{\mathrm{Dol}}(\mathbf{PGL}_r)$, where $\mathcal{J}_r$ is the stack of μ_r-torsors on $\mathcal{X}$. Hence, $\mathcal{M}^{\alpha}_{\mathrm{Dol}}(\mathbf{PGL}_r)$ is locally of finite type over $\mathbb{C}$ (see [Lie09, Lemma 3.4]). **Case II**. Suppose that the image of α in $H^2(\mathcal{X}, \mathbb{G}_m)$ is nonzero. We consider the μ_r-gerbe $p_\alpha : \mathcal{G}_\alpha \to \mathcal{X}$ corresponding to α. Then, we introduce the notion of twisted Higgs bundles and the moduli stack $\mathcal{M}^{\alpha}_{\mathrm{Dol}}(\mathbf{SL}_r)$ of $\mathbf{SL}_r$-Higgs bundles with trivial determinant. Then, we prove that $\mathcal{M}^{\alpha}_{\mathrm{Dol}}(\mathbf{SL}_r)$ is locally of finite type over $\mathbb{C}$. On the other hand, we show that $\mathcal{M}^{\alpha}_{\mathrm{Dol}}(\mathbf{SL}_r)$ is a $\mathcal{J}_r$-torsor over $\mathcal{M}^{\alpha}_{\mathrm{Dol}}(\mathbf{PGL}_r)$. Thus, $\mathcal{M}^{\alpha}_{\mathrm{Dol}}(\mathbf{PGL}_r)$ is also locally of finite type over $\mathbb{C}$ (see [Lie09, Lemma 3.4]). In Section 3.4, we consider the case of stacky curves and give a definition of the moduli space $M^{\alpha,s}_{\mathrm{Dol}}(\mathbf{PGL}_r)$ of stable $\mathbf{PGL}_r$-Higgs bundles with topological type α. For further applications, we also consider the moduli space $M^{s}_{\mathrm{Dol},\xi}(\mathbf{SL}_r)$ (resp. $M^{\alpha,s}_{\mathrm{Dol},\xi}(\mathbf{PGL}_r)$) of stable $\mathbf{SL}_r$-Higgs bundles (resp. stable $\mathbf{PGL}_r$-Higgs bundles) with fixed K-class $\xi \in K_0(\mathcal{X})_{\mathbb{Q}}$.

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Hitchin morphism was introduced by Hitchin in his study of two-dimensional reduction of Yang–Mills equations (see [Hit87]). We also introduce the Hitchin morphisms in our setup. If $\mathcal{X}$ is a hyperbolic stacky curve, then the Hitchin morphism is proper (see [Yok93]), where we use the correspondence between the Higgs bundles on a stacky curve and the parabolic Higgs bundles on its coarse moduli space (this correspondence is called orbifold-parabolic correspondence in this paper). In Appendix B, we will give a direct proof of the properness of the Hitchin morphisms, following the argument of [Nit91]. As an immediate corollary, the Hitchin morphism $h_{\mathbf{SL}_r} : M^{ss}_{\mathrm{Dol},\xi}(\mathbf{SL}_r) \to \mathbb{H}^o(r, K_{\mathcal{X}})$ is also proper, where $\mathbb{H}^o(r, K_{\mathcal{X}})$ is the affine space associated with the vector space $\bigoplus_{i=2}^{r} H^0(\mathcal{X}, K^i_{\mathcal{X}})$.

For hyperbolic Riemann surfaces, if the rank of Higgs bundles is at least 2, then a general spectral curve is integral (see [BNR89, Remark 3.1]). But, for hyperbolic Deligne–Mumford curves, it is not so. Indeed, there is a hyperbolic Deligne–Mumford curve $\mathbb{E}_5$ such that for any $\boldsymbol{a} \in H^0(\mathbb{E}_5, K_{\mathbb{E}_5}) \oplus H^0(\mathbb{E}_5, K^2_{\mathbb{E}_5})$, the associated spectral curve is reducible (see Example 4.21). With regard to this, we find an optimal criterion for the integrality of spectral curves (see Proposition 4.3 and Remark 4.2).

A partial classification of spectral curves is obtained (Theorem 4.18). We also construct an example satisfying the last conclusion of the above theorem, i.e., for hyperbolic stacky curve $\mathbb{P}^1_{4,2,2,2}$, we show that for a general element $\boldsymbol{a}$ of $\bigoplus_{i=1}^{6} H^0(\mathbb{P}^1_{4,2,2,2}, K^i_{\mathbb{P}^1_{4,2,2,2}})$, the corresponding spectral curve $\mathcal{X}_{\boldsymbol{a}}$ is singular (see Example 4.22). On the other hand, we also show that the coarse moduli space of the spectral curve on a hyperbolic stacky curve $\mathcal{X}$ is the spectral curve of the corresponding parabolic Higgs bundle on X under some condition (see Theorem 4.13 and Remark 4.14).

In Section 5, we systematically study the norm theory on Deligne–Mumford stacks. Applying the general theory to the stacky curves, we obtain the Norm maps for stacky curves (see Proposition 5.21). And, there is a connection between the Norm map of a finite morphism of stacky curves and the Norm map of the induced finite morphism of coarse moduli spaces (see Lemma 5.23). With the help of it, the proof of the SYZ duality can be reduced to the usual case.

1.3 Main results

Let $\mathcal{X}$ be a hyperbolic stacky curve of genus g with coarse moduli space $\pi : \mathcal{X} \to X$. The stacky points of $\mathcal{X}$ are $p_1, \ldots, p_m$, and the stabilizer groups are $\mu_{r_1}, \ldots, \mu_{r_m}$, respectively. Assume that the assumptions of Corollary 4.19 (which ensure that a general spectral curve is irreducible and smooth) are satisfied. Suppose that the K-class ξ satisfies (82) and that $\xi = (r, d_\xi, (m_{1,i})_{i=1}^{r_1-1}, \ldots, (m_{m,i})_{i=1}^{r_m-1}) \in K_0(\mathcal{X})_{\mathbb{Q}}$. Fix a line bundle $L \in \mathrm{Pic}^{d',(j_1,\ldots,j_m)}(\mathcal{X})$, where $d', j_1, \ldots, j_m$ satisfy (83). Consider the moduli space of $M^{ss}_{\mathrm{Dol},\xi}(\mathbf{SL}_r)$ of semistable $\mathbf{SL}_r$-Higgs bundles with K-class ξ and determinant L. The Hitchin morphism $h_{\mathbf{SL}_r} : M^{ss}_{\mathrm{Dol},\xi}(\mathbf{SL}_r) \to \mathbb{H}^o(r, K_{\mathcal{X}})$ is surjective. Note that the stable locus $M_{\mathrm{Dol},\xi}$ of $M^{ss}_{\mathrm{Dol},\xi}(\mathbf{SL}_r)$ is nonempty. Therefore, the properness of $h_{\mathbf{SL}_r}$ implies that there is a nonempty open subset $\mathcal{U} \subseteq \mathbb{H}^o(r, K_{\mathcal{X}})$ such that the inverse image $h^{-1}_{\mathbf{SL}_r}(\mathcal{U})$ is contained in $M_{\mathrm{Dol},\xi}$. Then, $M_{\mathbf{SL}_r} := h^{-1}_{\mathbf{SL}_r}(\mathcal{U})$ is a hyperkähler manifold and $M_{\mathbf{PGL}_r} := h^{-1}_{\mathbf{PGL}_r}(\mathcal{U}) = [M_{\mathbf{SL}_r}/\Gamma_0]$ is a hyperkähler orbifold. Furthermore, we

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obtained two proper morphisms

$$\begin{tikzcd} M_{\mathbf{SL}_r} \arrow[dr, "h_{\mathbf{SL}_r,\mathcal{U}}"'] & & M_{\mathbf{PGL}_r} \arrow[dl, "h_{\mathbf{PGL}_r,\mathcal{U}}"] \\ & \mathcal{U} & \end{tikzcd} \tag{1}$$

where $h_{\mathbf{SL}_r,\mathcal{U}}$ and $h_{\mathbf{PGL}_r,\mathcal{U}}$ are complete algebraically integrable systems. If we perform a hyperkähler rotation, i.e., change to a different complex structure, the generic fiber of $h_{\mathbf{SL}_r,\mathcal{U}}$ (resp. $h_{\mathbf{PGL}_r,\mathcal{U}}$) is a special Lagrangian torus (see Proposition 6.6). Moreover, for a general point $\boldsymbol{a} \in \mathbb{H}^o(r, K_{\mathcal{X}})$, $h^{-1}_{\mathbf{SL}_r,\mathcal{U}}(\boldsymbol{a})$ and $h^{-1}_{\mathbf{PGL}_r,\mathcal{U}}(\boldsymbol{a})$ are dual (see Corollary 6.8). On the other hand, there are two flat unitary gerbes $\mathcal{B}$ and $\check{\mathcal{B}}$ on $M_{\mathbf{SL}_r}$ and $M_{\mathbf{PGL}_r}$, respectively (see Section 6.2). We can therefore state our main results (Theorem 6.13).

Theorem 1.1 (1) *Assume that $\lceil \frac{r}{r_k} \rceil \in \{\frac{r}{r_k}, \frac{r+1}{r_k}\}$ for all $1 \leq k \leq m$. Then $(M_{\mathbf{SL}_r}, \mathcal{B})$ and $(M_{\mathbf{PGL}_r}, \check{\mathcal{B}})$ are SYZ mirror partners if one of the following conditions is satisfied:*
(i) *$g \geq 2$;*
(ii) *$g = 1$ and $\sum_{k=1}^{m}(r - \lceil \frac{r}{r_k} \rceil) \geq 2$;*
(iii) *$g = 0$ and $\sum_{k=1}^{m}(r - \lceil \frac{r}{r_k} \rceil) \geq 2r + 1$;*
(iv) *$g = 0$, $\sum_{k=1}^{m}(r - \lceil \frac{r}{r_k} \rceil) \geq 2r$ and $\dim_{\mathbb{C}} H^0(\mathcal{X}, K^k_{\mathcal{X}}) \geq 2$ for some $2 \leq k \leq r$.*

(2) *Suppose that the assumption about $\lceil \frac{r}{r_k} \rceil$ in (1) does not hold. We make the following assumption: if $\lceil \frac{r}{r_k} \rceil \geq \frac{r+2}{r_k}$ for some $1 \leq k \leq m$, then $\lceil \frac{r-1}{r_k} \rceil = \frac{r-1}{r_k}$. Then $(M_{\mathbf{SL}_r}, \mathcal{B})$ and $(M_{\mathbf{PGL}_r}, \check{\mathcal{B}})$ are SYZ mirror partners if one of the following conditions is satisfied:*
(i) *$g \geq 2$;*
(ii) *$g = 1$ and $\sum_{k=1}^{m}(r - 1 - \lceil \frac{r-1}{r_k} \rceil) \geq 2$;*
(iii) *$g = 0$, $\sum_{k=1}^{m}(r - 1 - \lceil \frac{r-1}{r_k} \rceil) \geq 2r - 2$ and $K_{\mathcal{X}}$ satisfies the condition (43) in Section 4.1.*

Corollary 1.2 *If the K-class ξ satisfies the condition of Proposition A.4, then for a generic rational parabolic weight (Definition A.3), the moduli spaces $M^s_{\mathrm{Dol},\xi}(\mathbf{SL}_r)$ and $M^{\alpha,\xi}_{\mathrm{Dol},\xi}(\mathbf{PGL}_r)$ with natural flat unitary gerbes $\mathcal{B}$ and $\check{\mathcal{B}}$, respectively, are SYZ mirror partners.*

Remark 1.3 At the end of Section 6.2, we construct a pair of moduli spaces of Higgs bundles with structure group $\mathbf{SL}_3$ and $\mathbf{PGL}_3$, respectively, on the stacky curve $\mathbb{P}^1_{3,2,2,2,2}$ (see Example 6.15 in Section 6.2). Moreover, in this example, under the orbifold-parabolic correspondence, the quasi-parabolic flags of the corresponding parabolic Higgs bundles are not all full flags. Our theorem provides more examples for the SYZ duality.

1.4 Hausel–Thaddeus conjecture

For any two natural numbers d, e coprime to r, Hausel and Thaddeus [HT03] conjectured that the mixed Hodge numbers of the moduli space $M^d_{\mathrm{Dol}}(\mathbf{SL}_r)$ of stable $\mathbf{SL}_r$-Higgs bundles of degree d on a compact Riemann surface are equal to the stringy mixed Hodge numbers of the moduli space $M^e_{\mathrm{Dol}}(\mathbf{PGL}_r)$ of stable $\mathbf{PGL}_r$-Higgs bundles of degree e on the same compact Riemann surface. And, Hausel and Thaddeus

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proved the conjecture for $r = 2, 3$ by direct calculations in the same paper. Only recently, the conjecture was proved by Groechenig, Wyss, and Ziegler in [GWZ20] via p-adic integration. Maulik and Shen [MS21] gave a new proof of the conjecture using perverse filtration on the moduli space and Ngô's support theorem in [Ngo06, Ngo10]. The method in [MS21] has more applications in the area of Gopakumar–Vafa invariants. For the moduli spaces of parabolic Higgs bundles, Gothen and Oliveira [GO19] proved the Hausel–Thaddeus conjecture for ranks 2 and 3 but gave evidence that the same holds for any rank.

Notations and conventions

- All schemes and Deligne–Mumford stacks are defined over the complex field $\mathbb{C}$ throughout of the paper. A Deligne–Mumford stack is always assumed to be a global quotient stack with projective coarse moduli space unless otherwise specified.
- $K_0(\mathcal{X})_{\mathbb{Q}}$ is the rational K-group of coherent sheaves on the Deligne–Mumford stack $\mathcal{X}$.
- For a Deligne–Mumford stack $\mathcal{X}$, let $\mathcal{X}_{\text{ét}}$ denote the small étale site of $\mathcal{X}$.
- Let $U \to \mathcal{X}$ be a morphism from a scheme U to a Deligne–Mumford stack $\mathcal{X}$. We use $U[n]$ to represent the Cartesian product $U \times_{\mathcal{X}} U \times_{\mathcal{X}} \cdots \times_{\mathcal{X}} U$ of $n+1$ copies of U. Let $\mathrm{pr}_i : U[1] \to U$ be the projection to the ith factor, for $i = 1, 2$, and let $\mathrm{pr}_{12} : U[2] \to U[1]$, $\mathrm{pr}_{23} : U[2] \to U[1]$, and $\mathrm{pr}_{13} : U[2] \to U[1]$ be the three natural projections. For an étale covering $U \to \mathcal{X}$, let $Des(U/\mathcal{X})$ denote the category of pairs (E, σ), where E is a sheaf of $\mathcal{O}_U$-modules on $U_{\text{ét}}$ and $\sigma : \mathrm{pr}_1^* E \to \mathrm{pr}_2^* E$ is an isomorphism on $U[1]_{\text{ét}}$, which satisfies the cocycle condition $\mathrm{pr}_{23}^* \sigma \circ \mathrm{pr}_{12}^* \sigma = \mathrm{pr}_{13}^* \sigma$.
- $(\mathrm{Sch}/\mathbb{C})_{\text{ét}}$ is the category of schemes over the complex field $\mathbb{C}$ with big étale topology.
- For a Deligne–Mumford stack $\mathcal{X}$ and a $\mathbb{C}$-scheme T, we denote the fiber product $\mathcal{X} \times T$ by $\mathcal{X}_T$. Also, $\mathrm{pr}_{\mathcal{X}} : \mathcal{X}_T \to \mathcal{X}$ is the projection to $\mathcal{X}$ and $\mathrm{pr}_T : \mathcal{X}_T \to T$ is the projection to T.
- $\mathrm{Tot}(E)$ denotes the relative $\mathbf{Spec}_{\mathcal{X}}(\mathrm{Sym}^\bullet E^\vee)$, where $\mathrm{Sym}^\bullet E^\vee$ is the symmetric algebra of the dual $E^\vee$ of a locally free sheaf E on a Deligne–Mumford stack $\mathcal{X}$.
- For a locally free sheaf E on a Deligne–Mumford stack $\mathcal{X}$, the associated projective bundle $\mathbb{P}(E)$ is defined to be the relative $\mathbf{Proj}(\mathrm{Sym}^\bullet E^\vee)$, where $\mathrm{Sym}^\bullet E^\vee$ is the symmetric algebra of the dual $E^\vee$ of E.
- For any real number $c \in \mathbb{R}$, we use $\lceil c \rceil$ to denote the ceiling of c.

2 Preliminaries

2.1 Deligne–Mumford curves

We recall some basic definitions of Deligne–Mumford curves. For a detailed discussion of these topics, please refer to [BN06].

Definition 2.1 A *Deligne–Mumford curve* $\mathcal{X}$ is a one-dimensional Deligne–Mumford stack of finite type over $\mathbb{C}$. A *stacky curve* (or an *orbifold curve*) is a Deligne–Mumford curve with trivial generic stabilizers.

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Remark 2.2 For a smooth Deligne–Mumford curve $\mathcal{X}$, there is a smooth stacky curve $\widehat{\mathcal{X}}$ and a morphism $\mathcal{R} : \mathcal{X} \to \widehat{\mathcal{X}}$, where $\mathcal{R}$ is an H-gerbe for some finite group H (see [BN06, Proposition 4.6]).

Definition 2.3 A smooth irreducible Deligne–Mumford curve $\mathcal{X}$ is said to be *hyperbolic* if the degree $\deg(K_{\mathcal{X}})$ of the canonical line bundle $K_{\mathcal{X}}$ is positive.

Remark 2.4 A Deligne–Mumford curve $\mathcal{X}$ is hyperbolic if and only if the associated stacky curve is hyperbolic (see [BN06, proposition 7.4]).

2.2 Semistable sheaves on Deligne–Mumford stacks

On a Deligne–Mumford stack, there is no very ample line bundle on it unless it is an algebraic space. However, Olsson and Starr [OS03] have discovered that under certain hypothesis, there are locally free sheaves, called generating sheaves, which behave like very ample line bundles. In the following, $\mathcal{X}$ is a Deligne–Mumford stack with coarse moduli space $\pi : \mathcal{X} \to X$.

Definition 2.5 Let F be a coherent sheaf on $\mathcal{X}$. F is said to be a *pure sheaf* of dimension d if the support of every nonzero coherent subsheaf G of F is of dimension d.

Definition 2.6 A locally free sheaf $\mathcal{E}$ on $\mathcal{X}$ is said to be a *generating sheaf* if for any quasicoherent sheaf F on $\mathcal{X}$, the left adjoint of the identity morphism $\mathrm{id} : \pi_*(F \otimes \mathcal{E}^\vee) \to \pi_*(F \otimes \mathcal{E}^\vee)$, $\pi^*(\pi_*(\mathcal{E}^\vee \otimes F)) \otimes \mathcal{E} \longrightarrow F$ is surjective.

Remark 2.7 A smooth Deligne–Mumford curve $\mathcal{X}$ possesses a generating sheaf (see [Kre05, Theorem 5.3]).

Definition 2.8 A *polarization* on $\mathcal{X}$ is a pair $(\mathcal{E}, \mathcal{O}_X(1))$, where $\mathcal{E}$ is a generating sheaf on $\mathcal{X}$ and $\mathcal{O}_X(1)$ is a very ample line bundle on X.

Example 2.9 Suppose that $\mathcal{X}$ is a smooth irreducible stacky curve with coarse moduli space $\pi : \mathcal{X} \to X$. Let $\{p_1, \ldots, p_m\}$ be the set of stacky points of $\mathcal{X}$ with the orders of stabilizer groups $\{r_1, \ldots, r_m\}$. Then, the locally free sheaf

$$\mathcal{E}_u = \bigoplus_{i=1}^{m} \bigoplus_{j=0}^{r_i-1} \mathcal{O}_{\mathcal{X}}\left(\tfrac{j}{r_i} \cdot p_i\right) \tag{2}$$

is a generating sheaf, since it is π-very ample (see [Nir08, Definition 2.2 and Proposition 2.7]). Let $\mathcal{O}_X(1)$ be a very ample line bundle on X. Then, $(\mathcal{E}_u, \mathcal{O}_X(1))$ is a polarization on $\mathcal{X}$.

Definition 2.10 Let $(\mathcal{E}, \mathcal{O}_X(1))$ be a polarization on the $\mathcal{X}$, and let F be a coherent sheaf on it. The *modified Hilbert polynomial* P_F of F is defined by $P_F(m) = \chi(\pi_*(F \otimes \mathcal{E}^\vee) \otimes \mathcal{O}_X(m))$, where $\chi(\pi_*(F \otimes \mathcal{E}^\vee) \otimes \mathcal{O}_X(m))$ is the Euler characteristic of $\pi_*(F \otimes \mathcal{E}^\vee) \otimes \mathcal{O}_X(m)$ on X.

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Remark 2.11 In general, the modified Hilbert polynomial is

$$P_F(m) = \sum_{i=0}^{d} \frac{a_i(F)}{i!} \cdot m^i, \tag{3}$$

where d is the dimension of the support $\mathrm{supp}(F)$ and $a_i(F)$ are rationals.

Definition 2.12 If the modified Hilbert polynomial P_F of F is (3), then its *reduced Hilbert polynomial* p_F is defined to be $p_F(m) = \frac{P_F(m)}{a_d(F)}$.

Definition 2.13 The *modified slope* $\mu_{\mathcal{E}}(F)$ of F is defined by $\mu_{\mathcal{E}}(F) = \frac{a_{d-1}(F)}{a_d(F)}$.

Definition 2.14 Suppose that F is a pure sheaf on $\mathcal{X}$. F is said to be *semistable* (*stable*) if for every proper coherent subsheaf F' of F, we have

$$p_{F'}(m) \leq (<) p_F(m), \quad \text{for } m \gg 0.$$

2.3 Higgs bundles and stability

Let $\mathcal{X}$ be a smooth Deligne–Mumford curve with coarse moduli space $\pi : \mathcal{X} \to X$.

Definition 2.15 A rank n *Higgs bundle* (E, ϕ) on $\mathcal{X}$ consists of a rank n locally free sheaf E on $\mathcal{X}$ and a morphism $\phi : E \to E \otimes K_{\mathcal{X}}$ of $\mathcal{O}_{\mathcal{X}}$-modules, where ϕ is called the *Higgs field*.

Definition 2.16 For a scheme T, a T-family (E_T, ϕ_T) of Higgs bundles on $\mathcal{X}$ consists of a rank n locally free sheaf E_T on $\mathcal{X}_T$ and a morphism of $\mathcal{O}_{\mathcal{X}_T}$-modules $\phi_T : E_T \longrightarrow E_T \otimes \mathrm{pr}_{\mathcal{X}}^* \mathrm{K}_{\mathcal{X}}$.

Example 2.17 Let $\mathcal{X}$ be a smooth irreducible Deligne–Mumford curve with coarse moduli space $\pi : \mathcal{X} \to X$. The canonical line bundle is $K_{\mathcal{X}} = \pi^* K_X \otimes L_{\mathcal{X}}$, for some line bundle $L_{\mathcal{X}}$ on $\mathcal{X}$. In fact, if $\mathcal{X}$ is a stacky curve, it is so (see [VB22, Proposition 5.5.6]). Then, we can get the formula for a general Deligne–Mumford curve by Remark (2.2). Let $E = E_1 \oplus E_2$, where $E_1 = \pi^* K_X^{\frac{1}{2}}$ and $E_2 = \pi^* K_X^{-\frac{1}{2}} \otimes L_{\mathcal{X}}^{-1}$. With respect to the decomposition of E, there is a morphism of $\mathcal{O}_{\mathcal{X}}$-modules

$$\phi = \begin{pmatrix} 0 & 0 \\ \mathbf{1} & 0 \end{pmatrix} \in \mathrm{Hom}_{\mathcal{O}_{\mathcal{X}}}(E, E \otimes K_{\mathcal{X}}),$$

where $\mathbf{1}$ is the identity morphism in $\mathrm{Hom}_{\mathcal{O}_{\mathcal{X}}}(E_1, E_1)$. The pair (E, ϕ) is a Higgs bundle on $\mathcal{X}$.

Using the modified slope, we introduce the notions of semistable (stable) Higgs bundles.

Definition 2.18 Fix a polarization $(\mathcal{E}, \mathcal{O}_X(1))$ on $\mathcal{X}$. A Higgs bundle (E, ϕ) is said to be *semistable* (resp. *stable*) if for all proper nonzero ϕ-invariant locally free subsheaf $F \subset E$ (i.e., $\phi(F) \subseteq F \otimes K_{\mathcal{X}}$), we have

$$\mu_{\mathcal{E}}(F) \leq \mu_{\mathcal{E}}(E) \quad (\text{resp. } \mu_{\mathcal{E}}(F) < \mu_{\mathcal{E}}(E)).$$

If (E, ϕ) is not semistable, we say that (E, ϕ) is unstable.

Example 2.19 If $\mathcal{X}$ is a hyperbolic stacky curve, then the Higgs bundle (E,ϕ) in Example 2.17 is a stable Higgs bundle with respect to the polarization $(\mathcal{E}_u, \mathcal{O}_X(1))$ in Example 2.9.

3 Moduli spaces of Higgs bundles

In the following, $\mathcal{X}$ is supposed to be a hyperbolic Deligne–Mumford curve with coarse moduli space $\pi : \mathcal{X} \to X$.

3.1 Moduli stacks of Higgs bundles

The moduli functor of rank r Higgs bundles is

$$\mathcal{M}_{\mathrm{Dol}}(\mathbf{GL}_r) : (\mathrm{Sch}/\mathbb{C})^{o}_{\text{ét}} \longrightarrow (\text{groupoids}),$$

where $\mathcal{M}_{\mathrm{Dol}}(\mathbf{GL}_r)(T)$ is the groupoid of T-families of rank r Higgs bundles on $\mathcal{X}$ for a test scheme T. Similarly, we can also define the moduli functor $\mathcal{M}_{\mathrm{Dol},P}(\mathbf{GL}_r)$ of rank r Higgs bundles with modified Hilbert polynomial P on $\mathcal{X}$. Suppose that $\mathcal{M}_{\mathrm{Vec},r}$ is the moduli functor of rank r locally free sheaves on $\mathcal{X}$. There is a forgetful functor

$$\mathcal{F} : \mathcal{M}_{\mathrm{Dol}}(\mathbf{GL}_r) \longrightarrow \mathcal{M}_{\mathrm{Vec},r} \tag{4}$$

defined by forgetting the Higgs fields.

Proposition 3.1 *$\mathcal{M}_{\mathrm{Vec},r}$ is an algebraic stack locally of finite type over $\mathbb{C}$.*

Proof Since the stack $\mathcal{C}oh(\mathcal{X})$ of coherent sheaves on $\mathcal{X}$ is an algebraic stack locally of finite type over $\mathbb{C}$ (see [Nir08, Corollary 2.27]) and the inclusion of $\mathcal{M}_{\mathrm{Vec}}$ into $\mathcal{C}oh(\mathcal{X})$ is represented by open immersion (see [HL10, Lemma 2.1.8]), $\mathcal{M}_{\mathrm{Vec},r}$ is an algebraic stack locally of finite type over $\mathbb{C}$ (see [Ols16, Proposition 10.2.2]). ∎

Proposition 3.2 *$\mathcal{M}_{\mathrm{Dol}}(\mathbf{GL}_r)$ is an algebraic stack locally of finite type over $\mathbb{C}$.*

Proof The morphism (4) is representable, which is an abelian cone over $\mathcal{M}_{\mathrm{Vec},r}$. Hence, $\mathcal{M}_{\mathrm{Dol}}(\mathbf{GL}_r)$ is an algebraic stack locally of finite type over $\mathbb{C}$ (see [Ols16, Proposition 10.2.2]). ∎

Corollary 3.3 *$\mathcal{M}_{\mathrm{Dol},P}(\mathbf{GL}_r)$ is an algebraic stack locally of finite type over $\mathbb{C}$.*

Let $\mathcal{Y} = \mathbb{P}(K_{\mathcal{X}} \oplus \mathcal{O}_{\mathcal{X}})$ be the projective bundle associated with $K_{\mathcal{X}} \oplus \mathcal{O}_{\mathcal{X}}$, and let $\mathcal{O}_{\mathcal{Y}}(1)$ be the relative hyperplane bundle on $\mathcal{Y}$. Due to the universal property of coarse moduli spaces, we have the commutative diagram

$$\begin{tikzcd} \mathcal{Y} \arrow[r, "\pi'"] \arrow[d, "\Psi"'] & Y \arrow[d, "\widetilde{\Psi}"] \\ \mathcal{X} \arrow[r, "\pi"] & X \end{tikzcd} \tag{5}$$

where $\pi' : \mathcal{Y} \to Y$ is the coarse moduli space of $\mathcal{Y}$ and the first square is Cartesian. For a polarization $(\mathcal{E}_{\mathcal{X}}, \mathcal{O}_X(1))$ on $\mathcal{X}$, there is a polarization $(\mathcal{E}_{\mathcal{Y}}, \mathcal{O}_Y(1))$ on $\mathcal{Y}$, where $\mathcal{E}_{\mathcal{Y}} = \Psi^*\mathcal{E}_{\mathcal{X}}$ ($\Psi^*\mathcal{E}_{\mathcal{X}}$ is a generating sheaf on $\mathbb{P}(K_{\mathcal{X}} \oplus \mathcal{O}_{\mathcal{X}})$ (see [OS03, Proposition 5.3])) and $\pi'^*\mathcal{O}_Y(1) = (\pi \circ \Psi)^*\mathcal{O}_X(1) \otimes \mathcal{O}_{\mathcal{Y}}(m)$ for some $m \gg 0$. A Higgs bundle

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(E,ϕ) on $\mathcal{X}$ is equivalent to a compactly supported one-dimensional pure sheaf E_ϕ on $\mathrm{Tot}(K_{\mathcal{X}})$ (see Appendix C).

Proposition 3.4 *A Higgs bundle (E,ϕ) on $\mathcal{X}$ is semistable (resp. stable) with respect to $(\mathcal{E}_{\mathcal{X}},\mathcal{O}_X(1))$ if and only if E_ϕ is Gieseker semistable (resp. stable) with respect to $(\mathcal{E}_{\mathcal{Y}},\mathcal{O}_Y(1))$.*

Proof E_ϕ is a pure sheaf on $\mathcal{Y}$ with modified Hilbert polynomial $P_{E_\phi}(n) = \chi(\mathcal{Y},\mathcal{E}_{\mathcal{Y}}^\vee \otimes E_\phi \otimes \pi'^*\mathcal{O}_Y(n))$. Since the support of E_ϕ is contained in $\mathrm{Tot}(K_{\mathcal{X}})$, we have

$$\chi(\mathcal{Y},\mathcal{E}_{\mathcal{Y}}^\vee \otimes E_\phi \otimes \pi'^*\mathcal{O}_Y(n)) = \chi(\mathrm{Tot}(K_{\mathcal{X}}),\psi^*\mathcal{E}^\vee \otimes E_\phi \otimes (\pi\circ\psi)^*\mathcal{O}_X(n)),$$

where $\psi : \mathrm{Tot}(\mathcal{X}) \to \mathcal{X}$ is the natural projection. Note that ψ is an affine morphism. Hence, the pushforward functor ψ_* is exact on the category of quasicoherent sheaves. Then, we have the identity

$$\chi(\mathrm{Tot}(K_{\mathcal{X}}),\psi^*\mathcal{E}^\vee \otimes E_\phi \otimes (\pi\circ\psi)^*\mathcal{O}_X(n)) = \chi(\mathcal{X},\mathcal{E}^\vee \otimes E \otimes \pi^*\mathcal{O}_X(n))),$$

where we use the fact $\psi_* E_\phi = E$. So, $P_{E_\phi} = P_E$. On the other hand, the ϕ-invariant locally free subsheaves of E are equivalent to the coherent subsheaves of E_ϕ. We complete the proof. ∎

In order to construct the moduli space of semistable Higgs bundles on $\mathcal{X}$, we recall a lemma (see [FGIKN05, Section 5.6]).

Lemma 3.5 [FGIKN05] *Let $f : \widehat{X} \to S$ be a proper morphism of Noetherian schemes. Suppose that $\widehat{Y}$ is a closed subscheme of $\widehat{X}$ and F is a coherent sheaf on $\widehat{X}$. Then, there exists an open subscheme S' of S with the universal property that a morphism $T \to S$ factors through S' if and only if the support of the pullback F_T on $\widehat{X}\times_S T$ is disjoint from $\widehat{Y}\times_S T$.*

We need a stacky version of the above lemma. First, we state a technical lemma.

Lemma 3.6 *Suppose that $\widehat{\mathcal{X}}$ is a proper Deligne–Mumford stack over a Noetherian scheme S and E is a coherent sheaf on $\widehat{\mathcal{X}}$. If $\mathcal{E}$ is a generating sheaf on $\widehat{\mathcal{X}}$, then we have*

$$\mathrm{supp}(F_{\mathcal{E}}(E)) \subseteq \pi(\mathrm{supp}(E)),$$

where $\pi : \widehat{\mathcal{X}} \to \widehat{X}$ is the coarse moduli space of $\widehat{\mathcal{X}}$ and $F_{\mathcal{E}}(E) = \pi_*(\mathcal{E}^\vee \otimes E)$. Moreover, $F_{\mathcal{E}}(E) = 0$ if and only if $E = 0$.

Proof The proof of this lemma is the same as Lemma 3.4 in [Nir08]. ∎

Lemma 3.7 *Let $\widehat{\mathcal{X}}$ be a proper Deligne–Mumford stack over a Noetherian scheme S, and let $\mathcal{W}$ be a closed substack of $\widehat{\mathcal{X}}$. For a coherent sheaf E on $\widehat{\mathcal{X}}$, there exists an open subscheme S' of S with the universal property: a morphism $T \to S$ factors through S' if and only if the support of the pullback E_T of E to $\widehat{\mathcal{X}}_T = \widehat{\mathcal{X}}\times_S T$ is disjoint from $\mathcal{W}_T = \mathcal{W}\times_S T$.*

Proof Let $\pi_{\mathcal{W}} : \mathcal{W} \to W$ and $\pi_{\widehat{\mathcal{X}}} : \widehat{\mathcal{X}} \to \widehat{X}$ be the coarse moduli spaces of $\mathcal{W}$ and $\widehat{\mathcal{X}}$, respectively. Then, by the universal property of coarse moduli spaces, there is a commutative diagram

$$
\begin{tikzcd}
\mathcal{W} \arrow[r, "i"] \arrow[d, "\pi_{\mathcal{W}}"'] & \widehat{\mathcal{X}} \arrow[d, "\pi_{\widehat{\mathcal{X}}}"] \\
W \arrow[r, "i'"] & \widehat{X}
\end{tikzcd}
\tag{6}
$$

where i is the closed immersion and i' is the induced morphism.

Claim *The morphism $i' : W \to \widehat{X}$ in (6) is a closed immersion. In fact, there is a short exact sequence of coherent sheaves*

$$0 \longrightarrow \mathcal{I}_{\mathcal{W}} \longrightarrow \mathcal{O}_{\widehat{\mathcal{X}}} \longrightarrow i_*\mathcal{O}_{\mathcal{W}} \longrightarrow 0\ , \tag{7}$$

where $\mathcal{I}_{\mathcal{W}}$ is the ideal sheaf of $\mathcal{W}$ in $\widehat{X}$. By the tameness of $\widehat{\mathcal{X}}$ (see Definition 1.1 and Theorem 1.2 in [Nir08]), the pushforward of (7) to $\widehat{X}$ is

$$0 \longrightarrow \pi_{\widehat{\mathcal{X}}*}\mathcal{I}_{\mathcal{W}} \longrightarrow \pi_{\widehat{\mathcal{X}}*}\mathcal{O}_{\widehat{\mathcal{X}}} \longrightarrow \pi_{\widehat{\mathcal{X}}*}(i_*(\mathcal{O}_{\mathcal{W}})) \longrightarrow 0\ . \tag{8}$$

By (6), the following two compositions

$$\mathcal{O}_{\widehat{X}} \to \pi_{\widehat{\mathcal{X}}*}\mathcal{O}_{\widehat{\mathcal{X}}} \to \pi_{\widehat{\mathcal{X}}*}(i_*(\mathcal{O}_{\mathcal{W}})) \text{ and } \mathcal{O}_{\widehat{X}} \to i'_*\mathcal{O}_W \to i'_*(\pi_{\mathcal{W}*}(\mathcal{O}_{\mathcal{W}})) \tag{9}$$

are the same. By (8) and the isomorphism $\mathcal{O}_{\widehat{X}} \to \pi_{\widehat{\mathcal{X}}*}\mathcal{O}_{\widehat{\mathcal{X}}}$, we have the commutative diagram of short exact sequences

$$
\begin{tikzcd}
0 \arrow[r] & \pi_{\widehat{\mathcal{X}}*}\mathcal{I}_{\mathcal{W}} \arrow[r] & \pi_{\widehat{\mathcal{X}}*}\mathcal{O}_{\widehat{\mathcal{X}}} \arrow[r] & \pi_{\widehat{\mathcal{X}}*}(i_*(\mathcal{O}_{\mathcal{W}})) \arrow[r] & 0\ , \\
0 \arrow[r] & \mathcal{I}_W \arrow[r] \arrow[u] & \mathcal{O}_{\widehat{X}} \arrow[r] \arrow[u] & \pi_{\widehat{\mathcal{X}}*}(i_*(\mathcal{O}_{\mathcal{W}})) \arrow[r] \arrow[u, "="'] & 0
\end{tikzcd}
\tag{10}
$$

where $\mathcal{I}_W$ is the kernel of the composition in (9). Note that the naturel morphism $\mathcal{O}_W \to \pi_{\mathcal{W}*}\mathcal{O}_{\mathcal{W}}$ is an isomorphism. By (6) and (10), we have the short exact sequence

$$0 \longrightarrow \mathcal{I}_W \longrightarrow \mathcal{O}_{\widehat{X}} \longrightarrow i'_*\mathcal{O}_W \longrightarrow 0\ . \tag{11}$$

On the other hand, the morphism of topological spaces induced by i' is a closed embedding. Thus, $W \to X$ is a closed immersion. By Lemma 3.5, for the coherent sheaf $F_{\mathcal{E}}(E)$, there is an open subscheme S' of S with the universal property: a morphism $T \to S$ factors through S' if and only if the support of the pullback $F_{\mathcal{E}}(E)_T$ of $F_{\mathcal{E}}(E)$ on $\widehat{X}_T = \widehat{X} \times_S T$ is disjoint from $W \times_S T$. On the other hand, for a morphism $T \to S$, we consider the Cartesian diagram

$$
\begin{tikzcd}
\widehat{\mathcal{X}}_T \arrow[r] \arrow[d, "\mathrm{id}_T \times \pi_{\widehat{\mathcal{X}}}"'] & \widehat{\mathcal{X}} \arrow[d, "\pi_{\widehat{\mathcal{X}}}"] \\
\widehat{X}_T \arrow[r] & \widehat{X}
\end{tikzcd}
$$

By Proposition 1.5 in [Nir08], we have $F_{\mathcal{E}}(E)_T \simeq F_{\mathcal{E}_T}(E_T)$, where the pullback $\mathcal{E}_T$ of $\mathcal{E}$ to $T \times_S \widehat{\mathcal{X}}$ is a generating sheaf (see [OS03, Theorem 5.5]). Then, the support of the pullback E_T of E on $\widehat{\mathcal{X}}_T$ is disjoint from $\mathcal{W}_T$ if and only if the morphism $T \to S$ factors through the S', by Lemma 3.6. ■

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Recall Theorem 5.1 in [Nir08]:

Theorem 3.8 *There is an open subscheme R_1^{ss} of $\mathrm{Quot}_{\mathcal{X}/\mathbb{C}}(V\otimes_{\mathbb{C}}\pi'^{*}\mathcal{O}_Y(-n)\otimes\mathcal{E}_{\mathcal{Y}},P)$ such that the moduli stack of semistable purely one-dimensional sheaves with modified Hilbert polynomial P on $\mathcal{Y}$ is the quotient stack $[R_1^{ss}/\mathbf{GL}_N]$, where $\mathbf{GL}_N$ is the general linear group over $\mathbb{C}$ with $N=P(n)$.*

Let $\mathcal{M}_{\mathrm{Dol},P}^{ss}(\mathbf{GL}_r)$ be the moduli stack of rank r semistable Higgs bundles with modified Hilbert polynomial P on $\mathcal{X}$. By Proposition 3.4 and Lemma 3.7, we have the following corollary.

Corollary 3.9 *There is an open subscheme R^{ss} of R_1^{ss} such that $\mathcal{M}_{\mathrm{Dol},P}^{ss}(\mathbf{GL}_r)$ is quotient stack $\left[R^{ss}/\mathbf{GL}_N\right]$, where $N=P(n)$.*

We need the following proposition to define the S-equivalence of semistable Higgs bundles.

Proposition 3.10 *Suppose that (E,ϕ) is a semistable Higgs bundle on $\mathcal{X}$. Then, there is a sequence of ϕ-invariant locally free subsheaves $0\subset E_1\subset E_2\subset\cdots\subset E_s=E$ such that $\mu\left(E_i/E_{i-1}\right)=\mu(E)$ and $(E_i/E_{i-1},\phi_i)$ is stable for each $i=1,\ldots,s$, where $\phi_i: E_i/E_{i-1}\to E_i/E_{i-1}\otimes K_{\mathcal{X}}$ is induced by ϕ. Moreover, the associated graded Higgs bundle $gr(E,\phi)=\bigoplus_{i=1}^{l}\left(E_i/E_{i-1},\phi_i\right)$ is uniquely determined up to an isomorphism by (E,ϕ).*

Proof This proposition can be proved following the steps in the proof of [Nit91, Proposition 4.1]. ∎

Remark 3.11 Under the equivalence (C.1) (see Appendix C), the coherent sheaf corresponding to $gr(E,\phi)=\bigoplus_{i=1}^{l}\left(E_i/E_{i-1},\phi_i\right)$ is isomorphic to a Jordan–Hölder filtration of E_ϕ.

Definition 3.12 Suppose that (E,ϕ) and (E',ϕ') are two semistable Higgs bundles on $\mathcal{X}$. They are said to be *S-equivalent* if the associated graded Higgs bundles are isomorphic.

In general, the algebraic stacks without finite inertia rarely admit coarse moduli spaces. Alper introduced the notion of good moduli spaces in [Alp13].

Definition 3.13 Let $\varpi:\mathcal{S}\to S$ be a morphism from an algebraic stack to an algebraic space. We say that $\varpi:\mathcal{S}\to S$ is a *good moduli space* if the following properties are satisfied:

(i) The pushforward functor ϖ_* on the categories of quasicoherent sheaves is exact.
(ii) The morphism of sheaves $\mathcal{O}_S\to\varpi_*\mathcal{O}_{\mathcal{S}}$ is an isomorphism.

Theorem 3.14 *$\mathcal{M}_{\mathrm{Dol},P}^{ss}(\mathbf{GL}_r)$ has a good moduli space $\mathcal{Q}:\mathcal{M}_{\mathrm{Dol},P}^{ss}(\mathbf{GL}_r)\to M_{\mathrm{Dol},P}^{ss}(\mathbf{GL}_r)$. More precisely, the following hold:*

(i) *Universal property: for a scheme Z and a morphism $g:\mathcal{M}_{\mathrm{Dol},P}^{ss}(\mathbf{GL}_r)\to Z$, there is a unique morphism $\theta: M_{\mathrm{Dol},P}^{ss}(\mathbf{GL}_r)\to Z$ such that the following diagram*

$$\begin{tikzcd} \mathcal{M}^{ss}_{\mathrm{Dol},P}(\mathbf{GL}_r) \arrow[rr, "\mathcal{Q}"] \arrow[dr, "g"'] & & M^{ss}_{\mathrm{Dol},P}(\mathbf{GL}_r) \arrow[dl, "\theta"] \\ & Z & \end{tikzcd}$$

commutes.

(ii) $M^{ss}_{\mathrm{Dol},P}(\mathbf{GL}_r)$ *is a quasiprojective scheme over* $\mathbb{C}$.

Proof According to Theorem 6.22 in [Nir08], the moduli stack $[R^{ss}_1/\mathbf{GL}_N]$ of semistable one-dimensional pure sheaves with modified Hilbert polynomial P on $\mathbb{P}(K_{\mathfrak{X}} \oplus \mathcal{O}_{\mathfrak{X}})$ has a good moduli space $\mathcal{Q}_1 : [R^{ss}_1/\mathbf{GL}_N] \longrightarrow M^{ss}_1$, where M^{ss}_1 is the GIT quotient of R^{ss}_1 with respect to the $\mathbf{SL}_N$-action. It also satisfies the properties:

- Universal property: for every scheme Z and every morphism $g_1 : [R^{ss}_1/\mathbf{GL}_N] \to Z$, there is a unique morphism $\theta_1 : M^{ss}_1 \to Z$ such that $g_1 = \theta_1 \circ \mathcal{Q}_1$.
- M^{ss}_1 is a projective scheme over $\mathbb{C}$.

Recall $\mathcal{M}^{ss}_{\mathrm{Dol},P}(\mathbf{GL}_r) = [R^{ss}/\,\mathbf{GL}_N]$ (see Corollary 3.9). Since R^{ss} is a $\mathbf{GL}_N$-invariant open subscheme of R^{ss}_1, $R^{ss}_1 \backslash R^{ss}$ is a $\mathbf{GL}_N$-invariant closed subset. Let $Q : R^{ss}_1 \to M^{ss}_1$ be the GIT quotient, which is a good quotient. Hence, the image $Q(R^{ss}_1 \backslash R^{ss})$ is a closed subset of M^{ss}_1. And, $Q(R^{ss}) \cap Q(R^{ss}_1 \backslash R^{ss}) = \varnothing$. In fact, two semistable sheaves on $\mathcal{Y}$ represent the same point in M^{ss}_1 if and only if they are S-equivalent (see [Nir08, Theorem 6.20]). By Remark 3.11, for a semistable (E, ϕ) on $\mathfrak{X}$, the support of the graded sheaf associated with some Jordan–Hölder filtration of E_ϕ is contained in $\mathrm{Tot}(K_{\mathfrak{X}})$. Denote $Q(R^{ss})$ by M^{ss}. The following diagram is Cartesian:

$$\begin{tikzcd} {[R^{ss}/\mathbf{GL}_N]} \arrow[r] \arrow[d, "\mathcal{Q}"'] & {[R^{ss}_1/\mathbf{GL}_N]} \arrow[d, "\mathcal{Q}_1"] \\ M^{ss} \arrow[r] & M^{ss}_1. \end{tikzcd}$$

The universal property of $\mathcal{Q} : [R^{ss}/\mathbf{GL}_N] \to M^{ss}$ is an immediate conclusion, since Q is a universal categorical quotient. M^{ss} is the good moduli space of $[R^{ss}/\mathbf{GL}_N]$ (see [Alp13, Remark 6.2]). ■

3.2 Moduli stack of $\mathbf{SL}_r$-Higgs bundles

Definition 3.15 Fix a line bundle L on $\mathfrak{X}$. An $\mathbf{SL_r}$-*Higgs bundle* (E, ϕ) is a rank r Higgs bundle with $\det(E) \simeq L$ and $\mathrm{tr}(\phi) = 0$. The *stability* of $\mathbf{SL}_r$-Higgs bundles is the same as Definition 2.18.

The moduli stack $\mathcal{M}_{\mathrm{Dol}}(\mathbf{SL}_r)$ of $\mathbf{SL}_r$-Higgs bundles is the stack whose fiber over a test scheme T is the groupoid of T-families of $\mathbf{SL}_r$-Higgs bundles on $\mathfrak{X}$. Similarly, we have the moduli stack $\mathcal{M}_{\mathrm{Dol},P}(\mathbf{SL}_r)$ (resp. $\mathcal{M}^{ss}_{\mathrm{Dol},P}(\mathbf{SL}_r)$) of (resp. semistable) $\mathbf{SL}_r$-Higgs bundles with fixed modified Hilbert polynomial P.

Proposition 3.16 $\mathcal{M}_{\mathrm{Dol}}(\mathbf{SL}_r)$ *is an algebraic stack locally of finite type over* $\mathbb{C}$.

Proof The Picard stack $\mathcal{P}ic(\mathfrak{X})$ of $\mathfrak{X}$ is an algebraic stack locally of finite type over $\mathbb{C}$ (see [Aok06]). There is a morphism of algebraic stacks $\mathcal{D}et : \mathcal{M}_{\mathrm{Dol}}(\mathbf{GL}_r) \to \mathcal{P}ic(\mathfrak{X})$, which is defined by taking determinants. By taking the traces of Higgs fields, we can define a morphism $\mathcal{T}r : \mathcal{M}_{\mathrm{Dol}}(\mathbf{GL}_r) \to \mathbb{H}^0(\mathfrak{X}, K_{\mathfrak{X}})$, where $\mathbb{H}^0(\mathfrak{X}, K_{\mathfrak{X}})$ is the

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affine space associated with $H^0(\mathcal{X}, K_{\mathcal{X}})$. On the other hand, L defines a geometric point $[L] : \mathrm{Spec}(\mathbb{C}) \to \mathcal{P}ic(\mathcal{X})$ and the origin of $H^0(\mathcal{X}, K_{\mathcal{X}})$ defines a closed point $o : \mathrm{Spec}(\mathbb{C}) \to \mathbb{H}^0(\mathcal{X}, K_{\mathcal{X}})$. We therefore have the Cartesian diagram

$$\begin{tikzcd} \mathcal{M}_{\mathrm{Dol}}(\mathbf{SL}_r) \arrow[r] \arrow[d] & \mathcal{M}_{\mathrm{Dol}}(\mathbf{GL}_r) \arrow[d, "(\mathcal{D}et, \mathcal{T}r)"] \\ \mathrm{Spec}(\mathbb{C}) \arrow[r, "({[L]}, o)"] & \mathcal{P}ic(\mathcal{X}) \times \mathbb{H}^0(\mathcal{X}, K_{\mathcal{X}}). \end{tikzcd} \tag{12}$$

Hence, $\mathcal{M}_{\mathrm{Dol}}(\mathbf{SL}_r)$ is an algebraic stack locally of finite type over $\mathbb{C}$ by Proposition 3.2. ■

Corollary 3.17 $\mathcal{M}_{\mathrm{Dol},P}(\mathbf{SL}_r)$ *is an algebraic stack locally of finite type over* $\mathbb{C}$.

Proof Since the moduli stack $\mathcal{M}_{\mathrm{Dol},P}(\mathbf{SL}_r)$ is an open and closed substack of $\mathcal{M}_{\mathrm{Dol}}(\mathbf{SL}_r)$, $\mathcal{M}_{\mathrm{Dol},P}(\mathbf{SL}_r)$ is an algebraic stack of locally finite type over $\mathbb{C}$ by Proposition 3.16. ■

In the following, we will show that the moduli stack $\mathcal{M}^{ss}_{\mathrm{Dol},P}(\mathbf{SL}_r)$ is a quotient stack. Let $(E_{R^{ss}}, \phi_{R^{ss}})$ be the $\mathbf{GL}_N$-equivariant Higgs bundle on $\mathcal{X} \times R^{ss}$, which is the pushforward of the universal quotient sheaf on $\mathrm{Tot}(K_{\mathcal{X}}) \times R^{ss}$. Let $\det : R^{ss} \to \mathrm{Pic}(\mathcal{X})$ be the classifying morphism defined by $\det(E_{R^{ss}})$, where $\mathrm{Pic}(\mathcal{X})$ is the Picard scheme of $\mathcal{X}$ (see [Bro12, Corollary 2.3.7(i)]). On the other hand, the trace of the morphism $\phi_{R^{ss}} : E_{R^{ss}} \to E_{R^{ss}} \otimes \mathrm{pr}^*_{\mathcal{X}} K_{\mathcal{X}}$ defines a section $\mathrm{tr}(\phi_{R^{ss}})$ of $\mathrm{pr}^*_{\mathcal{X}} K_{\mathcal{X}}$. It defines a morphism $\mathrm{tr} : R^{ss} \to \mathbb{H}^0(\mathcal{X}, K_{\mathcal{X}})$. Consider the Cartesian diagram

$$\begin{tikzcd} R^{ss}_{\mathrm{SL_r}} \arrow[r] \arrow[d] & R^{ss} \arrow[d, "(\det, \mathrm{tr})"] \\ \mathrm{Spec}(\mathbb{C}) \arrow[r, "({[L]}, o)"] & \mathrm{Pic}(\mathcal{X}) \times \mathbb{H}^0(\mathcal{X}, K_{\mathcal{X}}). \end{tikzcd} \tag{13}$$

Theorem 3.18 *There exists a* $\mathbf{GL}_N$*-equivariant line bundle* W *on* $R^{ss}_{\mathbf{SL}_r}$ *such that the moduli stack* $\mathcal{M}^{ss}_{\mathrm{Dol},P}(\mathbf{SL}_r)$ *can be represented by* $[W^*/\mathbf{GL}_N]$, *where* W^* *is the frame bundle associated with* W.

Proof First, we consider the Cartesian diagram

$$\begin{tikzcd} \mathcal{M}^{ss,o}_{\mathrm{Dol},P}(\mathbf{GL}_r) \arrow[r] \arrow[d] & \mathrm{Spec}(\mathbb{C}) \arrow[d, "o"] \\ \mathcal{M}^{ss}_{\mathrm{Dol},P}(\mathbf{GL}_r) \arrow[r, "\mathcal{T}r"] & \mathbb{H}^0(\mathcal{X}, K_{\mathcal{X}}). \end{tikzcd} \tag{14}$$

Hence, $\mathcal{M}^{ss,o}_{\mathrm{Dol},P}(\mathbf{GL}_r)$ can be presented by $[R^{ss,o}/\mathbf{GL}_N]$, where $R^{ss,o} = R^{ss} \times_{\mathbb{H}^0(\mathcal{X}, K_{\mathcal{X}})} \mathrm{Spec}(\mathbb{C})$. Moreover, the moduli stack $\mathcal{M}^{ss}_{\mathrm{Dol},P}(\mathbf{SL}_r)$ is the fiber product

$$\begin{tikzcd} \mathcal{M}^{ss}_{\mathrm{Dol},P}(\mathbf{SL}_r) \arrow[r] \arrow[d] & \mathcal{M}^{ss,o}_{\mathrm{Dol},P}(\mathbf{GL}_r) \arrow[d, "\mathcal{D}et"] \\ \mathrm{Spec}(\mathbb{C}) \arrow[r, "{[L]}"] & \mathcal{P}ic(\mathcal{X}). \end{tikzcd} \tag{15}$$

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On the other hand, we have the following Cartesian diagram:

(16)
$$\begin{tikzcd} \mathcal{M}^{ss,o}_{\mathrm{Dol},P}(\mathbf{SL}_r) \arrow[r] \arrow[d] & \mathcal{M}^{ss,o}_{\mathrm{Dol},P}(\mathbf{GL}_r) \arrow[d, "\det"] \\ \mathrm{Spec}(\mathbb{C}) \arrow[r, "{[L]}"] & \mathrm{Pic}(\mathcal{X}), \end{tikzcd}$$

where det is the classifying morphism of the determinant line bundle of the universal Higgs bundle on $\mathcal{X} \times \mathcal{M}^{ss,o}_{\mathrm{Dol},P}(\mathbf{GL}_r)$. The closed substack $\mathcal{M}^{ss,o}_{\mathrm{Dol},P}(\mathbf{SL}_r)$ of $\mathcal{M}^{ss,o}_{\mathrm{Dol},P}(\mathbf{GL}_r)$ can be presented as $[R^{ss}_{\mathbf{SL}_r}/\mathbf{GL}_N]$, where $R^{ss}_{\mathbf{SL}_r}$ is the fiber product in the Cartesian diagram (13). Since $\mathcal{M}^{ss}_{\mathrm{Dol},P}(\mathbf{SL}_r) \to \mathcal{M}^{ss,o}_{\mathrm{Dol},P}(\mathbf{GL}_r)$ in the diagram (15) factors through the closed immersion $\mathcal{M}^{ss,o}_{\mathrm{Dol},P}(\mathbf{SL}_r) \to \mathcal{M}^{ss,o}_{\mathrm{Dol},P}(\mathbf{GL}_r)$ in the diagram (16), it is easy to check that the following commutative diagram is Cartesian

(17)
$$\begin{tikzcd} \mathcal{M}^{ss}_{\mathrm{Dol},P}(\mathbf{SL}_r) \arrow[r] \arrow[d] & \mathcal{M}^{ss,o}_{\mathrm{Dol},P}(\mathbf{SL}_r) \arrow[d, "\mathcal{D}et"] \\ \mathrm{Spec}(\mathbb{C}) \arrow[r, "{[L]}"] & \mathcal{P}ic(\mathcal{X}), \end{tikzcd}$$

where $\mathcal{D}et$ is the restriction of $\mathcal{D}et : \mathcal{M}^{ss,o}_{\mathrm{Dol},P}(\mathbf{GL}_r) \to \mathcal{P}ic(\mathcal{X})$ to $\mathcal{M}^{ss,o}_{\mathrm{Dol},P}(\mathbf{SL}_r)$. On the other hand, $\det(E_{R^{ss}})|_{\mathcal{X}\times R^{ss}_{\mathbf{SL}_r}} \simeq \mathrm{pr}^*_{\mathcal{X}} L \otimes \mathrm{pr}^*_{R^{ss}_{\mathbf{SL}_r}} W$ for some line bundle W on $R^{ss}_{\mathbf{SL}_r}$, where $\mathrm{pr}_{\mathcal{X}}$ and $\mathrm{pr}_{R^{ss}_{\mathbf{SL}_r}}$ are the projections to $\mathcal{X}$ and $R^{ss}_{\mathbf{SL}_r}$, respectively. Moreover, W is a $\mathbf{GL}_N$-equivariant line bundle on $R^{ss}_{\mathbf{SL}_r}$, since $\mathrm{pr}^*_{\mathcal{X}} L$ is a $\mathbf{GL}_N$-equivariant line bundle with the trivial equivariant structure. By the Cartesian diagram (17), $\mathcal{M}^{ss}_{\mathrm{Dol},P}(\mathbf{SL}_r)$ can be represented by $[W^*/\mathbf{GL}_N]$, where W^* is the frame bundle associated with W. ■

Corollary 3.19 *$\mathcal{M}^{ss}_{\mathrm{Dol},P}(\mathbf{SL}_r)$ has a good moduli space $M^{ss}_{\mathrm{Dol},P}(\mathbf{SL}_r)$, which is a closed subscheme of $M^{ss}_{\mathrm{Dol},P}(\mathbf{GL}_r)$.*

Proof As the center $\mathbb{C}^*$ of $\mathbf{GL}_N$ acts trivially on $R^{ss}_{\mathbf{SL}_r}$, the $\mathbf{GL}_N$-equivariant morphism $W^* \to R^{ss}_{\mathbf{SL}_r}$ induces a morphism of quotient stacks

(18)
$$[W^*/\mathbf{GL}_N] \longrightarrow [R^{ss}_{\mathbf{SL}_r}/\mathbf{PGL}_N].$$

On the other hand, there is a Cartesian diagram:

(19)
$$\begin{tikzcd} {[W^*/\mathbb{C}^*]} \arrow[r] \arrow[d] & R^{ss}_{\mathbf{SL}_r} \arrow[d] \\ {[W^*/\mathbf{GL}_N]} \arrow[r] & {[R^{ss}_{\mathbf{SL}_r}/\mathbf{PGL}_N]}. \end{tikzcd}$$

Note that the top morphism in (19) is a μ_r-gerbe. It follows that the bottom morphism (18) is also a μ_r-gerbe. So, the good moduli space of $[R^{ss}_{\mathrm{SL}_r}/\mathbf{PGL}_N]$ coincides with the good moduli space of $[W^*/\mathbf{GL}_N]$. Note the good moduli space of $[R^{ss}_{\mathrm{SL}_r}/\mathbf{PGL}_N]$ is a closed subscheme of $M^{ss}_{\mathrm{Dol},P}(\mathbf{GL}_r)$. This completes the proof. ■

3.3 Moduli stack of PGL_r-Higgs bundles

We first recall some basic facts about principal bundles (or torsors) on $\mathcal{X}$. Our main reference is [Gir71]. For an algebraic group G, the set of isomorphism classes of principal G-bundles is denoted by $H^1(\mathcal{X}, G)$ (if G is abelian, $H^1(\mathcal{X}, G)$ is equivalent

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to the étale cohomology group with values in G). For a morphism of algebraic groups $G \to H$, we have a morphism of pointed sets

$$H^1(\mathcal{X}, G) \longrightarrow H^1(\mathcal{X}, H), \quad [\mathcal{P}_G] \longmapsto [\mathcal{P}_G \wedge^G H], \tag{20}$$

where $\mathcal{P}_G \wedge^G H$ is also denoted by $\mathcal{P}_G \times^G H$ in some literatures. In general, the morphism (20) is not surjective. We say a principal H-bundle $\mathcal{P}_H$ can be lifted to a principal G-bundle if $\mathcal{P}_H \simeq \mathcal{P}_G \wedge^G H$ for some principal G-bundle $\mathcal{P}_G$. For simplicity, we only consider the case when G is a central extension of H by C, i.e., there is an exact sequence of algebraic groups $0 \longrightarrow C \longrightarrow G \longrightarrow H \longrightarrow 0$. The obstruction of lifting $\mathcal{P}_H$ to a principal G-bundle is the so-called *lifting gerbe* $\mathcal{G}_{\mathcal{P}_H}$ (or G-*lifting gerbe*). Recall that there is a natural morphism of classifying stacks $BG \to BH$ defined by

$$BG(T) \longrightarrow BH(T), \quad (\mathcal{P}_G \to T) \longmapsto (\mathcal{P}_G \wedge^G H \to T),$$

for a test scheme T. The lifting gerbe $\mathcal{G}_{\mathcal{P}_H}$ is the fiber product $\mathcal{X} \times_{BH} BG$ for the Cartesian diagram

$$\begin{tikzcd} \mathcal{G}_{\mathcal{P}_H} \arrow[r] \arrow[d] & BG \arrow[d] \\ \mathcal{X} \arrow[r] & BH, \end{tikzcd}$$

where $\mathcal{X} \to BH$ is the classifying morphism of $\mathcal{P}_H$.

Remark 3.20 The lifting gerbe $\mathcal{G}_{\mathcal{P}_H} \to \mathcal{X}$ is a C-gerbe on $\mathcal{X}$ (see [Ols16, Definition12.2.2]), since the morphism $BG \to BH$ is a C-gerbe.

The set of isomorphism classes of C-gerbes is equal to $H^2_{\text{ét}}(\mathcal{X}, C)$. Then, we have a morphism of pointed sets

$$\partial : H^1(\mathcal{X}, H) \longrightarrow H^2_{\text{ét}}(\mathcal{X}, C), \quad [\mathcal{P}_H] \longmapsto [\mathcal{G}_{\mathcal{P}_H}], \tag{21}$$

which maps the trivial principal H-bundle to the trivial C-gerbe. Indeed, according to the general theory of [Gir71], we have:

Proposition 3.21 *A principal H-bundle $\mathcal{P}_H$ can be lifted to a principal G-bundle if and only if the lifting gerbe $\mathcal{G}_{\mathcal{P}_H}$ is trivial. Moreover, there is an associated exact sequence of pointed sets*

$$\begin{tikzcd} 1 \arrow[r] & H^0_{\acute{e}t}(\mathcal{X}, C) \arrow[r] & H^0(\mathcal{X}, G) \arrow[r] & H^0(\mathcal{X}, H) \arrow[dll, rounded corners] & \\ & H^1_{\acute{e}t}(\mathcal{X}, C) \arrow[r] & H^1(\mathcal{X}, G) \arrow[r] & H^1(\mathcal{X}, H) \arrow[r, "\partial"] & H^2_{\acute{e}t}(\mathcal{X}, C). \end{tikzcd} \tag{22}$$

Recall the Kummer sequence

$$\cdots \longrightarrow H^1_{\text{ét}}(\mathcal{X}, \mathbb{G}_m) \xrightarrow{[r]} H^1_{\text{ét}}(\mathcal{X}, \mathbb{G}_m) \xrightarrow{\delta} H^2_{\text{ét}}(\mathcal{X}, \mu_r) \longrightarrow H^2_{\text{ét}}(\mathcal{X}, \mathbb{G}_m) \longrightarrow \cdots. \tag{23}$$

For a line bundle L on $\mathcal{X}$, we use $\mathcal{G}_L$ to denote the μ_r-gerbe defined by the cohomology class $\delta([L])$. Consider the central extension

$$1 \longrightarrow \mu_r \longrightarrow \mathbf{SL}_r \longrightarrow \mathbf{PGL}_r \longrightarrow 1 \,. \tag{24}$$

By Proposition 3.21, we have the following exact sequences of pointed sets:

$$\cdots \longrightarrow H^1_{\text{ét}}(\mathcal{X}, \mu_r) \longrightarrow H^1(\mathcal{X}, \mathbf{SL}_r) \longrightarrow H^1(\mathcal{X}, \mathbf{PGL}_r) \xrightarrow{\partial} H^2_{\text{ét}}(\mathcal{X}, \mu_r). \tag{25}$$

The $\mathbf{SL}_r$-lifting gerbe of $\mathcal{P}_{\mathbf{PGL}_r}$ is denoted by $\mathcal{G}_{\mathcal{P}_{\mathbf{PGL}_r}}$.

Proposition 3.22 *Let E be a rank r locally free sheaf on $\mathcal{X}$, and let $\mathcal{P}_E$ be the associated frame bundle of E. Then, the two gerbes are equivalent*

$$\mathcal{G}_{\det(E)^\vee} \simeq \mathcal{G}_{\mathcal{P}_{\mathbf{PGL}_r}},$$

where $\mathcal{P}_{\mathbf{PGL}_r} = \mathcal{P}_E \wedge^{\mathbf{GL}_r} \mathbf{PGL}_r$.

Proof By repeating the proof of [HS03, Lemma 2.5], but with replacing the analytic topology with the étale topology, the conclusion of the proposition is immediate. ■

Definition 3.23 A $\mathbf{PGL}_r$-*Higgs bundle* $(\mathcal{P}_{\mathbf{PGL}_r}, \phi)$ consists of a principal $\mathbf{PGL}_r$-bundle $\mathcal{P}_{\mathbf{PGL}_r}$ and a section ϕ of $\mathrm{ad}(\mathcal{P}_{\mathbf{PGL}_r}) \otimes K_{\mathcal{X}}$, where $\mathrm{ad}(\mathcal{P}_{\mathbf{PGL}_r})$ is the adjoint bundle of $\mathcal{P}_{\mathbf{PGL}_r}$. For a scheme T, a *T-family of $\mathbf{PGL}_r$-Higgs bundles* $(\mathcal{P}_{\mathbf{PGL}_r,T}, \phi_T)$ is a T-family of principal $\mathbf{PGL}_r$-bundles $\mathcal{P}_{\mathbf{PGL}_r,T}$ with a section of $\mathrm{ad}(\mathcal{P}_{\mathbf{PGL}_r,T}) \otimes \mathrm{pr}_{\mathcal{X}}{}^* K_{\mathcal{X}}$.

In order to construct the moduli stack of $\mathbf{PGL}_r$-Higgs bundles, we introduce the following notion:

Definition 3.24 Suppose that α is a cohomology class in $H^2_{\text{ét}}(\mathcal{X}, \mu_r)$. Let k be an algebraically closed field containing $\mathbb{C}$. A principal $\mathbf{PGL}_r$-bundle $\mathcal{P}_{\mathbf{PGL}_r,k}$ on $\mathcal{X}_k$ is said to have *topological type* $\alpha \in H^2_{\text{ét}}(\mathcal{X}, \mu_r)$ if the $\mathbf{SL}_r$-lifting gerbe of $\mathcal{P}_{\mathbf{PGL}_r,k}$ in $H^2_{\text{ét}}(\mathcal{X}_k, \mu_r)$ is $\mathrm{pr}_{\mathcal{X}}{}^*\alpha$, where $\mathrm{pr}_{\mathcal{X}} : \mathcal{X}_k = \mathcal{X} \times_{\mathrm{Spec}\,\mathbb{C}} \mathrm{Spec}\, k \to \mathcal{X}$ is the natural projection. For a scheme T, a T-family of principal $\mathbf{PGL}_r$-bundles with topological type α is a principal $\mathbf{PGL}_r$-bundle $\mathcal{P}_{\mathbf{PGL}_r,T}$ on $\mathcal{X}_T$, which restricts to every geometric fiber of $\mathcal{X}_T \to T$ is with topological type α.

The moduli stack $\mathcal{M}^\alpha_{\mathrm{Dol}}(\mathbf{PGL}_r)$ of $\mathbf{PGL}_r$-Higgs bundles with topological type α is defined by: for a test scheme T, $\mathcal{M}^\alpha_{\mathrm{Dol}}(\mathbf{PGL}_r)(T)$ is the groupoid of T-families of $\mathbf{PGL}_r$-Higgs bundles, in which the principal bundles with topological type α. Suppose that there is a principal $\mathbf{PGL}_r$-bundle $\mathcal{P}_{\mathbf{PGL}_r}$ satisfying $\partial([\mathcal{P}_{\mathbf{PGL}_r}]) = \alpha$. Consider the exact sequence:

$$1 \longrightarrow \mathbb{G}_m \longrightarrow \mathbf{GL}_r \longrightarrow \mathbf{PGL}_r \longrightarrow 1.$$

The cohomology class in $H^2_{\text{ét}}(\mathcal{X}, \mathbb{G}_m)$ corresponding the $\mathbf{GL}_r$-lifting gerbe of $\mathcal{P}_{\mathbf{PGL}_r}$ is the image of α in $H^2_{\text{ét}}(\mathcal{X}, \mathbb{G}_m)$ (see the proof of [Mil80, Proposition 2.7 in Chapter IV]).

Case I: Assume that the image of α in $H^2_{\text{ét}}(\mathcal{X}, \mathbb{G}_m)$ is zero. Then, there is a locally free sheaf E on $\mathcal{X}$ such that the associated frame bundle is a $\mathbf{GL}_r$-lifting of $\mathcal{P}_{\mathbf{PGL}_r}$ and $\delta([\det(E)^\vee]) = \alpha$. Moreover, by the Kummer sequence (23), we see that $\det(E)$ is uniquely determined up to an rth power of some line bundle. We therefore have the proposition:

Proposition 3.25 *Suppose that α is zero in $H^2(\mathcal{X}, \mathbb{G}_m)$ and that L is a line bundle with $\delta([L]) = -\alpha$ in the Kummer sequence (23). For every principal $\mathbf{PGL}_r$-bundle $\mathcal{P}_{\mathbf{PGL}_r}$ with $\partial([\mathcal{P}_{\mathbf{PGL}_r}]) = \alpha$, there is a locally free sheaf E with $\det(E) \simeq L$, whose associated frame bundle $\mathcal{P}_E$ is a $\mathbf{GL}_r$-lifting of $\mathcal{P}_{\mathbf{PGL}_r}$.*

Let $\mathcal{M}_{\mathrm{Dol}}(\mathbf{SL}_r)$ be the moduli stack of $\mathbf{SL}_r$-Higgs bundles with fixed determinant L. By Proposition 3.25, there is a surjective morphism of stacks

$$\mathcal{M}_{\mathrm{Dol}}(\mathbf{SL}_r) \longrightarrow \mathcal{M}^{\alpha}_{\mathrm{Dol}}(\mathbf{PGL}_r), \tag{26}$$

defined by

$$\mathcal{M}_{\mathrm{Dol}}(\mathbf{SL}_r)(T) \longrightarrow \mathcal{M}^{\alpha}_{\mathrm{Dol}}(\mathbf{PGL}_r)(T), \quad (E_T, \phi_T) \longmapsto (\mathcal{P}_{E_T} \wedge^{\mathbf{GL}_r} \mathbf{PGL}_r, \phi_T),$$

where T is any test scheme and $\mathcal{P}_{E_T}$ is the frame bundle associated with E_T.

Proposition 3.26 *The morphism (26) is a $\mathcal{J}_r$-torsor, where $\mathcal{J}_r$ is the stack of μ_r-torsors on $\mathcal{X}$.*

Proof The proof is the same as [Las97, Lemma 5.1]. ■

Case II: Assume that $\alpha \in H^2_{\text{ét}}(\mathcal{X}, \mu_r)$ is not zero in $H^2_{\text{ét}}(\mathcal{X}, \mathbb{G}_m)$. The corresponding μ_r-gerbe is denoted by $p_\alpha : \mathcal{G}_\alpha \to \mathcal{X}$, which is a Deligne–Mumford curve. By the universal property of $\mathcal{G}_\alpha$, for any $\mathcal{P}_{\mathbf{PGL}_r}$ with topological type α, $p^*_\alpha \mathcal{P}_{\mathbf{PGL}_r}$ has an $\mathbf{SL}_r$-lifting on $\mathcal{G}_\alpha$, i.e., there exists a locally free sheaf E of rank r with $\det(E) \simeq \mathcal{O}_{\mathcal{G}_\alpha}$ on $\mathcal{G}_\alpha$ such that the associated principal bundle is an $\mathbf{SL}_r$-lifting of $p^*_\alpha \mathcal{P}_{\mathbf{PGL}_r}$. The E is a twisted vector bundle. In what follows, we will give the definition of twisted vector bundles. For a quasicoherent sheaf F on $\mathcal{G}_\alpha$, it admits an eigendecomposition

$$F = \bigoplus_{\lambda \in \mathbb{Z}/r\mathbb{Z}} F_\lambda, \tag{27}$$

where F_λ is the eigensheaf on $\mathcal{G}_\alpha$ with respect to the character λ of μ_r (see [Lie08, Proposition 3.1.1.4]).

Definition 3.27 A quasicoherent sheaf F on $\mathcal{G}_\alpha$ is called a *twisted quasicoherent sheaf* if $F = F_{\bar{1}}$ in the eigendecomposition (27). In particular, if the aforementioned F is a locally free sheaf, we say that F is a *twisted vector bundle*. A *twisted Higgs bundle* is a pair (E, ϕ), where E is a twisted vector bundle and $\phi : E \to E \otimes K_{\mathcal{G}_\alpha}$ is a μ_r-equivariant morphism of $\mathcal{O}_{\mathcal{G}_\alpha}$-modules.

Consider the moduli stack $\mathcal{M}^{\alpha}_{\mathrm{Dol}}(\mathbf{GL}_r)$ of twisted Higgs bundles on $\mathcal{G}_\alpha$, whose fiber over a test scheme T is the groupoid of T-families of rank r twisted Higgs bundles on $\mathcal{G}_\alpha$. $\mathcal{M}^{\alpha}_{\mathrm{Dol}}(\mathbf{GL}_r)$ is an open and closed substack of the moduli stack of rank r Higgs bundles on $\mathcal{G}_\alpha$ for the decomposition (27). For a modified Hilbert polynomial P, we can also consider the moduli stack $\mathcal{M}^{\alpha}_{\mathrm{Dol},P}(\mathbf{GL}_r)$ of rank r twisted Higgs bundles with modified Hilbert polynomial P, which is an open and closed substack of $\mathcal{M}_{\mathrm{Dol},P}(\mathbf{GL}_r)$ on $\mathcal{G}_\alpha$. If there is a polarization on $\mathcal{G}_\alpha$, we can also introduce the notion of stability for twisted Higgs bundles as usual. The moduli stack of semistable twisted Higgs bundle with modified Hilbert polynomial P is denoted by $\mathcal{M}^{\alpha,ss}_{\mathrm{Dol},P}(\mathbf{GL}_r)$.

Proposition 3.28 *$\mathcal{M}^{\alpha}_{\mathrm{Dol}}(\mathbf{GL}_r)$ and $\mathcal{M}^{\alpha}_{\mathrm{Dol},P}(\mathbf{GL}_r)$ are algebraic stacks locally of finite type over $\mathbb{C}$. Moreover, $\mathcal{M}^{\alpha,ss}_{\mathrm{Dol},P}(\mathbf{GL}_r)$ of semistable twisted Higgs bundles with modified Hilbert polynomial P is a quotient stack, whose good moduli space $M^{\alpha,ss}_{\mathrm{Dol},P}(\mathbf{GL}_r)$ is a quasiprojective scheme.*

Proof Since "twisted," "with fixed modified Hilbert polynomial," and "semistable" are open conditions, the conclusion of the proposition is immediate by the counterparts in Section 3.1. ∎

Definition 3.29 A *twisted $\mathbf{SL}_r$-Higgs bundle* is a twisted Higgs bundle (E, ϕ) with $\det(E) \simeq \mathcal{O}_{\mathcal{G}_\alpha}$ and $\mathrm{tr}(\phi) = 0$.

The moduli stack $\mathcal{M}^{\alpha}_{\mathrm{Dol}}(\mathbf{SL}_r)$ of twisted $\mathbf{SL}_r$-Higgs bundles is an open and closed substack of the moduli stack of $\mathbf{SL}_r$-Higgs bundle on $\mathcal{G}_\alpha$. As Proposition 3.28, we have:

Proposition 3.30 *$\mathcal{M}^{\alpha}_{\mathrm{Dol}}(\mathbf{SL}_r)$ is an algebraic stack locally of finite type over $\mathbb{C}$. Furthermore, $\mathcal{M}^{\alpha,ss}_{\mathrm{Dol},P}(\mathbf{SL}_r)$ of semistable twisted $\mathbf{SL}_r$-Higgs bundles with modified Hilbert polynomial P is a quotient stack of finite type over $\mathbb{C}$. Its good moduli space $M^{\alpha,ss}_{\mathrm{Dol},P}(\mathbf{SL}_r)$ is a quasiprojective scheme.*

For a twisted $\mathbf{SL}_r$-Higgs bundle on $\mathcal{G}_\alpha$, the associated $\mathbf{PGL}_r$-Higgs bundle is the pullback of a $\mathbf{PGL}_r$-Higgs bundle with topological data α on $\mathcal{X}$. Then, we have a surjective morphism of algebraic stacks

$$\mathcal{M}^{\alpha}_{\mathrm{Dol}}(\mathbf{SL}_r) \longrightarrow \mathcal{M}^{\alpha}_{\mathrm{Dol}}(\mathbf{PGL}_r). \tag{28}$$

Similar to Proposition 3.26, we also have:

Proposition 3.31 *The morphism (28) is a $\mathcal{J}_r$-torsor, where $\mathcal{J}_r$ is the stack of μ_r-torsors on $\mathcal{X}$.*

By [Lie09, Lemma 3.4] and Propositions 3.26 and 3.31, we have the following theorem.

Theorem 3.32 *For any $\alpha \in H^2_{ét}(\mathcal{X}, \mu_r)$, the moduli stack $\mathcal{M}^{\alpha}_{\mathrm{Dol}}(\mathbf{PGL}_r)$ of $\mathbf{PGL}_r$-Higgs bundles with topological type α is a locally finite type algebraic stack over $\mathbb{C}$.*

3.4 Application to the case of stacky curves

In this subsection, $\mathcal{X}$ is assumed to be a genus g hyperbolic stacky curve with coarse moduli space $\pi : \mathcal{X} \to X$. Fix a polarization $(\mathcal{E}, \mathcal{O}_X(1))$ on $\mathcal{X}$. Since $H^2_{\text{ét}}(\mathcal{X}, \mathbb{G}_m)$ is trivial (see [Pom13, Proposition 5.3]), every cohomology class of $H^2_{\text{ét}}(\mathcal{X}, \mu_r)$ satisfies the assumption of **Case I** (see Section 3.3). Suppose that $\alpha \in H^2_{\text{ét}}(\mathcal{X}, \mu_r)$ is in the image of the ∂ in (25) and L is a line bundle on $\mathcal{X}$ such that $\delta([L]) = -\alpha$ in (23). Note that there are finitely many modified Hilbert polynomials if the rank and the determinant are fixed. Then, the moduli stack $\mathcal{M}^{s}_{\mathrm{Dol}}(\mathbf{SL}_r)$ of stable $\mathbf{SL}_r$-Higgs bundles with determinant L contains finitely many open-closed substacks indexed by the modified Hilbert polynomials. Therefore, $\mathcal{M}^{s}_{\mathrm{Dol}}(\mathbf{SL}_r)$ admits good moduli space $M^{s}_{\mathrm{Dol}}(\mathbf{SL}_r)$, which is finite type over $\mathbb{C}$. After rigidification, we get an action of the group Γ of r-torsion points of $\mathrm{Pic}(\mathcal{X})$ on the moduli space $M^{s}_{\mathrm{Dol}}(\mathbf{SL}_r)$ of stable $\mathbf{SL}_r$-Higgs bundles (see Proposition 3.26). Specifically, the group Γ acts on $M^{s}_{\mathrm{Dol}}(\mathbf{SL}_r)$ via

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the tensor product

$$W\cdot(E,\phi)=(W\otimes E,\phi),\qquad W\in\Gamma.$$

We give the definition of moduli space of stable $\mathbf{PGL}_r$-Higgs bundles with topological type α.

Definition 3.33 The moduli space of stable $\mathbf{PGL}_r$-Higgs bundles with topological type α is defined to be the quotient stack

$$M^{\alpha,s}_{\mathrm{Dol}}(\mathbf{PGL}_r)=[M^s_{\mathrm{Dol}}(\mathbf{SL}_r)/\Gamma].$$

Remark 3.34 For a modified Hilbert polynomial P, the moduli stack $\mathcal{M}^s_{\mathrm{Dol},P}(\mathbf{SL}_r)$ (resp. $M^s_{\mathrm{Dol},P}(\mathbf{SL}_r)$) may have many open-closed substacks (resp. subschemes) indexed by K-classes in $K_0(\mathcal{X})_{\mathbb{Q}}$.

Suppose that the set of stacky points of $\mathcal{X}$ is $\{p_1,\dots,p_m\}$ and the corresponding stabilizer groups are $\mu_{r_1},\dots,\mu_{r_m}$. For each p_i, the residue gerbe $\iota_i: B\mu_{r_i}\to\mathcal{X}$ is a closed immersion. On the other hand, $K_0(B\mu_{r_i})$ is isomorphic to the representation ring $\mathbf{R}\mu_{r_i}=\mathbb{Z}[x]/(x^{r_i}-1)$ where x represents the representation defined by the inclusion $\mu_{r_i}\hookrightarrow\mathbb{C}^*$. The following proposition is well known (see [AR03, Example 5.9] or [MM99, p. 563]).

Proposition 3.35 *We have an isomorphism*

$$K_0(\mathcal{X})_{\mathbb{Q}}\simeq\mathbb{Q}\times\mathbb{Q}\times\mathbb{Q}^{r_1-1}\times\cdots\times\mathbb{Q}^{r_m-1}. \tag{29}$$

Suppose that E is a locally free sheaf on $\mathcal{X}$. If the K-class $[\iota_i^*E]=\sum_{k=0}^{r_i-1}m_{i,k}\cdot x^k$ for every i, then the image of $[E]$ under (29) is

$$(\mathrm{rk}(E),\deg(\pi_*(E)),(m_{1,i})_{i=1}^{r_1-1},\dots,(m_{m,i})_{i=1}^{r_m-1}).$$

According to the rational K-classes of line bundles on $\mathcal{X}$, the Picard group $\mathrm{Pic}(\mathcal{X})$ is the disjoint union:

$$\mathrm{Pic}(\mathcal{X})=\textstyle\coprod_{d\in\mathbb{Z}}\coprod_{i_1=0}^{r_1-1}\coprod_{i_2=0}^{r_2-1}\cdots\coprod_{i_m=0}^{r_m-1}\mathrm{Pic}^{d,(i_1,\dots,i_m)}(\mathcal{X}). \tag{30}$$

Then, the line bundle L belongs to a unique $\mathrm{Pic}^{d,(i_1,\dots,i_m)}(\mathcal{X})$. Suppose $\xi=(r,d',(m_{1,i})_{i=1}^{r_1-1},\dots,(m_{m,i})_{i=1}^{r_m-1})\in K_0(\mathcal{X})_{\mathbb{Q}}$ and r, d', $m_{1,i},\dots,m_{m,i}$ are all integers. We further assume that ξ satisfies:

- i_k is the remainder, when $\sum_{i=1}^{r_1-1}m_{k,i}$ divided by r_k for every $1\le k\le m$;
- $d'=d+\sum_{k=1}^m\sum_{i=1}^{r_k-1}m_{k,i}\frac{i}{r_k}-\sum_{k=1}^m i_k$.

Consider the moduli stack $\mathcal{M}^s_{\mathrm{Dol},\xi}(\mathbf{SL}_r)$ (with good moduli space $M^s_{\mathrm{Dol},\xi}(\mathbf{SL}_r)$) of stable $\mathbf{SL}_r$-Higgs bundles with K-class ξ. We give the following definition.

Definition 3.36 The moduli space $M^{\alpha,s}_{\mathrm{Dol},\xi}(\mathbf{PGL}_r)$ of stable $\mathbf{PGL}_r$-Higgs bundles with topological type α and K-class ξ is defined to be the quotient stack $M^{\alpha,s}_{\mathrm{Dol},\xi}(\mathbf{PGL}_r)=[M^s_{\mathrm{Dol},\xi}/\Gamma_0]$, where Γ_0 is the group of r-torsion points of $\mathrm{Pic}^0(X)$ and the action Γ_0 on $M^s_{\mathrm{Dol},\xi}(\mathbf{SL}_r)$ is given by

$$W\cdot(E,\phi)=(\pi^*W\otimes E,\phi),\qquad W\in\Gamma_0.$$

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Remark 3.37 By the decomposition of $K_0(\mathcal{X})_{\mathbb{Q}}$ (see, for example, [AR03, Example 5.9]), the subgroup of Γ which preserves the K-class ξ is the image of Γ_0 under the morphism π^* in (73) (see Section 5.3). Then, $M^{\alpha,s}_{\mathrm{Dol},\xi}(\mathbf{PGL}_r)$ is an open-closed substack of $M^{\alpha,s}_{\mathrm{Dol}}(\mathbf{PGL}_r)$.

Remark 3.38 By the orbifold-parabolic correspondence, for a rational parabolic weight, the corresponding parabolic slope can also define a stability condition on the moduli stack $\mathcal{M}_{\mathrm{Dol},\xi}(\mathbf{GL}_r)$ of Higgs bundles with K-class ξ. In fact, this way supplies more abundant stability conditions than using modified slopes (see Proposition A.1 and Remark A.2). For stacky curve, we will use parabolic slopes to define stability hereafter.

By the standard infinitesimal deformation theory of Higgs bundles on stacky curves (see [KSZ20, Proposition 3.2 and Corollaries 3.3 and 3.4]), we have following proposition.

Proposition 3.39 *If (E,ϕ) is a stable $\mathbf{SL}_r$-Higgs bundle with K-class ξ, the dimension of the tangent space of $M^s_{\mathrm{Dol},\xi}(\mathrm{SL_r})$ at (E,ϕ) is*

$$r^2(2g-2)+2-2g+\sum_{i=1}^m\Big(r^2-\big(r-\sum_{k=1}^{r_i-1}m_{i,k}\big)^2-\sum_{k=1}^{r_i-1}m_{i,k}^2\Big).$$

Moreover, $M^s_{\mathrm{Dol},\xi}(\mathbf{SL}_r)$ is smooth at (E,ϕ).

4 Spectral curves and Hitchin morphisms

4.1 Spectral curves

Let (E,ϕ) be a Higgs bundle on a hyperbolic Deligne Mumford curve $\mathcal{X}$. The characteristic polynomial of ϕ is $\det(\lambda-\phi)=\lambda^r+a_1\lambda^{r-1}+\cdots+a_r$, where λ is an indeterminate variable and $a_i=(-1)^i\mathrm{tr}(\wedge^i\phi)$ for $1\le i\le r$. It defines the so-called spectral curve associated with the Higgs bundle (E,ϕ). More precisely, the spectral curve is the zero locus of the section

$$\tau^{\otimes r}+\psi^*a_1\otimes\tau^{\otimes r-1}+\cdots+\psi^*a_{r-1}\otimes\tau+\psi^*a_r, \tag{31}$$

where $\psi:\mathrm{Tot}(K_{\mathcal{X}})\to\mathcal{X}$ is the total space of $K_{\mathcal{X}}$ and τ is the tautological section of $\psi^*K_{\mathcal{X}}$. Since the spectral curve is only dependent on the coefficients of the characteristic polynomial, we can define a spectral curve $\mathcal{X}_{\boldsymbol{a}}$ for any element $\boldsymbol{a}=(a_1,\ldots,a_r)\in\bigoplus_{i=1}^r H^0(\mathcal{X},K^i_{\mathcal{X}})$. In general, a spectral curve is neither smooth nor integral. Nevertheless, under some mild conditions, for a general element $\boldsymbol{a}\in\bigoplus_{i=1}^r H^0(\mathcal{X},K^i_{\mathcal{X}})$, the associated spectral curve $\mathcal{X}_{\boldsymbol{a}}$ is integral (see Proposition 4.3). It is easy to check the following proposition.

Proposition 4.1 *Suppose that $f:\mathcal{X}_{\boldsymbol{a}}\to\mathcal{X}$ is the projection. Then, $f_*(\mathcal{O}_{\mathcal{X}_{\boldsymbol{a}}})\simeq\bigoplus_{i=0}^{r-1}K^{-i}_{\mathcal{X}}$ and the arithmetic genus of $\mathcal{X}_{\boldsymbol{a}}$ is $\sum_{i=1}^r\dim_{\mathbb{C}}H^0(\mathcal{X},K^i_{\mathcal{X}})$.*

There is another method to construct spectral curves, which is used in [BNR89]. Recall that $\Psi:\mathbb{P}(K_{\mathcal{X}}\oplus\mathcal{O}_{\mathcal{X}})\to\mathcal{X}$ is the projective bundle associated with $K_{\mathcal{X}}\oplus\mathcal{O}_{\mathcal{X}}$. Since $\Psi_*\mathcal{O}_{\mathbb{P}(K_{\mathcal{X}}\oplus\mathcal{O}_{\mathcal{X}})}(1)=K^{-1}_{\mathcal{X}}\oplus\mathcal{O}_{\mathcal{X}}$, the section $(0,1)$ of $K^{-1}_{\mathcal{X}}\oplus\mathcal{O}_{\mathcal{X}}$ gives a section y of $\mathcal{O}_{\mathbb{P}(K_{\mathcal{X}}\oplus\mathcal{O}_{\mathcal{X}})}(1)$. Meanwhile, since $\Psi_*(\Psi^*K_{\mathcal{X}}\otimes\mathcal{O}_{\mathbb{P}(K_{\mathcal{X}}\oplus\mathcal{O}_{\mathcal{X}})}(1))=\mathcal{O}_{\mathcal{X}}\oplus K_{\mathcal{X}}$,

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$\Psi_*(\Psi^* K_{\mathcal{X}} \otimes \mathcal{O}_{\mathbb{P}(K_{\mathcal{X}} \oplus \mathcal{O}_{\mathcal{X}})}(1))$ has a section $(1,0)$. It gives a section x of $\Psi^* K_{\mathcal{X}} \otimes \mathcal{O}_{\mathbb{P}(K_{\mathcal{X}} \oplus \mathcal{O}_{\mathcal{X}})}(1)$. For $\boldsymbol{a} = (a_1, \ldots, a_r) \in \bigoplus_{i=1}^r H^0(\mathcal{X}, K_{\mathcal{X}}^i)$, there is a section

$$s := x^{\otimes r} + \Psi^* a_1 \otimes x^{\otimes r-1} \otimes y + \cdots + \Psi^* a_r \otimes y^{\otimes r} \tag{32}$$

of $\Psi^* K_{\mathcal{X}}^r \otimes \mathcal{O}_{\mathbb{P}(K_{\mathcal{X}} \oplus \mathcal{O}_{\mathcal{X}})}(r)$. Note that the zero locus of x and y are $\mathbb{P}(\mathcal{O}_{\mathcal{X}})$ and $\mathbb{P}(K_{\mathcal{X}})$, respectively. Hence, the zero locus of section (32) is the spectral curve $\mathcal{X}_{\boldsymbol{a}}$ associated with $\boldsymbol{a}$.

Remark 4.2 There exist a stacky curve $\widehat{\mathcal{X}}$ and a morphism $\mathcal{R} : \mathcal{X} \to \widehat{\mathcal{X}}$ which is an H-gerbe on $\widehat{\mathcal{X}}$ for some finite group H (see Remark 2.2). Since $\mathcal{R}^* K_{\widehat{\mathcal{X}}} = K_{\mathcal{X}}$, the spectral curves on $\mathcal{X}$ are H-gerbes on the corresponding spectral curves on $\widehat{\mathcal{X}}$.

Proposition 4.3 *Let $\mathcal{X}$ be a hyperbolic Deligne–Mumford curve, and let $r \geq 2$ be an integer. Suppose that $K_{\mathcal{X}}$ satisfies*

$$\dim_{\mathbb{C}} H^0(\mathcal{X}, K_{\mathcal{X}}^k) \geq 2 \text{ for some } 1 \leq k \leq r \quad \text{and} \quad \dim_{\mathbb{C}} H^0(\mathcal{X}, K_{\mathcal{X}}^r) \neq 0. \tag{33}$$

Then, for a general element of $\bigoplus_{i=1}^r H^0(\mathcal{X}, K_{\mathcal{X}}^i)$, the associated spectral curve is integral.

Proof Recall a basic fact: for a gerbe $\mathcal{X}_1 \to \mathcal{X}_2$, $\mathcal{X}_1$ is integral if and only if $\mathcal{X}_2$ is so. By Remark 4.2, we can assume that $\mathcal{X}$ is a stacky curve in the following discussion. Since $\mathcal{X}$ is a hyperbolic stacky curve, there is a smooth projective algebraic curve Σ with an action of a finite group G such that $\mathcal{X} = [\Sigma/G]$ (see [BN06, Corollary 7.7]). Suppose that $g : \Sigma \to \mathcal{X}$ is the morphism defined by the trivial G-torsor on Σ and the G-action. Then, g is a G-torsor over $\mathcal{X}$. As before, let $\Psi : \mathbb{P}(K_{\mathcal{X}} \oplus \mathcal{O}_{\mathcal{X}}) \to \mathcal{X}$ be the projective bundle of $K_{\mathcal{X}} \oplus \mathcal{O}_{\mathcal{X}}$. Then, there is a Cartesian diagram:

$$\begin{tikzcd} \mathbb{P}(K_\Sigma \oplus \mathcal{O}_\Sigma) \arrow[r, "g'"] \arrow[d, "\Psi'"'] & \mathbb{P}(K_{\mathcal{X}} \oplus \mathcal{O}_{\mathcal{X}}) \arrow[d, "\Psi"] \\ \Sigma \arrow[r, "g"] & \mathcal{X}. \end{tikzcd} \tag{34}$$

Similar to the second method of the construction of a spectral, we use x' to denote the section of $\Psi'^* K_\Sigma \otimes \mathcal{O}_{\mathbb{P}(K_\Sigma \oplus \mathcal{O}_\Sigma)}(1)$ corresponding to the section $(1,0)$ of $\mathcal{O}_\Sigma \oplus K_\Sigma$. And, let y' be the section of $\mathcal{O}_{\mathbb{P}(K_\Sigma \oplus \mathcal{O}_\Sigma)}(1)$ corresponding to the section $(0,1)$ of $K_\Sigma^{-1} \oplus \mathcal{O}_\Sigma$. For any $\boldsymbol{a} = (a_1, \ldots, a_r) \in \bigoplus_{i=1}^r H^0(\mathcal{X}, K_{\mathcal{X}}^i)$, the associated spectral curve $\mathcal{X}_{\boldsymbol{a}}$ is the zero locus of the section s defined by (32). Since $(g')^* x = x'$ and $(g')^* y = y'$, the pullback section of s is

$$s' := g'^* s = x'^{\otimes r} + \Psi'^* a_1' \otimes x'^{\otimes r-1} \otimes y' + \cdots + \Psi'^* a_r' \otimes y'^{\otimes r}, \tag{35}$$

where $a_i' = g^* a_i$ for all $1 \leq i \leq r$. The zero locus $\Sigma_{\boldsymbol{a}}$ of s' fits into the Cartesian diagram

$$\begin{tikzcd} \Sigma_{\boldsymbol{a}} \arrow[r, "\widehat{g}"] \arrow[d] & \mathcal{X}_{\boldsymbol{a}} \arrow[d] \\ \mathbb{P}(K_\Sigma \oplus \mathcal{O}_\Sigma) \arrow[r, "g'"] & \mathbb{P}(K_{\mathcal{X}} \oplus \mathcal{O}_{\mathcal{X}}), \end{tikzcd} \tag{36}$$

where the vertical morphisms are closed immersions. Then, $\widehat{g} : \Sigma_{\boldsymbol{a}} \to \mathcal{X}_{\boldsymbol{a}}$ is a G-torsor and $\mathcal{X}_a = [\Sigma_{\boldsymbol{a}}/G]$. Under the hypothesis of Proposition 4.3, we will show that the $\Sigma_{\boldsymbol{a}}$

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is integral, for a general $\boldsymbol{a} \in \bigoplus_{i=1}^r H^0(\mathcal{X}, K_{\mathcal{X}}^i)$. Consider the injective linear map of complex vector spaces

$$\begin{aligned} \bigoplus_{i=1}^r H^0(\mathcal{X}, K_{\mathcal{X}}^i) &\to H^0(\mathbb{P}(K_\Sigma \oplus \mathcal{O}_\Sigma), \Psi'^* K_\Sigma^r \otimes \mathcal{O}_{\mathbb{P}(K_\Sigma \oplus \mathcal{O}_\Sigma)}(r)), \\ (a_1, \dots, a_r) &\mapsto \textstyle\sum_{i=1}^r (\Psi' \circ g)^* a_i \otimes x'^{\otimes(r-i)} \otimes y'^{\otimes i}. \end{aligned} \tag{37}$$

Let V be the vector subspace of $H^0(\mathbb{P}(K_\Sigma \oplus \mathcal{O}_\Sigma), \Psi'^* K_\Sigma^r \otimes \mathcal{O}_{\mathbb{P}(K_\Sigma \oplus \mathcal{O}_\Sigma)}(r))$ generated by the section $x'^{\otimes r}$ and the image of (37). Note that the zero loci of x' and y' are disjoint. Since $H^0(\mathcal{X}, K_{\mathcal{X}}^r) \neq 0$, the base locus $\mathcal{B}$ of the linear system corresponding to V is codimension 2. Then, there is a morphism

$$\Phi_V : \mathbb{P}(K_\Sigma \oplus \mathcal{O}_\Sigma) \backslash \mathcal{B} \to \mathbb{P}(V^\vee), \tag{38}$$

where $\mathbb{P}(V^\vee)$ is the projective space associated with the dual $V^\vee$ of V.

Claim *The dimension of the image of Φ_V is 2. We only need to show that the dimension of the image of the restriction*

$$\Phi_V|_\Pi : \Pi \to \mathbb{P}(V^\vee) \tag{39}$$

is 2, where $\Pi = \mathbb{P}(K_\Sigma \oplus \mathcal{O}_\Sigma) \backslash (\mathbb{P}(K_\Sigma) \cup \mathbb{P}(\mathcal{O}_\Sigma))$. For any closed point $x \in \Sigma$, the fiber of $\Psi'|_\Pi : \Pi \to \Sigma$ over x is

$$(\Psi'|_\Pi)^{-1}(x) = \mathbb{A}^1 \backslash \{0\}.$$

And, the restriction of the morphism $\Phi_V|_\Pi$ to $(\Psi'|_\Pi)^{-1}(x)$ is

$$\mathbb{A}^1 \backslash \{0\} \to \mathbb{P}(V^\vee), \quad z \mapsto [1, c_{11} z, \dots, c_{1n_1} z, \dots, c_{r1} z^r, \dots, c_{rn_r} z^r], \tag{40}$$

where all the $c_{\bullet\bullet} \in \mathbb{C}$ and $n_i = \dim_{\mathbb{C}} H^0(\mathcal{X}, K_{\mathcal{X}}^i)$ for all $1 \leq i \leq r$. If the image $g(x)$ of x in $\mathcal{X}$ is not in the base locus $\widetilde{\mathcal{B}}$ of the complete linear system $|K_{\mathcal{X}}^r|$, the coefficients of z^r in (40) are not all zero. In this case, the image of the fiber $(\Psi'|_\Pi)^{-1}(x)$ under the morphism $\Phi_V|_\Pi$ is dimension 1. On the other hand, if $K_{\mathcal{X}}$ satisfies the condition (33), there exist two closed points $y_1, y_2 \in \mathcal{X}^o \backslash (\widetilde{\mathcal{B}} \cup \widehat{\mathcal{B}})$ and a section $a \in H^0(\mathcal{X}, K_{\mathcal{X}}^k)$ such that

$$a(y_1) = 0 \quad \text{and} \quad a(y_2) \neq 0, \tag{41}$$

where $\mathcal{X}^o$ is the non-stacky locus of $\mathcal{X}$ and $\widehat{\mathcal{B}}$ is the base locus of the complete linear system $|K_{\mathcal{X}}^k|$ (if $r = k$, then $\widehat{\mathcal{B}} = \widetilde{\mathcal{B}}$). Therefore, for any $x_1 \in g^{-1}(y_1)$ and $x_2 \in g^{-1}(y_2)$, we have

$$\begin{aligned} (\Psi' \circ g)^* a \otimes x'^{\otimes k} \otimes y'^{\otimes(r-k)}|_{(\Psi'|_\Pi)^{-1}(x_1)} &= 0 \quad \text{and} \\ (\Psi' \circ g)^* a \otimes x'^{\otimes k} \otimes y'^{\otimes(r-k)}|_{(\Psi'|_\Pi)^{-1}(x_2)} &\neq 0. \end{aligned} \tag{42}$$

It means that the images of the two fibers $(\Psi'|_\Pi)^{-1}(x_1)$ and $(\Psi'|_\Pi)^{-1}(x_2)$ do not coincide. Hence, the image of Φ_V has dimension 2. ■

By Theorem 3.3.1 in [Laz04], for a general element $\boldsymbol{a} = (a_1, \dots, a_r) \in \bigoplus_{i=1}^r H^0(\mathcal{X}, K_{\mathcal{X}}^i)$, the zero locus $\Sigma_{\boldsymbol{a}}$ of

$$x'^{\otimes r} + (\Psi' \circ g)^* a_1 \otimes x'^{\otimes(r-1)} \otimes y' + \cdots + (\Psi' \circ g)^* a_r \otimes y'^{\otimes r}$$

is integral. Therefore, $\mathcal{X}_{\boldsymbol{a}} = [\Sigma_{\boldsymbol{a}}/G]$ is integral for a general $\boldsymbol{a} \in \bigoplus_{i=1}^r H^0(\mathcal{X}, K_{\mathcal{X}}^i)$.

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Remark 4.4 In general, the conclusion of Proposition 4.3 does not hold if the condition (33) is not satisfied (see Example 4.21 in Section 4.3).

By the proof of Proposition 4.3, we get an immediate corollary.

Corollary 4.5 *If a hyperbolic Deligne–Mumford curve $\mathcal{X}$ satisfies the conditions*

$$\dim_{\mathbb{C}} H^0(\mathcal{X}, K_{\mathcal{X}}^k) \geq 2 \text{ for some } 2 \leq k \leq r \quad \text{and} \quad H^0(\mathcal{X}, K_{\mathcal{X}}^r) \neq 0, \tag{43}$$

then for a general element of $\bigoplus_{i=2}^r H^0(\mathcal{X}, K_{\mathcal{X}}^i)$, the corresponding spectral curve is integral.

4.2 The Hitchin morphism

Let (E_T, ϕ_T) be a T-family of rank r Higgs bundles on $\mathcal{X}$ for a scheme T. Its characteristic polynomial is

$$\det(\lambda - \phi_T) = \lambda^r + a_1(T)\lambda^{r-1} + \cdots + a_r(T), \tag{44}$$

where $a_i(T) = (-1)^i \wedge^i \phi_T \in H^0(\mathcal{X}_T, \mathrm{pr}_{\mathcal{X}}{}^* K_{\mathcal{X}}^i)$. The zero locus of (44) in the total space of $\mathrm{pr}_{\mathcal{X}}{}^* K_{\mathcal{X}}$ is a flat family of spectral curves over T. The affine space $\mathbb{H}(r, K_{\mathcal{X}})$ associated with vector space $\bigoplus_{i=1}^r H^0(\mathcal{X}, K_{\mathcal{X}}^i)$ parametrizes the universal family of spectral curves. Indeed, it represents the functor

$$(\mathrm{Sch}/\mathbb{C})^o \to (sets) \quad T \mapsto \bigoplus_{i=1}^r H^0(\mathcal{X}_T, \mathrm{pr}_{\mathcal{X}}{}^* K_{\mathcal{X}}^i), \tag{45}$$

since there is a canonical isomorphism

$$H^0(T, H^0(\mathcal{X}, K_{\mathcal{X}}^i) \otimes_{\mathbb{C}} \mathcal{O}_T) \xrightarrow{\sim} H^0(\mathcal{X}_T, \mathrm{pr}_{\mathcal{X}}{}^* K_{\mathcal{X}}^i) \quad \text{for each } 1 \leq i \leq r$$

(see [Bro12, Corollary A.2.2]). We therefore have a morphism of stacks

$$\mathcal{H} : \mathcal{M}_{\mathrm{Dol},P}(\mathbf{GL}_r) \to \mathbb{H}(r, K_{\mathcal{X}}), \quad (E_T, \phi_T) \mapsto \det(\lambda - \phi_T) \quad \text{for any test scheme } T, \tag{46}$$

which is called the *Hitchin morphism*. The following proposition describes the fibers of the Hitchin morphism.

Proposition 4.6 *For a nonzero element $\boldsymbol{a} \in \bigoplus_{i=1}^r H^0(\mathcal{X}, K_{\mathcal{X}}^i)$, let $\underline{\boldsymbol{a}} : \mathrm{Spec}(\mathbb{C}) \to \mathbb{H}(r, K_{\mathcal{X}})$ be the closed point defined by $\boldsymbol{a}$. Consider the Cartesian diagram*

$$\begin{tikzcd} \mathcal{M}_{\mathrm{Dol},P}(\mathbf{GL}_r) \times_{\mathbb{H}(r,K_{\mathcal{X}})} \mathrm{Spec}(\mathbb{C}) \arrow[r] \arrow[d] & \mathrm{Spec}(\mathbb{C}) \arrow[d, "\underline{\boldsymbol{a}}"] \\ \mathcal{M}_{\mathrm{Dol},P}(\mathbf{GL}_r) \arrow[r, "\mathcal{H}"] & \mathbb{H}(r, K_{\mathcal{X}}). \end{tikzcd} \tag{47}$$

If the spectral curve $\mathcal{X}_{\boldsymbol{a}}$ associated with $\boldsymbol{a}$ is integral, then $\mathcal{M}_{\mathrm{Dol},P}(\mathbf{GL}_r) \times_{\mathbb{H}(r,K_{\mathcal{X}})} \mathrm{Spec}(\mathbb{C})$ is the moduli stack of rank one torsion-free sheaves on $\mathcal{X}_{\boldsymbol{a}}$ with modified Hilbert polynomial P.

Proof This proposition follows from Proposition C.2 in Appendix C. ∎

Since the moduli stack $\mathcal{M}_{\mathrm{Dol},P}^{ss}(\mathbf{GL}_r)$ of semistable Higgs bundles is an open substack of $\mathcal{M}_{\mathrm{Dol},P}(\mathbf{GL}_r)$, we can restrict the Hitchin morphism $\mathcal{H} : \mathcal{M}_{\mathrm{Dol}}(\mathbf{GL}_r) \to$

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$\mathbb{H}(r, K_{\mathcal{X}})$ to $\mathcal{M}^{ss}_{\mathrm{Dol},P}(\mathbf{GL}_r)$, which is also denoted by $\mathcal{H}$. Let $\mathcal{Q} : \mathcal{M}^{ss}_{\mathrm{Dol},P}(\mathbf{GL}_r) \to M^{ss}_{\mathrm{Dol},P}(\mathbf{GL}_r)$ be the good moduli space of $\mathcal{M}^{ss}_{\mathrm{Dol},P}(\mathbf{GL}_r)$. By the universal property of $M^{ss}_{\mathrm{Dol},P}(\mathbf{GL}_r)$, there exists a unique morphism $h : M^{ss}_{\mathrm{Dol},P}(\mathbf{GL}_r) \to \mathbb{H}(r, K_{\mathcal{X}})$ satisfying $\mathcal{H} = h \circ \mathcal{Q}$. h is also called *Hitchin morphism*.

Theorem 4.7 *If $\mathcal{X}$ is a hyperbolic stacky curve, then the Hitchin morphism h is proper.*

Proof The proof of the theorem is given in Appendix B. ∎

Restricting the Hitchin morphism h to $M^{ss}_{\mathrm{Dol},P}(\mathbf{SL}_r)$ of $M^{ss}_{\mathrm{Dol},P}(\mathbf{GL}_r)$, we get the Hitchin morphism for the moduli space of semistable $\mathbf{SL}_r$-Higgs bundles $h_{\mathbf{SL}_r} : M^{ss}_{\mathrm{Dol},P}(\mathbf{SL}_r) \to \mathbb{H}^o(r, K_{\mathcal{X}})$, where $\mathbb{H}^o(r, K_{\mathcal{X}})$ is the affine space associated with $\bigoplus_{i=2}^r H^0(\mathcal{X}, K^i_{\mathcal{X}})$. Let $\xi \in K_0(\mathcal{X})_{\mathbb{Q}}$ be a K-class such that the modified Hilbert polynomial is P. The restriction of $h_{\mathbf{SL}_r}$ to $M^{ss}_{\mathrm{Dol},\xi}(\mathbf{SL}_r)$ is also denoted by $h_{\mathbf{SL}_r}$. Since $M^{ss}_{\mathrm{Dol},\xi}(\mathbf{SL}_r)$ is an open and closed subscheme of $M^{ss}_{\mathrm{Dol},P}(\mathbf{SL}_r)$, the restriction $h_{\mathbf{SL}_r} : M^{ss}_{\mathrm{Dol},\xi}(\mathbf{SL}_r) \to \mathbb{H}^o(r, K_{\mathcal{X}})$ is also proper. Since $h_{\mathbf{SL}_r}$ is invariant under the action of the group Γ_0 of r-torsion points of $\mathrm{Pic}^0(X)$, we have the morphism $h_{\mathbf{PGL}_r} : M^{\alpha,s}_{\mathrm{Dol},\xi}(\mathbf{PGL}_r) \to \mathbb{H}^o(r, K_{\mathcal{X}})$, which is called the Hitchin morphism of $M^{\alpha,s}_{\mathrm{Dol},\xi}(\mathbf{PGL}_r)$.

Corollary 4.8 *If $\mathcal{X}$ is a hyperbolic stacky curve, $h_{\mathbf{SL}_r}$ is proper. Furthermore, if $M^{ss}_{\mathrm{Dol},\xi}(\mathbf{SL}_r)$ has no strictly semistable objects, then $h_{\mathbf{PGL}_r}$ is also proper.*

Proof Due to the properness of h, $h_{\mathbf{SL}_r}$ is also proper. If $M^{ss}_{\mathrm{Dol},\xi}$ has no strictly semistable objects, then $M^{ss}_{\mathrm{Dol},\xi}(\mathbf{SL}_r) = M^{s}_{\mathrm{Dol},\xi}(\mathbf{SL}_r)$. By Proposition 10.1.6(v) in [Ols16], $h_{\mathbf{PGL}_r}$ is proper. ∎

4.3 Classification of spectral curves

In the following, $\mathcal{X}$ is a stacky curve with coarse moduli space $\pi : \mathcal{X} \to X$. The set of stacky points is $\{p_1, \dots, p_m\}$ and the stabilizer groups are $\mu_{r_1}, \dots, \mu_{r_m}$. Every smooth stacky curve can be obtained by applying root constructions (see [Cad07]) to its coarse moduli space. Recall Theorem 3.63 in [Beh14].

Theorem 4.9 [Beh14] $\mathcal{X} = \sqrt[r_1]{p_1} \times_X \sqrt[r_2]{p_2} \times_X \cdots \times_X \sqrt[r_m]{p_m}$, *where $\sqrt[r_k]{p_k}$ is the r_k-th root stack associated with the divisor p_k for every $1 \le k \le m$.*

For each $1 \le k \le m$, let (L_k, s_k) be the pair, which consists of the universal line bundle L_k and section s_k of L_k on $\sqrt[r_k]{p_k}$. And, let s be the section $\otimes_{k=1}^m \mathrm{pr}_k^* s_k$ of $\otimes_{k=1}^m \mathrm{pr}_k^* L_k$, where $\mathrm{pr}_k : \mathcal{X} \to \sqrt[r_k]{p_k}$ is the projection to $\sqrt[r_k]{p_k}$ for every $1 \le k \le m$.

Corollary 4.10 [VB22] *Under the hypothesis of Theorem 4.9, we have*

$$K_{\mathcal{X}} = \pi^* K_X \otimes \otimes_{k=1}^m \mathrm{pr}_k^* L_k^{r_k - 1} \quad \text{and} \quad \pi^* \mathcal{O}_X(D)) = \otimes_{k=1}^m \mathrm{pr}_k^* L_k^{r_k},$$

where $D = \sum_{k=1}^m p_k$.

Proof By Proposition 5.5.6 in [VB22], we get the first formula. The second is obvious. ∎

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Lemma 4.11 *If the natural number r satisfies $r \le r_k$ for every $1 \le k \le m$, then for any element $\boldsymbol{a} = (a_1, \dots, a_r) \in \bigoplus_{i=1}^r H^0(\mathcal{X}, K_{\mathcal{X}}^i)$, there is an element*

$$\overline{\boldsymbol{a}} = (\overline{a}_1, \dots, \overline{a}_r) \in \bigoplus_{i=1}^r H^0(X, K_X^i \otimes \mathcal{O}((i-1)D)),$$

satisfying

$$a_i = \pi^* \overline{a}_i \otimes \bigotimes_{k=1}^m \mathrm{pr}_k^* s_k^{\otimes(r_k - i)} \quad \textit{for each} \quad 1 \le i \le m. \tag{48}$$

Proof By Corollary 4.10, we have $K_{\mathcal{X}}^i = \pi^* K_X^i \otimes \bigotimes_{k=1}^m \mathrm{pr}_k^* L_k^{(i-1)r_k + (r_k - i)}$ for any integer i. If i satisfies $i \le r_k$ for every $1 \le k \le m$, then we have

$$H^0(X, K_X^i \otimes \mathcal{O}_X((i-1)D) \to H^0(\mathcal{X}, K_{\mathcal{X}}^i), \quad \overline{a} \mapsto \pi^* \overline{a} \otimes \bigotimes_{k=1}^m \mathrm{pr}_k^* s_k^{\otimes(r_k - i)}$$

is an isomorphism, where $\overline{a} \in H^0(X, K_X^i \otimes \mathcal{O}_X((i-1)D)$. ■

Let t be the global section of the line bundle $\mathcal{O}_X(D)$ such that $\pi^* t = \bigotimes_{k=1}^m \mathrm{pr}_k^* s_k^{\otimes r_k}$. Then, we have the following corollary.

Corollary 4.12 *Assume that the natural number r satisfies $r \le r_k$ for every $1 \le k \le m$. Then there is an injection of vector spaces*

$$\bigoplus_{i=1}^r H^0(\mathcal{X}, K_{\mathcal{X}}^i) \longrightarrow \bigoplus_{i=1}^r H^0(X, (K_X(D))^i), \quad \boldsymbol{a} = (a_1, \dots, a_r) \longmapsto \widetilde{\boldsymbol{a}} = (\widetilde{a}_1, \dots, \widetilde{a}_r), \tag{49}$$

where $\widetilde{a}_i = \overline{a}_i \otimes t$ and $\overline{a}_i \in H^0(X, K_X^i \otimes \mathcal{O}_X((i-1)D))$ is the section associated with a_i in Lemma 4.11, for every $1 \le i \le r$.

Proof The section t defines an injection $K_X^i \otimes \mathcal{O}_X((i-1)D) \hookrightarrow (K_X(D))^i$ for every $1 \le i \le r$. Then, we get the injective linear map

$$\bigoplus_{i=1}^r H^0(X, K_X^i \otimes \mathcal{O}_X((i-1)D)) \hookrightarrow \bigoplus_{i=1}^r H^0(X, (K_X(D))^i). \tag{50}$$

Under the morphism (50), the image of $\overline{\boldsymbol{a}} = (\overline{a}_1, \dots, \overline{a}_r) \in \bigoplus_{i=1}^r H^0(X, K_X^i \otimes \mathcal{O}_X((i-1)D))$ is

$$\widetilde{\boldsymbol{a}} = (\widetilde{a}_1, \dots, \widetilde{a}_m) \in \bigoplus_{i=1}^r H^0(X, (K_X(D))^i),$$

where $\widetilde{a}_i = \overline{a}_i \otimes t$ for every $1 \le i \le m$. ■

Theorem 4.13 *Suppose that the natural number r satisfies $2 \le r \le r_i$ for all $1 \le i \le m$ and $\boldsymbol{a} = (a_1, \dots, a_r)$ is an element of $\bigoplus_{i=1}^r H^0(\mathcal{X}, K_{\mathcal{X}}^i)$. Then the coarse moduli space of $\mathcal{X}_{\boldsymbol{a}}$ is the curve $X_{\widetilde{\boldsymbol{a}}}$, which is the zero locus of the section $\overline{\tau}^{\otimes r} + \varphi^* \widetilde{a}_1 \otimes \overline{\tau}^{\otimes(r-1)} + \cdots + \varphi^* \widetilde{a}_r$ on the total space $\varphi : \mathrm{Tot}(K_X(D)) \to X$, where $\widetilde{\boldsymbol{a}} = (\widetilde{a}_1, \dots, \widetilde{a}_r)$ is the image of $\boldsymbol{a}$ under the morphism (49) and $\overline{\tau}$ is the tautological section of $\varphi^* K_X(D)$.*

Proof The section $s = \bigotimes_{k=1}^m \mathrm{pr}_k^* s_k$ of $\bigotimes_{k=1}^m \mathrm{pr}_k^* L_k$ defines an injection $K_{\mathcal{X}} \hookrightarrow \pi^* K_X(D)$. Let $\pi''' : \mathrm{Tot}(K_{\mathcal{X}}) \to \mathrm{Tot}(\pi^* K_X(D))$ be the corresponding morphism between total spaces. In general, π''' is not injective. It satisfies the commutative diagram

(51)
$$\begin{tikzcd} \mathrm{Tot}(K_{\mathcal{X}}) \arrow[rr, "\pi'''"] \arrow[dr, "\psi"'] & & \mathrm{Tot}(\pi^* K_X(D)) \arrow[dl, "\psi'"] \\ & \mathcal{X} & \end{tikzcd}.$$

On the other hand, there is a Cartesian diagram

(52)
$$\begin{tikzcd} \mathrm{Tot}(\pi^* K_X(D)) \arrow[r, "\pi''"] \arrow[d, "\psi'"'] & \mathrm{Tot}(K_X(D)) \arrow[d, "\varphi"] \\ \mathcal{X} \arrow[r, "\pi"] & X \end{tikzcd}.$$

Composing the diagrams (51) and (52), we get a new commutative diagram

(53)
$$\begin{tikzcd} \mathrm{Tot}(K_{\mathcal{X}}) \arrow[r, "\pi'"] \arrow[d, "\psi"'] & \mathrm{Tot}(K_X(D)) \arrow[d, "\varphi"] \\ \mathcal{X} \arrow[r, "\pi"] & X \end{tikzcd}$$

where $\pi' = \pi'' \circ \pi'''$. The curve $X_{\widetilde{a}}$ is the zero locus of the section $\overline{\tau}^{\otimes r} + \varphi^* \widetilde{a}_1 \otimes \overline{\tau}^{\otimes (r-1)} + \cdots + \varphi^* \widetilde{a}_r$ on the total space $\mathrm{Tot}(K_X(D))$. And, the spectral curve $\mathcal{X}_\alpha$ is the zero locus of section $\tau^{\otimes r} + \psi^* a_1 \otimes \tau^{\otimes (r-1)} + \cdots + \psi^* a_r$, where τ is the tautological section of $\psi^* K_{\mathcal{X}}$. Since $(\pi')^* \overline{\tau} = \tau \otimes \psi^* s$, we have

$$\begin{aligned} &(\pi')^* (\overline{\tau}^{\otimes r} + \varphi^* \widetilde{a}_1 \otimes \overline{\tau}^{\otimes (r-1)} + \cdots + \varphi^* \widetilde{a}_r) \\ &\quad = (\tau^{\otimes r} + \psi^* a_1 \otimes \tau^{\otimes (r-1)} + \cdots + \psi^* a_r) \otimes \psi^* s^{\otimes r}. \end{aligned}$$

We get the commutative diagram

$$\begin{tikzcd} \mathcal{X}_\alpha \arrow[r, "\pi'|_{\mathcal{X}_\alpha}"] \arrow[d, "\psi|_{\mathcal{X}_\alpha}"'] & X_{\widetilde{\alpha}} \arrow[d, "\varphi|_{X_{\widetilde{\alpha}}}"] \\ \mathcal{X} \arrow[r, "\pi"] & X \end{tikzcd}.$$

In order to show that $X_{\widetilde{\alpha}}$ is the coarse moduli space of $\mathcal{X}_\alpha$, we only need to check this locally. For each stacky point p_i, there is an affine open subset $U_i = \mathrm{Spec}(A_i)$ of X, such that U_i contains only p_i and $p_i = (f_i)$ as a divisor on U_i for some $f_i \in A_i$. In the following, we consider the commutative diagram

$$\begin{tikzcd} \mathcal{X}_\alpha \times_X U_i \arrow[r] \arrow[d] & X_{\widetilde{\alpha}} \times_X U_i \arrow[d] \\ \mathcal{X} \times_X U_i \arrow[r] & U_i \end{tikzcd},$$

where

$$X_{\widetilde{\alpha}} \times_X U_i = \mathrm{Spec}\left(\tfrac{A_i[x]}{(x^r + \overline{a}_1 f_i x^{r-1} + \overline{a}_2 f_i x^+ \cdots + \overline{a}_r f_i)} \right).$$

Since

$$\mathcal{X} \times_X U_i = \left[\mathrm{Spec}\left(\tfrac{A_i[t]}{(t^{r_i} - f_i)} \right) \middle/ \mu_{r_i} \right]$$

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(see [Ols16, Theorem 10.3.10(ii)]), we have

$$\mathcal{X}_\alpha \times_X U_i = \left[\mathrm{Spec}\left(\frac{A_i[t,y]}{(y^r+\overline{a}_1 t^{r_i-1}y^{r-1}+\cdots+\overline{a}_r t^{r_i-r}, t^{r_i}-f_i)}\right)\Big/\mu_{r_i}\right],$$

where the action of $\mu_{r_i} = \mathrm{Spec}(\mathbb{C}[z]/(z^{r_i}-1))$ is defined by

$$t \mapsto z \otimes t, \quad y \mapsto z^{-1} \otimes y.$$

Hence, the coarse moduli space of $\mathcal{X}_\alpha \times_X U_i$ is $\mathrm{Spec}\left(\frac{A_i[ty]}{((ty)^r+\overline{a}_1 f_i (ty)^{r-1}+\overline{a}_2 f_i (ty)^{r-2}+\cdots+\overline{a}_r f_i)}\right)$. ■

Remark 4.14 Biswas–Majumder–Wong [BMW13], Borne [BNR89], and Nasatyr–Steer [NS95] established the orbifold-parabolic correspondence, i.e., there is a one-to-one correspondence between the Higgs bundles on stacky curves $\mathcal{X}$ and the strongly parabolic Higgs bundles (see [BMW13, Section 3.1]) on its coarse moduli space X with marked points $\{p_1,\dots,p_m\}$. This theorem explain the relationship between the corresponding spectral curves.

Suppose that j is a natural number. It can be written uniquely in the form $j = h_{jk}\cdot r_k - q_{jk}$, where $h_{jk}, q_{jk} \in \mathbb{Z}$ satisfy $0 \le q_{jk} < r_k$ for every $0 \le k \le m$. More precisely, we have

$$h_{jk} = \left\lceil \frac{j}{r_k} \right\rceil \quad \text{and} \quad q_{jk} = r_k\left(\left\lceil \frac{j}{r_k}\right\rceil - \frac{j}{r_k}\right) \tag{54}$$

for every $0 \le k \le m$. Let $\widetilde{h}_{jk} = j - h_{jk}$ for every $1 \le k \le m$. Then, we have

$$\widetilde{h}_{jk} = j - \left\lceil \frac{j}{r_k}\right\rceil \quad \text{for every } 0 \le k \le m. \tag{55}$$

We therefore have $K^j_{\mathcal{X}} = \pi^* K^j_X \otimes \bigotimes_{k=1}^m \mathrm{pr}_k^* L_k^{j(r_k-1)} = \pi^* K^j_X \otimes \bigotimes_{k=1}^m \mathrm{pr}_k^* L_k^{\widetilde{h}_{jk}\cdot r_k + q_{jk}}$. Then, the pushforward of $K^j_{\mathcal{X}}$ is

$$\pi_* K^j_{\mathcal{X}} = K^j_X \otimes \mathcal{O}_X\left(\textstyle\sum_{k=1}^m \widetilde{h}_{jk}\cdot p_k\right). \tag{56}$$

Hence, there is an isomorphism

(57)

$$H^0(X, K^j_X \otimes \mathcal{O}_X(\textstyle\sum_{k=1}^m \widetilde{h}_{jk}\cdot p_k)) \longrightarrow H^0(\mathcal{X}, K^j_{\mathcal{X}}), \quad a \longmapsto \pi^* a \otimes \bigotimes_{k=1}^m \pi_1^* s_k^{\otimes q_{jk}}.$$

Lemma 4.15 *Suppose that the aforementioned $\mathcal{X}$ is hyperbolic with genus g and the natural number $r \ge 2$. Furthermore, we assume that*

$$q_{rk} = 0 \quad \text{or} \quad 1 \quad \text{for all } 1 \le k \le m. \tag{58}$$

If $\deg(\pi_*(K^r_{\mathcal{X}})) \ge 2g$ and the condition (33) hold, then the spectral curve $\mathcal{X}_{\boldsymbol{a}}$ is integral and smooth for a general element $\boldsymbol{a} = (a_1,\dots,a_r) \in \bigoplus_{i=1}^r H^0(\mathcal{X}, K^i_{\mathcal{X}})$.

Proof By the uniformization of Deligne–Mumford curves, there is a smooth complex projective algebraic curve Σ with a finite group G action such that $\mathcal{X} = [\Sigma/G]$. As before, $g : \Sigma \to \mathcal{X}$ is the étale covering defined by the trivial G-torsor and the G-action on Σ. For a general element $\boldsymbol{a} = (a_1,\dots,a_r) \in \bigoplus_{i=1}^r H^0(\mathcal{X}, K^i_{\mathcal{X}})$, the spectral curve $\Sigma_{\boldsymbol{a}}$ defined by $(g^*a_1,\dots,g^*a_r) \in \bigoplus_{i=1}^r H^0(\Sigma, K^i_\Sigma)$ is integral (see the proof of

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Proposition 4.3). On the other hand, since $\pi_*(K^r_{\mathcal{X}}) \geq 2g$, the linear system $|\pi_*(K^r_{\mathcal{X}})|$ has no base points (see [Har77, Corollary 3.2]). There are two cases: $\pi_*(K^r_{\mathcal{X}}) = \mathcal{O}_X$ and $\dim|\pi_*(K^r_{\mathcal{X}})| \geq 1$.

Case 1 *$\pi_*(K^r_{\mathcal{X}}) = \mathcal{O}_X$. By (57), the $H^0(\mathcal{X}, K^r_{\mathcal{X}})$ is a one-dimensional complex vector space generated by the section $\otimes_{k=1}^m \mathrm{pr_k}^* s_k^{\otimes q_{rk}}$. By assumption, $g^*(\otimes_{k=1}^m \mathrm{pr_k}^* s_k^{\otimes q_{rk}})$ only has simple zeros. Therefore, for a general $\boldsymbol{a} \in \bigoplus_{i=1}^m H^0(\mathcal{X}, K^i_{\mathcal{X}})$, the spectral curve $\Sigma_{\boldsymbol{a}}$ is integral and smooth, by the Jacobian criterion (see [Mat86, Theorem 30.3(5)]). Thus, a general spectral curve $\mathcal{X}_{\boldsymbol{a}} = [\Sigma_{\boldsymbol{a}}/G]$ is integral and smooth.* ∎

Case 2 *$\dim|\pi_*(K^r_{\mathcal{X}})| \geq 1$. In this case, a general element of $|\pi_*(K^r_{\mathcal{X}})|$ is a reduced divisor, whose support is disjoint with the stacky locus $\{p_1, \ldots, p_m\}$. By (57) and the assumptions about q_{rk} for all $1 \leq k \leq m$, for a general section $a_r \in H^0(\mathcal{X}, K^r_{\mathcal{X}})$, $g^* a_r$ only has simple zeros. As in **Case 1**, by Jacobian criterion, for a general $\boldsymbol{a} \in \bigoplus_{i=1}^r H^0(\mathcal{X}, K^i_{\mathcal{X}})$, $\Sigma_{\boldsymbol{a}}$ is integral and smooth. Therefore, a general spectral curve $\mathcal{X}_{\boldsymbol{a}}$ is also integral and smooth.*

Lemma 4.16 *We assume that the aforementioned hyperbolic stacky curve $\mathcal{X}$ and the natural number r do not satisfy the condition (58). For that, we make the assumption:*

$$\text{(59)} \qquad \textit{If } q_{rk} \geq 2 \textit{ for some } 1 \leq k \leq m, \textit{ then } q_{(r-1)k} = 0.$$

If $\deg(\pi_*(K^{r-1}_{\mathcal{X}})) \geq 2g$ and the condition (33) holds, then a general spectral curve $\mathcal{X}_{\boldsymbol{a}}$ is integral and smooth.

Proof As before, $\mathcal{X} = [\Sigma/G]$ and $g : \Sigma \to \mathcal{X}$ is the natural étale covering, where Σ is a smooth complex projective algebraic curve with an finite group G action. And, for a general $\boldsymbol{a} \in \bigoplus_{i=1}^r H^0(\mathcal{X}, K^i_{\mathcal{X}})$, the spectral curve $\Sigma_{\boldsymbol{a}}$ defined by $(g^* a_1, \ldots, g^* a_r) \in \bigoplus_{i=1}^r H^0(\Sigma, K^i_{\Sigma})$ is integral (see the proof of Proposition 4.3). Note that $\deg(\pi_*(K^r_{\mathcal{X}})) \geq 2g$. Hence, the linear system $|\pi_*(K^r_{\mathcal{X}})|$ is base-point-free (see [Har77, Corollary 3.2]). By the proof of Lemma 4.15, for a general $a_r \in H^0(\mathcal{X}, K^r_{\mathcal{X}})$, the multiple zeros of $g^* a_r$ are contained in the preimages of those stacky points p_k for which $q_{rk} \geq 2$. In order to show that the spectral curve $\Sigma_{\boldsymbol{a}}$ is smooth for a general $\boldsymbol{a} \in \bigoplus_{i=1}^r H^0(\mathcal{X}, K^i_{\mathcal{X}})$, we only need to prove that a general section $a_{r-1} \in H^0(\mathcal{X}, K^{r-1}_{\mathcal{X}})$ does not vanish at those stacky points p_k for which $q_{rk} \geq 2$ by Jacobian criterion (see [Mat86, Theorem 30.3(5)] or [BNR89, Remark 3.5]). Since the linear system $|\pi_*(K^{r-1}_{\mathcal{X}})|$ is base-point-free, we have $\pi_*(K^{r-1}_{\mathcal{X}}) = \mathcal{O}_X$ or $\dim|\pi_*(K^{r-1}_{\mathcal{X}})| \geq 1$.

Case 1 *$\pi_*(K^{r-1}_{\mathcal{X}}) = \mathcal{O}_X$. By (57), $H^0(\mathcal{X}, K^{r-1}_{\mathcal{X}})$ is a one-dimensional complex vector space generated by the section $\otimes_{k=1}^m \mathrm{pr_k}^* s_k^{\otimes q_{(r-1)k}}$. The assumptions about q_{rk} for all $1 \leq k \leq m$ imply that the zero locus of $g^*(\otimes_{k=1}^m \mathrm{pr_k}^* s_k^{\otimes q_{(r-1)k}})$ does not intersect with the preimage of those stacky points p_k for which $q_{rk} \geq 2$. Therefore, for a general $\boldsymbol{a} \in \bigoplus_{i=1}^r H^0(\mathcal{X}, K^i_{\mathcal{X}})$, the spectral curve $\Sigma_{\boldsymbol{a}}$ is integral and smooth. Then, a general spectral curve $\mathcal{X}_{\boldsymbol{a}} = [\Sigma_{\boldsymbol{a}}/G]$ is also integral and smooth.* ∎

Case 2 *$\dim|\pi_*(K^{r-1}_{\mathcal{X}})| \geq 1$. In this case, a general element of $|\pi_*(K^{r-1}_{\mathcal{X}})|$ is a reduced divisor, whose support is disjoint with the stacky locus $\{p_1, \ldots, p_m\}$. By (57) and the assumptions about q_{rk} for all $1 \leq k \leq m$, for a general section $a_{r-1} \in H^0(\mathcal{X}, K^{r-1}_{\mathcal{X}})$, $g^* a_{r-1}$ does not vanish at those points whose images are the stacky points p_k for which*

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*$q_{rk} \geq 2$. As **Case 1**, $\Sigma_{\boldsymbol{a}}$ is integral and smooth, for a general $\boldsymbol{a} \in \bigoplus_{i=1}^{r} H^0(\mathfrak{X}, K_{\mathfrak{X}}^i)$. Then, a general spectral curve $\mathfrak{X}_{\boldsymbol{a}}$ is integral and smooth.*

Lemma 4.17 *Suppose that the aforementioned hyperbolic stacky curve $\mathfrak{X}$ and the natural number r do not satisfy the conditions (58) and (59). If the condition (33) holds, then a general spectral curve $\mathfrak{X}_{\boldsymbol{a}}$ is singular.*

Proof Recall that $\mathfrak{X} = [\Sigma/G]$ and $g : \Sigma \to \mathfrak{X}$ is the natural étale covering, where Σ is a smooth complex projective algebraic curve with a finite group G action. For a general $\boldsymbol{a} \in \bigoplus_{i=1}^{r} H^0(\mathfrak{X}, K_{\mathfrak{X}}^i)$, the spectral curve $\Sigma_{\boldsymbol{a}}$ defined by $(g^*a_1, \ldots, g^*a_r) \in \bigoplus_{i=1}^{r} H^0(\Sigma, K_{\Sigma}^i)$ is integral (see the proof of Proposition 4.3). We will show that for a general $\boldsymbol{a} = (a_1, \ldots, a_r) \in \bigoplus_{i=1}^{r} H^0(\mathfrak{X}, K_{\mathfrak{X}}^i)$, a_{r-1} vanishes at the multiple zeros of a_r. Then, for a general $\boldsymbol{a} \in \bigoplus_{i=1}^{r} H^0(\mathfrak{X}, K_{\mathfrak{X}}^i)$, $\Sigma_{\boldsymbol{a}}$ is singular by Jacobian criterion (see [Mat86, Theorem 30.3(5)] or [BNR89, Remark 3.5]). If the conditions (58) and (59) do not hold, then we have

$$q_{rk} \geq 2 \quad \text{and} \quad q_{(r-1)k} \geq 1 \quad \text{for some } 1 \leq k \leq m.$$

By (57), for a general $\boldsymbol{a} = (a_1, \ldots, a_r) \in \bigoplus_{i=1}^{r} H^0(\mathfrak{X}, K_{\mathfrak{X}}^i)$, the closed points in the preimage of p_k are multiple zeros of g^*a_r and zeros of g^*a_{r-1}. We complete the proof of the lemma. ∎

Theorem 4.18 *Suppose that $\mathfrak{X}$ is a hyperbolic stacky curve of genus g. Let r be a natural number with $r \geq 2$, and let $\mathfrak{X}_{\boldsymbol{a}}$ be the spectral curve associated with $\boldsymbol{a} \in \bigoplus_{i=1}^{r} H^0(\mathfrak{X}, K_{\mathfrak{X}}^i)$.*

(1) *Assume that $\lceil \frac{r}{r_k} \rceil = \frac{r}{r_k}$ or $\lceil \frac{r}{r_k} \rceil = \frac{r+1}{r_k}$ for all $1 \leq k \leq m$. A general spectral curve $\mathfrak{X}_{\boldsymbol{a}}$ is integral and smooth if one of the following conditions is satisfied:*
   - (i) *$g \geq 2$;*
   - (ii) *$g = 1$ and $\sum_{k=1}^{m}(r - \lceil \frac{r}{r_k} \rceil) \geq 2$;*
   - (iii) *$g = 0$ and $\sum_{k=1}^{m}(r - \lceil \frac{r}{r_k} \rceil) \geq 2r + 1$;*
   - (iv) *$g = 0$, $\sum_{k=1}^{m}(r - \lceil \frac{r}{r_k} \rceil) \geq 2r$ and $\dim_{\mathbb{C}} H^0(\mathfrak{X}, K_{\mathfrak{X}}^i) \geq 2$ for some $1 \leq i \leq r$.*

(2) *Suppose that the assumption in (1) does not hold. We make the following assumption: if $\lceil \frac{r}{r_k} \rceil \geq \frac{r+2}{r_k}$ for some $1 \leq k \leq m$, then $\lceil \frac{r-1}{r_k} \rceil = \frac{r-1}{r_k}$. A general spectral curve $\mathfrak{X}_{\boldsymbol{a}}$ is integral and smooth if any of the following conditions is satisfied:*
   - (i) *$g \geq 2$;*
   - (ii) *$g = 1$ and $\sum_{k=1}^{m}(r - 1 - \lceil \frac{r-1}{r_k} \rceil) \geq 2$;*
   - (iii) *$g = 0$, $\sum_{k=1}^{m}(r - 1 - \lceil \frac{r-1}{r_k} \rceil) \geq 2r - 2$ and $K_{\mathfrak{X}}$ satisfies (33).*

(3) *If $\lceil \frac{r}{r_k} \rceil \geq \frac{r+2}{r_k}$ and $\lceil \frac{r-1}{r_k} \rceil \geq \frac{r}{r_k}$ for some $1 \leq k \leq m$, then the general spectral curve $\mathfrak{X}_{\boldsymbol{a}}$ is integral and singular if one of the following conditions occurs:*
   - (i) *$g \geq 2$;*
   - (ii) *$g = 1$ and $K_{\mathfrak{X}}$ satisfies (33);*
   - (iii) *$g = 0$ and $K_{\mathfrak{X}}$ satisfies (33).*

Proof (1). By (54), the assumption: $\lceil \frac{r}{r_k} \rceil = \frac{r}{r_k}$ or $\lceil \frac{r}{r_k} \rceil = \frac{r+1}{r_k}$ for all $1 \leq k \leq m$, is equivalent to the condition (58). And, by (56), we have $\deg(\pi_*(K_{\mathfrak{X}}^r)) = (2g-2)r + \sum_{k=1}^{m} \widetilde{h}_{rk}$. On the other hand, by the orbifold Riemann–Roch formula (see [AGA08, Theorem 7.21]) and Serre duality, we get

$$\dim_{\mathbb{C}} H^0(\mathfrak{X}, K_{\mathfrak{X}}^r) = (g-1)(2r-1) + \sum_{k=1}^{m} \widetilde{h}_{rk}. \tag{60}$$

By some elementary computations, we can show that if one of the conditions (i)–(iv) is satisfied, then $\deg(\pi_*(K^r_{\mathcal{X}})) \geq 2g$ and $\dim_{\mathbb{C}} H^0(\mathcal{X}, K^r_{\mathcal{X}}) \geq 2$. Hence, a general spectral curve is integral and smooth by Lemma 4.15.

(2). By (54), the assumption: if $\lceil \frac{r}{r_k} \rceil \geq \frac{r+2}{r_k}$ for some $1 \leq k \leq m$, then $\lceil \frac{r-1}{r_k} \rceil = \frac{r-1}{r_k}$ is equivalent to the condition (59). Moreover, the assumption of (2) implies $r \geq 3$. Then, by $\deg(\pi_*(K^{r-1}_{\mathcal{X}})) = (2g-2)(r-1) + \sum_{k=1}^m \widetilde{h}_{(r-1)k}$ (see (56)) and $\sum_{k=1}^m \widetilde{h}_{rk} \geq \sum_{k=1}^m \widetilde{h}_{(r-1)k}$, we have that: if either one of the conditions (i)–(iii) holds, then $\deg(\pi_*(K^{r-1}_{\mathcal{X}})) \geq 2g$ and the condition (33) holds. By Lemma 4.16, a general spectral curve is integral and smooth.

(3). As the above discussions, it is easy to check that the assumption of (3) satisfies the hypothesis of Lemma 4.17. The conclusion is immediately obtained. ■

Corollary 4.19 *With the same hypothesis as Theorem 4.18, we have:*

(1) *Under the assumption of (1) in Theorem 4.18, for a general* $\boldsymbol{a} \in \bigoplus_{i=2}^r H^0(\mathcal{X}, K^i_{\mathcal{X}})$, *the spectral curve* $\mathcal{X}_{\boldsymbol{a}}$ *is integral and smooth if one of the following conditions is satisfied:*
 (i) $g \geq 2$;
 (ii) $g = 1$ *and* $\sum_{k=1}^m (r - \lceil \frac{r}{r_k} \rceil) \geq 2$;
 (iii) $g = 0$ *and* $\sum_{k=1}^m (r - \lceil \frac{r}{r_k} \rceil) \geq 2r + 1$;
 (iv) $g = 0$, $\sum_{k=1}^m (r - \lceil \frac{r}{r_k} \rceil) \geq 2r$ *and* $\dim_{\mathbb{C}} H^0(\mathcal{X}, K^k_{\mathcal{X}}) \geq 2$ *for some* $2 \leq k \leq r$.

(2) *Under the assumption of (2) in Theorem 4.18, for a general* $\boldsymbol{a} \in \bigoplus_{i=2}^r H^0(\mathcal{X}, K^i_{\mathcal{X}})$, *the spectral curve* $\mathcal{X}_{\boldsymbol{a}}$ *is integral and smooth if any of the following conditions is satisfied:*
 (i) $g \geq 2$;
 (ii) $g = 1$ *and* $\sum_{k=1}^m (r - 1 - \lceil \frac{r-1}{r_k} \rceil) \geq 2$;
 (iii) $g = 0$, $\sum_{k=1}^m (r - 1 - \lceil \frac{r-1}{r_k} \rceil) \geq 2r - 2$ *and* $K_{\mathcal{X}}$ *satisfies (43).*

(3) *Under the assumption of (3) in Theorem 4.18, for a general* $\boldsymbol{a} \in \bigoplus_{i=2}^r H^0(\mathcal{X}, K^i_{\mathcal{X}})$, *the spectral curve* $\mathcal{X}_{\boldsymbol{a}}$ *is integral and singular if one of the following conditions occurs:*
 (i) $g \geq 2$;
 (ii) $g = 1$ *and* $K_{\mathcal{X}}$ *satisfies (43);*
 (iii) $g = 0$ *and* $K_{\mathcal{X}}$ *satisfies (43).*

Lemma 4.20 *Suppose that* $f : \mathcal{X}_{\boldsymbol{a}} \to \mathcal{X}$ *is the projection from the spectral curve* $\mathcal{X}_{\boldsymbol{a}}$ *to* $\mathcal{X}$. *Under the assumptions of Theorem 4.18 (resp. Corollary 4.19), which ensure a general spectral curve is smooth, for a general* $\mathcal{X}_{\boldsymbol{a}}$, *the stacky points of* $\mathcal{X}_{\boldsymbol{a}}$ *are contained in* $f^{-1}(\{p_1, \ldots, p_m\} \backslash \Omega)$, *where* Ω *consists of these stacky points* $p_k \in \{p_1, \ldots, p_m\}$ *satisfying*

$$r \equiv 0 \mod r_k.$$

Moreover, for any $p_k \in \{p_1, \ldots, p_m\} \backslash \Omega$, *there is a unique stacky point* $\widetilde{p}_k$ *in* $f^{-1}(p_k)$ *with stabilizer group* μ_{r_k}.

Proof Under these assumptions (which ensure a general spectral curve is smooth), $\deg(\pi_*(K^r_{\mathcal{X}})) \geq 2$. Therefore, the linear system $|\pi_*(K^r_{\mathcal{X}})|$ is base-point-free (see [Har77, Corollary 3.2]). Then, $\pi_*(K^r_{\mathcal{X}}) = \mathcal{O}_{\mathcal{X}}$ or $\dim |\pi_*(K^r_{\mathcal{X}})| \geq 1$. By (57), the general section $a_r \in H^0(\mathcal{X}, K^r_{\mathcal{X}})$ does not vanish at any stacky point in Ω. If $\mathcal{X}$ is be viewed as

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the zero locus of $\mathrm{Tot}(K_{\mathcal{X}})$, then the set of the stacky points of $\mathrm{Tot}(\mathcal{X})$ is $\{p_1,\dots,p_m\}$ with stabilizer groups $\mu_{r_1},\dots,\mu_{r_m}$. We complete the proof. ∎

Example 4.21 The condition (33) is an indispensable hypothesis for Theorem 4.18. For example, let $\mathbb{E}$ be an elliptic curve, and let p be a closed point of $\mathbb{E}$. Consider the stacky curve $\mathbb{E}_5 = \sqrt[5]{p}$. The projection from $\mathbb{E}_5$ to $\mathbb{E}$ is denoted by $\pi : \mathbb{E}_5 \to \mathbb{E}$. The canonical line bundle of $\mathbb{E}_5$ is $\mathcal{O}_{\mathbb{E}_5}(\frac{4}{5}p)$. Its degree $\deg(K_{\mathbb{E}_5})$ is $\frac{4}{5}$. So, it is a hyperbolic stacky curve. It is easy to check that

$$\pi_*(K_{\mathbb{E}_5}) = \mathcal{O}_{\mathbb{E}} \quad \text{and} \quad \pi_*(K^2_{\mathbb{E}_5}) = \mathcal{O}_{\mathbb{E}}(p).$$

Then, $\dim_{\mathbb{C}} H^0(\mathbb{E}_5, K_{\mathbb{E}_5}) = 1$ and $\dim_{\mathbb{C}} H^0(\mathbb{E}_5, K^2_{\mathbb{E}_5}) = 1$. Hence, we have

$$H^0(\mathbb{E}_5, K_{\mathbb{E}_5}) = \mathbb{C}\cdot\tau_1^{\otimes 4} \quad \text{and} \quad H^0(\mathbb{E}_5, K^2_{\mathbb{E}_5}) = \mathbb{C}\cdot\tau_1^{\otimes 8},$$

where τ_1 is the universal section of $\mathcal{O}_{\mathbb{E}_5}(\frac{1}{5}p)$. For a general $\boldsymbol{a} = (a\tau_1^{\otimes 4}, b\tau_1^{\otimes 8}) \in H^0(\mathbb{E}_5, K_{\mathbb{E}_5}) \oplus H^0(\mathbb{E}_5, K^2_{\mathbb{E}_5})$, the spectral curve $\mathcal{X}_{\boldsymbol{a}}$ is the zero locus of the section

$$\tau^{\otimes 2} + a\tau_1^{\otimes 4}\otimes\tau + b\tau_1^{\otimes 8}, \tag{61}$$

where $a, b \in \mathbb{C}$. The section (61) can be represented as a product of two sections

$$\left(\tau + (a/2 - \sqrt{a^2/4 - b})\tau_1^{\otimes 4}\right)\otimes\left(\tau + (a/2 + \sqrt{a^2/4 - b})\tau_1^{\otimes 4}\right).$$

Hence, a general spectral curve is not irreducible.

Example 4.22 We will construct an example satisfying the last conclusion of Theorem 4.18. Taking four distinct points $\{p_1, p_2, p_3, p_4\}$ on the projective line $\mathbb{P}^1$, we construct a stacky curve $\mathbb{P}^1_{4,2,2,2}$ as follows:

$$\mathbb{P}^1_{4,2,2,2} = \sqrt[4]{p_1}\times_{\mathbb{P}^1}\sqrt[2]{p_2}\times_{\mathbb{P}^1}\sqrt[2]{p_3}\times_{\mathbb{P}^1}\sqrt[2]{p_4}.$$

The canonical line bundle $K_{\mathbb{P}^1_{4,2,2,2}} = \pi^*K_{\mathbb{P}^1}\otimes\mathcal{O}_{\mathbb{P}^1_{4,2,2,2}}(\frac{3}{4}p_1 + \frac{1}{2}p_2 + \frac{1}{2}p_3 + \frac{1}{2}p_4)$, where $\pi : \mathbb{P}^1_{4,2,2,2} \to \mathbb{P}^1$ is the coarse moduli space. And, the degree of $K_{\mathbb{P}^1_{4,2,2,2}}$ is $\frac{1}{4}$. Hence, it is a hyperbolic stacky curve. Since $\dim_{\mathbb{C}} H^0(\mathbb{P}^1_{4,2,2,2}, K^6_{\mathbb{P}^1_{4,2,2,2}}) \geq 2$, the condition (33) holds. Suppose that τ_1, τ_2, τ_3, and τ_4 are the sections of $\mathcal{O}_{\mathbb{P}^1_{4,2,2,2}}(\frac{1}{4}p_1)$, $\mathcal{O}_{\mathbb{P}^1_{4,2,2,2}}(\frac{1}{2}p_2)$, $\mathcal{O}_{\mathbb{P}^1_{4,2,2,2}}(\frac{1}{2}p_3)$, and $\mathcal{O}_{\mathbb{P}^1_{4,2,2,2}}(\frac{1}{2}p_4)$, respectively, such that they are the pullback sections of the universal sections on the corresponding root stacks. By Lemma 4.11, any section of $K^6_{\mathbb{P}^1_{4,2,2,2}}$ can be represented by

$$\pi^*\widehat{s}\otimes\tau_1^{\otimes 2}, \quad \text{where } \widehat{s} \text{ is a section of } \pi_*(K^6_{\mathbb{P}^1_{4,2,2,2}}). \tag{62}$$

Let $\psi : \mathrm{Tot}(K_{\mathbb{P}^1_{4,2,2,2}}) \to \mathbb{P}^1_{4,2,2,2}$ be the projection from the total space of $K_{\mathbb{P}^1_{4,2,2,2}}$ to $\mathbb{P}^1_{4,2,2,2}$. For a general element $\boldsymbol{a}$ of $\bigoplus_{i=1}^6 H^0(\mathbb{P}^1_{4,2,2,2}, K^i_{\mathbb{P}^1_{4,2,2,2}})$, the spectral curve $\mathcal{X}_{\boldsymbol{a}}$ is the zero locus of the section

$$\tau^{\otimes 6} + \psi^*a_2\otimes\tau^{\otimes 4} + \psi^*a_4\otimes\tau^{\otimes 2} + \psi^*a_6, \tag{63}$$

where τ is the tautological section of $\psi^*K_{\mathbb{P}^1_{4,2,2,2}}$. By the GAGA for Deligne–Mumford curves (see [BN06]), we can assume that $\mathbb{P}^1_{4,2,2,2}$ is equipped with complex analytic

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topology. Then, there is a unit disk $\mathbb{D} \subset \mathbb{P}^1$ around p_1 such that $\pi : \mathbb{P}^1_{4,2,2,2} \to \mathbb{P}^1$ restricting to $\mathbb{D}$ is isomorphic to $\pi_{\mathbb{D}} : [\mathbb{D}/\mu_4] \longrightarrow \mathbb{D}$, where the action of μ_4 on $\mathbb{D}$ is multiplication and the morphism $\pi_{\mathbb{D}}$ is induced by the morphism $q : \mathbb{D} \longrightarrow \mathbb{D}$, $z \longmapsto z^4$.

Consider the commutative diagram

$$\begin{tikzcd} \mathbb{D} \arrow[r, "g_{\mathbb{D}}"] \arrow[dr, "q"'] & {[\mathbb{D}/\mu_4]} \arrow[d, "\pi_{\mathbb{D}}"] \\ & \mathbb{D} \end{tikzcd}$$

where $g_{\mathbb{D}}$ is the natural projection. Pulling back the spectral curve defined by (62) along $g_{\mathbb{D}} : \mathbb{D} \to [\mathbb{D}/\mu_4]$, we get

$$\{(z,t) \in \mathbb{D} \times \mathbb{C} | t^6 + \widehat{a}_2(z) \cdot t^4 + \widehat{a}_4(z) \cdot t^2 + \widehat{a}_6(z^4) \cdot z^2 = 0\}, \tag{64}$$

where $\widehat{a}_2(z)$, $\widehat{a}_4(z)$, and $\widehat{a}_6(z)$ are holomorphic functions on $\mathbb{D}$. It is easy to check that $(0,0)$ is a singular point of (64).

5 Norm maps

In this section, we systematically study the norm theory on Deligne–Mumford stacks. As an application, we apply the general theory to the case of stacky curves which plays a central role in studying the Hitchin fiber of the moduli space of $\mathbf{SL}_r$-Higgs bundles.

5.1 Norms of invertible sheaves on Deligne–Mumford stacks

Let $\mathcal{X}$ be a Deligne–Mumford stack, and let $\mathcal{A}$ be a commutative $\mathcal{O}_{\mathcal{X}}$-algebra with unit. Then, $\mathcal{A}$ is canonically identified with an $\mathcal{O}_{\mathcal{X}}$-subalgebra of $\mathscr{H}om_{\mathcal{O}_{\mathcal{X}}}(\mathcal{A},\mathcal{A})$. In fact, for an object $(T \to \mathcal{X})$ in $\mathcal{X}_{\text{ét}}$, a section $s \in \mathcal{A}(T \to \mathcal{X})$ defines a morphism of $\mathcal{O}_T$-modules $\mathcal{A}|_T \to \mathcal{A}|_T$ by multiplication. If $\mathcal{A}$ is a locally free $\mathcal{O}_{\mathcal{X}}$-module of finite rank, then there is a morphism $\det : \mathscr{H}om_{\mathcal{O}_{\mathcal{X}}}(\mathcal{A},\mathcal{A}) \to \mathcal{O}_{\mathcal{X}}$ defined by

$$\mathscr{H}om_{\mathcal{O}_{\mathcal{X}}}(\mathcal{A},\mathcal{A})(T \to \mathcal{X}) = \mathrm{Hom}_{\mathcal{O}_T}(\mathcal{A}|_T, \mathcal{A}|_T) \longrightarrow \mathcal{O}_T(T), \quad \phi \longmapsto \det(\phi).$$

The composition $\mathcal{A} \hookrightarrow \mathscr{H}om_{\mathcal{O}_{\mathcal{X}}}(\mathcal{A},\mathcal{A}) \xrightarrow{\det} \mathcal{O}_{\mathcal{X}}$ is denoted by $\mathrm{N}_{\mathcal{A}/\mathcal{O}_{\mathcal{X}}}$. Obviously, $\mathrm{N}_{\mathcal{A}/\mathcal{O}_{\mathcal{X}}}$ is a morphism of sheaves of multiplicative monoids. Following [EGA2, Section 6.5], it is easy to verify the following proposition.

Proposition 5.1 *For an étale morphism $T \to \mathcal{X}$, we have:*

(i) $\mathrm{N}_{\mathcal{A}/\mathcal{O}_{\mathcal{X}}}(s_1 \cdot s_2) = \mathrm{N}_{\mathcal{A}/\mathcal{O}_{\mathcal{X}}}(s_1) \cdot \mathrm{N}_{\mathcal{A}/\mathcal{O}_{\mathcal{X}}}(s_2)$, *for* $s_1, s_2 \in \mathcal{A}(T \to \mathcal{X})$;
(ii) $\mathrm{N}_{\mathcal{A}/\mathcal{O}_{\mathcal{X}}}(1_{\mathcal{A}}) = 1$;
(iii) $\mathrm{N}_{\mathcal{A}/\mathcal{O}_{\mathcal{X}}}(t \cdot 1_{\mathcal{A}}) = t^n$ *if* $t \in \mathcal{O}_{\mathcal{X}}(T \to \mathcal{X})$ *and the rank of* $\mathcal{A}$ *is* n.

Therefore, $\mathrm{N}_{\mathcal{A}/\mathcal{O}_{\mathcal{X}}}$ induces a morphism of sheaves of abelian groups

$$\mathrm{N}_{\mathcal{A}/\mathcal{O}_{\mathcal{X}}} : \mathcal{A}^* \longrightarrow \mathcal{O}^*_{\mathcal{X}}, \tag{65}$$

where $\mathcal{A}^*$ is the sheaf of invertible elements of $\mathcal{A}$.

Definition 5.2 An *$\mathcal{A}$-invertible sheaf* L on $\mathcal{X}$ is an $\mathcal{A}$-module on $\mathcal{X}_{\text{ét}}$ whose restriction $L|_U$ to some étale covering $U \to \mathcal{X}$ is isomorphic to $\mathcal{A}|_U$ as an $\mathcal{A}|_U$-module.

We will introduce the notion of norm of an $\mathcal{A}$-invertible sheaf L. For the notations used in the following, we refer the reader to the section on Notations and conventions. Since L is a coherent sheaf on $\mathcal{X}$, there is an object $(\mathcal{A}|_U, \sigma)$ in $Des(U/\mathcal{X})$ representing L for an étale covering $U \to \mathcal{X}$. Then the morphism $\widetilde{\sigma} = \phi_2 \circ \sigma \circ \phi_1^{-1} : \mathcal{A}|_{U[1]} \to \mathcal{A}|_{U[1]}$ is an isomorphism of $\mathcal{A}|_{U[1]}$-modules, where $\phi_i : \mathrm{pr}_i^*(\mathcal{A}|_U) \to \mathcal{A}|_{U[1]}$ are the natural isomorphisms of $\mathcal{O}_{U[1]}$-algebras, for $i = 1, 2$. Let a be the image of the unit $1 \in \mathcal{A}^*(U[1] \to \mathcal{X})$ under the morphism $\widetilde{\sigma}$. On the other hand, there are three isomorphisms of $\mathcal{A}|_{U[2]}$-modules

$$\widetilde{\sigma}_{12} = \phi_{12} \circ \mathrm{pr}_{12}^* \widetilde{\sigma} \circ \phi_{12}^{-1} : \mathcal{A}|_{U[2]} \to \mathcal{A}|_{U[2]}, \quad \widetilde{\sigma}_{23} = \phi_{23} \circ \mathrm{pr}_{23}^* \widetilde{\sigma} \circ \phi_{23}^{-1} : \mathcal{A}|_{U[2]} \to \mathcal{A}|_{U[2]},$$
$$\widetilde{\sigma}_{13} = \phi_{13} \circ \mathrm{pr}_{13}^* \widetilde{\sigma} \circ \phi_{13}^{-1} : \mathcal{A}|_{U[2]} \to \mathcal{A}|_{U[2]},$$

where $\phi_{12} : \mathrm{pr}_{12}^*(\mathcal{A}|_{U[1]}) \to \mathcal{A}|_{U[2]}$, $\phi_{23} : \mathrm{pr}_{23}^*(\mathcal{A}|_{U[1]}) \to \mathcal{A}|_{U[2]}$, and $\phi_{13} : \mathrm{pr}_{13}^*(\mathcal{A}|_{U[1]}) \to \mathcal{A}|_{U[2]}$ are three natural isomorphisms of $\mathcal{O}_{U[2]}$-algebras. It is easy to check that the cocycle condition: $\widetilde{\sigma}_{23} \circ \widetilde{\sigma}_{12} = \widetilde{\sigma}_{13}$ is satisfied. Then, we have $\phi_{23}(\mathrm{pr}_{23}{}^* a) \cdot \phi_{12}(\mathrm{pr}_{12}{}^* a) = \phi_{13}(\mathrm{pr}_{13}{}^* a)$ in $\mathcal{A}^*(U[2] \to \mathcal{X})$. Since $\mathrm{N}_{\mathcal{A}/\mathcal{O}_\mathcal{X}}$ is a morphism of sheaves of abelian groups, we have

$$\mathrm{pr}_{23}^* \mathrm{N}_{\mathcal{A}/\mathcal{O}_\mathcal{X}}(a) \cdot \mathrm{pr}_{12}^* \mathrm{N}_{\mathcal{A}/\mathcal{O}_\mathcal{X}}(a) = \mathrm{pr}_{13}^* \mathrm{N}_{\mathcal{A}/\mathcal{O}_\mathcal{X}}(a) \tag{66}$$

in $\mathcal{O}_\mathcal{X}^*(U[2] \to \mathcal{X})$ by Proposition 5.1. Therefore, $(\mathcal{O}_U, \mathrm{N}_{\mathcal{A}/\mathcal{O}_\mathcal{X}}(a))$ is an object of $\mathcal{D}es(U/\mathcal{X})$ which defines a line bundle $\mathrm{N}_{\mathcal{A}/\mathcal{O}_\mathcal{X}}(L)$ on $\mathcal{X}$.

Definition 5.3 For an $\mathcal{A}$-invertible sheaf L, the line bundle $\mathrm{N}_{\mathcal{A}/\mathcal{O}_\mathcal{X}}(L)$ is called the *norm* of L.

We summarize some basic properties of the norms of $\mathcal{A}$-invertible sheaves.

Proposition 5.4 *The norms of $\mathcal{A}$-invertible sheaves satisfy the following properties (up to a canonical isomorphism):*

(i) $\mathrm{N}_{\mathcal{A}/\mathcal{O}_\mathcal{X}}(L_1 \otimes_\mathcal{A} L_2) = \mathrm{N}_{\mathcal{A}/\mathcal{O}_\mathcal{X}}(L_1) \otimes_{\mathcal{O}_\mathcal{X}} \mathrm{N}_{\mathcal{A}/\mathcal{O}_\mathcal{X}}(L_2)$, *for any two $\mathcal{A}$-invertible sheaves L_1 and L_2 on $\mathcal{X}$;*
(ii) $\mathrm{N}_{\mathcal{A}/\mathcal{O}_\mathcal{X}}(\mathcal{A}) = \mathcal{O}_\mathcal{X}$;
(iii) $\mathrm{N}_{\mathcal{A}/\mathcal{O}_\mathcal{X}}(L^{-1}) = \mathrm{N}_{\mathcal{A}/\mathcal{O}_\mathcal{X}}(L)^{-1}$, *for an $\mathcal{A}$-invertible sheaf L on $\mathcal{X}$;*
(iv) $\mathrm{N}_{\mathcal{A}/\mathcal{O}_\mathcal{X}}(L \otimes_{\mathcal{O}_\mathcal{X}} \mathcal{A}) = \mathrm{N}_{\mathcal{A}/\mathcal{O}_\mathcal{X}}(L)^n$, *for an $\mathcal{O}_\mathcal{X}$-invertible sheaf L on $\mathcal{X}$.*

Proof By Proposition 5.1 and the definition of norm, the proposition is immediate. ∎

5.2 Norm maps of finite morphisms of Deligne–Mumford stacks

Suppose that $f : \mathcal{X}_1 \to \mathcal{X}_2$ is a finite morphism of Deligne–Mumford stacks and that $f_*\mathcal{O}_{\mathcal{X}_1}$ is a locally free sheaf of rank n. Then, for any invertible sheaf L on $\mathcal{X}_1$, the pushforward f_*L is a $f_*\mathcal{O}_{\mathcal{X}_1}$-invertible sheaf. In fact, for an étale covering $U \to \mathcal{X}_2$, there is a Cartesian diagram

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$$\begin{tikzcd} U \times_{\mathcal{X}_2} \mathcal{X}_1 \arrow[r] \arrow[d, "f_U"'] & \mathcal{X}_1 \arrow[d, "f"] \\ U \arrow[r] & \mathcal{X}_2. \end{tikzcd}$$

${f_U}_*(L|_{U\times_{\mathcal{X}_2}\mathcal{X}_1})$ is an ${f_U}_*(\mathcal{O}_{U\times_{\mathcal{X}_2}\mathcal{X}_1})$-invertible sheaf on U (see [EGA2, Proposition 6.I.I2.I]). Then, we can introduce the notion of the norm map of f.

Definition 5.5 The *norm map* Nm_f of f is $\mathrm{Nm}_f : \mathrm{Pic}(\mathcal{X}_1) \to \mathrm{Pic}(\mathcal{X}_2)$, $L \mapsto \mathrm{N}_{f_*\mathcal{O}_{\mathcal{X}_1}/\mathcal{O}_{\mathcal{X}_2}}(f_*L)$.

Proposition 5.6 *The norm map* Nm_f *satisfies the following properties:*

(i) $\mathrm{Nm}_f(L_1 \otimes L_2) = \mathrm{Nm}_f(L_1) \otimes \mathrm{Nm}_f(L_2)$, *for any two line bundles* L_1 *and* L_2 *on* $\mathcal{X}_1$.
(ii) $\mathrm{Nm}_f(\mathcal{O}_{\mathcal{X}_1}) = \mathcal{O}_{\mathcal{X}_2}$.
(iii) $\mathrm{Nm}_f(L^{-1}) = \mathrm{Nm}_f(L)^{-1}$, *for a line bundle* L *on* $\mathcal{X}_1$.
(iv) $\mathrm{Nm}_f(f^*L) = \mathrm{Nm}_f(L)^n$, *for a line bundle* L *on* $\mathcal{X}_2$.
(v) *For a morphism of line bundles* $\alpha : L_1 \to L_2$ *on* $\mathcal{X}_1$, *there is a morphism of line bundles* $\mathrm{Nm}_f(\alpha) : \mathrm{Nm}_f(L_1) \to \mathrm{Nm}_f(L_2)$. *And, it satisfies:*
   - *If there is another morphism of line bundles* $\beta : L_2 \to L_3$, *then we have* $\mathrm{Nm}_f(\beta) \circ \mathrm{Nm}_f(\alpha) = \mathrm{Nm}_f(\beta \circ \alpha)$.
   - *For two morphism of line bundles* $\alpha_1 : L_1 \to L_2$ *and* $\alpha_2 : L_3 \to L_4$, *we have* $\mathrm{Nm}_f(\alpha_1) \otimes \mathrm{Nm}_f(\alpha_2) = \mathrm{Nm}_f(\alpha_1 \otimes \alpha_2)$.

Proof By Proposition 5.4, the conclusions of this proposition are immediate. ■

Remark 5.7 In Proposition 5.6 (v), if $L_1 = \mathcal{O}_{\mathcal{X}_1}$, we obtain a canonical map

$$\mathrm{Nm}_f : H^0(\mathcal{X}_1, L) \longrightarrow H^0(\mathcal{X}_2, \mathrm{Nm}_f(L)) \tag{67}$$

for any line bundle L on $\mathcal{X}_1$.

Proposition 5.8 *Suppose that* $f : \mathcal{X}_1 \to \mathcal{X}_2$ *is a finite morphism of Deligne–Mumford stacks such that* $f_*\mathcal{O}_{\mathcal{X}_1}$ *is a rank n locally free sheaf. For a morphism of Deligne–Mumford stacks* $g : \mathcal{Y}_2 \to \mathcal{X}_2$ *and the Cartesian diagram*

$$\begin{tikzcd} \mathcal{Y}_1 \arrow[r, "g'"] \arrow[d, "f'"'] & \mathcal{X}_1 \arrow[d, "f"] \\ \mathcal{Y}_2 \arrow[r, "g"] & \mathcal{X}_2, \end{tikzcd} \tag{68}$$

we have $\mathrm{Nm}_{f'}(g'^*L) = g^*\mathrm{Nm}_f(L)$ *for any line bundle* L *on* $\mathcal{X}_1$.

Proof Using descent theory, we can prove this proposition following the proof of the counterpart in [EGA2]. ■

Proposition 5.9 *Let* $f : \mathcal{X}_1 \to \mathcal{X}_2$ *be a finite morphism of Deligne–Mumford stacks, and let* L *be a line bundle on* $\mathcal{X}_1$. *Assume that* $f_*\mathcal{O}_{\mathcal{X}_1}$ *is a locally free sheaf of rank n. Then, we have*

$$\mathrm{Nm}_f(L) = \det(f_*L) \otimes \det(f_*\mathcal{O}_{\mathcal{X}_1})^{-1}.$$

Proof There exists an étale covering $U_2 \to \mathcal{X}_2$ such that $L|_{U_1} = \mathcal{O}_{U_1}$ where $U_1 = U_2 \times_{\mathcal{X}_2} \mathcal{X}_1$. Hence, there exists $a \in \mathcal{O}^*_{U_1}(U_1)$ such that L is defined by the object $(\mathcal{O}_{U_1}, a)$ of $Des(U_1/\mathcal{X}_1)$. Consider the commutative diagram

(69)
$$
\begin{tikzcd}
 & U_1 \arrow[rr] \arrow[dd, "f_1" near end] & & \mathcal{X}_1 \arrow[dd, "f"] \\
U_1[1] \arrow[ur, "pr_2"] \arrow[rr, "pr_1" near start] \arrow[dd, "f_2"'] & & U_1 \arrow[ur] \arrow[dd, "f_1" near start] & \\
 & U_2 \arrow[rr] & & \mathcal{X}_2 \\
U_2[1] \arrow[ur, "pr_2"] \arrow[rr, "pr_1"] & & U_2 \arrow[ur] &
\end{tikzcd}
$$

in which every square is Cartesian. The pushforward f_*L is represented by the object $\big(f_{1*}\mathcal{O}_{U_1}, f_{2*}(a\cdot \mathrm{id})\big)$ of $Des(U_2/\mathcal{X}_2)$ where $f_{2*}(a\cdot\mathrm{id})$ is identified with the composition

$$\mathrm{pr}_1^* f_{1*}\mathcal{O}_{U_1} \tilde{\to} f_{2*}\mathrm{pr}_1^*\mathcal{O}_{U_1} = f_{2*}\mathcal{O}_{U_1[1]} \xrightarrow{f_{2*}(a\cdot\mathrm{id})} f_{2*}\mathcal{O}_{U_1[1]} = f_{2*}\mathrm{pr}_2^*\mathcal{O}_{U_1} \tilde{\to} \mathrm{pr}_2^* f_{1*}\mathcal{O}_{U_1}.$$

Therefore, $\det(f_*L)$ is defined by the object $\big(\det(f_{1*}\mathcal{O}_{U_1}), \det(f_{2*}(a\cdot\mathrm{id}))\big)$ of $Des(U_2/\mathcal{X}_2)$, where $\det(f_{2*}(a\cdot\mathrm{id}))$ is the composition

$$\mathrm{pr}_1^*\det(f_{1*}\mathcal{O}_{U_1}) \tilde{\to} \det(f_{2*}\mathcal{O}_{U_1[1]}) \xrightarrow{\det(f_{2*}(a\cdot\mathrm{id}))} \det(f_{2*}\mathcal{O}_{U_1[1]}) \tilde{\to} \mathrm{pr}_2^*\det(f_{1*}\mathcal{O}_{U_1}).$$

In addition, the dual $\det(f_*\mathcal{O}_{\mathcal{X}_1})^{-1}$ of $f_*\mathcal{O}_{\mathcal{X}_1}$ is represented by the object $\big(\det(f_{1*}\mathcal{O}_{U_1})^{-1}, \mathrm{id}\big)$ of $Des(U_2/\mathcal{X}_2)$, where id denotes the composition

$$\mathrm{pr}_1^*\det(f_{1*}\mathcal{O}_{U_1})^{-1} \tilde{\to} \det(f_{2*}\mathcal{O}_{U_1[1]})^{-1} \xrightarrow{\mathrm{id}} \det(f_{2*}\mathcal{O}_{U_1[1]})^{-1} \tilde{\to} \mathrm{pr}_2^*\det(f_{1*}\mathcal{O}_{U_1})^{-1}.$$

Therefore, the line bundle $\det(f_*L)\otimes\det(f_*\mathcal{O}_{\mathcal{X}_1})^{-1}$ is represented by the object $\big(\mathcal{O}_{U_2}, \mathrm{N}_{f_*\mathcal{O}_{\mathcal{X}_1}/\mathcal{O}_{\mathcal{X}_2}}(a)\big)$ of $Des(U_2/\mathcal{X}_2)$. By the definition of $\mathrm{Nm}_f(L)$, we have $\mathrm{Nm}_f(L) = \det(f_*L)\otimes\det(f_*\mathcal{O}_{\mathcal{X}_1})^{-1}$. ∎

In the following, for simplicity, we always assume that $\mathcal{X}$ is a smooth irreducible Deligne–Mumford stack of finite type over $\mathbb{C}$.

Definition 5.10 (i) A *prime divisor* on $\mathcal{X}$ is a codimension one closed integral substack of $\mathcal{X}$.

(ii) A *Weil divisor* is an element of the free abelian group $\mathrm{Div}(\mathcal{X})$ generated by the prime divisors on $\mathcal{X}$

(iii) Let $\mathcal{D} = \sum_i n_i\mathcal{Y}_i$ be a Weil divisor, where the $\mathcal{Y}_i$ are prime divisors and the n_i are integers. If all the coefficients $n_i \geq 0$, then $\mathcal{D}$ is said to be *effective*.

(iv) A *rational function* on $\mathcal{X}$ is a morphism $\mathcal{U}\to\mathbb{A}^1_{\mathbb{C}}$ from a nonempty open substack to the affine line. The rational functions of $\mathcal{X}$ form a field $k(\mathcal{X})$, which is called the *quotient field* of $\mathcal{X}$ (see [Vis89, Definition 3.4]). By [Vis89, Lemma 3.3], there is a morphism of abelian groups $\partial_{\mathcal{X}} : k^*(\mathcal{X}) \longrightarrow \mathrm{Div}(\mathcal{X})$, where $k^*(\mathcal{X})$ is the group of nonzero elements of $k(\mathcal{X})$. By convention, we use the notation **div** to denote $\partial_{\mathcal{X}}$. A Weil divisor is said to be a *principal divisor* if it is in the image of **div**.

(v) Two Weil divisors $\mathcal{D}, \mathcal{D}' \in \mathrm{Div}(\mathcal{X})$ are *linearly equivalent* if $\mathcal{D}-\mathcal{D}'$ is in the image of **div**.

(vi) The cokernel of **div** is called the *divisor class group* $\mathrm{Cl}(\mathcal{X})$ of $\mathcal{X}$.

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Remark 5.11 In the intersection theory of Deligne–Mumford stacks (see [Gil84, Vis89]), the group $\mathrm{Div}(\mathcal{X})$ of Weil divisors is the same as the group $Z_{n-1}(\mathcal{X})$ of $(n-1)$-dimensional cycles, where n is the dimension of $\mathcal{X}$. And, the divisor class group $\mathrm{Cl}(\mathcal{X})$ is the Chow group $A_{n-1}(\mathcal{X})$.

The following definition is a modified version of [Vis89, Definition 3.6].

Definition 5.12 Let $f : \mathcal{X}_1 \to \mathcal{X}_2$ be a morphism of n-dimensional Deligne–Mumford stacks, and let $\mathcal{Y}$ be any closed integral substack of $\mathcal{X}_1$.

(i) If f is proper and representable, the proper pushforward is $f_* : \mathrm{Div}(\mathcal{X}_1) \to \mathrm{Div}(\mathcal{X}_2) \quad \mathcal{Y} \mapsto \deg(\mathcal{Y}/\mathcal{Y}')\mathcal{Y}'$, where $\mathcal{Y}'$ is image of $\mathcal{Y}$ in $\mathcal{X}_2$ and $\deg(\mathcal{Y}/\mathcal{Y}')$ is the degree of the restriction of f to $\mathcal{Y}$ and $\mathcal{Y}'$ [Vis89, Definition 1.15].
(ii) If f is flat, the flat pullback is $f^* : \mathrm{Div}(\mathcal{X}_2) \to \mathrm{Div}(\mathcal{X}_1) \quad f^*(\mathcal{Y}) \mapsto \mathcal{D}_{\mathcal{Y}}$, where $\mathcal{D}_{\mathcal{Y}}$ is the cycle associated with the closed substack $\mathcal{Y} \times_{\mathcal{X}_2} \mathcal{X}_1$ [Vis89, Definition 3.5].

The following proposition is immediately.

Proposition 5.13 *There is a morphism of abelian groups*

$$\mathrm{Div}(\mathcal{X}) \to \mathrm{Pic}(\mathcal{X}), \quad \mathcal{D} \longmapsto \mathcal{O}_{\mathcal{X}}(\mathcal{D}). \tag{70}$$

If $\mathcal{D}$ is a principal divisor, then $\mathcal{O}_{\mathcal{X}}(\mathcal{D}) \simeq \mathcal{O}_{\mathcal{X}}$. Then, we have a morphism from the divisor class group $\mathrm{Cl}(\mathcal{X})$ to $\mathrm{Pic}(\mathcal{X})$.

Remark 5.14 In Proposition 5.13, the homomorphism $\mathrm{Cl}(\mathcal{X}) \to \mathrm{Pic}(\mathcal{X})$ is injective. In general, it is not surjective if the generic stabilizer of $\mathcal{X}$ is not trivial.

Proposition 5.15 *Assume that $f : \mathcal{X}_1 \to \mathcal{X}_2$ is a finite morphism of smooth irreducible Deligne–Mumford stacks such that $f_*\mathcal{O}_{\mathcal{X}}$ is a locally free sheaf. If a line bundle $L \simeq \mathcal{O}_{\mathcal{X}_1}(\mathcal{D})$ for some Weil divisor $\mathcal{D}$, then $\mathrm{Nm}_f(L) \simeq \mathcal{O}_{\mathcal{X}_2}(f_*(\mathcal{D}))$, i.e., the diagram*

$$\begin{tikzcd}
\mathrm{Div}(\mathcal{X}_1) \arrow[r] \arrow[d, "f_*"'] & \mathrm{Pic}(\mathcal{X}_1) \arrow[d, "\mathrm{Nm}_f"] \\
\mathrm{Div}(\mathcal{X}_2) \arrow[r] & \mathrm{Pic}(\mathcal{X}_2)
\end{tikzcd}$$

is commutative.

Proof The proof is divided into two steps. First, we show the conclusion for an effective Weil divisor $\mathcal{D}$. Finally, we check the general case.

Case 1 *Let $\mathcal{D}$ be an effective Weil divisor. Then, $\mathcal{D} = (s)$ for some section $s \in H^0(\mathcal{X}_1, L)$. There is an étale morphism $p_2 : U_2 \to \mathcal{X}_2$ such that L is represented by an object $(\mathcal{O}_{U_1}, a)$ of $Des(U_1/\mathcal{X}_1)$ where $U_1 = U_2 \times_{\mathcal{X}_2} \mathcal{X}_1$ and $a \in \mathcal{O}^*_{U_1[1]}(U_1[1])$. Thus, s is represented by an element $h \in \mathcal{O}_{U_1}(U_1)$ which satisfies $\mathrm{pr}_1^* h \cdot a = \mathrm{pr}_2^* h$ on $U_1[1]$. By (67), the restriction of the norm $\mathrm{Nm}_f(s)$ to U_2 is $\mathrm{N}_{f_*\mathcal{O}_{\mathcal{X}_1}/\mathcal{O}_{\mathcal{X}_2}}(h)$. Consider the Cartesian diagram*

$$\begin{tikzcd}
U_1 \arrow[r, "p_1"] \arrow[d, "f_1"'] & \mathcal{X}_1 \arrow[d, "f"] \\
U_2 \arrow[r, "p_2"] & \mathcal{X}_2.
\end{tikzcd}$$

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By the proof of [Vis89, Lemma 3.9], we have

$$f_{1*} \circ p_1^* = p_2^* \circ f_* : \mathrm{Div}(\mathcal{X}_1) \longrightarrow \mathrm{Div}(U_2). \tag{71}$$ ■

Claim $f_{1*}(\mathrm{div}(h)) = \mathrm{div}(\mathrm{N}_{f_*\mathcal{O}_{\mathcal{X}_1}/\mathcal{O}_{\mathcal{X}_2}}(h))$. *Without loss of generality, we can assume that U_2 is irreducible. And, U_1 is the disjoint union of its irreducible components. Due to the irreducibility of $\mathcal{X}_1$ and $\mathcal{X}_2$, the restriction of the morphism f_1 to each irreducible component of U_1 is a surjective finite morphism to U_2. Therefore,* $f_{1*}(\mathrm{div}(h)) = \mathrm{div}(\mathrm{N}_{f_*\mathcal{O}_{U_1}/\mathcal{O}_{U_2}}(h))$ *(see [Ful98, Proposition 1.4]). The flat pullback* $p_1^*(\mathcal{D}) = \mathrm{div}(h)$ *and (71) implies* $p_2^*(f_*(\mathcal{D})) = \mathrm{div}(\mathrm{N}_{f_*\mathcal{O}_{\mathcal{X}_1}/\mathcal{O}_{\mathcal{X}_2}}(h))$. *In addition, the flat pullback* $p_2^*((\mathrm{Nm}_f(s))) = \mathrm{div}(\mathrm{N}_{f_*\mathcal{O}_{\mathcal{X}_1}/\mathcal{O}_{\mathcal{X}_2}}(h))$. *Thus,* $f_*(\mathcal{D}) = (\mathrm{Nm}_f(s))$ *(see [Gil84, Lemma 4.2]). As a result,* $\mathrm{Nm}_f(L) \simeq \mathcal{O}_{\mathcal{X}_2}(f_*(\mathcal{D}))$.

Case 2 *If $\mathcal{D}$ is a Weil divisor on $\mathcal{X}_1$, then there are two effective Weil divisors* $\mathcal{D}_1, \mathcal{D}_2 \in \mathrm{Div}(\mathcal{X}_1)$ *such that $\mathcal{D} = \mathcal{D}_1 - \mathcal{D}_2$. So,* $\mathcal{O}_{\mathcal{X}_1}(\mathcal{D}) = \mathcal{O}_{\mathcal{X}_1}(\mathcal{D}_1) \otimes \mathcal{O}_{\mathcal{X}_1}(\mathcal{D}_2)^{-1}$. *Thus,* $\mathrm{Nm}_f(L) \simeq \mathrm{Nm}_f(\mathcal{O}_{\mathcal{X}_1}(\mathcal{D}_1) \otimes \mathcal{O}_{\mathcal{X}_1}(\mathcal{D}_2)^{-1})$. *By Proposition 5.6, we have* $\mathrm{Nm}_f(\mathcal{O}_{\mathcal{X}_1}(\mathcal{D}_1) \otimes \mathcal{O}_{\mathcal{X}_1}(\mathcal{D}_2)^{-1}) = \mathrm{Nm}_f(\mathcal{O}_{\mathcal{X}_1}(\mathcal{D}_1)) \otimes \mathrm{Nm}_f(\mathcal{O}_{\mathcal{X}_1}(\mathcal{D}_2))^{-1}$. *Therefore, we have* $\mathrm{Nm}_f(L) \simeq \mathcal{O}_{\mathcal{X}_2}(f_*(\mathcal{D}_1)) \otimes \mathcal{O}_{\mathcal{X}_2}(-f_*(\mathcal{D}_2)) = \mathcal{O}_{\mathcal{X}_2}(f_*(\mathcal{D}))$.

5.3 The case of stacky curves

In this subsection, all stacky curves are assumed to be irreducible and smooth. For a stacky curve $\mathcal{X}$ with coarse moduli space $\pi : \mathcal{X} \to X$, the group of Weil divisors of $\mathcal{X}$ is

$$\mathrm{Div}(\mathcal{X}) = \oplus_{x \in X(\mathbb{C})} \mathbb{Z} \cdot \tfrac{1}{r_x} \cdot x, \tag{72}$$

where $X(\mathbb{C})$ is the set of closed points of X and r_x is the order of the stabilizer group of x. Suppose that the stacky points of $\mathcal{X}$ are $p_1, \dots, p_m$ and that the stabilizer groups are $\mu_{r_1}, \dots, \mu_{r_m}$, respectively. We have the following lemma.

Lemma 5.16 *For every line bundle L on $\mathcal{X}$, it can be uniquely (up to isomorphism) expressed as* $L = \pi^* W \otimes \mathcal{O}_{\mathcal{X}}(\sum_{k=1}^m \frac{i_k}{r_k} p_k)$, *where W is a line bundle on X and* $0 \le i_k \le r_k - 1$ *for all* $1 \le k \le m$.

Proof For any line bundle L on $\mathcal{X}$, there is a Weil divisor $\mathcal{D} \in \mathrm{Div}(\mathcal{X})$ such that $L = \mathcal{O}_{\mathcal{X}}(\mathcal{D})$ (see [NS95, Proposition 1.3]). Note that $\mathcal{D}$ can be written as $\sum_{x \in X(\mathbb{C})} n_x \cdot x + \sum_{k=1}^m \frac{i_k}{r_k} \cdot p_k$, where $0 \le i_k \le r_k - 1$ for all $1 \le k \le m$ and $n_x \in \mathbb{Z}$. We therefore have $L = \pi^* W \otimes \mathcal{O}_{\mathcal{X}}(\sum_{k=1}^m \frac{i_k}{r_k} \cdot p_k)$, where $W = \mathcal{O}_X(\sum_{x \in X(\mathbb{C})} n_x \cdot x)$.

If two Weil divisors $\mathcal{D}_1, \mathcal{D}_2 \in \mathrm{Div}(\mathcal{X})$ are linearly equivalent, then there is a rational function c on X such that $\mathcal{D}_1 = \mathcal{D}_2 + \mathrm{div}(\pi^* c)$. Hence, the line bundle W is unique up to an isomorphism. ■

Moreover, we also have the following lemma (see [Bro09, Section 5.4]).

Lemma 5.17 [Bro09] *There is an exact sequences of group schemes*

$$0 \longrightarrow \mathrm{Pic}(X) \xrightarrow{\pi^*} \mathrm{Pic}(\mathcal{X}) \longrightarrow \textstyle\prod_{i=1}^m \mathbb{Z}/r_i\mathbb{Z} \longrightarrow 0\ . \tag{73}$$

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Remark 5.18 For every m-tuple $(i_1,\ldots,i_m)$ of integers, we have the translation

(74) $$T_{(i_1,\ldots,i_m)}:\mathrm{Pic}(\mathcal{X})\longrightarrow \mathrm{Pic}(\mathcal{X}),\quad L\longmapsto L\otimes \mathcal{O}_{\mathcal{X}}(\textstyle\sum_{k=1}^{m}\frac{i_k}{r_k}\cdot p_k)$$

defined by the line bundle $\mathcal{O}_{\mathcal{X}}(\sum_{k=1}^{m}\frac{i_k}{r_k}\cdot p_k)$. By Lemmas 5.16 and 5.17, $\mathrm{Pic}(\mathcal{X})$ is the disjoint union of open and closed subschemes

$$\mathrm{Pic}(\mathcal{X})=\textstyle\coprod_{i_1=0}^{r_1-1}\coprod_{i_2=0}^{r_2-1}\cdots\coprod_{i_m=0}^{r_m-1}\mathrm{Pic}^{(i_1,\ldots,i_m)}(\mathcal{X}),$$

where $\mathrm{Pic}^{(i_1,\ldots,i_m)}(\mathcal{X})=T_{(i_1,\ldots,i_m)}(\pi^*(\mathrm{Pic}(X)))$ for all $(i_1,\ldots,i_m)$. For any integer d, let $\mathrm{Pic}^d(X)$ be the moduli space of line bundles with degree d on X. It is a connected component of $\mathrm{Pic}(X)$. Then, the connected components of $\mathrm{Pic}(\mathcal{X})$ are

(75) $$\mathrm{Pic}^{d,(i_1,\ldots,i_m)}(\mathcal{X}):=T_{(i_1,\ldots,i_m)}(\pi^*(\mathrm{Pic}^d(X))),$$

where $d\in\mathbb{Z}$ and $(i_1,\ldots,i_m)$ satisfy $0\le i_k\le r_k-1$ for all $1\le k\le m$. We therefore have the decomposition of $\mathrm{Pic}(\mathcal{X})$ into connected components

(76) $$\mathrm{Pic}(\mathcal{X})=\textstyle\coprod_{d\in\mathbb{Z}}\coprod_{i_1=0}^{r_1-1}\coprod_{i_2=0}^{r_2-1}\cdots\coprod_{i_m=0}^{r_m-1}\mathrm{Pic}^{d,(i_1,\ldots,i_m)}(\mathcal{X}),$$

which coincide with the decomposition (30).

In the following, we will consider norm maps for stacky curves. Let $\mathcal{X}_1$ and $\mathcal{X}_2$ be two stacky curves with coarse moduli spaces $\pi_i:\mathcal{X}_i\to X_i$ for $i=1,2$. The set of stacky points of $\mathcal{X}_1$ is $\{p_1,\ldots,p_{m_1}\}$ and $\mathcal{X}_2$'s is $\{\widetilde{p}_1,\ldots,\widetilde{p}_{m_2}\}$. The stabilizer groups of $\mathcal{X}_1$ and $\mathcal{X}_2$ are $\{\mu_{r_1},\ldots,\mu_{r_{m_1}}\}$ and $\{\mu_{\widetilde{r}_1},\ldots,\mu_{\widetilde{r}_{m_2}}\}$, respectively. Suppose that $f:\mathcal{X}_1\to\mathcal{X}_2$ is a finite morphism and $f':X_1\to X_2$ is the induced morphism between coarse moduli spaces.

Lemma 5.19 *The proper pushforward of f is*

(77) $$f_*:\mathrm{Div}(\mathcal{X}_1)\to\mathrm{Div}(\mathcal{X}_2),\quad \frac{1}{r_x}\cdot x\mapsto\frac{r_{f'(x)}}{r_x}\cdot\frac{1}{r_{f'(x)}}\cdot f'(x),$$

Proof By Definition 5.12, the conclusion is immediate. ∎

Remark 5.20 Since the finite morphism f is representable, the stabilizer group of x is isomorphic to a subgroup of stabilizer group of $f'(x)$. Hence, $r_{f'(x)}/r_x$ is an integer.

Proposition 5.21 *The norm map of f is*

(78)
$$\mathrm{Nm}_f:\mathrm{Pic}(\mathcal{X}_1)\to\mathrm{Pic}(\mathcal{X}_2),\quad \mathcal{O}_{\mathcal{X}_1}\Big(\textstyle\sum_i n_i\frac{1}{r_{x_i}}x_i\Big)\mapsto\mathcal{O}_{\mathcal{X}_2}\Big(\textstyle\sum_i n_i\frac{r_{f'(x_i)}}{r_{x_i}}\frac{1}{r_{f'(x_i)}}f'(x_i)\Big).$$

Proof For smooth stacky curves, the homomorphism (70) is surjective (see [NS95, Proposition 1.3]). By Proposition 5.15 and Lemma 5.19, we complete the proof. ∎

Corollary 5.22 *Assume that $f'(p_i)=\widetilde{p}_i$ for all $1\le i\le m_1$. For any $d\in\mathbb{Z}$ and any $(i_1,\ldots,i_{m_1})\in\mathbb{Z}^{m_1}\cap[0,r_1]\times\cdots\times[0,r_{m_1}]$, the restriction of Nm_f to $\mathrm{Pic}^{d,(i_1,\ldots,i_{m_1})}(\mathcal{X}_1)$ is*

(79) $$\mathrm{Nm}_f:\mathrm{Pic}^{d,(i_1,\ldots,i_{m_1})}(\mathcal{X}_1)\to\mathrm{Pic}^{d,(\widetilde{i}_1,\ldots,\widetilde{i}_{m_2})}(\mathcal{X}_2),$$

where

(80)
$$\widetilde{i}_k = \begin{cases} i_k \frac{\widetilde{r}_k}{r_k}, & \text{if } 0 \le k \le m_1, \\ 0, & \text{if } m_1 < m_2 \text{ and } m_1 + 1 \le k \le m_2. \end{cases}$$

Proof For any $L \in \mathrm{Pic}^{d,(i_1,\dots,i_{m_1})}(\mathcal{X}_1)$, there is a Weil divisor $\mathcal{D} = \sum_{x \in X_1(\mathbb{C})} n_x \cdot x + \sum_{k=1}^{m_1} \frac{i_k}{r_k} \cdot p_k$ with $n_x \in \mathbb{Z}$ such that $L = \mathcal{O}_{\mathcal{X}_1}(\mathcal{D})$. Then, $\mathrm{Nm}_f(L) = \mathcal{O}_{\mathcal{X}_2}(\sum_{x \in X(\mathbb{C})} n_x f'(x) + \sum_{k=1}^{m_1} i_k \frac{\widetilde{r}_k}{r_k} \frac{1}{\widetilde{r}_k} f'(p_k))$ (see Proposition 5.21). ■

Lemma 5.23 *There is a commutative diagram*

$$\begin{array}{ccc} \mathrm{Pic}^{d,(i_1,\dots,i_{m_1})}(\mathcal{X}_1) & \xrightarrow{\mathrm{Nm}_f} & \mathrm{Pic}^{d,(\widetilde{i}_1,\dots,\widetilde{i}_{m_2})}(\mathcal{X}_2) \\ \Big\downarrow{\scriptstyle \pi_{1*}} & & \Big\downarrow{\scriptstyle \pi_{2*}} \\ \mathrm{Pic}^d(X_1) & \xrightarrow{\mathrm{Nm}_{f'}} & \mathrm{Pic}^d(X_2), \end{array}$$

in which the pushforward morphisms π_{1*} and π_{2*} are isomorphisms.

Proof Without loss of generality, we only show that π_{1*} is an isomorphism. For any $L \in \mathrm{Pic}^{d,(i_1,\dots,i_{m_1})}(\mathcal{X}_1)$, there is a unique $W \in \mathrm{Pic}^d(X_1)$ such that

$$L = \pi_1^* W \otimes \mathcal{O}_{\mathcal{X}_1}(\textstyle\sum_{k=1}^{m_1} \frac{i_k}{r_k} \cdot p_k).$$

Then, $\pi_{1*}L = W \otimes \pi_{1*}\mathcal{O}_{\mathcal{X}_1}(\sum_{k=1}^{m_1} \frac{i_k}{r_k} \cdot p_k)$. On the other hand, $\pi_{1*}\mathcal{O}_{\mathcal{X}_1}(\sum_{k=1}^{m_1} \frac{i_k}{r_k} \cdot p_k) = \mathcal{O}_{X_1}$ (see [Beh14, Theorem 3.64]). Hence, π_{1*} is an isomorphism. As the proof of Corollary 5.22, we can directly verify $\pi_{2*} \mathrm{Nm}_f(L) = \mathrm{Nm}_{f'}(\pi_{1*}L)$. ■

6 SYZ duality

In this section, $\mathcal{X}$ is a hyperbolic stacky curve with coarse moduli space $\pi : \mathcal{X} \to X$. The stacky points are $p_1, \dots, p_m$, and the stabilizer groups are $\mu_{r_1}, \dots, \mu_{r_m}$, respectively. For each stacky point p_k, its residue gerbe $\iota_k : B\mu_{r_k} \hookrightarrow \mathcal{X}$ is a closed immersion.

6.1 BNR correspondence

For $\boldsymbol{a} \in \bigoplus_{i=1}^r H^0(\mathcal{X}, K_{\mathcal{X}}^i)$, let $\pi' : \mathcal{X}_{\boldsymbol{a}} \to X_{\boldsymbol{a}}$ be the coarse moduli space of $\mathcal{X}_{\boldsymbol{a}}$. There is a commutative diagram

(81)
$$\begin{array}{ccc} \mathcal{X}_{\boldsymbol{a}} & \xrightarrow{f} & \mathcal{X} \\ \Big\downarrow{\scriptstyle \pi'} & & \Big\downarrow{\scriptstyle \pi} \\ X_{\boldsymbol{a}} & \xrightarrow{f'} & X, \end{array}$$

where $f : \mathcal{X}_{\boldsymbol{a}} \to \mathcal{X}$ is the natural projection and $f' : X_{\boldsymbol{a}} \to X$ is the induced morphism between coarse moduli spaces. Assume that the assumptions of Theorem 4.18 (which ensure that a general spectral curve is irreducible and smooth) are satisfied. Hence, we can assume that $\mathcal{X}_{\boldsymbol{a}}$ is an irreducible smooth stacky curve and satisfies the conclusion of Lemma 4.20 in the following discussion. Without loss of generality, suppose that

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the set of stacky points of $\mathcal{X}_{\boldsymbol{a}}$ is $\{\widetilde{p}_1, \dots, \widetilde{p}_{m_1}\}$ such that $f(\widetilde{p}_k) = p_k$ for all $1 \le k \le m_1$. Note that $K_0(B\mu_{r_k})$ is isomorphic to the representation ring $\mathbf{R}\mu_{r_k}$ for every stacky point p_k and $\mathbf{R}\mu_{r_k} = \mathbb{Z}[x_k]/(x_k^{r_k} - 1)$, where x_k represents the representation defined by the inclusion $\mu_{r_k} \hookrightarrow \mathbb{C}^*$.

Lemma 6.1 *For each $1 \le k \le m_1$, the decomposition of the K-class $[\iota_k^*(f_*\mathcal{O}_{\mathcal{X}_{\boldsymbol{a}}})]$ in $\mathbf{R}\mu_{r_k}$ only consists of the following two cases:*

(i) *If $\lceil \frac{r}{r_k} \rceil = \frac{r+1}{r_k}$, then $[\iota_k^*(f_*\mathcal{O}_{\mathcal{X}_{\boldsymbol{a}}})] = m_k x_k^0 + (m_k - 1)x_k^1 + \cdots + m_k x_k^{r_k-1}$, where $m_k = \frac{r+1}{r_k}$.*

(ii) *If $\lceil \frac{r-1}{r_k} \rceil = \frac{r-1}{r_k}$, then $[\iota_k^*(f_*\mathcal{O}_{\mathcal{X}_{\boldsymbol{a}}})] = (m_k + 1)x_k^0 + m_k x_k^1 + \cdots + m_k x_k^{r_k-1}$, where $m_k = \frac{r-1}{r_k}$.*

Proof By Proposition 4.1, we have $f_*\mathcal{O}_{\mathcal{X}_{\boldsymbol{a}}} = \bigoplus_{i=0}^{r-1} K_{\mathcal{X}}^{-i}$. Since $\mathcal{X}_{\boldsymbol{a}}$ satisfies the conclusion of Lemma 4.20, by some elementary computation, we get the decomposition of $[\iota_k^*(f_*\mathcal{O}_{\mathcal{X}_{\boldsymbol{a}}})]$ in $\mathbf{R}\mu_{r_k}$ for every $1 \le k \le m_1$. ∎

If (E, ϕ) is a rank r Higgs bundle with spectral curve $\mathcal{X}_{\boldsymbol{a}}$, then there is a line bundle W in some $\mathrm{Pic}^{d_1,(i_1,\dots,i_{m_1})}(\mathcal{X}_{\boldsymbol{a}})$ such that $f_*(W) = E$ (see Proposition C.2).

Lemma 6.2 *The K-class $[W] \in K_0(\mathcal{X}_{\boldsymbol{a}})_{\mathbb{Q}}$ is uniquely determined by the K-class $[E] \in K_0(\mathcal{X})_{\mathbb{Q}}$.*

Proof By Proposition 3.35, we only need to show that $\deg(W)$ and $\{i_1, \dots, i_{m_1}\}$ are uniquely determined by $[E]$. First, note that there is a line bundle $W' \in \mathrm{Pic}(X_{\boldsymbol{a}})$ such that

$$W = \pi'^* W' \otimes f^*\mathcal{O}_{\mathcal{X}}\left(\sum_{k=1}^{m_1} \frac{i_k}{r_k} \cdot p_k\right)$$

(see Lemma 5.16). Hence, $E = f_*(\pi'^* W') \otimes \mathcal{O}_{\mathcal{X}}(\sum_{k=1}^{m_1} \frac{i_k}{r_k} \cdot p_k)$. We therefore have

$$[\iota_k^* E] = [\iota_k^*(f_*(\pi'^* W'))] \cdot x_k^{i_k} \quad \text{in } \mathbf{R}\mu_{r_k},$$

for each $1 \le k \le m_1$. Note that $[\iota_k^*(f_*(\pi'^* W'))] = [\iota_k^*(f_*\mathcal{O}_{\mathcal{X}_{\boldsymbol{a}}})]$ in $\mathbf{R}\mu_{r_k}$ for all $1 \le k \le m$. By Lemma 6.1, i_k are uniquely determined. On the other hand, by Propositions 5.9 and 5.21, $\deg(W) = \deg(E) - \deg(f_*\mathcal{O}_{\mathcal{X}_{\boldsymbol{a}}})$, where $\deg(f_*\mathcal{O}_{\mathcal{X}_{\boldsymbol{a}}}) = \frac{r(1-r)}{r}(2g - 2 + \sum_{k=1}^{m} \frac{r_i - 1}{r_i})$. ∎

Corollary 6.3 *If the spectral curve of (E, ϕ) is irreducible and smooth, then there exists $(i_1, \dots, i_{m_1}) \in \mathbb{Z}^{m_1} \cap [0, r_1 - 1] \times \cdots \times [0, r_{m_1} - 1]$ such that $[E] \in K_0(\mathcal{X})_{\mathbb{Q}}$ satisfies*

$$[\iota_k^* E] = \left[\iota_k^*\left(\left(\bigoplus_{i=0}^{r-1} K_{\mathcal{X}}^{-i}\right) \otimes \mathcal{O}_{\mathcal{X}}\left(\sum_{k=1}^{m_1} \frac{i_k}{r_k} \cdot p_k\right)\right)\right], \tag{82}$$

for all $1 \le k \le m$.

Denote the K-class $[E] \in K_0(\mathcal{X})_{\mathbb{Q}}$ by ξ. Consider the moduli space $M_{\mathrm{Dol},\xi}^{ss}(\mathbf{GL}_r)$ of moduli space of semistable Higgs bundles with K-class ξ. By Proposition C.2, the following lemma is immediate.

Lemma 6.4 *The fiber $h^{-1}(\boldsymbol{a})$ of the Hitchin morphism $h : M_{\mathrm{Dol},\xi}^{ss}(\mathbf{GL}_r) \to \mathbb{H}(r, K_{\mathcal{X}})$ at $\boldsymbol{a}$ is isomorphic to $\mathrm{Pic}^{d,(i_1,\dots,i_{m_1})}(\mathcal{X}_{\boldsymbol{a}})$.*

Suppose that $\xi = (r, d_\xi, (m_{1,i})_{i=1}^{r_1-1}, \dots, (m_{m,i})_{i=1}^{r_m-1}) \in K_0(\mathcal{X})_{\mathbb{Q}}$. Fix a line bundle $L \in \mathrm{Pic}^{d',(j_1,\dots,j_m)}(\mathcal{X})$, where $d', j_1, \dots, j_m$ satisfy

(83)
$$j_k = \text{the remainder, when } \textstyle\sum_{i=1}^{r_1-1} i \cdot m_{k,i} \text{ divided by } r_k \text{ for every } 1 \le k \le m \quad \text{and}$$
$$d' = d_\xi + \sum_{k=1}^{m}\Big(\sum_{i=1}^{r_k-1} \frac{i \cdot m_{k,i}}{r_k} - j_k\Big).$$

We consider the moduli space $M_{\mathrm{Dol},\xi}^{ss}(\mathbf{SL}_r)$ of semistable $\mathbf{SL}_r$-Higgs bundles with K-class ξ and determinant L. Assume that the assumptions of Corollary 4.19 (which ensure that a general spectral curve is irreducible and smooth) are satisfied, then for a general $\boldsymbol{a} \in \bigoplus_{i=2}^r H^0(\mathcal{X}, K_{\mathcal{X}}^i)$, the spectral curve $\mathcal{X}_{\boldsymbol{a}}$ satisfies the conclusion in Lemma 4.20. We also assume that $\mathcal{X}_{\boldsymbol{a}}$ satisfies the conclusion in Lemma 4.20.

Lemma 6.5 *The fiber of the Hitchin morphism $h_{\mathbf{SL}_r} : M_{\mathrm{Dol},\xi}^{ss}(\mathbf{SL}_r) \to \mathbb{H}^o(r, K_{\mathcal{X}})$ at $\boldsymbol{a}$ is*

$$h_{\mathbf{SL}_r}^{-1}(\boldsymbol{a}) = \{W \in \mathrm{Pic}^{d,(i_1,\dots,i_{m_1})}(\mathcal{X}_{\boldsymbol{a}}) \,|\, \mathrm{Nm}_f(W) = L \otimes K_{\mathcal{X}}^{r(r-1)/r}\}.$$

Proof Since $\mathcal{X}_{\boldsymbol{a}}$ satisfies the conclusion in Lemma 4.20, we have

$$h_{\mathbf{SL}_r}^{-1}(\boldsymbol{a}) = \{W \in \mathrm{Pic}^{d,(i_1,\dots,i_{m_1})}(\mathcal{X}_{\boldsymbol{a}}) \,|\, \det(f_*(W)) = L\}.$$

Therefore,

$$h_{\mathbf{SL}_r}^{-1}(\boldsymbol{a}) = \{W \in \mathrm{Pic}^{d,(i_1,\dots,i_{m_1})}(\mathcal{X}_{\boldsymbol{a}}) \,|\, \mathrm{Nm}_f(W) = \det(f_*(W)) \otimes \det(f_*(\mathcal{O}_{\mathcal{X}_{\boldsymbol{a}}}))^{-1}\}$$

(see Proposition 5.9). Note that $\det(f_*(\mathcal{O}_{\mathcal{X}_{\boldsymbol{a}}}))^{-1} = K_{\mathcal{X}}^{r(r-1)/r}$. This completes the proof. ∎

For $f' : X_{\boldsymbol{a}} \to X$ in the diagram (81), the Prym varieties $\mathrm{Prym}_{f'}(X_{\boldsymbol{a}})$ is defined by

$$\mathrm{Prym}_{f'}(X_{\boldsymbol{a}}) = \mathrm{Ker}(\mathrm{Nm}_{f'}) = \{W \in \mathrm{Pic}^0(X_{\boldsymbol{a}}) \,|\, \mathrm{Nm}_{f'}(W) = \mathcal{O}_X\}.$$

Then, $h_{\mathbf{SL}_r}^{-1}(\boldsymbol{a})$ is a $\mathrm{Prym}_{f'}(X_{\boldsymbol{a}})$-torsor (see Lemma 5.23). The fiber $h_{\mathbf{PGL}_r}^{-1}(\boldsymbol{a})$ of the Hitchin morphism $h_{\mathbf{PGL}_r} : M_{\mathrm{Dol},\xi}^{\alpha,s}(\mathbf{PGL}_r) \to \mathbb{H}^o(r, K_{\mathcal{X}})$ at $\boldsymbol{a}$ is a $\mathrm{Prym}_{f'}(X_{\boldsymbol{a}})/\Gamma_0$-torsor, where

$$\Gamma_0 = \{W \in \mathrm{Pic}^0(X) \,|\, W^{\otimes r} = \mathcal{O}_X\}.$$

Thus, we have the following proposition.

Proposition 6.6 *$h_{\mathbf{SL}_r}^{-1}(\boldsymbol{a})$ is a $\mathrm{Prym}_{f'}(X_{\boldsymbol{a}})$-torsor and $h_{\mathbf{PGL}_r}^{-1}(\boldsymbol{a})$ is a $\mathrm{Prym}_{f'}(X_{\boldsymbol{a}})/\Gamma_0$-torsor.*

For brevity, we introduce the following notations:

- $\mathcal{P}^{d,(i_1,\dots,i_{m_1})} = \{W \in \mathrm{Pic}^{d,(i_1,\dots,i_{m_1})}(\mathcal{X}_{\boldsymbol{a}}) \,|\, \mathrm{Nm}_f(W) \simeq L \otimes K_{\mathcal{X}}^{r(r-1)/2}\}$.
- $P^d = \{W \in \mathrm{Pic}^d(X_{\boldsymbol{a}}) \,|\, \mathrm{Nm}_{f'}(W) \simeq \pi'_*(L \otimes K_{\mathcal{X}}^{r(r-1)/2})\}$.
- $\mathcal{P}^{0,(0,\dots,0)} = \{W \in \mathrm{Pic}^{0,(0,\dots,0)}(\mathcal{X}_{\boldsymbol{a}}) \,|\, \mathrm{Nm}_f(W) \simeq \mathcal{O}_{\mathcal{X}}\}$.
- $P^0 = \{W \in \mathrm{Pic}^0(X_{\boldsymbol{a}}) \,|\, \mathrm{Nm}_{f'}(W) \simeq \mathcal{O}_X\}$.
- $\widehat{\mathcal{P}}^{d,(i_1,\dots,i_{m_1})} = \mathcal{P}^{d,(i_1,\dots,i_{m_1})}/\Gamma_0$.
- $\widehat{P}^d = P^d/\Gamma_0$.

- $\widehat{\mathcal{P}^0} = \mathcal{P}^{0,(0,\ldots,0)}/\Gamma_0$.
- $\widehat{P}^0 = P^0/\Gamma_0$.

Obviously, $\mathcal{P}^{d,(i_1,\ldots,i_{m_1})}$ ($\widehat{\mathcal{P}}^{d,(i_1,\ldots,i_{m_1})}$) is a $\mathcal{P}^0$ ($\widehat{\mathcal{P}}^0$)-torsor. By Lemma 5.23, we have the following lemma.

Lemma 6.7 *The pushforward*

$$\pi'_* : \mathcal{P}^{0,(0,\ldots,0)} \to P^0, \quad W \mapsto \pi'_* W \tag{84}$$

is an isomorphism of abelian varieties. And,

$$\pi'_* : \mathcal{P}^{d,(i_1,\ldots,i_{m_1})} \longrightarrow P^d \tag{85}$$

is an isomorphism of torsors with respect to the isomorphism (84). Moreover, (84) induces an isomorphism of abelian varieties

$$\widehat{\pi}'_* : \widehat{\mathcal{P}}^0 \longrightarrow \widehat{P}^0. \tag{86}$$

The morphism (85) gives an isomorphism

$$\widehat{\pi}'_* : \widehat{\mathcal{P}}^{d,(i_1,\ldots,i_{m_1})} \longrightarrow \widehat{P}^d \tag{87}$$

of torsors with respect to (86).

Corollary 6.8 *The dual of $\mathcal{P}^0$ is $\widehat{\mathcal{P}}^0$.*

Proof The dual of P^0 is $\widehat{P}^0$ (see [HT03, Lemma 2.3]). Then, the dual of $\mathcal{P}^0$ is $\widehat{\mathcal{P}}^0$, by the isomorphisms (84) and (86) in Lemma 6.7. ∎

6.2 The proof of SYZ duality

For convenience, the moduli space $M^s_{\mathrm{Dol},\xi}(\mathbf{SL}_r)$ of stable $\mathbf{SL}_r$-Higgs bundles is denoted by $M_{\mathrm{Dol},\xi}$. In general, the universal Higgs bundle $(\boldsymbol{E},\boldsymbol{\Phi})$ does not exist. But we can construct a universal projective bundle $\mathbb{P}(\boldsymbol{E})$ and a universal endomorphism bundle $\mathcal{E}nd(\boldsymbol{E})$, even though $\boldsymbol{E}$ does not exist. There is a universal Higgs field $\boldsymbol{\Phi} \in H^0(\mathcal{E}nd(\boldsymbol{E}) \otimes K_{\mathcal{X}})$. Fix a closed point $c \in \mathcal{X}$. Restricting $\mathbb{P}(\boldsymbol{E})$ to $M_{\mathrm{Dol},\xi} \times \{c\}$, we get a projective bundle $\mathbb{P}$ on $M_{\mathrm{Dol},\xi}$. The obstruction to lift the $\mathbf{PGL}_r$-bundle $\mathbb{P}$ to an $\mathbf{SL}_r$-bundle defines a $\mathbb{Z}_r$-gerbe $\boldsymbol{B}$ on $M_{\mathrm{Dol},\xi}$.

Lemma 6.9 *The restriction of $\boldsymbol{B}$ to each regular fiber of the Hitchin morphism $h_{\mathbf{SL}_r} : M_{\mathrm{Dol},\xi} \to \mathbb{H}^0(r, K_{\mathcal{X}})$ is trivial as a $\mathbb{Z}_r$-gerbe.*

Proof Suppose that $a \in \mathbb{H}^0(r, K_{\mathcal{X}})$ is a closed point such that the associated spectral curve $\mathcal{X}_a$ is integral and smooth. Recall that the fiber of the Hitchin morphism $h_{\mathbf{SL}_r}$ at a is $\mathcal{P}^{d,(i_1,\ldots,i_{m_1})}$, where $0 \le i_k \le r_k - 1$ for all k. Let $\boldsymbol{L}$ be a universal line bundle on $\mathcal{P}^{d,(i_1,\ldots,i_{m_1})} \times \mathcal{X}_a$. The projection of $\mathcal{X}_{\boldsymbol{a}}$ to $\mathcal{X}$ is $f : \mathcal{X}_a \longrightarrow \mathcal{X}$. The pushforward $((\mathrm{id} \times f)_*(\boldsymbol{L}), (\mathrm{id} \times f)_*(\widetilde{\boldsymbol{\phi}}))$ is a $\mathcal{P}^{d,(i_1,\ldots,i_{m_1})}$-family of Higgs bundles on $\mathcal{X}$, where $\widetilde{\phi} : \boldsymbol{L} \to \boldsymbol{L} \otimes_{\mathcal{O}_{\mathcal{X}_a}} f^* K_{\mathcal{X}}$ is defined by the tautological section of $f^* K_{\mathcal{X}}$. It induces an inclusion $\mathcal{P}^{d,(i_1,\ldots,i_{m_1})} \subseteq M_{\mathrm{Dol},\xi}$. Hence, we have

$$\mathbb{P}((\mathrm{id} \times p)_* \boldsymbol{L})|_{\mathcal{P}^{d,(i_1,\ldots,i_{m_1})} \times \{c\}} = \mathbb{P}(\boldsymbol{E})|_{\mathcal{P}^{d,(i_1,\ldots,i_{m_1})} \times \{c\}}.$$

Since $f : \mathcal{X}_a \to \mathcal{X}$ is a finite morphism, we can choose a closed point $c \in \mathcal{X}$ such that $f^{-1}(c)$ does not contain any branched points. So, we have

$$\begin{aligned}(\mathrm{id} \times f)_*(\boldsymbol{L})|_{\mathcal{P}^{d,(i_1,\ldots,i_{m_1})} \times \{c\}} &= (\mathrm{id} \times f)_*(\boldsymbol{L}|_{\mathcal{P}^{d,(i_1,\ldots,i_{m_1})} \times f^{-1}(c)}) \\ &= \oplus_{y \in f^{-1}(c)} \boldsymbol{L}|_{\mathcal{P}^{d,(i_1,\ldots,i_{m_1})} \times \{y\}}.\end{aligned} \tag{88}$$

Thus, $\det((\mathrm{id} \times p)_*(\boldsymbol{L})|_{\mathcal{P}^{d,(i_1,\ldots,i_{m_1})} \times \{c\}}) = \otimes_{y \in p^{-1}(c)} \boldsymbol{L}|_{\mathcal{P}^{d,(i_1,\ldots,i_{m_1})} \times \{y\}} = V$. On the other hand, the Néron–Severi class of V is divisible by r. Therefore, there exists a line bundle W on $\mathcal{P}^{d,(i_1,\ldots,i_{m_1})}$ such that $W^{\otimes r} \simeq V$. We have

$$\det((\mathrm{id} \times p)_*(\mathrm{pr}^*_{\mathcal{P}^{d,(i_1,\ldots,i_{m_1})}} W^{-1} \otimes \boldsymbol{L})|_{\mathcal{P}^{d,(i_1,\ldots,i_{m_1})} \times \{c\}}) \simeq \mathcal{O}_{\mathcal{P}^{d,(i_1,\ldots,i_{m_1})} \times \{c\}} \quad \text{and}$$
$$\mathbb{P}((\mathrm{id} \times p)_*(\mathrm{pr}^*_{\mathcal{P}^{d,(i_1,\ldots,i_{m_1})}} W^{-1} \otimes \boldsymbol{L})|_{\mathcal{P}^{d,(i_1,\ldots,i_{m_1})} \times \{c\}}) = \mathbb{P}(\boldsymbol{E})|_{\mathcal{P}^{d,(i_1,\ldots,i_{m_1})} \times \{c\}},$$

i.e., $(\mathrm{id} \times p)_*(\mathrm{pr}^*_{\mathcal{P}^{d,(i_1,\ldots,i_{m_1})}} W^{-1} \otimes \boldsymbol{L})|_{\mathcal{P}^{d,(i_1,\ldots,i_{m_1})} \times \{c\}}$ is an $\mathbf{SL}_r$-lifting of $\mathbb{P}|_{\mathcal{P}^{d,(i_1,\ldots,i_{m_1})}}$. So, the restriction of $\boldsymbol{B}$ to $\mathcal{P}^{d,(i_1,\ldots,i_{m_1})}$ is a trivial $\mathbb{Z}_r$-gerbe. ∎

Remark 6.10 Recall the commutative diagram (81). Therefore, we have the commutative diagram

$$\begin{tikzcd} \mathcal{P}^{d,(i_1,\ldots,i_m)} \times \mathcal{X}_a \arrow[r, "\mathrm{id} \times f"] \arrow[d, "\pi'_* \times \pi'"'] & \mathcal{P}^{d,(i_1,\ldots,i_m)} \times \mathcal{X} \arrow[d, "\pi'_* \times \pi"] \\ P^d \times X_a \arrow[r, "\mathrm{id} \times f'"] & P^d \times X \end{tikzcd} \tag{89}$$

where $\pi'_* : \mathcal{P}^{d,(i_1,\ldots,i_{m_1})} \to P^d$ is the isomorphism in Lemma 6.7. For the universal line bundle $\boldsymbol{L}$ in the proof of Lemma 6.9, there exists a universal line bundle $\boldsymbol{W}$ on $P^d \times X_a$ such that

$$\boldsymbol{L} \simeq ((\pi'_*) \times \pi')^* \boldsymbol{W} \otimes \mathrm{pr}^*_{\mathcal{X}_a} W_{i_1,\ldots,i_{m_1}}, \tag{90}$$

where $W_{i_1,\ldots,i_{m_1}} = \mathcal{O}_{\mathcal{X}_a}(\sum_{k=1}^{m_1} \frac{i_k}{r_k} \cdot \widetilde{p}_k)$. Let c' be the image of c in the coarse moduli space X. Since c is a closed point in $\mathcal{X}$, it is easy to check that

$$(\pi'_*)^* \mathbb{P}((\mathrm{id} \times f')_* \boldsymbol{W})|_{P^d \times \{c'\}} = \mathbb{P}|_{\mathcal{P}^{d,(i_1,\ldots,i_{m_1})} \times \{c\}}. \tag{91}$$

Then, the two sets of trivializations are isomorphic

$$\mathrm{Triv}^{\mathbb{Z}_r}(\mathcal{P}^{d,(i_1,\ldots,i_{m_1})}, \boldsymbol{B}) \simeq \mathrm{Triv}^{\mathbb{Z}_r}(P^d, \boldsymbol{B}'), \tag{92}$$

where $\boldsymbol{B}'$ is the $\mathbf{SL}_r$-lifting gerbe of $\mathbb{P}((\mathrm{id} \times f')_* \boldsymbol{W})|_{P^d \times \{c'\}}$.

By the proof of Lemma 6.9, we see that a trivialization of $\boldsymbol{B}$ on $\mathcal{P}^{d,(i_1,\ldots,i_{m_1})}$ is equivalent to give a universal line bundle $\boldsymbol{L}$ on $\mathcal{P}^{d,(i_1,\ldots,i_{m_1})} \times \mathcal{X}_a$ such that

$$\det((\mathrm{id} \times f)_*(\boldsymbol{L})|_{\mathcal{P}^{d,(i_1,\ldots,i_{m_1})} \times \{c\}}) \simeq \mathcal{O}_{\mathcal{P}^{d,(i_1,\ldots,i_{m_1})} \times \{c\}}. \tag{93}$$

Then the set $\mathrm{Triv}^{\mathbb{Z}_r}(\mathcal{P}^{d,(i_1,\ldots,i_{m_1})}, \boldsymbol{B})$ of trivialization of $\boldsymbol{B}$ on $\mathcal{P}^{d,(i_1,\ldots,i_{m_1})}$ is identified with the set $\mathfrak{T}$ of universal line bundles $\boldsymbol{L}$ on $\mathcal{P}^{d,(i_1,\ldots,i_{m_1})} \times \mathcal{X}_a$ satisfying (93). Let

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$\widehat{\mathcal{P}}^0[r]$ be the group of torsion points of order r in $\widehat{\mathcal{P}}^0$. The set $\mathfrak{T}$ is naturally a $\widehat{\mathcal{P}}^0[r]$-torsor. On the other hand, we have

$$H^1(\mathcal{P}^{d,(i_1,\ldots,i_{m_1})}, \mathbb{Z}_r) = H^1(\mathcal{P}^0, \mathbb{Z}_r) = \widehat{\mathcal{P}}^0[r].$$

Then, the $H^1(\mathcal{P}^{d,(i_1,\ldots,i_{m_1})}, \mathbb{Z}_r)$-torsor $\mathrm{Triv}^{\mathbb{Z}_r}(\mathcal{P}^{d,(i_1,\ldots,i_{m_1})}, \boldsymbol{B})$ is isomorphic to the $\widehat{\mathcal{P}}^0[r]$-torsor $\mathfrak{T}$.

Since $\mathbb{Z}_r$ is a subgroup of $U(1)$, any $\mathbb{Z}_r$-gerbe extends to a $U(1)$-gerbe. Let $\mathcal{B}$ be the $U(1)$-gerbe defined by the $\mathbb{Z}_r$-gerbe $\boldsymbol{B}$. The triviality of the $\mathbb{Z}_r$-gerbe $\boldsymbol{B}$ implies that the $U(1)$-gerbe $\mathcal{B}$ is also trivial. The set of all trivialization of $\mathcal{B}$ on $\mathcal{P}^{d,(i_1,\ldots,i_{m_1})}$ is denoted by $\mathrm{Triv}^{U(1)}(\mathcal{P}^{d,(i_1,\ldots,i_{m_1})}, \mathcal{B})$, which is an $H^1(\mathcal{P}^{d,(i_1,\ldots,i_{m_1})}, U(1))$-torsor. Similarly, we have

$$H^1(\mathcal{P}^{d,(i_1,\ldots,i_{m_1})}, U(1)) = H^1(\mathcal{P}^0, U(1)) = \widehat{\mathcal{P}}^0.$$

We have a natural identification

$$\mathrm{Triv}^{U(1)}(\mathcal{P}^{d,(i_1,\ldots,i_{m_1})}, \mathcal{B}) = \frac{\mathrm{Triv}^{\mathbb{Z}_r}(\mathcal{P}^{d,(i_1,\ldots,i_{m_1})}, \boldsymbol{B}) \times \widehat{\mathcal{P}}^0}{\widehat{\mathcal{P}}^0[r]}.$$

Proposition 6.11 *For any $d, e \in \mathbb{Z}$, there is a smooth isomorphism of $\widehat{\mathcal{P}}^0$-torsors*

$$\mathrm{Triv}^{U(1)}(\mathcal{P}^{d,(i_1,\ldots,i_{m_1})}, \mathcal{B}^e) \simeq \widehat{\mathcal{P}}^e.$$

Proof By the isomorphism (92), [HT03, Proposition 3.2], and [BD12, Theorem 4.2], we complete the proof. ∎

Now consider the reverse direction. We need a gerbe $\widehat{\boldsymbol{B}}$ on the global quotient stack $[M_{\mathrm{Dol},\xi}/\Gamma_0]$, i.e., a Γ_0-equivariant gerbe on $M_{\mathrm{Dol},\xi}$. In fact, this is just $\boldsymbol{B}$ equipped with a Γ_0-equivariant structure. For $\gamma \in \Gamma_0$, we use L_γ to indicate the line bundle on X corresponding to $\gamma \in \Gamma$. Then the action of Γ_0 on $M_{\mathrm{Dol},\xi}$ is given by

$$\gamma : M_{\mathrm{Dol},\xi} \longrightarrow M_{\mathrm{Dol},\xi} \quad (E, \phi) \longrightarrow (E \otimes \pi^* L_\gamma, \phi),$$

for $\gamma \in \Gamma_0$. Let $(\boldsymbol{E}, \phi)$ be the universal Higgs bundle on $M_{\mathrm{Dol},\xi} \times \mathcal{X}$ (if the moduli space $M_{\mathrm{Dol},\xi}$ is not fine, $\boldsymbol{E}$ is a twisted vector bundle on $M_{\mathrm{Dol},\xi}$). We have a canonical isomorphism

$$\boldsymbol{f}_\gamma : (\gamma \times \mathrm{id})^* \mathbb{P}(\boldsymbol{E}) = \mathbb{P}(\boldsymbol{E} \otimes \mathrm{pr}_{\mathcal{X}}{}^*(\pi^* L_\gamma)) \longrightarrow \mathbb{P}(\boldsymbol{E})$$

on $M_{\mathrm{Dol},\xi} \times \mathcal{X}$, for every $\gamma \in \Gamma_0$. And, for $\gamma_1, \gamma_2 \in \Gamma_0$,

$$\boldsymbol{f}_{\gamma_1} \circ (\gamma_1 \times \mathrm{id})^* \boldsymbol{f}_{\gamma_2} = \boldsymbol{f}_{\gamma_1\gamma_2}.$$

Hence, $\mathbb{P}(\boldsymbol{E})$ is a Γ_0-equivariant projective bundle on $M_{\mathrm{Dol},\xi} \times \mathcal{X}$. The restriction $\mathbb{P}$ of $\mathbb{P}(\boldsymbol{E})$ to $M_{\mathrm{Dol},\xi} \times \{c\}$ is also a Γ_0-equivariant projective bundle on $M_{\mathrm{Dol},\xi}$. It determines a Γ_0-equivariant structure on the $\mathbb{Z}_r$-gerbe $\boldsymbol{B}$ on $M_{\mathrm{Dol},\xi}$. Then, it defines a $\mathbb{Z}_r$-gerbe $\widehat{\boldsymbol{B}}$ on $[M_{\mathrm{Dol},\xi}/\Gamma_0]$. Specifically, the Γ_0-equivariant structure of $\mathbb{P}|_{\mathcal{P}^{d,(i_1,\ldots,i_m)}}$ is

$$\begin{aligned} \boldsymbol{f}_\gamma|_{\mathcal{P}^{d,(i_1,\ldots,i_{m_1})}} &: \gamma^* \mathbb{P}(\boldsymbol{E}|_{\mathcal{P}^{d,(i_1,\ldots,i_{m_1})} \times \{c\}}) \\ &= \mathbb{P}(\boldsymbol{E}|_{\mathcal{P}^{d,(i_1,\ldots,i_{m_1})} \times \{c\}} \otimes_{\mathbb{C}} (\pi^*(L_\gamma)|_{\{c\}})) \longrightarrow \mathbb{P}(\boldsymbol{E}|_{\mathcal{P}^{d,(i_1,\ldots,i_{m_1})} \times \{c\}}), \end{aligned}$$

for every $\gamma \in \Gamma_0$. By Remark 6.10, there exists a locally free sheaf $(\mathrm{id} \times f')_* \boldsymbol{W}|_{P^d \times \{c'\}}$ on $P^d \times \{c'\}$ such that

$$(\pi'_*)^* \mathbb{P}((\mathrm{id} \times f')_* \boldsymbol{W})|_{P^d \times \{c'\}} = \mathbb{P}(\boldsymbol{E})|_{\mathcal{P}^{d,(i_1,\ldots,i_{m_1})} \times \{c\}},$$

where $\boldsymbol{W}$ is a universal line bundle on $P^d \times X_a$. On the other hand, the projective bundle $\mathbb{P}((\mathrm{id} \times f')_* \boldsymbol{W}|_{P^d \times \{c'\}})$ admits a Γ_0-equivariant structure

$$\begin{aligned} \boldsymbol{g}_\gamma &: \gamma^* \mathbb{P}((\mathrm{id} \times f')_* \boldsymbol{W}|_{P^d \times \{c'\}}) \\ &= \mathbb{P}((\mathrm{id} \times f')_* \boldsymbol{W}|_{P^d \times \{c'\}} \otimes_{\mathbb{C}} L_\gamma|_{\{c\}}) \longrightarrow \mathbb{P}((\mathrm{id} \times f')_* \boldsymbol{W}|_{P^d \times \{c'\}}) \end{aligned}$$

for every $\gamma \in \Gamma_0$, which is induced by the natural Γ_0-equivariant structure of $\mathbb{P}((\mathrm{id} \times f')_* \boldsymbol{W})$ on $P^d \times X$. Obviously, the Γ_0-equivariant projective bundle $\mathbb{P}|_{\mathcal{P}^{d,(i_1,\ldots,i_{m_1})}}$ is isomorphic to the pullback of the Γ_0-equivariant projective bundle $\mathbb{P}((\mathrm{id} \times f')_* \boldsymbol{W}|_{P^d \times \{c'\}})$, along the Γ_0-equivariant morphism $\pi'_* : \mathcal{P}^{d,(i_1,\ldots,i_{m_1})} \to P^d$. We therefore have the following proposition (see [HT03, Lemma 3.5 and Proposition 3.6]).

Proposition 6.12 *The restriction of $\widehat{\boldsymbol{B}}$ to $\widehat{\mathcal{P}}^{d,(i_1,\ldots,i_{m_1})}$ is trivial as a $\mathbb{Z}_r$-gerbe. Moreover, there is a smooth isomorphism of $\mathcal{P}^0$-torsors*

$$\mathrm{Triv}^{U(1)}(\widehat{\mathcal{P}}^{d,(i_1,\ldots,i_{m_1})}, \widehat{\mathcal{B}}^e) \simeq \mathcal{P}^e,$$

where $\widehat{\mathcal{B}}$ is the $U(1)$-gerbe obtained by the extension of $\widehat{\boldsymbol{B}}$ and $d, e \in \mathbb{Z}$.

Assume that the assumptions of Corollary 4.19 (which ensure that a general spectral curve is irreducible and smooth) are satisfied. Suppose that the K-class ξ satisfies (82) and $\xi = (r, d_\xi, (m_{1,i})_{i=1}^{r_1-1}, \ldots, (m_{m,i})_{i=1}^{r_m-1}) \in K_0(\mathcal{X})_{\mathbb{Q}}$. Fix a line bundle $L \in \mathrm{Pic}^{d',(j_1,\ldots,j_m)}(\mathcal{X})$, where $d', j_1, \ldots, j_m$ satisfy (83). Consider the moduli space $M^{ss}_{\mathrm{Dol},\xi}(\mathbf{SL}_r)$ of semistable $\mathbf{SL}_r$-Higgs bundles with K-class ξ and determinant L. The Hitchin morphism $h_{\mathbf{SL}_r} : M^{ss}_{\mathrm{Dol},\xi}(\mathbf{SL}_r) \to \mathbb{H}^o(r, K_{\mathcal{X}})$ is surjective. Note that the stable locus $M_{\mathrm{Dol},\xi}$ of $M^{ss}_{\mathrm{Dol},\xi}(\mathbf{SL}_r)$ is nonempty. By Proposition 3.39, we have

$$\dim M_{\mathrm{Dol},\xi} = (r^2-1)(2g-2) + \textstyle\sum_{i=1}^m (r^2 - (r - \sum_{k=1}^{r_i-1} m_{i,k})^2 - \sum_{k=1}^{r_i-1} m_{i,k}^2).$$

On the other hand, by some elementary computation, we have

$$\dim \mathbb{H}^o(r, K_{\mathcal{X}}) = (r^2-1)(g-1) + \tfrac{1}{2}\textstyle\sum_{i=1}^m (r^2 - (r - \sum_{k=1}^{r_i-1} m_{i,k})^2 - \sum_{k=1}^{r_i-1} m_{i,k}^2),$$

i.e., $\dim \mathbb{H}^o(r, K_{\mathcal{X}}) = \frac{1}{2} \dim M_{\mathrm{Dol},\xi}$. Hence, $h_{\mathbf{SL}_r}$ is surjective, since the restriction of $h_{\mathbf{SL}_r}$ to a nonempty open subsect of $M_{\mathrm{Dol},s}$ is an algebraically integrable systems. Therefore, the properness of $h_{\mathbf{SL}_r}$ implies that there is a nonempty open subset $\mathcal{U} \subseteq \mathbb{H}^o(r, K_{\mathcal{X}})$ such that the inverse image $h^{-1}_{\mathbf{SL}_r}(\mathcal{U})$ is contained in $M_{\mathrm{Dol},\xi}$. Note that $h^{-1}_{\mathbf{SL}_r}(\mathcal{U})$ is Γ_0-invariant, due to the Γ_0-equivariantness of $h_{\mathbf{SL}_r}$ and the trivial action of Γ_0 on $\mathbb{H}^o(r, K_{\mathcal{X}})$. Then, we obtain two proper morphisms

$$h_{\mathbf{SL}_r,\mathcal{U}} : h^{-1}_{\mathbf{SL}_r}(\mathcal{U}) \to \mathcal{U} \quad \text{and} \quad h_{\mathbf{PGL}_r,\mathcal{U}} : h^{-1}_{\mathbf{PGL}_r}(\mathcal{U}) = [h^{-1}_{\mathbf{SL}_r}(\mathcal{U})/\Gamma_0] \to \mathcal{U},$$

where $h_{\mathbf{SL}_r,\mathcal{U}}$ and $h_{\mathbf{PGL}_r,\mathcal{U}}$ are complete algebraically integrable systems (see [LM10, Mar94]). Moreover, $M_{\mathbf{SL}_r} := h^{-1}_{\mathbf{SL}_r}(\mathcal{U})$ is a hyperkähler manifold and

$M_{\mathbf{PGL}_r} := h^{-1}_{\mathbf{PGL}_r}(\mathcal{U}) = [M_{\mathbf{SL}_r}/\Gamma_0]$ is a hyperkähler orbifold (see [Kon93]). Summarizing the above discussion, we get our main result.

Theorem 6.13 (1) *Assume that* $\lceil \frac{r}{r_k} \rceil = \frac{r}{r_k}$ *or* $\lceil \frac{r}{r_k} \rceil = \frac{r+1}{r_k}$ *for all* $1 \leq k \leq m$. $(M_{\mathbf{SL}_r}, \mathcal{B})$ *and* $(M_{\mathbf{PGL}_r}, \widehat{\mathcal{B}})$ *are SYZ mirror partners if one of the following conditions is satisfied:*

(i) $g \geq 2$;
(ii) $g = 1$ *and* $\sum_{k=1}^{m}(r - \lceil \frac{r}{r_k} \rceil) \geq 2$;
(iii) $g = 0$ *and* $\sum_{k=1}^{m}(r - \lceil \frac{r}{r_k} \rceil) \geq 2r + 1$;
(iv) $g = 0$, $\sum_{k=1}^{m}(r - \lceil \frac{r}{r_k} \rceil) \geq 2r$ *and* $\dim_{\mathbb{C}} H^0(\mathcal{X}, K_{\mathcal{X}}^k) \geq 2$ *for some* $2 \leq k \leq r$.

(2) *Suppose that the assumption in* (1) *does not hold. We make the following assumption: if* $\lceil \frac{r}{r_k} \rceil \geq \frac{r+2}{r_k}$ *for some* $1 \leq k \leq m$, *then* $\lceil \frac{r-1}{r_k} \rceil = \frac{r-1}{r_k}$. *Hence,* $(M_{\mathbf{SL}_r}, \mathcal{B})$ *and* $(M_{\mathbf{PGL}_r}, \widehat{\mathcal{B}})$ *are SYZ mirror partners if any of the following conditions is satisfied:*

(i) $g \geq 2$;
(ii) $g = 1$ *and* $\sum_{k=1}^{m}(r - 1 - \lceil \frac{r-1}{r_k} \rceil) \geq 2$;
(iii) $g = 0$, $\sum_{k=1}^{m}(r - 1 - \lceil \frac{r-1}{r_k} \rceil) \geq 2r - 2$ *and* $K_{\mathcal{X}}$ *satisfies the condition* (43) *in Section* 4.1.

Proof From Proposition 6.6, Lemma 6.7, Corollary 6.8, Proposition 6.11, and Proposition 6.12, we conclude the conclusions of the theorem. ∎

Corollary 6.14 *If there are no strictly semistable* $\mathbf{SL}_r$*-Higgs bundles with K-class* ξ, *then the* $(M^s_{\mathrm{Dol},\xi}(\mathbf{SL}_r), \mathcal{B})$ *and* $(M^{\alpha,s}_{\mathrm{Dol},\xi}(\mathbf{PGL}_r), \widehat{\mathcal{B}})$ *are mirror partners.*

Example 6.15 For five distinct points $\{p_1, p_2, p_3, p_4, p_5\}$ on the projective line $\mathbb{P}^1$, we can construct the stacky curve

$$\mathcal{X} = \mathbb{P}^1_{3,2,2,2,2} = \sqrt[3]{p_1} \times_{\mathbb{P}^1} \sqrt[2]{p_2} \times_{\mathbb{P}^1} \sqrt[2]{p_3} \times_{\mathbb{P}^1} \sqrt[2]{p_4} \times_{\mathbb{P}^1} \sqrt[2]{p_5}.$$

Its coarse moduli space is $\pi : \mathcal{X} \to \mathbb{P}^1$. The canonical line bundle of $\mathcal{X}$ is $K_{\mathcal{X}} = \pi^* K_{\mathbb{P}^1} \otimes \mathcal{O}_{\mathcal{X}}(\frac{2}{3}p_1 + \frac{1}{2}\sum_{k=2}^{5} p_k)$. Note that the degree of $K_{\mathcal{X}}$ is $\frac{2}{3}$. Hence, it is a hyperbolic stacky curve. We can show that

$$\pi_*(K_{\mathcal{X}}) = \mathcal{O}_{\mathbb{P}^1}(-2), \quad \pi_*(K^2_{\mathcal{X}}) = \mathcal{O}_{\mathbb{P}^1}(1) \quad \text{and} \quad \pi_*(K^3_{\mathcal{X}}) = \mathcal{O}_{\mathbb{P}^1}.$$

Since $\dim_{\mathbb{C}} H^0(\mathcal{X}, K^2_{\mathcal{X}}) = 2$ and $\dim_{\mathbb{C}} H^0(\mathcal{X}, K^3_{\mathcal{X}}) = 1$, the condition (33) is satisfied. Note that $K^3_{\mathcal{X}} = \mathcal{O}_{\mathcal{X}}(\frac{1}{2}\sum_{k=2}^{5} p_k)$. So, $H^0(\mathcal{X}, K^3_{\mathcal{X}})$ is generated by the section $s = \tau_2 \otimes \tau_3 \otimes \tau_4 \otimes \tau_5$, where τ_i is the pullback section of the universal section on root stack $\sqrt[2]{p_i}$, for each i. Consider the spectral curve $\mathcal{X}_s$ define by s, i.e., it is the zero locus of section $\tau^{\otimes 3} + \psi^* s$, where $\psi : \mathrm{Tot}(K_{\mathcal{X}}) \to \mathcal{X}$ is the projection and τ is the tautological section. According to the uniformization of Deligne–Mumford curves (see [BN06]), there exists a smooth projective curve Σ with an action of a finite group G such that $\mathcal{X}$ is $[\Sigma/G]$. More precisely, we have the commutative diagram

$$\begin{tikzcd} \Sigma \arrow[rr, "g"] \arrow[dr, "f"'] & & \mathcal{X} \arrow[dl, "\pi"] \\ & \mathbb{P}^1 & \end{tikzcd}$$

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where $g : \Sigma \to \mathcal{X}$ is the natural étale covering of $\mathcal{X}$ and f is a ramified finite covering. By the discussion in Section 4.1, $\mathcal{X}_s = [\Sigma_{s'}/G]$, where $\Sigma_{s'}$ is the spectral curve on Σ defined by the section $s' = g^* s$. The divisor defined by s is $(s) = \sum_{k=2}^{5} \frac{1}{2} p_k$. Thus, the divisor associated with s' is a reduced divisor on Σ. Hence, the spectral curve $\Sigma_{s'}$ is a smooth irreducible curve. Then, $\mathcal{X}_s$ is a smooth irreducible stacky curve. By Proposition 4.1, the genus of $\mathcal{X}_s$ is $g(\mathcal{X}_s) = 3$. The coarse moduli space of $\mathcal{X}_s$ is denoted by X_s. Obviously, X_s has four stacky points $\{\widehat{p}_2, \widehat{p}_3, \widehat{p}_4, \widehat{p}_5\}$, whose images in $\mathbb{P}^1$ are $\{p_2, \dots, p_5\}$. Their stabilizer groups are μ_2. Let W be the line bundle $\mathcal{O}_{\mathcal{X}_s}(d \cdot \widehat{p} + \frac{1}{2}\widehat{p}_2)$, where $\widehat{p} \in X_s$ is not a stacky point and $d \in \mathbb{Z}$. Let $f : \mathcal{X}_s \to \mathcal{X}$ be the projection. Then, we obtain a rank 3 Higgs bundle (E, ϕ_1) on $\mathcal{X}$, where $E = f_* W$ and ϕ_1 is the pushforward of the tautological section τ. In the following, we will determine the representations defined by the action of stabilizer groups on the fibers of E. Consider the Cartesian diagram

$$\begin{array}{ccc} B\mu_3 \times_{\mathcal{X}} \mathcal{X}_s & \longrightarrow & \mathcal{X}_s \\ \downarrow & & \downarrow f \\ B\mu_3 & \xrightarrow{\iota_1} & \mathcal{X} \end{array}$$

where $\iota_1 : B\mu_3 \to \mathcal{X}$ is the residue gerbe of p_1. We use $\mathcal{Y}_1$ to denote $B\mu_3 \times_{\mathcal{X}} \mathcal{X}_s$. Then, $\mathcal{Y}_1$ is isomorphic to the quotient stack

$$[\mathrm{Spec}(\mathbb{C}[x]/(x^3 - 1))/\mu_3],$$

where the action of μ_3 is defined by multiplication. It is a free action. Hence, any locally free sheaf of rank r on $\mathcal{Y}_1$ is isomorphic to $\mathcal{O}_{\mathcal{Y}_1}^{\oplus r}$. Then, the μ_3-representation corresponding to $\iota_1^* E$ is $\rho_1^{\otimes 0} \oplus \rho_1 \oplus \rho_1^{\otimes 2}$, where ρ_1 is the representation defined by the inclusion $\mu_3 \hookrightarrow \mathbb{C}^*$. Similarly, the μ_2-representation corresponding to $\iota_2^* E$ ($\iota_2 : B\mu_2 \to \mathcal{X}$ is the residue gerbe of p_2) is $\rho_2^{\otimes 0} \oplus \rho_2 \oplus \rho_2$, where ρ_2 is the representation defined by the inclusion $\mu_2 \hookrightarrow \mathbb{C}^*$. For another three stacky points $\{p_3, p_4, p_5\} \subset X$, the corresponding representations are isomorphic to $\rho_2^{\otimes 0} \oplus \rho_2^{\otimes 0} \oplus \rho_2$. Denote $\pi_* E$ by F. There is a strongly parabolic Higgs bundle (F, ϕ_2) on $\mathbb{P}^1$ with marked points $\{p_1, p_2, p_3, p_4, p_5\}$, corresponding to (E, ϕ_1). The quasi-parabolic structure on F is given by:

- $F_{p_1} = F_{p_1,0} \supset F_{p_1,1} \supset F_{p_1,2} \supset F_{p_1,3} = \{0\}$ at p_1;
- $F_{p_2} = F_{p_2,0} \supset F_{p_2,1} \supset F_{p_2,2} = \{0\}$ at p_2;
- $F_{p_3} = F_{p_3,0} \supset F_{p_3,1} \supset F_{p_3,2} = \{0\}$ at p_3;
- $F_{p_4} = F_{p_4,0} \supset F_{p_4,1} \supset F_{p_5,2} = \{0\}$ at p_4;
- $F_{p_5} = F_{p_5,0} \supset F_{p_5,1} \supset F_{p_5,2} = \{0\}$ at p_5.

And, the multiplicities are:

- $\dim_{\mathbb{C}}(F_{p_1,0}/F_{p_1,1}) = 1$, $\dim_{\mathbb{C}}(F_{p_1,1}/F_{p_1,2}) = 1$ and $\dim_{\mathbb{C}}(F_{p_1,2}/F_{p_1,3}) = 1$;
- $\dim_{\mathbb{C}}(F_{p_2,0}/F_{p_2,1}) = 1$ and $\dim_{\mathbb{C}}(F_{p_2,1}/F_{p_2,2}) = 2$;
- $\dim_{\mathbb{C}}(F_{p_3,0}/F_{p_3,1}) = 2$ and $\dim_{\mathbb{C}}(F_{p_3,1}/F_{p_3,2}) = 1$;
- $\dim_{\mathbb{C}}(F_{p_4,0}/F_{p_4,1}) = 2$ and $\dim_{\mathbb{C}}(F_{p_4,1}/F_{p_4,2}) = 1$;
- $\dim_{\mathbb{C}}(F_{p_5,0}/F_{p_5,1}) = 2$ and $\dim_{\mathbb{C}}(F_{p_5,1}/F_{p_5,2}) = 1$.

Denote the K-class of E in $K_0(\mathcal{X})_{\mathbb{Q}}$ by ξ_E, and denote the determinant line bundle of E by L_E. By Proposition A.4, for a generic rational parabolic weight (see Defini-

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tion A.3), the moduli stack $\mathcal{M}_{\mathrm{Dol},\xi_E}(\mathbf{SL}_3)$ of $\mathbf{SL}_3$-Higgs bundles with determinant L_E has no strictly semistable object. With a generic parabolic weight, the moduli spaces $M^s_{\mathrm{Dol},\xi_E}(\mathbf{SL}_3)$ and $M^{\alpha_E,s}_{\mathrm{Dol},\xi_E}(\mathbf{PGL}_3)$ with natural flat unitary gerbes are SYZ mirror partners, where $\alpha_E \in H^2(\mathcal{X},\mu_3)$ is the image of L_E^{-1} under the morphism δ in the Kummer sequence (23).

A Comparison of the modified slope and the parabolic slope

Suppose that $\mathcal{X}$ is a smooth irreducible stacky curve and that $\pi : \mathcal{X} \to X$ is its coarse moduli space. The stacky points of X are $p_1,\dots,p_m$, and the corresponding stabilizer groups are $\mu_{r_1},\dots,\mu_{r_m}$. Fix a stacky point $p_i \in X$. The residue gerbe of p_i is a closed immersion $\iota_i : B\mu_{r_i} \to \mathcal{X}$. Let $(\mathcal{E},\mathcal{O}_X(1))$ be a polarization on $\mathcal{X}$, and let E be a locally free sheaf on $\mathcal{X}$. The decompositions of $\iota_i^* E$ and $\iota_i^*\mathcal{E}$ in the representation ring $\mathbf{R}\mu_{r_i}$ are

$$[\iota_i^* E] = \textstyle\sum_{k=0}^{r_i-1} m_{i,k}x_i^k \text{ and } \quad [\iota_i^*\mathcal{E}] = \textstyle\sum_{k=0}^{r_i-1} n_{i,k}x_i^k,$$

where x_i represents the representation corresponding to the natural inclusion $\mu_{r_i} \hookrightarrow \mathbb{C}^*$. By orbifold-parabolic correspondence, E corresponds to $\pi_*(E)$ with quasi-parabolic structure defined by the stacky structure of E at marked points $p_1,\dots,p_m$. The multiplicities of the quasi-parabolic structure at p_i are $(m_{i,0},\dots,m_{i,r_i-1})$ for every $1 \le i \le m$. The aforementioned quasi-parabolic structure with the parabolic weights

$$\text{(A.1)} \qquad \alpha_{i,0} := 0 \quad \text{and} \quad \alpha_{i,j} := \frac{\sum_{h=1}^{j} n_{i,h}}{\mathrm{rk}(\mathcal{E})} \quad \text{when } 1 \le j \le r_i - 1$$

for each $1 \le i \le m$ is a parabolic structure on $\pi_*(E)$. At this time, $\pi_*(E)$ is called a parabolic bundle. The parabolic degree is

$$\text{(A.2)} \qquad \text{par-deg}(\pi_* E) := \deg(\pi_* E) + \sum_{i=1}^{m}\sum_{j=1}^{r_i-1} \alpha_{i,j} m_{i,j}.$$

Its parabolic slope is

$$\text{(A.3)} \qquad \text{par-}\mu(\pi_* E) = \frac{\text{par-deg}(\pi_* E)}{\mathrm{rk}(\pi_* E)}.$$

With the parabolic slope, we can introduce the stability condition for parabolic bundle $\pi_*(E)$. By some elementary computations, we have the following proposition.

Proposition A.1 *The modified slope $\mu_{\mathcal{E}}(E)$ is equivalent to the parabolic slope of par-$\mu(\pi_* E)$ with weights $\{\alpha_{i,j}\}$. Furthermore, the modified slope $\mu_{\mathcal{E}}$ and the parabolic slope par-μ define the same stability condition on E and $\pi_*(E)$, respectively.*

Remark A.2 For abundant stability conditions, we can directly use rational parabolic weights to define the stability condition of Higgs bundles on stacky curve $\mathcal{X}$. And, for any rational parabolic weight, all the results in Section 3 about moduli stacks (spaces) of Higgs bundles on hyperbolic stacky curve hold, by orbifold-parabolic correspondence.

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Definition A.3 A rational parabolic weight is said to be generic if the induced stability condition on the moduli stack $\mathcal{M}_{\mathrm{Dol},\xi}(\mathbf{GL}_r)$ of Higgs bundles with K-class ξ has no strictly semistable objects.

Recall Proposition 3.2 in [BY99].

Proposition A.4 [BY99] *For a K-class $\xi \in K_0(\mathcal{X})$, there is a generic rational parabolic weight if and only if d and the set of multiplicities $\{m_{i,j} | 1 \le i \le m, \quad 0 \le j \le r_i - 1\}$ have greatest common divisor equal to one, where d is the degree of the K-theoretical pushforward of ξ under the morphism π.*

B Proof of the properness of the Hitchin morphism

Let $\mathcal{X}$ be a smooth irreducible stacky curve. For a DVR R over $\mathbb{C}$ with maximal ideal $m = (\pi)$ and residue field $k = \mathbb{C} \subset R$, there is a Cartesian diagram

$$\begin{tikzcd}
\mathcal{X}_K \arrow[r, hook, "i"] \arrow[d] & \mathcal{X}_R \arrow[d] & \mathcal{X}_k \arrow[l, hook', "j"'] \arrow[d] \\
\mathrm{Spec}(K) \arrow[r, hook] & \mathrm{Spec}(R) & \mathrm{Spec}(k) \arrow[l, hook']
\end{tikzcd}$$

where $\mathcal{X}_R = \mathcal{X} \times \mathrm{Spec}(R)$; $\mathcal{X}_K = \mathcal{X} \times \mathrm{Spec}(K)$; $\mathcal{X}_k = \mathcal{X} \times \mathrm{Spec}(k)$; $i : \mathcal{X}_K \hookrightarrow \mathcal{X}_R$ is the open immersion; $j : \mathcal{X}_k \hookrightarrow \mathcal{X}_R$ is the closed immersion.

Theorem B.1 *Suppose (E_K, ϕ_K) is a semistable Higgs bundle on $\mathcal{X}_K$ with characteristic polynomial $f_K \in \bigoplus_{i=1}^r H^0(\mathcal{X}_K, K^i_{\mathcal{X}_K})$. If f_K is the restriction of some $f_R \in \bigoplus_{i=1}^r H^0(\mathcal{X}_R, K^i_{\mathcal{X}_R})$ to $\mathcal{X}_K$, then there exists a family (E_R, ϕ_R) of Higgs bundles parametrized by* $\mathrm{Spec}(R)$ *such that:*

- *(E_K, ϕ_K) is the restriction of (E_R, ϕ_R) to $\mathcal{X}_K$.*
- *The characteristic polynomial of (E_R, ϕ_R) is f_R.*
- *The restriction of (E_R, ϕ_R) to $\mathcal{X}_k$ is a semistable Higgs bundle.*

If Theorem B.1 is true, then the proof of [Nit91, Theorem 6.1] also works in our case, which shows that Theorem 4.2 is true. The rest of this section is devoted to proving Theorem B.1. $\mathcal{X}$ has an open substack $\mathcal{X}^o$ such that it is a smooth irreducible curve over k. For convenience, we introduce the following notations: $\mathcal{X}^o_K = \mathcal{X}^o \times \mathrm{Spec}(K)$ and $\mathcal{X}^o_k = \mathcal{X}^o \times \mathrm{Spec}(k)$; $\beta_2 : \Xi \to \mathcal{X}^o_K$ is the generic point of $\mathcal{X}^o_K$ and ξ is the generic point of $\mathcal{X}^o_k$; $\mathcal{O}_\xi$ is the stalk of $\mathcal{O}_{\mathcal{X}^o_R}$ at ξ and $\beta_1 : \mathrm{Spec}(\mathcal{O}_\xi) \to \mathcal{X}^o_R$ is the natural morphism; $\alpha : \Xi \to \mathrm{Spec}(\mathcal{O}_\xi)$ is the open immersion and $\gamma : \mathcal{X}^o \hookrightarrow \mathcal{X}$ is the open immersion. Then, there is a Cartesian diagram

$$\begin{tikzcd}
\mathrm{Spec}(\mathcal{O}_\xi) \arrow[r, "\beta_1"] & \mathcal{X}^o_R \arrow[r, hook, "\gamma_R"] & \mathcal{X}_R \\
\Xi \arrow[u, hook, "\alpha"] \arrow[r, "\beta_2"] & \mathcal{X}^o_K \arrow[u, hook, "i^o"] \arrow[r, hook, "\gamma_K"] & \mathcal{X}_K. \arrow[u, hook, "i"']
\end{tikzcd}$$

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Lemma B.2 *Let (E_K, ϕ_K) be a Higgs bundle of rank r on $\mathcal{X}_K$. Suppose that M is a $(\phi_K)_\Xi$-invariant free rank r $\mathcal{O}_\xi$-submodule of $(\gamma_K^* E_K)_\Xi$ with $M \otimes_{\mathcal{O}_\xi} \mathcal{O}_\Xi = (\gamma_K^* E_K)_\Xi$. Then, there exists a unique family (E_R, ϕ_R) of Higgs bundles parametrized by* $\mathrm{Spec}(R)$ *such that $E_R \subseteq i_* E_K$ and ϕ_R is the restriction to E_R of $i_* \phi_K$.*

Proof Using Lemma 3.3 in [Hua22], this lemma can be proved following the same steps as in the proof of [Nit91, Proposition 6.5]. ∎

Fixing a semistable Higgs bundle (E_K, ϕ_K) of rank r on $\mathcal{X}_K$, we can introduce the so-called Bruhat–Tits complex for it. Let $\mathfrak{M}$ be the set of all rank n free (ϕ_K)-invariant $\mathcal{O}_\xi$-submodules of $(E_K)_\Xi$. $\mathfrak{M}$ is not empty (see [Nit91, Lemma 6.6]). An equivalence relation $\sim$ on $\mathfrak{M}$ is given by: for $M \in \mathfrak{M}$, $M \sim \pi^p M$ for $p \in \mathbb{Z}$. By Lemma B.2, equivalent modules in $\mathfrak{M}$ induce isomorphic extensions of (E_K, ϕ_K) to $\mathcal{X}_R$. Let $\mathfrak{Q}$ be the quotient set $\mathfrak{M}/\sim$. We can define a structure of an r-dimensional simplicial complex on $\mathfrak{Q}$. $\mathfrak{Q}$ with the simplicial complex structure is called the *Bruhat–Tits complex*. Two equivalent classes $[M]$ and $[M']$ are said to be adjacent if M has a direct decomposition $M = N \oplus P$ such that $M' = N + \pi M$. In other words, $[M]$ and $[M']$ are adjacent if and only if M has a basis $\{e_1, e_2, \ldots, e_r\}$ over $\mathcal{O}_\xi$ such that $\{e_1, \ldots, e_s, \pi e_{s+1}, \ldots, \pi e_r\}$ is a basis of M' over $\mathcal{O}_\xi$. If $0 \subset N_1 \subset N_2 \subset \cdots \subset N_t \subset M$ is a sequence of submodules of M such that each N_i is a direct factor of M and $M_i = N_i + \pi M$ is $(\phi_K)_\Xi$-invariant, then the $t+1$ mutually adjacent vertices $[M], [M_1], \ldots, [M_t]$ form a t-simplex in $\mathfrak{Q}$. To prove Theorem B.1, we only need to find a vertex $[E_\xi]$ of $\mathfrak{Q}$ such that the reduction (E_k, ϕ_k) of the corresponding extension (E_R, ϕ_R) is semistable.

Proposition B.3 *Suppose that $[E_\xi]$ is a vertex in $\mathfrak{Q}$ and (E_k, ϕ_k) is the restriction of the corresponding extension (E_R, ϕ_R) to $\mathcal{X}_k$. Then there is a one-to-one correspondence between edges in $\mathfrak{Q}$ at $[E_\xi]$ and proper ϕ_k-invariant subbundles of E_k. Furthermore, if $F \subseteq E_k$ is a ϕ_k-invariant subbundle corresponds to the edge $[E_\xi] - [E'_\xi]$ at $[E_\xi]$ and $Q' \subseteq E'_k$ is the ϕ'_k-invariant subbundle corresponds to the edge $[E'_\xi] - [E_\xi]$ at $[E'_\xi]$, then there is a homomorphism $(E_k, \phi_k) \to (E'_k, \phi'_k)$ of Higgs bundles with kernel F and image Q', and a homomorphism $(E', \phi'_k) \to (E_k, \phi_k)$ of Higgs bundles with kernel Q' and image F.*

Proof **Part 1.** Suppose that $E_\xi = (e_1, \ldots, e_r)$ represents the vertex $[E_\xi]$ and $E'_\xi = (e_1, \ldots, e_s, \pi e_{s+1}, \ldots, \pi e_r)$ represents an adjacent vertex. Since $E'_\xi \subseteq E_\xi$, there is an injection of the corresponding extensions

$$(E'_R, \phi'_R) \hookrightarrow (E_R, \phi_R) \tag{B.1}$$

(see [Hua22, Lemma 3.3]). Consider the exact sequence of Higgs bundles

$$0 \longrightarrow (E'_R, \phi'_R) \longrightarrow (E_R, \phi_R) \longrightarrow (Q, \overline{\phi}_k) \longrightarrow 0\,, \tag{B.2}$$

where $(Q, \overline{\phi}_k)$ is a Higgs bundle on $\mathcal{X}_k$ (see the proof of [Hua22, Proposition 3.6]). Restricting (B.2) to $\mathcal{X}_k$, we get an exact sequence

$$0 \longrightarrow (F, \phi_k|_F) \longrightarrow (E_k, \phi_k) \longrightarrow (Q, \overline{\phi}_k) \longrightarrow 0\,, \tag{B.3}$$

where F is the image of the restriction of (B.1) to $\mathcal{X}_k$. We therefore get a Higgs subbundle $(F, \phi_k|_F)$ of (E_k, ϕ_k). Conversely, if F is a ϕ_k-invariant subbundle of E_k and $Q = E_k/F$ is a bundle on $\mathcal{X}_k$, then we have an exact sequence of Higgs bundles

$$0 \longrightarrow (F, \phi_k|_F) \longrightarrow (E_k, \phi_k) \longrightarrow (Q, \overline{\phi}_k) \longrightarrow 0\,. \tag{B.4}$$

Composing the restriction $(E_R, \phi_R) \to (E_k, \phi_k)$ with the surjective morphism $(E_k, \phi_k) \to (Q, \overline{\phi}_k)$ in (B.4), we get a new surjective morphism $(E_R, \phi_R) \to (Q, \overline{\phi}_k)$, i.e., there is an exact sequence

$$0 \longrightarrow (E'_R, \phi'_R) \longrightarrow (E_R, \phi_R) \longrightarrow (Q, \overline{\phi}_k) \longrightarrow 0\,, \tag{B.5}$$

where ϕ'_R is the restriction of ϕ_R to E'_R. Consider the exact sequence of $\mathcal{O}_\xi$-modules

$$\begin{tikzcd} 0 \arrow[r] & (E'_R)_\xi \arrow[r] & (E_R)_\xi \arrow[rr] \arrow[dr] & & Q_\xi \arrow[r] & 0\,. \\ & & & (E_k)_\xi \arrow[ur] & & \end{tikzcd}$$

Suppose that $(E_k)_\xi$ is generated by $\{\overline{e}_1, \ldots, \overline{e}_r\}$ and $\{\overline{e}_1, \ldots, \overline{e}_s\}$ is a basis of F_ξ over $\mathcal{O}_{\mathcal{X}^o_k, \xi}$. Moreover, $\{\overline{e}_1, \ldots, \overline{e}_r\}$ lifts to a basis $\{e_1, \ldots, e_r\}$ of $(E_R)_\xi$ over $\mathcal{O}_\xi$. Then, $(E'_R)_\xi$ is generated by $\{e_1, \ldots, e_s, \pi e_{s+1}, \ldots, \pi e_r\}$ and $(E'_R)_\xi$ is ϕ_K-invariant. So, it represents a vertex $[E'_\xi]$ of $\mathbf{\Omega}$ adjacent to $[E_\xi]$.

Part 2. Since $\pi E_\xi \subseteq E'_\xi$, there is another injection $(\pi E_R, \phi_R|_{\pi E_R}) \to (E_R, \phi_R)$. Composing it with the isomorphism $(E_R, \phi_R) \xrightarrow{\pi} (\pi E_R, \phi_R|_{\pi E_R})$, we get the injection

$$(E_R, \phi_R) \hookrightarrow (E'_R, \phi'_R). \tag{B.6}$$

By (B.1) and (B.6), we have

$$(E'_R, \phi'_R) \hookrightarrow (E_R, \phi_R) \hookrightarrow (E'_R, \phi'_R)\,, \tag{B.7}$$

$$(E_R, \phi_R) \hookrightarrow (E'_R, \phi'_R) \hookrightarrow (E_R, \phi_R)\,. \tag{B.8}$$

The restriction of (B.7) to the special fiber $\mathcal{X}_k$ is

$$(E'_k, \phi'_k) \longrightarrow (E_k, \phi_k) \longrightarrow (E'_k, \phi'_k). \tag{B.9}$$

The composition of the two morphisms in (B.9) is zero. In fact, the composition of

$$(E'_k)_\xi \longrightarrow (E_k)_\xi \longrightarrow (E'_k)_\xi \tag{B.10}$$

is zero and E'_k is torsion-free. Obviously, the sequence (B.10) is exact at the middle term. By [Hua22, Proposition 2.23], the sequence (B.9) is exact at the middle term. Similarly, restricting (B.8) to $\mathcal{X}_k$, we get

$$(E_k, \phi_k) \longrightarrow (E'_k, \phi'_k) \longrightarrow (E_k, \phi_k). \tag{B.11}$$

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We can also show that (B.11) is exact at the middle term. Therefore, we have the following exact sequence:

$$0 \longrightarrow (Q, \overline{\phi}_k) \longrightarrow (E'_k, \phi'_k) \longrightarrow (F, \phi_k|_F) \longrightarrow 0\,. \tag{B.12}$$ ■

Definition B.4 Let E be a locally free sheaf with modified Hilbert polynomial $P_E(m) = a_1 \cdot m + a_0$ on $\mathcal{X}$. For every locally free sheaf E_1 on $\mathcal{X}$, we define the $\boldsymbol{\beta}$*-invariant* $\beta(E_1)$ of E_1 with respect to E as follows: $\beta(E_1) = a_1 \cdot a_0(E_1) - a_0 \cdot a_1(E_1)$, where $P_{E_1}(m) = a_1(E_1) \cdot m + a_0(E_1)$ is the modified Hilbert polynomial of E_1.

Remark B.5 (E, ϕ) is semistable if and only if $\beta(F) \leq 0$ for all ϕ-invariant subsheaf $F \subseteq E$.

Recall some properties of β-invariants (see [Hua22, Proposition 2.27]).

Proposition B.6 (i) *If E_1 and E_2 are two ϕ-invariant subsheaves of locally free sheaf E on $\mathcal{X}$, then*

$$\beta(E_1) + \beta(E_2) \leq \beta(E_1 \vee E_2) + \beta(E_1 \cap E_2),$$

with equality if and only if $E_1 \vee E_2 = E_1 + E_2$.

(ii) *If* $0 \longrightarrow F \longrightarrow G \longrightarrow K \longrightarrow 0$ *is an exact sequence of locally free sheaves on $\mathcal{X}$, then $\beta(F) + \beta(K) = \beta(G)$.*

Proposition B.7 *For a Higgs bundle (E, ϕ) on $\mathcal{X}$, there exists a unique ϕ-invariant proper subsheaf $B \subset E$ such that:*

(i) *For every ϕ-invariant subsheaf G of B with* $\mathrm{rk}(G) < \mathrm{rk}(B)$, *we have $\beta(G) < \beta(B)$.*
(ii) *For every ϕ-invariant subsheaf H of E, we have $\beta(H) \leq \beta(B)$.*

Proof The claim can be proved following the same steps as in the proof of Proposition 2.31 in [Hua22] ■

If the Higgs bundle (E, ϕ) is unstable, then the ϕ-invariant subsheaf B in the above proposition, will be called the $\boldsymbol{\beta}$*-subbundle* of (E, ϕ). Now, assume that we are given a vertex $[E_\xi]$ of $\mathfrak{Q}$ such that the corresponding Higgs bundle (E_k, ϕ_k) on $\mathcal{X}_k$ is unstable. Let $B \subset E_k$ be the β-subbundle of (E_k, ϕ_k). Thus, $\beta(B) > 0$ (See Proposition B.7). By Proposition B.3, there is an edge in $\mathfrak{Q}$ at $[E_\xi]$ corresponding to B. Let $[E_\xi^{(1)}]$ be the vertex in $\mathfrak{Q}$ determined by the edge corresponding to B, and let $(E_k^{(1)}, \phi_k^{(1)})$ be the corresponding Higgs bundle on $\mathcal{X}_k$. Let $F_1 \subseteq E_k^{(1)}$ be the image of the canonical homomorphism $E_k \to E_k^{(1)}$(= the kernel of the homomorphism $E_k^{(1)} \to E_k$).

Following similar steps as in the proof of [Lan75, Lemma 1], we can show the following lemma:

Lemma B.8 *If $G \subset E_k^{(1)}$ is a $\phi_k^{(1)}$-invariant subbundle of $E_k^{(1)}$, then $\beta(G) \leq \beta(B)$, with equality possible only if $G + F_1 = E_k^{(1)}$.*

Now, we are going to define a path $\mathcal{P}$ in $\mathfrak{Q}$, starting with a vertex $[E_\xi]$, whose corresponding Higgs bundle (E_k, ϕ_k) is unstable. The succeeding vertex is the vertex determined by the edge corresponding to the β-subbundle B of (E_k, ϕ_k). If $\mathcal{P}$ reaches

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a vertex $[E_\xi^{(m)}]$ such that the corresponding Higgs bundle $(E_k^{(m)}, \phi_k^{(m)})$ is semistable, then the process stops automatically and Theorem B.1 is proved. If the path $\mathcal{P}$ never reaches a vertex corresponding to a semistable reduction, then the process continuous indefinitely. We have to show that the second alternative is impossible.

Denote the β-subbundle of $(E_k^{(m)}, \phi_k^{(m)})$ by $B^{(m)}$, and let $\beta_m = \beta(B^{(m)})$. By Lemma B.8, $\beta_{m+1} \leq \beta_m$ and we must have $\beta_m > 0$ unless $E_k^{(m)}$ is semistable. Thus, if the path $\mathcal{P}$ is continuous indefinitely, we have $\beta_m = \beta_{m+1} = \cdots$, for sufficiently large m. Also, by Lemma B.8, for sufficiently large m, $B^{(m)} + F^{(m)} = E_k^{(m)}$, where $F^{(m)} = \mathrm{Im}(E_k^{(m-1)} \to E_k^{(m)})$ $(\mathrm{Ker}(E_k^{(m)} \to E_k^{(m-1)}))$. So, $\mathrm{rank}(B^{(m)}) + \mathrm{rank}(F^{(m)}) \geq r$. On the other hand, $\mathrm{rank}(B^{(m-1)}) + \mathrm{rank}(F^{(m)}) = r$. Therefore, $\mathrm{rank}(B^{(m)}) \geq \mathrm{rank}(B^{(m-1)})$, for sufficiently large m. Since $\mathrm{rank}(B^{(m)}) \leq r$, we must have $\mathrm{rank}(B^{(m)}) = \mathrm{rank}(B^{(m+1)}) = \cdots$, for sufficiently large m. Thus, $\mathrm{rank}(B^{(m)}) + \mathrm{rank}(F^{(m)}) = r$. So, $B^{(m)} \cap F^{(m)} = 0$ and $B^{(m)} \oplus F^{(m)} = E_k^{(m)}$. Consequently, the canonical homomorphism $E_k^{(m)} \to E_k^{(m-1)}$ induces isomorphism $B^{(m)} \to B^{(m-1)}$. Also, the canonical homomorphism $E_k^{(m-1)} \to E_k^{(m)}$ induces isomorphism $F^{(m-1)} \to F^{(m)}$. If R is a complete discrete valuation ring, the following lemma leads us to a contradiction.

Lemma B.9 *Assume that the discrete valuation ring R is complete and $\mathcal{P}$ is an infinite path in $\mathfrak{Q}$ with vertices $[E_\xi]$, $[E_\xi^{(1)}]$, $[E_\xi^{(2)}]$, ... Let $F^{(m)} = Im(E_k^{(m+1)} \to E_k^{(m)})$. If the canonical homomorphism $E^{(m+1)} \to E^{(m)}$ induces isomorphism $F^{(m+1)} \to F^{(m)}$ for every m, then $\beta(F) \leq 0$.*

Proof The lemma can be checked step by step as Lemma 6.11 in [Nit91]. ■

Hence, Theorem B.1 is proved under the assumption that R is complete. The general case can be proved as [Nit91].

C Spectral construction

In this subsection, we recall the spectral construction. Suppose that $\mathcal{X}$ is a hyperbolic Deligne–Mumford curve and $\psi : \mathrm{Tot}(K_{\mathcal{X}}) \to \mathcal{X}$ is the natural projection. For a Higgs bundle (E, ϕ) on $\mathcal{X}$, the Higgs field ϕ defines a morphism of $\mathcal{O}_{\mathcal{X}}$-algebras $\mathrm{Sym}^\bullet(K_{\mathcal{X}}^\vee) \to \mathcal{E}nd_{\mathcal{O}_{\mathcal{X}}}(E)$. Then, E is endowed with an $\mathrm{Sym}^\bullet(K_{\mathcal{X}}^\vee)$-module structure. It defines a compactly supported $\mathcal{O}_{\mathrm{Tot}(K_{\mathcal{X}})}$-module E_ϕ over $\mathrm{Tot}(K_{\mathcal{X}})$. Moreover, E_ϕ is a pure sheaf of dimension one (see Proposition C.1). Conversely, if F is a compactly supported pure sheaf of dimension one on $\mathrm{Tot}(K_{\mathcal{X}})$, then there is a Higgs bundle (E, ϕ) on $\mathcal{X}$ such that $E_\phi = F$, where $E = \psi_*(F)$ and ϕ is defined by the tautological section of $\psi^* K_{\mathcal{X}}$. There is an equivalence of two categories

$$\mathbf{Higgs}(\mathcal{X}) \simeq \mathbf{Coh}_c(\mathrm{Tot}(K_{\mathcal{X}})), \tag{C.1}$$

where $\mathbf{Higgs}(\mathcal{X})$ is the category of Higgs sheaves on $\mathcal{X}$ and $\mathbf{Coh}_c(\mathrm{Tot}(K_{\mathcal{X}}))$ is the category of compactly supported coherent sheaves on $\mathrm{Tot}(K_{\mathcal{X}})$ (see [JK21, Proposition 2.18]).

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Proposition C.1 *The equivalence (C.1) gives a one-to-one correspondence between Higgs bundles on $\mathcal{X}$ and compactly supported pure sheaves of dimension one on* $\mathrm{Tot}(K_{\mathcal{X}})$.

Proof The conclusion of this proposition can be proved locally in étale topology as Proposition 2.18 in [JK21]. ∎

As Remark 3.7 in [BNR89], we have the following proposition.

Proposition C.2 *Suppose that $f : \mathcal{X}_{\boldsymbol{a}} \to \mathcal{X}$ is an integral spectral curve and (E, ϕ) is a rank r Higgs bundle on $\mathcal{X}$ with spectral curve $\mathcal{X}_{\boldsymbol{a}}$. Then, the rank one torsion-free sheaf E_ϕ on $\mathcal{X}_{\boldsymbol{a}}$ corresponding to (E, ϕ) satisfies*

$$0 \longrightarrow E_\phi \otimes f^* K_{\mathcal{X}}^{1-r} \longrightarrow f^*E \xrightarrow{f^*\phi - \tau} f^*(E \otimes K_{\mathcal{X}}) \longrightarrow E_\phi \otimes f^* K_{\mathcal{X}} \longrightarrow 0\ , \tag{C.2}$$

where τ is the restriction of the tautological section of $\psi^* K_{\mathcal{X}}$ to $\mathcal{X}_{\boldsymbol{a}}$.

Proof Consider the total space of the canonical line bundle $\psi : \mathrm{Tot}(K_{\mathcal{X}}) \to \mathcal{X}$. Similar to [TT20, Proposition 2.11], it is easy to show that there is an exact sequence

$$0 \longrightarrow \psi^*E \xrightarrow{\psi^*\phi - \tau} \psi^*(E \otimes K_{\mathcal{X}}) \longrightarrow E_\phi \otimes \psi^* K_{\mathcal{X}} \longrightarrow 0 \tag{C.3}$$

on $\mathrm{Tot}(K_{\mathcal{X}})$, where τ is the tautological section of $\psi^* K_{\mathcal{X}}$. On the other hand, there is an exact sequence

$$0 \longrightarrow \psi^* K_{\mathcal{X}}^{-r} \longrightarrow \mathcal{O}_{\mathrm{Tot}(K_{\mathcal{X}})} \longrightarrow \mathcal{O}_{\mathcal{X}_{\boldsymbol{a}}} \longrightarrow 0\ . \tag{C.4}$$

Then, we have the commutative diagram

$$\begin{tikzcd}
& 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & \psi^*(E \otimes K_{\mathcal{X}}^{-r}) \arrow[r] \arrow[d] & \psi^*(E \otimes K_{\mathcal{X}}^{1-r}) \arrow[r] \arrow[d] & E_\phi \otimes \psi^* K_{\mathcal{X}}^{1-r} \arrow[r] \arrow[d, "0"'] & 0 \\
0 \arrow[r] & \psi^*E \arrow[r] \arrow[d] & \psi^*(E \otimes K_{\mathcal{X}}) \arrow[r] \arrow[d] & E_\phi \otimes \psi^* K_{\mathcal{X}} \arrow[r] \arrow[d, "="'] & 0 \\
& \psi^*E|_{\mathcal{X}_a} \arrow[r] \arrow[d] & \psi^*(E \otimes K_{\mathcal{X}})|_{\mathcal{X}_a} \arrow[r] \arrow[d] & E_\phi \otimes \psi^* K_{\mathcal{X}}|_{\mathcal{X}_a} \arrow[r] \arrow[d] & 0 \\
& 0 & 0 & 0. &
\end{tikzcd}$$

By diagram chasing, we have the exact sequence

$$0 \longrightarrow E_\phi \otimes f^* K_{\mathcal{X}}^{1-r} \longrightarrow f^*E \xrightarrow{f^*\phi - \tau} f^*(E \otimes K_{\mathcal{X}}) \longrightarrow E_\phi \otimes f^* K_{\mathcal{X}} \longrightarrow 0\ . \qquad ∎$$

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