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Toric Geometry of $SL_2(\mathbb{C})$ Free Group Character Varieties from Outer Space

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Abstract. Culler and Vogtmann defined a simplicial space $O(g)$, called *outer space*, to study the outer automorphism group of the free group F_g. Using representation theoretic methods, we give an embedding of $O(g)$ into the analytification of $\mathcal{X}(F_g, SL_2(\mathbb{C}))$, the $SL_2(\mathbb{C})$ character variety of F_g, reproving a result of Morgan and Shalen. Then we show that every point v contained in a maximal cell of $O(g)$ defines a flat degeneration of $\mathcal{X}(F_g, SL_2(\mathbb{C}))$ to a toric variety $X(P_\Gamma)$. We relate $\mathcal{X}(F_g, SL_2(\mathbb{C}))$ and $X(v)$ topologically by showing that there is a surjective, continuous, proper map $\Xi_v: \mathcal{X}(F_g, SL_2(\mathbb{C})) \to X(v)$. We then show that this map is a symplectomorphism on a dense open subset of $\mathcal{X}(F_g, SL_2(\mathbb{C}))$ with respect to natural symplectic structures on $\mathcal{X}(F_g, SL_2(\mathbb{C}))$ and $X(v)$. In this way, we construct an integrable Hamiltonian system in $\mathcal{X}(F_g, SL_2(\mathbb{C}))$ for each point in a maximal cell of $O(g)$, and we show that each v defines a topological decomposition of $\mathcal{X}(F_g, SL_2(\mathbb{C}))$ derived from the decomposition of $X(P_\Gamma)$ by its torus orbits. Finally, we show that the valuations coming from the closure of a maximal cell in $O(g)$ all arise as divisorial valuations built from an associated projective compactification of $\mathcal{X}(F_g, SL_2(\mathbb{C}))$.

1 Introduction

In their seminal paper [23], Morgan and Shalen introduced a piecewise-linear compactification construction for any complex affine variety V, where the points at infinity correspond to valuations on the coordinate ring of V. By applying this construction to the $SL_2(\mathbb{C})$ character varieties of surface fundamental groups, Morgan and Shalen were able to produce Thurston's piecewise linear compactification of Teichmüller spaces [32]. As part of their program, they showed that isometric actions of the free group F_g on metric trees are limit points of the character variety $\mathcal{X}(F_g, SL_2(\mathbb{C}))$, and consequently can be understood as equivalence classes of valuations on the coordinate ring $\mathbb{C}[\mathcal{X}(F_g, SL_2(\mathbb{C}))]$ (Allesandrini [1] also has an account of this construction). We explore an alternative construction of such a complex of valuations that uses a compactification of $SL_2(\mathbb{C})$ and Vinberg's enveloping monoid as a centerpiece (see [13, 33, 34] and Section 4). Built into this method is a way to directly relate the symplectic geometry of $\mathcal{X}(F_g, SL_2(\mathbb{C}))$ to that of the flat degeneration $X(P_\Gamma)$ associated with any valuation we construct. Generally, the flat degenerations $X(P_\Gamma)$ are toric, so we are able to create a dense open integrable system with globally defined continuous momentum maps in $\mathcal{X}(F_g, SL_2(\mathbb{C}))$ for each maximal cell in the complex of valuations we consider (this construction should be compared to the main result of Harada

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and Kaveh in [11]). In short, from the perspectives of both algebraic and symplectic geometry, $\mathcal{X}(F_g, SL_2(\mathbb{C}))$ is "almost" a toric variety in many ways.

Our first main result is a reformulation of Morgan and Shalen's construction of a valuation on $\mathbb{C}[\mathcal{X}(F_g, SL_2(\mathbb{C}))]$ from an isometric F_g action on a metric tree. In order to state it we recall spaces from geometric group theory and the theory of Berkovich analytification. In [6], Culler and Vogtmann introduced a space $O(g)$ with an action of the outer automorphism group $\mathrm{Out}(F_g)$ called *outer space*. Points of $O(g)$ are graphs Γ with no leaves and first Betti number equal to g, along with choice of metric ℓ and a "marking" (see Section 2), which provides an isomorphism $\phi : \pi(\Gamma) \cong F_g$ (see [6, 35] and Section 2). The metric is subject to a normalization condition, stipulating that the sum of the lengths of the edges of Γ must be 1. Culler and Vogtmann show that $O(g)$ has a simplicial structure, with a simplicial action by $\mathrm{Out}(F_g)$ and use this to prove that $\mathrm{Out}(F_g)$ has a number of group theoretic properties (see [35] for a survey of these results). Points in $O(g)$ can be viewed as isometric actions of F_g on trees by replacing (Γ, ℓ) with its universal cover, a metric tree with an isometric action of F_g defined by the isomorphism ϕ. This places $O(g)$ in the context studied by Morgan and Shalen in [23]. We drop the normalization condition on the metric and consider the resulting complex of simplicial cones, $\widehat{O}(g)$, referred to as the cone over outer space.

Each reduced word $\omega \in F_g$ defines a continuous function $d_\omega : \widehat{O}(g) \to \mathbb{R}$, where $d_\omega(\Gamma, \ell, \phi)$ is the length of the minimal length path in (Γ, ℓ) which represents ω in $\pi_1(\Gamma)$. Reduced words $\omega \in F_g$ also define regular functions $tr_\omega \in \mathbb{C}[\mathcal{X}(F_g, SL_2(\mathbb{C}))]$ called *trace-word functions*; see Sections 2 and 6. A point in $\mathcal{X}(F_g, SL_2(\mathbb{C}))$ is an $SL_2(\mathbb{C})$ representation of F_g, this is a choice of g matrices $A_1, \dots, A_g \in SL_2(\mathbb{C})$. Two choices of matrices define the same representation if and only if they are related by simultaneous conjugation by an element of $SL_2(\mathbb{C})$. More precisely, $\mathcal{X}(F_g, SL_2(\mathbb{C}))$ is defined as a Geometric Invariant Theory (GIT) quotient.

$$\mathcal{X}\big(F_g, SL_2(\mathbb{C})\big) = \big[SL_2(\mathbb{C}) \times \cdots \times SL_2(\mathbb{C})\big] /\!/ SL_2(\mathbb{C})$$

The function tr_ω is computed on

$$[A_1, \dots, A_g] \in \mathcal{X}\big(F_g, SL_2(\mathbb{C})\big)$$

by evaluating $\omega(A_1, \dots, A_g)$ and taking the trace of the resulting matrix.

The Berkovich analytification V^{an} of an affine variety V (see [3, 27] and Section 3) is a Hausdorff topological space composed of all the rank 1 valuations on the coordinate ring of V. Every regular function $f \in \mathbb{C}[V]$ defines a function on the analytification $ev_f : V^{an} \to \mathbb{R}$ called the evaluation function; it is computed by taking a valuation v to $v(f)$. The topology on V^{an} is the weakest one that makes the evaluation functions continuous.

Theorem 1.1 *There is an embedding* $\Sigma : \widehat{O}(g) \to \mathcal{X}(F_g, SL_2(\mathbb{C}))^{an}$*; furthermore, for any reduced word* $\omega \in F_g$, $ev_{tr_\omega} \circ \Sigma = d_\omega$.

Theorem 1.1 is analogous to [31, Theorem 3.4], which identifies the tropical variety of the Grassmannian variety $Gr_2(\mathbb{C}^n)$ with the space of phylogenetic trees with

n ordered leaves. In [20, Proposition 3.1], the author uses combinatorial and representation theoretic methods to construct an embedding of the space of metric trees into the analytification $Gr_2(\mathbb{C}^n)^{an}$. We use similar methods to give a new proof of Theorem 1.1, employing a theory of valuations stemming from compactifications of $SL_2(\mathbb{C})$, Vinberg's remarkable enveloping monoid construction, and a noncommutative quiver variety construction.

In [7], Florentino and Lawton study varieties $M_\Gamma(G)$ for G a reductive group, built as noncommutative analogues of quiver varieties. They show that each $M_\Gamma(G)$ for Γ a directed graph with first Betti number equal to g is isomorphic to $\mathcal{X}(F_g, G)$ (see also [21]). We extend this construction to a functor $M_-(SL_2(\mathbb{C}))$ from the category of graphs with certain finite cellular topological maps to the category of complex schemes. We show that an isomorphism $\phi\colon \pi_1(\Gamma) \to F_g$ defines an isomorphism $M_\Gamma(SL_2(\mathbb{C})) \cong \mathcal{X}(F_g, SL_2(\mathbb{C}))$, and that the variety $M_\Gamma(SL_2(\mathbb{C}))$ comes with a naturally defined simplicial cone C_Γ of valuations on its coordinate ring. The pullbacks $C_{\Gamma,\phi}$ of these cones in $\mathcal{X}(F_g, SL_2(\mathbb{C}))^{an}$ paste together to give the embedding Σ.

The graph Γ also plays a fundamental role in our analysis of the coordinate ring of $\mathcal{X}(F_g, SL_2(\mathbb{C}))$. For a trivalent directed graph Γ, a spin diagram of topology Γ is an assignment of non-negative integers $a\colon E(\Gamma) \to \mathbb{Z}_{\geq 0}$ to the edges of Γ that satisfy the inequalities of a polyhedral cone $\mathcal{P}_\Gamma$ and belong to a certain lattice $L_\Gamma \subset \mathbb{R}^{E(\Gamma)}$ (see Figure 1 and Section 6).

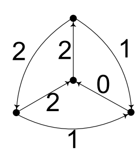

Figure 1: A spin diagram.

There is a basis of regular functions $\Phi_a, a \in P_\Gamma = \mathcal{P}_\Gamma \cap L_\Gamma$ in $\mathbb{C}[\mathcal{X}(F_g, SL_2(\mathbb{C}))]$ for each marked Γ that does not depend on the directed structure on Γ (see Section 6). These "spin diagram" functions have a rich combinatorial theory, and have been studied before by Lawton and Peterson [17]. Spin diagrams are a beautiful combinatorial tool in topological field theory and knot theory; see [2] for a survey of some of these topics.

The set P_Γ is an affine semigroup with associated affine semigroup algebra $\mathbb{C}[P_\Gamma]$. For $a, b \in P_\Gamma$ the product $[a][b] \in \mathbb{C}[P_\Gamma]$ is computed naively, $[a][b] = [a+b]$. We let $X(P_\Gamma)$ be the affine toric variety $\mathrm{Spec}(\mathbb{C}[P_\Gamma])$. Our second main theorem says that the associated graded multiplication operation in $\mathbb{C}[\mathcal{X}(F_g, SL_2(\mathbb{C}))]$ with respect to the filtration defined by the valuation $\Sigma(\Gamma, \ell, \phi) = v_{\Gamma,\ell,\phi}$ is this naive multiplication operation.

Theorem 1.2 *The associated graded algebra* $gr_{v_{\Gamma,\ell,\phi}}(\mathbb{C}[\mathcal{X}(F_g, SL_2(\mathbb{C}))])$ *is isomorphic to* $\mathbb{C}[P_\Gamma]$.

Let $S_{2,g} \subset \mathbb{C}[\mathcal{X}(F_g, SL_2(\mathbb{C}))]$ be the set of traceword functions τ_w for reduced words w in which any letter appears at most twice, counting inverses and multiplicity. In Section 6 we use Theorem 1.2 to show that the functions $S_{2,g}$ generate $\mathbb{C}[\mathcal{X}(F_g, SL_2(\mathbb{C}))]$ and that the tropical variety $\mathbb{T}(I_{2,g})$ of the ideal of forms that vanish on $S_{2,g}$ contains the polyhedral complex Υ_g of "metric spanned graphs" with first Betti number g (see Subsection 2.3 and Theorem 6.21).

Next we analyze a relationship between $\mathcal{X}(F_g, SL_2(\mathbb{C}))$ and $X(P_\Gamma)$ in symplectic geometry. We relate $\mathcal{X}(F_g, SL_2(\mathbb{C}))$ to $X(P_\Gamma)$ topologically by constructing a continuous map $\Xi_\Gamma: \mathcal{X}(F_g, SL_2(\mathbb{C})) \to X(P_\Gamma)$ in Section 7. We recall that as a toric variety, $X(P_\Gamma)$ is stratified by toric subvarieties $X(F)$ for $F \subset P_\Gamma$ the affine semigroups of $\mathcal{L}_\Gamma$ points in a face $\mathcal{F} \subset \mathcal{P}_\Gamma$."

Theorem 1.3 *The map Ξ_Γ is surjective, continuous, and proper. If a face $\mathcal{F} \subset \mathcal{P}_\Gamma$ is not contained in a coordinate hyperplane of $\mathbb{R}^{E(\Gamma)}$, then $\Xi_\Gamma: \Xi_\Gamma^{-1}(X(F)) \to X(F)$ is a homeomorphism. If $\mathcal{F}$ is contained in a coordinate hyperplane of $\mathbb{R}^{E(\Gamma)}$, then $\Xi_\Gamma: \Xi_\Gamma^{-1}(X(F)) \to X(F)$ has connected fibers, and if $\mathcal{F}$ is the origin of $\mathbb{R}^{E(\Gamma)}$, then the fiber of this bundle is the compact character variety $\mathcal{X}(F_g, SU(2))$.*

In Section 7 we place symplectic structures on both $\mathcal{X}(F_g, SL_2(\mathbb{C}))$ and $X(P_\Gamma)$, derived from the symplectic form on $SL_2(\mathbb{C})$ obtained from an identification with the cotangent bundle $T^*(SU(2))$.

Theorem 1.4 *The map Ξ_Γ is a symplectomorphism on a dense, open subset of $\mathcal{X}(F_g, SL_2(\mathbb{C}))$.*

The map Ξ_Γ transfers the integrable system defined by the open torus orbit of $X(P_\Gamma)$ to the character variety $\mathcal{X}(F_g, SL_2(\mathbb{C}))$. This result makes use of recent work by Hilgert, the author, and Martens [13] on the symplectic geometry of Vinberg's enveloping monoid construction.

Our last theorem shows that the valuations $v_{\Gamma,\ell,\phi}$ come from divisorial valuations on a compactification of $\mathcal{X}(F_g, SL_2(\mathbb{C}))$. We bring in a compactification X of the group $SL_2(\mathbb{C})$ and use it to build a space $M_\Gamma(X)$ by carrying out the the construction $M_\Gamma(-)$ for the $SL_2(\mathbb{C}) \times SL_2(\mathbb{C})$ space X; see Section 8.

Theorem 1.5 *For each isomorphism $\phi: \pi_1(\Gamma) \to F_g$, there is a compactification of $\mathcal{X}(F_g, SL_2(\mathbb{C}))$ by the projective scheme $M_\Gamma(X)$. The boundary divisor of this compactification is of combinatorially normal crossings type, and the divisorial valuations of its irreducible components generate the extremal rays of the cone $C_{\Gamma,\phi}$.*

In Section 8 we extend the toric degenerations of Theorem 1.2 to the compactification $M_\Gamma(X)$, obtaining a degeneration $M_\Gamma(X) \Rightarrow X(Q_\Gamma)$ to a toric variety associated with a polytope $\mathcal{Q}_\Gamma \subset \mathcal{P}_\Gamma$. The strata of the boundary divisor in Theorem 1.5 are shown to degenerate to toric varieties corresponding to the faces of $\mathcal{Q}_\Gamma$.

A degeneration $\mathbb{C}[\mathcal{X}(F_g, SL_2(\mathbb{C}))] \Rightarrow \mathbb{C}[P_\Gamma]$ was studied by the author in [22] in the context of the theory of Newton–Okounkov bodies (see [15]). In particular,

$\mathbb{C}[P_\Gamma]$ was realized as the associated graded algebra of $\mathbb{C}[\mathcal{X}(F_g, SL_2(\mathbb{C}))]$ with respect to a full rank (*i.e.*, dimension of $\mathcal{X}(F_g, SL_2(\mathbb{C}))$) valuation. With Theorem 8.12 we show that this maximal rank valuation can be defined using the boundary divisor of $M_\Gamma(X)$.

Theorems 1.3 and 1.4 are inspired by earlier work of the author with Howard and Millson [12] and the recent work of Harada and Kaveh [11] and Nishinou, Nohara, and Ueda [26]. In [11], the authors use a construction of Ruan [29] to show that a degeneration of a smooth, projective variety $Y \subset \mathbb{P}^N$ to a toric variety $Y(C) \subset \mathbb{P}^N$ guarantees the existence of a surjective, continuous, proper map $\Phi: Y \to Y(C)$ that is a symplectomorphism on a dense, open subset of Y with respect to the standard Kähler form on $\mathbb{P}^N$; this defines a dense, open integrable system in Y with momentum image C. Theorems 1.1 and 1.4 combine to produce an explicitly computable map for the variety $\mathcal{X}(F_g, SL_2(\mathbb{C}))$, which is notably both affine and singular in general. The combinatorial organization of the toric degenerations of $\mathcal{X}(F_g, SL_2(\mathbb{C}))$ in this paper are remeniscient of the work of Gross, Hacking, Keel, and Kontsevich [8] on cluster algebras. In the spirit of their work and this paper, it would be interesting to see an explicit construction of an integrable system in a cluster variety for each choice of seed in the associated cluster algebra.

In [21] the author shows that the character variety $\mathcal{X}(F_g, SL_2(\mathbb{C}))$ is a dense open subspace in a degeneration $M_{C_\Gamma}(SL_2(\mathbb{C}))$ of the moduli space of semistable $SL_2(\mathbb{C})$ principal bundles on a smooth curve. In particular, the projective varietes $M_\Gamma(SL_2(\mathbb{C}))$ and $M_{C_\Gamma}(SL_2(\mathbb{C}))$ are birational. We note that Theorem 8.12 and results from [21] then imply that the toric degenerations of $M_{C_\Gamma}(SL_2(\mathbb{C}))$ studied in [19,21] emerge from the Newton–Okounkov body construction defined by the boundary divisor in $M_\Gamma(SL_2(\mathbb{C}))$; see Subsection 8.4.

2 Metric Graphs and Outer Space

In this section we review material on metric graphs and cellular maps, and we recall Culler and Vogtmann's ([6, 35]) definition of $O(g)$ and the length functions

$$d_\omega: \widehat{O}(g) \longrightarrow \mathbb{R}.$$

2.1 Graph Notation and $\widehat{O}(g)$

Throughout the paper Γ denotes a graph with edge set $E(\Gamma)$, vertex set $V(\Gamma)$, and leaf set $L(\Gamma)$. We require all non-leaf vertices $v \in V(\Gamma)$ to have valence at least 3. Many of our constructions make use of an orientation on a graph; this is an ordering $\delta(e) = (u, v)$ of the endpoints of each edge $e \in E(\Gamma)$. An orientation also defines a partition $\epsilon(v) = i(v) \cup o(v)$ of the edges that contain a given vertex into incoming and outgoing edges. For any vertex $v \in V(\Gamma)$, we let the link $\Gamma_v \subset \Gamma$ be the subgraph induced by those edges in $\epsilon(v)$. For each genus g we fix a distinguished graph Γ_g with a single vertex and no leaves.

The points of both outer space $O(g)$ and the cone over outer space $\widehat{O}(g)$ are represented by marked $\mathbb{R}$-graphs. Topologically, graphs are considered as finite CW complexes; in particular, edges are 1-cells, and vertices are 0-cells. An $\mathbb{R}$-graph structure on a graph Γ is a metric that makes each edge locally isometric to a bounded interval in $\mathbb{R}$, such that the distance between any two vertices is the length of the shortest edge-path that connects them. A continuous map $\psi\colon \Gamma_g \to \Gamma$ is called a *marking* if $\pi_1(\psi)$ is an isomorphism of fundamental groups. Two markings $\psi_1, \psi_2\colon \Gamma_g \to \Gamma$ are said to be equivalent if there is an isometry $h\colon \Gamma \to \Gamma$ so that the diagram in Figure 2 commutes up to free homotopy (see [6, Section 0]).

$$\begin{tikzcd} \Gamma \arrow[rr, "h"] & & \Gamma \\ & \Gamma_g \arrow[ul, "\psi_1"] \arrow[ur, "\psi_2"'] & \end{tikzcd}$$

Figure 2: Equivalence of markings

From now on, a map of graphs $\phi\colon \Gamma \to \Gamma'$ is taken to be a finite cellular map on the underlying topological spaces. In particular, for any edge $e \in E(\Gamma)$ with $\delta(e) = (u, v)$, there is a finite subset of points $S \subset e$ (considered as a compact interval) such that the sub interval between two consecutive members $s, t \in S$ maps homeomorphically to an edge in $e' \in E(\Gamma')$ through π, with s and t mapping to the endpoints of e'. Any such $\pi\colon \Gamma \to \Gamma'$ defines a set map on vertices $\pi_v\colon V(\Gamma) \to V(\Gamma')$.

Culler and Vogtmann observe that every marking can be represented with a finite cellular map $\psi\colon \Gamma_g \to \Gamma$ [6, 1.2] as follows. For a spanning tree $\mathcal{T} \subset \Gamma$, collapsing $\mathcal{T}$ to a single vertex defines a homotopy equivalence $\psi_{\mathcal{T}}\colon \Gamma \to \Gamma_g$. Notice that for each $e_i \in E(\Gamma_g)$ there is a unique edge $\overline{e}_i \in E(\Gamma) \smallsetminus E(\mathcal{T})$ that maps to e_i under $\psi_{\mathcal{T}}$. An inverse $\phi_{\mathcal{T},V}$ to $\psi_{\mathcal{T}}$ can be constructed from a choice of vertex $V \in V(\Gamma)$ by sending e_i to the path formed by concatenating the unique path in $\mathcal{T}$ from V to the source endpoint of $\overline{e}_i$ with $\overline{e}_i$ and the unique path in $\mathcal{T}$ from the sink point in $\overline{e}_i$ back to V. Changing V to a different vertex induces an equivalent inverse map on free homotopy. We say that $\phi_{\mathcal{T},V}\colon \Gamma_g \to \Gamma$ is a distinguished map. Any outer automorphism α of F_g also defines a map on Γ_g by sending an edge e_i to the cellular path in Γ_g dictated by the image of the i-th generator of F_g under α.

Lemma 2.1 (Culler, Vogtmann) *Any marking $\phi\colon \Gamma_g \to \Gamma$ is equivalent to a composition of a distinguished marking $\phi_{\mathcal{T},V}\colon \Gamma_g \to \Gamma$ with a map $\alpha\colon \Gamma_g \to \Gamma_g$, where $\pi_1(\alpha)\colon F_g \to F_g$ is an outer automorphism.*

Proof For $\phi\colon \Gamma_g \to \Gamma$, choose a spanning tree $\mathcal{T} \subset \Gamma$ and consider $\psi_{\mathcal{T}} \circ \phi\colon \Gamma_g \to \Gamma_g$. Let $w\colon F_g \to F_g$ be the inverse of $\pi_1[\psi_{\mathcal{T}} \circ \phi]$, and let $\alpha_w\colon \Gamma_g \to \Gamma_g$ be the corresponding map. Then for any $V \in V(\Gamma)$, $\phi_{\mathcal{T},V} \circ \alpha_w^{-1}\colon \Gamma_g \to \Gamma$ represents the inverse to $\alpha_w \circ \phi_{\mathcal{T}}$ in homotopy, and is therefore equivalent to ϕ. ∎

As a set, the cone over outer space $\widehat{O}(g)$ is the collection of equivalence classes of markings of a metric graphs of genus g. The volume of a metric graph is defined in [6] to be the sum of the lengths of the edges. Outer space $O(g) \subset \widehat{O}(g)$ is then the set of those marked metric graphs with volume 1. For a fixed graph Γ, the collection of supported metrics forms a simplicial cone C_Γ equal to the positive orthant in $\mathbb{R}^{E(\Gamma)}$. The points on the boundary of C_Γ can be viewed as metrics on degenerations of Γ, where some edges with a 0 length are collapsed. Each marking ϕ defines a cone $C_{\phi,\Gamma} \subset \widehat{O}(g)$ isomorphic to C_Γ.

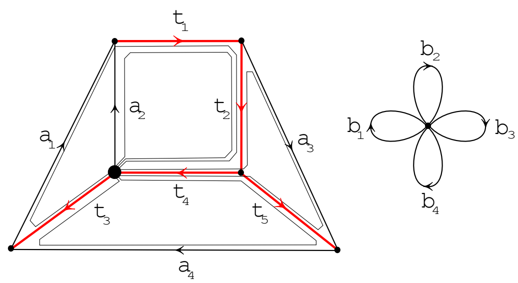

Figure 3: A spanning tree $\mathcal{T}$ in Γ. The associated maps ϕ, ψ are $\psi(a_i) = b_i, \psi(t_i) = 1, \phi(b_1) = t_3a_1t_1t_2t_4, \phi(b_2) = a_2t_1t_2t_4, \phi(b_3) = t_4^{-1}t_2^{-1}a_3t_5^{-1}t_4, \phi(b_4) = t_4^{-1}t_5a_4t_3^{-1}$.

2.2 Minimal Length Homotopy Classes and Length Functions

The cellular length of a path $e_1 \cdots e_n = \gamma \subset \Gamma$ is equal to the edge count n (note that this is different than the length assigned to this path by a metric). A backtrack in a path $\gamma \subset \Gamma$ is a sequence of edges of the form $e_1 \cdots e_k e_k^{-1} \cdots e_1^{-1}$ (inverse is taken with respect to any background orientation). When a path γ has no backtracks, we say that it is reduced. For any metric graph Γ and free homotopy class $[\gamma]$, there is a unique, reduced, cellular representative $\gamma \sim \gamma_{\min}$ in Γ. In particular, $\gamma_{\min}$ is the unique representative of $[\gamma_{\min}]$ without a backtrack.

Proposition 2.2 *Fix a graph Γ and a class $[\alpha] \in \pi_1(\Gamma)$ then there is a unique minimal cellular length loop γ with $[\gamma] = [\alpha]$ such that any cellular loop γ' with $[\gamma'] = [\alpha]$ can be transformed into γ by eliminating backtracks. Furthermore, γ also has the smallest metric length in its class according to any metric ℓ on Γ.*

Proof First we pick a spanning tree $\mathcal{T} \subset \Gamma$ with the associated map $\phi_{\mathcal{T}}: \Gamma \to \Gamma_g$. Since $\pi_1[\phi_{\mathcal{T}}]$ is an isomorphism, there is a reduced word $w \in F_g$ such that any γ' as above must have $\pi_1(\phi_{\mathcal{T}})[\gamma']$ equal to the homotopy class determined by w. In particular if $\overline{e}_i$ is the edge in $E(\Gamma) \setminus E(\mathcal{T})$ that maps to $e_i \in E(\Gamma_g)$, the $\overline{e}_i$ appear in γ' in the same cyclic order as the e_i appear in w, and in between the corresponding e_i in $\phi_{\mathcal{T}}(\gamma')$ we must have backtracks that can be eliminated.

Now notice that if e_i and e_j appear consecutively in $\phi_{\mathcal{T}}(\gamma')$, then the relevant piece of γ' must be of the form $\overline{e}_i\rho\overline{e}_j$ for ρ the unique path in $\mathcal{T}$ that connects the endpoints of $\overline{e}_i$ and $\overline{e}_j$.

Now by these two observations, if e_i and e_j appear consecutively in the class $[\phi_{\mathcal{T}}(\gamma')]$, then in γ' we must have $\overline{e}_i\rho'\overline{e}_j$, where ρ' reduces to ρ after eliminating all backtracks. Carrying out all of these reductions results in a path γ in Γ constructed by taking the edges $\overline{e}_i$ in w along with the unique paths in $\mathcal{T}$ between their consecutive endpoints. By the observations above, this path is the unique path of shortest cellular length in Γ that can map to w.

By construction, every edge that appears in γ also appears in any γ', it follows that the length of γ is less than or equal to that of γ' with respect to any metric ℓ. ■

The cone over outer space has a natural set of coordinate functions defined by reduced words in the free group F_g. Each element $\omega \in F_g$ defines a homotopy class in $\pi_1(\Gamma_g)$, which can then be pushed forward to Γ by the marking ϕ. By passing to the unique reduced representative of $\pi_1(\phi)(\omega)$, we obtain a function $d_\omega\colon C_{\phi,\Gamma} \to \mathbb{R}_{\geq 0}$ by measuring the total length of this representative in (Γ, ℓ); see [6, p. 2], [35, p. 6]. Since this function is an invariant of the homotopy class, conjugate words define the same length function. Let $\langle F_g\rangle$ denote the equivalence classes under conjugation. The collection of length functions $d_\omega, \omega \in F_g$ define an embedding $\iota\colon \widehat{O}(g) \to \mathbb{R}^{\langle F_g\rangle}$; in particular, the length function values determine the $\mathbb{R}$-graph structure on Γ (see [5]). Furthermore, the cones $C_{\Gamma,\phi}$ define a decomposition of $\widehat{O}(g)$ into simplicial cones in this topology, [35]. From now on a marked metric graph is a triple (Γ, ϕ, ℓ) of a graph Γ, a marking ϕ, and a metric $\ell \in C_{\Gamma,\phi}$.

2.3 The Space Υ_g of Metric Spanned Graphs

Let Γ be a graph with $\beta_1(\Gamma) = g$, and let $\mathcal{T} \subset \Gamma$ be a spanning tree. The complement $E(\Gamma) \setminus E(\mathcal{T})$ is a set of size g; we label these edges $1, \dots, g$ and give each one an orientation. Taken together, we say this information makes Γ into a spanned graph. The data of a spanned graph structure on Γ is equivalent to giving a map $\psi_{\mathcal{T}}\colon \Gamma \to \Gamma_g$ as in Subsection 2.1. We say two spanned graphs Γ, Γ' are equivalent if there is a graph isomorphism $h\colon \Gamma \to \Gamma'$ such that $\psi_{\mathcal{T}'} \circ h = \psi_{\mathcal{T}}$. We let Υ_g be the set of triples $(\Gamma, \psi_{\mathcal{T}}, \ell)$, where $\psi_{\mathcal{T}}$ defines a spanned graph structure on Γ and ℓ is a metric on Γ.

Recall the space $\mathfrak{T}_n$ of phylogenetic trees with n leaves introduced by Billera, Holmes, and Vogtmann [4]. A point in $\mathfrak{T}_n$ is a tree $\mathcal{T}$ with n leaves labeled $1, \dots, n$ and a metric represented as a function $\ell\colon E(\mathcal{T}) \to \mathbb{R}_{>0}$. For a phylogenetic tree $\mathcal{T}$ we let $f_i \in E(\mathcal{T})$ be the edge connected to the i-th leaf.

Proposition 2.3 *The set Υ_g can be identified with the subset of $\mathfrak{T}_{2g}$ of phylogenetic trees with $\ell(f_{2i}) = \ell(f_{2i-1})$ for $1 \leq i \leq g$.*

Proof Let $(\Gamma, \psi_{\mathcal{T}}, \ell) \in \Upsilon_g$, and let $\{e_1, \dots, e_g\} = E(\Gamma) \setminus E(\mathcal{T})$. Each e_i can be split into two edges f_{2i}, f_{2i-1}, where f_{2i} is connected to the vertex at the head of the orientation on e_i; this defines a labeled tree $\mathcal{T}'$ with $2g$ leaves. By stipulating that $\ell'(f_{2i}) = \ell'(f_{2i-1}) = \ell(e_i)$ and $\ell'(h) = \ell(h)$ for all edges in $\mathcal{T}$, we define the structure of a phylogenetic tree on $\mathcal{T}'$. This procedure can be reversed by replacing f_{2i} and

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f_{2i-1} with an oriented edge e_i, and the resulting graph is clearly isomorphic to Γ as a spanned graph. As this latter procedure can be performed with any point in $\mathfrak{T}_{2g}$, we have $\Upsilon_g \cong \mathfrak{T}_{2g}$. ∎

The polyhedral complex structure on $\mathfrak{T}_{2g}$ defines the structure of a polyhedral complex on Υ_g. For each spanned graph $(\Gamma, \psi_{\mathcal{T}})$, we choose a basepoint $V \in V(\Gamma)$; this defines a homotopy inverse $\phi_{\mathcal{T},V}$ to $\psi_{\mathcal{T}}$, and a marked graph structure on Γ. Any two choices of basepoint define the same marking by definition. In this way, Υ_g can be identified with a subset of $\widehat{O}_g$.

Proposition 2.4 *Any point* $(\Gamma, \phi, \ell) \in \widehat{O}_g$ *is in the image of* Υ_g *under the action of* $\mathrm{Out}(F_g)$.

Proof This follows from Lemma 2.1. ∎

3 Valuations, Filtrations, and Analytification

In this section we review technical details of valuations that we will use in Section 5. We also recall some of the structure of the analytification X^{an} of a complex affine variety $X = \mathrm{Spec}(A)$.

3.1 Valuations on Tensor Products

Throughout the paper we use the MAX convention in the definition of a valuation. A function $v: A \to \mathbb{R} \cup \{-\infty\}$ on a commutative $\mathbb{C}$ domain is known as a rank 1 valuation that lifts the trivial valuation on $\mathbb{C}$ if it satisfies:

(a) $v(ab) = v(a) + v(b)$,
(b) $v(a + b) \le \max\{v(a), v(b)\}$,
(c) $v(C) = 0$ for all $C \in \mathbb{C}^*$, and
(d) $v(0) = -\infty$.

A valuation v defines a filtration on A by the spaces $v_{\le k} = \{f \in A | v(f) \le k\} \subset A$; we let $gr_v(A)$ be the associated graded algebra of this filtration. An increasing algebraic $\mathbb{R}$-filtration F by complex vector spaces likewise defines a real-valued function $v_F: A \to \mathbb{R} \cup \{-\infty\}$ by sending f to $inf\{r | f \in F_{\le r}(A)\}$. This function is a valuation as above if and only if $gr_F(A)$ is a domain. More generally, v_F is said to be a quasivaluation.

Let v and w be valuations on commutative $\mathbb{C}$-domains A and B; these can be used to define filtrations on the tensor product $A \otimes B$ (taken over $\mathbb{C}$):

$$v_{\le k}(A \otimes B) = v_{\le k}(A) \otimes B, \quad w_{\le k}(A \otimes B) = A \otimes w_{\le k}(B).$$

The sum $v \oplus w$ of two valuations v, w is the quasivaluation corresponding to the filtration defined by the following subspaces:

$$(v \oplus w)_{\le m} = \sum_{i+j \le m} v_{\le i} \cap w_{\le j}.$$

Similarly, for $R \in \mathbb{R}_{\ge 0}$, the multiple $R \circ v$ of a valuation is defined by setting

$$(R \circ v)_{\le m} = \sum_{k \le \frac{m}{R}} v_k(A).$$

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Lemma 3.1 *Let $v, w: A \otimes B \to \mathbb{R} \cup \{-\infty\}$ be as above. Any linear combination $S \circ v \oplus T \circ w$, $S, T \in \mathbb{R}_{\geq 0}$ defines a valuation on $A \otimes B$.*

Proof The associated graded algebra of v as a valuation on $A \otimes B$ is $gr_v(A) \otimes B$, which is manifestly a domain. This implies that v and w define valuations on $A \otimes B$.

Now we consider the sum valuation $S \circ v \oplus T \circ w$. By the first part, without loss of generality, we can take $S, T = 1$. We must prove that the associated graded algebra is a domain; we claim that it is in fact isomorphic to $gr_v(A) \otimes gr_w(B)$. We compute the quotient $(v \oplus w)_{\leq m}/(v \oplus w)_{<m}$:

$$(v \oplus w)_{\leq m}/(v \oplus w)_{<m} = \sum_{i+j \leq m} v_{\leq i} \cap w_{\leq j} \Big/ \sum_{i+j<m} v_{\leq i} \cap w_{\leq j}$$

The space $v_{\leq i} \cap w_{\leq j}$ is equal to $v_{\leq i} \otimes w_{\leq j}$ and $v_{\leq i} \otimes w_{\leq j} \subset v_{\leq i'} \otimes w_{\leq j'}$ if and only if $i \leq i'$ and $j \leq j'$. The space $(v \oplus w)_{\leq m}/(v \oplus w)_{<m}$ is spanned by the images of the spaces $v_{\leq i} \otimes w_{\leq j}$ with $i + j = m$, and the intersection of two of these spaces $v_{\leq i} \otimes w_{\leq j} \cap v_{\leq i'} \otimes w_{\leq j'}$ is $v_{\leq min\{i,i'\}} \otimes w_{\leq min\{j,j'\}}$, which is contained in $(v \oplus w)_{<m}$. It follows that the above quotient simplifies to the direct sum of the spaces $v_{\leq i} \otimes w_{\leq j}/v_{<i} \otimes w_{\leq j} + v_{\leq i} \otimes w_{<j} = v_{\leq i}/v_{<i} \otimes w_{\leq j}/w_{<j}$; this proves the claim. ∎

Lemma 3.1 can be applied to two valuations $v, w: C \to \mathbb{R} \cup \{-\infty\}$ induced from an inclusion of algebras $C \subset A \otimes B$; this is how we use it in Section 5.

We will also need the following lemma, which relates the associated graded algebra of a filtration by representations of a reductive group G to the associated graded algebra of the invariant ring.

Lemma 3.2 *Let R be a $\mathbb{C}$ algebra with a rational action by a reductive group G, and let v be a G-stable valuation; then the following holds.*

$$gr_v(R^G) = (gr_v(R))^G$$

Proof This is a consequence of the fact that passing to invariants by a reductive group is an exact functor. This fact is applied to the short exact sequence $0 \to v_{<m} \to v_{\leq m} \to v_{\leq m}/v_{<m} \to 0$. ∎

3.2 Analytification

The set of all rank 1 valuations on a domain A that lift the trivial valuation on $\mathbb{C}$ can be given the structure of a topological space X^{an}, called the analytification of the associated affine variety $X = \mathrm{Spec}(A)$; see [3,27]. The analytification is given coarsest topology that makes the evaluation maps $ev_f: X^{an} \to \mathbb{R}$, $ev_f(v) = v(f) \in \mathbb{R}$, $f \in A$ continuous with respect to the usual topology on $\mathbb{R}$. In particular, the topology on X^{an} is generated by the pullbacks of open subsets of $\mathbb{R}$ under the maps ev_f; this makes X^{an} into a path connected, locally compact Hausdorff space. It follows that for a topological space Y, a set map $\Phi: Y \to X^{an}$ is continuous if and only if for each $f \in A$, the map $[f] \circ \Phi$ defines a continuous map from Y to $\mathbb{R}$.

4 Degeneration, Compactification, and Hamiltonian Systems in $SL_2(\mathbb{C})$

In this section we use a valuation to construct a flat degeneration of $SL_2(\mathbb{C})$ to the singular matrices $SL_2(\mathbb{C})^c \subset M_2(\mathbb{C})$. We place symplectic structures on both $SL_2(\mathbb{C})$ and $SL_2(\mathbb{C})^c$ and recall a surjective, continuous map $\Xi: SL_2(\mathbb{C}) \to SL_2(\mathbb{C})^c$ of Hamiltonian $SU(2)\times SU(2)$ spaces that is a symplectomorphism on a dense open subset of $SL_2(\mathbb{C})$.

4.1 The Coordinate Ring of $SL_2(\mathbb{C})$

The algebraic aspects of $\mathbb{C}[\mathcal{X}(F_g, SL_2(\mathbb{C}))]$ covered in this paper are derived from the interplay between two different ways to view the coordinate ring $\mathbb{C}[SL_2(\mathbb{C})]$. As an algebraic variety, the group $SL_2(\mathbb{C})$ is most familiar as the locus of the equation $AD - BC = 1$ in the space of 2×2 matrices $M_{2\times 2}(\mathbb{C})$:

$$SL_2(\mathbb{C}) = \Big\{ A, B, C, D \mid \det\Big[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\Big] = 1 \Big\} \subset M_{2\times 2}(\mathbb{C}).$$

The Peter–Weyl theorem grants another description of this ring as the direct sum of the endomorphism spaces of the irreducible representations of $SL_2(\mathbb{C})$. Recall that there is one such irreducible $V(i)$ for each non-negative integer $i \in \mathbb{Z}_{\geq 0}$, with $V(i) \cong \mathrm{Sym}^i(\mathbb{C}^2)$. For any $f \in \mathrm{End}(V(i))$, there is a regular function on $SL_2(\mathbb{C})$ defined by $g \to \mathrm{Tr}(g^{-1} \circ f)$. This produces an inclusion of $SL_2(\mathbb{C}) \times SL_2(\mathbb{C})$ representations: $\mathrm{End}(V(i)) \subset \mathbb{C}[SL_2(\mathbb{C})]$, and defines the $SL_2(\mathbb{C}) \times SL_2(\mathbb{C})$ isotypical decomposition of $\mathbb{C}[SL_2(\mathbb{C})]$:

$$\mathbb{C}[SL_2(\mathbb{C})] = \bigoplus_{i\in\mathbb{Z}_{\geq 0}} \mathrm{End}(V(i)).$$

An element $(g, h) \in SL_2(\mathbb{C}) \times SL_2(\mathbb{C})$ acts on $f: V(a) \to V(a)$ by pre and post composition, respectively:

$$(g, h) \circ [V \to f(V)] = [V \to g \circ f(h^{-1}V)].$$

These descriptions are connected by identifying the generators A, B, C, D above with matrix elements in $\mathrm{End}(V(1)) \cong M_{2\times 2}(\mathbb{C})$. Let $X_{ij} \in \mathrm{End}(\mathbb{C}^2)$ be the matrix that has ij entry equal to 1 and all other entries 0. The generators of $\mathbb{C}[SL_2(\mathbb{C})]$ are computed on an element $M \in SL_2(\mathbb{C})$ as follows:

$$A(M) = \mathrm{Tr}(M^{-1}X_{11}), \qquad B(M) = \mathrm{Tr}(M^{-1}X_{01}),$$
$$C(M) = \mathrm{Tr}(M^{-1}X_{10}), \qquad D(M) = \mathrm{Tr}(M^{-1}X_{00}).$$

Let $U_-, U_+ \subset SL_2(\mathbb{C})$ be the groups of upper (resp. lower) triangular matrices in $SL_2(\mathbb{C})$. It will be necessary to use the coordinate ring of the GIT quotients $SL_2(\mathbb{C})/\!\!/U_-$, $U_+\backslash\!\!\backslash SL_2(\mathbb{C})$, both of which are identified with $\mathbb{C}^2$. The algebras $\mathbb{C}[SL_2(\mathbb{C})]^{U_-}$, $\mathbb{C}[SL_2(\mathbb{C})]^{U_+}$ are the subspaces of right highest (resp. left lowest) weight vectors in the representation $\bigoplus_{i\in\mathbb{Z}_{\geq 0}} \mathrm{End}(V(i))$:

$$\mathbb{C}[SL_2(\mathbb{C})]^{U_-} = \mathbb{C}[SL_2(\mathbb{C})]^{U_+} = \bigoplus_{i\in\mathbb{Z}_{\geq 0}} V(i).$$

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With actions taken on the right-hand side, the algebra $\mathbb{C}[SL_2(\mathbb{C})]^{U_-} \subset \mathbb{C}[SL_2(\mathbb{C})]$ is a polynomial ring, generated by A, C. Under the automorphism $(-)^{-1}: SL_2(\mathbb{C}) \to SL_2(\mathbb{C})$, $\mathbb{C}[U_-\backslash\backslash SL_2(\mathbb{C})]$ is likewise identified with a polynomial ring in two variables.

The isotypical decomposition of $\mathbb{C}[SL_2(\mathbb{C})]$ allows an $SL_2(\mathbb{C}) \times SL_2(\mathbb{C})$-stable filtration defined by the spaces $E_{\leq k} = \bigoplus_{k-i\in 2\mathbb{Z}_{\geq 0}} \mathrm{End}(V(i)) \subset \mathbb{C}[SL_2(\mathbb{C})]$. The even numbers $2\mathbb{Z}$ in this expression should be interpreted as the root lattice of $SL_2(\mathbb{C})$. Following [10, Chapter 7], the $E_{\leq k}$ define a filtration of algebras; in particular, the image of a product $\mathrm{End}(V(i))\,\mathrm{End}(V(j))$ lies in the space $\bigoplus_{i+j-k\in 2\mathbb{Z}_{\geq 0}} \mathrm{End}(V(k))$, and the associated graded algebra of this filtration is the algebra $[\mathbb{C}[SL_2(\mathbb{C})]^{U_-} \otimes \mathbb{C}[SL_2(\mathbb{C})]^{U_+}]^{\mathbb{C}^*}$. Taking invariants by the $\mathbb{C}^*$ action in this expression picks out the invariant subalgebra $\bigoplus_{i\in\mathbb{Z}_{\geq 0}} V(i) \otimes V(i) \subset \mathbb{C}[SL_2(\mathbb{C})]^{U_-} \otimes \mathbb{C}[SL_2(\mathbb{C})]^{U_+}$. Hidden in this statement is the important fact that for any non-zero $f \in \mathrm{End}(V(i)), g \in \mathrm{End}(V(j))$, the component $(fg)_{i+j} \in \mathrm{End}(V(i+j))$ is always non-zero; this is a consequence of the fact that $[\mathbb{C}[SL_2(\mathbb{C})]^{U_-} \otimes \mathbb{C}[SL_2(\mathbb{C})]^{U_+}]^{\mathbb{C}^*}$ is a domain.

We let $SL_2(\mathbb{C})^c$ be the GIT quotient $[SL_2(\mathbb{C})/\!/U_- \times U_+\backslash\backslash SL_2(\mathbb{C})]/\!/\mathbb{C}^*$, following the notation in [13], [18]; this space is known as the horospherical contraction of $SL_2(\mathbb{C})$. Multiplication in $\mathbb{C}[SL_2(\mathbb{C})^c]$ is graded by the components $V(i) \otimes V(i)$, and is $SL_2(\mathbb{C}) \times SL_2(\mathbb{C})$-equivariant. The isotypical components are irreducible representations of $SL_2(\mathbb{C}) \times SL_2(\mathbb{C})$; it follows that each product map $[V(i) \otimes V(i)] \otimes [V(j) \otimes V(j)] \to [V(i+j) \otimes V(i+j)]$ is surjective, and this algebra is generated by the subspace $V(1) \otimes V(1)$. The associated graded algebra can be presented by $\mathbb{C}[A, B, C, D]$ modulo the initial ideal of $\langle AD - BC - 1\rangle$ with respect to the filtration F. As A, B, C, D all have the same filtration level, and 1 is a scalar, we have

$$\mathbb{C}[SL_2(\mathbb{C})^c] = \mathbb{C}[A, B, C, D]/\langle AD - BC\rangle.$$

Notice that the equation $AD - BC = 0$ cuts out the set of singular matrices in $M_{2\times 2}(\mathbb{C})$. We let X, Y be generators of the polynomial coordinate ring $\mathbb{C}[\mathbb{C}^2]$. The algebra $\mathbb{C}[SL_2(\mathbb{C})^c]$ is isomorphic to $\mathbb{C}[(\mathbb{C}^2 \times \mathbb{C}^2)/\!/\mathbb{C}^*]$ by the map $A \to X \otimes X, B \to X \otimes Y, C \to Y \otimes X, D \to Y \otimes Y$.

Let $\mathcal{P}$ be the pointed, polyhedral cone in $\mathbb{R}^3$ defined by the origin $(0,0,0)$ and the rays through the points $\{(0,0,1), (1,0,1), (0,1,1), (1,1,1)\}$, and let P be the affine semigroup $\mathcal{P} \cap \mathbb{Z}^3$. The map $A \to (0,0,1)$, $B \to (0,1,1)$, $C \to (1,0,1)$, $D \to (1,1,1)$ defines an isomorphism of algebras,

$$\mathbb{C}[SL_2(\mathbb{C})^c] \cong \mathbb{C}[P].$$

We must also consider the Rees algebra of the filtration E:

$$R = \bigoplus_{k\in\mathbb{Z}_{\geq 0}} E_{\leq k}.$$

This algebra can be presented as $\mathbb{C}[A, B, C, D, t]/\langle AD - BC - t\rangle$, where $t \circ F_{\leq k} \subset F_{\leq k+2}$ identifies $F_{\leq k}$ with itself as a subspace of $F_{\leq k+2}$. The following are standard properties of Rees algebras; see [13, 3.1].

Proposition 4.1

(i) *R is flat over $\mathbb{C}[t]$ (The scheme* $\mathrm{Spec}(R)$ *is a flat family over the affine line.)*

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(ii) $\frac{1}{t}R = \mathbb{C}[SL_2(\mathbb{C})] \otimes \mathbb{C}[t, \frac{1}{t}]$ *(A general fiber of* $\mathrm{Spec}(R)$ *is isomorphic to* $SL_2(\mathbb{C})$*.)*
(iii) $R/t = \mathbb{C}[SL_2(\mathbb{C})^c]$ *(The scheme* $\mathrm{Spec}(R)$ *defines a degeneration of* $SL_2(\mathbb{C})$ *to* $SL_2(\mathbb{C})^c$*)*

The total family $t\colon \mathrm{Spec}(R) \to \mathbb{C}$ is the determinant family of 2×2 matrices $\det\colon M_{2\times 2}(\mathbb{C}) \to \mathbb{C}$. The space $M_{2\times 2}(\mathbb{C})$ is the Vinberg enveloping monoid of $SL_2(\mathbb{C})$, and $SL_2(\mathbb{C})^c$ is known as the asymptotic semigroup of this monoid; see [13, 33, 34].

4.2 Symplectic Structures

The discussion of integrable systems in Section 7 requires us to fix symplectic forms on $SL_2(\mathbb{C})$ and $SL_2(\mathbb{C})^c$. All of the forms we use stem from the canonical $SU(2) \times SU(2)$-invariant symplectic form ω on the cotangent bundle $T^*(SU(2))$. Recall that $T^*(SU(2))$ is an $SU(2) \times SU(2)$ Hamiltonian space, with right and left momentum maps computed as follows. Here $h \circ v$ denotes the coadjoint action of $SU(2)$ on $su(2)^*$:

$$h \circ_L (k, v) = (hk, v), \qquad h \circ_R (k, v) = (kh^{-1}, h \circ v)$$
$$\mu_L(k, v) = k \circ v, \qquad \mu_R(k, v) = -v.$$

Following [13, Subsection 2.3] and [25], we induce symplectic form on $SL_2(\mathbb{C})$ with a Hamiltonian $SU(2) \times SU(2)$ action from the embedding $SL_2(\mathbb{C}) \to M_{2\times 2}(\mathbb{C})$, which carries the Kähler form $\omega(X, Y) = -\mathrm{Im}(\mathrm{Tr}(XY^*))$. Appendix A of [25] and [30] then imply that there is an associated isomorphism $SL_2(\mathbb{C}) \cong T^*(SU(2))$ of Hamiltonian $SU(2) \times SU(2)$ manifolds. Letting $\pi\colon SL_2(\mathbb{C}) \to SU(2)$ be the projection derived from the Cartan polar decomposition, the isomorphism above is computed as below:

$$SL_2(\mathbb{C}) \cong T^*\big(SU(2)\big), \quad g \longrightarrow \big(\pi(g), \mu(g)\big),$$

where $\mu\colon M_{2\times 2}(\mathbb{C}) \to su(2)^*$ is the momentum map for the right action of $SU(2)$.

A symplectic form is placed on $SL_2(\mathbb{C})/\!/U_-$, $U_+\backslash\!\backslash SL_2(\mathbb{C})$ by viewing these spaces as the right (resp. left) *symplectic implosions* of the cotangent bundle with respect to the standard Weyl chamber of $SU(2)$ and its negative, respectively. We refer the reader to [9, 14] for the fundamentals of the symplectic implosion operation with respect to a Hamiltonian action by a compact group; a more detailed description appears below.

We let $\rho \in su(2)^*$ be the root vector in $su(2)^*$, with $\mathbb{R}_{\geq 0}\rho, \mathbb{R}_{\leq 0}\rho \subset su(2)^*$ the positive and negative Weyl chambers. The spaces $SL_2(\mathbb{C})/\!/U_+$ and $U_-\backslash\!\backslash SL_2(\mathbb{C})$ are identified with quotients of the subspaces $\mu_R^{-1}(\mathbb{R}_{\leq 0}\rho), \mu_L^{-1}(\mathbb{R}_{\geq 0}\rho) \subset T^*(SU(2))$ by the equivalence relation that collapses all (k, v) with $v = 0$ to a single point. The subspaces $\mu_R^{-1}(\mathbb{R}_{\leq 0}\rho)$ and $\mu_L^{-1}(\mathbb{R}_{\geq 0}\rho)$ are both topologically half-closed cylinders over a 3-sphere (homeomorphic to $SU(2)$). This equivalence relation collapses the closed sphere at the boundary, creating a topological copy of $\mathbb{R}^4$. Implosions of Hamiltonian spaces all come with induced, often singular symplectic structures; however, the forms on $SL_2(\mathbb{C})/\!/U_-$ and $U_+\backslash\!\backslash SL_2(\mathbb{C})$ both agree with the standard form on $\mathbb{C}^2$; see [9]. These isomorphisms are S^1 equivariant with respect to the standard action of S^1 on $\mathbb{C}^2$ and the residual S^1 action on $SL_2(\mathbb{C})/\!/U$ through $\mathbb{C}^*$. In imploded cotangent coordinates, this action is $t \circ_R [k, v] = [kt^{-1}, v]$ and $t \circ_L [k, v] = [tk, v]$. The

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momentum maps for these actions are given simply by taking the length of the v coordinate. We place a symplectic form on $SL_2(\mathbb{C})^c$ by taking the symplectic reduction of $SL_2(\mathbb{C})//U_- \times U_+\backslash\backslash SL_2(\mathbb{C})$ at level 0 by the diagonal S^1 action. The momentum 0 subspace $\mu^{-1}(0) \subset SL_2(\mathbb{C})//U_- \times U_+\backslash\backslash SL_2(\mathbb{C})$ is given by those elements $[k, w][h, v]$ with $w = h \circ v$. By results of Kirwan [14] and Kempf and Ness [16], this symplectic reduction can be identified with $SL_2(\mathbb{C})^c$. Notice that this space carries a residual Hamiltonian S^1 action, and that the momentum map for this action takes $[k, w][h, v]$ to the length $|v|$. By [13, Subsection 5.3.3] this form coincides with the form on $SL_2(\mathbb{C})^c$ induced from the Kähler structure on $M_{2\times 2}(\mathbb{C})$.

4.3 The Map Ξ

Using the identification $SL_2(\mathbb{C}) \cong T^*(SU(2))$, we can define a continuous function on $SL_2(\mathbb{C})$ by setting $\xi(g)$ equal to $|\mu_R(g)|$, the length with respect to the Killing form on $su(2)^*$. This can be computed from the entries of g itself by as follows:

$$\xi(g) = \sqrt{\det\big(g^* g - \tfrac{1}{2}\operatorname{Tr}(g^* g) I\big)}.$$

The map ξ is smooth on $SL_2(\mathbb{C})_o \subset SL_2(\mathbb{C})$, the space of matrices g with $\mu_R(g) \neq 0$, where it generates a Hamiltonian S^1 action. However, ξ is singular along the subgroup $SU(2) \subset SL_2(\mathbb{C})$, where the S^1 flow is consequently not well defined. Next we introduce a map that "repairs" the undefined flow by relating $SL_2(\mathbb{C})$ to $SL_2(\mathbb{C})^c$. We recall the contraction map $\Xi\colon SL_2(\mathbb{C}) \to SL_2(\mathbb{C})^c$ from [18] and [13, Section 4]. For $g \to (k, v)$, let $h \in SU(2)$ be an element that diagonalizes v, that is, $h \circ v \in \mathbb{R}_{\geq 0}\rho \subset su(2)^*$. We define $\Xi\colon SL_2(\mathbb{C}) \to SL_2(\mathbb{C})^c$ as follows:

$$\Xi(k, v) = [kh^{-1}, h \circ v][h, v].$$

It is straightforward to check that $\Xi(k, v)$ defines a point in $SL_2(\mathbb{C})^c$. The following proposition recalls all the properties of Ξ we will need.

Proposition 4.2 *The map $\Xi\colon SL_2(\mathbb{C}) \to SL_2(\mathbb{C})^c$ is well defined, surjective, continuous, and defines a symplectomorphism on $SL_2(\mathbb{C})_o$. The subgroup $SU(2) = SL_2(\mathbb{C}) \setminus SL_2(\mathbb{C})_o$ is collapsed to a single point in the image. Furthermore, the composition of Ξ with the residual S^1 momentum map $\mu\colon SL_2(\mathbb{C})^c \to \mathbb{R}$ is ξ.*

4.4 The Gradient Hamiltonian Flow

The determinant map $\det\colon M_{2\times 2}(\mathbb{C}) \to \mathbb{C}$ defines an algebraic family joining $SL_2(\mathbb{C}) = \det^{-1}(1)$ to $SL_2(\mathbb{C})^c = \det^{-1}(0)$. As explained in [11] and [13, Section 5], the Kähler structure on this family can be used to define a gradient Hamiltonian vector field:

$$V_{\det} = \frac{\nabla(\operatorname{Re}(\det))}{|\nabla(\operatorname{Re}(\det))|}.$$

The work of Harada and Kaveh (making use of a technique of Ruan [29]) implies the existence of a surjective, continuous map $\Xi_{\det}\colon SL_2(\mathbb{C}) \to SL_2(\mathbb{C})^c$, which is a symplectomorphism on $SL_2(\mathbb{C})_o$. Corollary 5.12 of [13] implies that $\Xi_{\det}$ coincides with the map Ξ above.

4.5 The Compactification $SL_2(\mathbb{C}) \subset X$

There is another, perhaps more natural, filtration on $\mathbb{C}[SL_2(\mathbb{C})]$ defined by the spaces $F_{\leq k} = \bigoplus_{i\leq k} \mathrm{End}(V(i))$. This is also a filtration of algebras, and the spectrum of the Rees algebra $\overline{R} = \bigoplus_{k\in\mathbb{Z}_{\geq 0}} F_{\leq k}$ is a double cover of $M_{2\times 2}(\mathbb{C}) = \mathrm{Spec}(R)$. The associated graded algebra of this filtration is also isomorphic to $\mathbb{C}[P]$.

The filtration F naturally defines a valuation $v: \mathbb{C}[SL_2(\mathbb{C})] \to \mathbb{Z} \cup \{-\infty\}$; for $f = \sum f_i \in \bigoplus_{i\in\mathbb{Z}_{\geq 0}} \mathrm{End}(V(i))$, one has $v(f) = \max\{i | f_i \neq 0\}$. This function is the divisorial valuation associated to a compactification X of $SL_2(\mathbb{C})$:

(a) $X = \mathrm{Proj}(\mathbb{C}[a,b,c,d,t]/\langle ad - bc - t^2\rangle) \subset \mathbb{P}^4$,
(b) $D = \mathrm{Proj}(\mathbb{C}[a,b,c,d]/\langle ad - bc\rangle) \subset X \subset \mathbb{P}^4$.

The divisor D is cut out by the principal ideal $\langle t\rangle = \bigoplus_{a<k} \mathrm{End}(V(a))t^k$ and is isomorphic to the projective toric variety $\mathrm{Proj}(\mathbb{C}[P]) = \mathbb{P}^1 \times \mathbb{P}^1$. The pullback of $O(1)$ (on $\mathbb{P}^4$) to $D \cong \mathbb{P}^1 \times \mathbb{P}^1$ is $\mathcal{O}(1) \boxtimes \mathcal{O}(1)$. It follows from the equation

$$\frac{1}{t}\mathbb{C}[a,b,c,d,t]/\langle ad - cb - t^2\rangle = \mathbb{C}[SL_2(\mathbb{C})] \otimes \mathbb{C}\Big[t, \frac{1}{t}\Big]$$

that the complement of D is the scheme

$$\mathrm{Proj}\big(\tfrac{1}{t}\mathbb{C}[a,b,c,d,t]/\langle ad - bc - t\rangle\big) = \mathrm{Spec}\big(\mathbb{C}[a,b,c,d]/\langle ad - bc - 1\rangle\big) = SL_2(\mathbb{C}).$$

We induce a valuation $\overline{v}$ on $\overline{R}$ from the valuation $v \oplus 0$ on $\mathbb{C}[SL_2(\mathbb{C})] \otimes \mathbb{C}[t]$. The associated graded algebra $T = gr_{\overline{v}}(\overline{R})$ is presented as $\mathbb{C}[a,b,c,d,t]/\langle ad - bc\rangle$. This defines a flat $SL_2(\mathbb{C}) \times SL_2(\mathbb{C})$-stable degeneration of X to a projective toric variety $X_0 = \mathrm{Proj}(T) = \mathrm{Proj}(\mathbb{C}[P \times \mathbb{Z}_{\geq 0}])$.

5 Valuations on $\mathbb{C}[\mathcal{X}(F_g, SL_2(\mathbb{C}))]$

In this section we construct a cone of valuations $C_{\Gamma,\phi} \subset \mathcal{X}(F_g, SL_2(\mathbb{C}))^{an}$ for each marking $\phi: \Gamma_g \to \Gamma$ in three steps. First, we define a space $M_\Gamma(SL_2(\mathbb{C}))$ for each oriented graph Γ, such that $M_{\Gamma_g}(SL_2(\mathbb{C}))$ is naturally isomorphic to $\mathcal{X}(F_g, SL_2(\mathbb{C}))$. Then we construct a simplicial cone of valuations $C_\Gamma \subset M_\Gamma(SL_2(\mathbb{C}))^{an}$ using the valuation $v: \mathbb{C}[SL_2(\mathbb{C})] \to \mathbb{Z} \cup \{-\infty\}$ constructed in Section 4. We conclude by showing that each marking $\phi: \Gamma_g \to \Gamma$ gives an isomorphism $\phi^*: M_\Gamma(SL_2(\mathbb{C})) \to M_{\Gamma_g}(SL_2(\mathbb{C}))$; this induces a cone of valuations $C_{\Gamma,\phi} = \phi^*(C_\Gamma) \subset \mathcal{X}(F_g, SL_2(\mathbb{C}))^{an}$.

5.1 The Space $M_\Gamma(X)$ and the Cone C_Γ

Let X be an affine $SL_2(\mathbb{C}) \times SL_2(\mathbb{C})$ space, and let Γ be any finite oriented graph with all vertices of valence at least three. We define an action of $SL_2(\mathbb{C})^{V(\Gamma)}$ on $X^{E(\Gamma)}$, by having an element $(\dots, g_w, \dots) \in SL_2(\mathbb{C})^{V(\Gamma)}$ act on $(\dots, x_e, \dots) \in X^{E(\Gamma)}$ by sending it to $(\dots, g_v x_e g_u^{-1}, \dots)$, where $\delta(e) = (v,u)$. We let $M_\Gamma(X)$ be the corresponding GIT quotient:

$$M_\Gamma(X) = SL_2(\mathbb{C})^{V(\Gamma)} \backslash\backslash X^{E(\Gamma)}.$$

Three cases of this construction are of special interest: $X = SL_2(\mathbb{C}), SL_2(\mathbb{C})^c$, and $M_{2\times 2}(\mathbb{C})$. When the graph is Γ_g (with any orientation), and $X = SL_2(\mathbb{C})$, each

$(h_1,\dots,h_g) \in SL_2(\mathbb{C})^{E(\Gamma_g)}$ is conjugated by $g_v \in SL_2(\mathbb{C})^{V(\Gamma_g)} = SL_2(\mathbb{C})$, so this construction reduces to the GIT definition of the character variety of F_g:

$$M_{\Gamma_g}\big(SL_2(\mathbb{C})\big) = \mathcal{X}\big(F_g, SL_2(\mathbb{C})\big).$$

The coordinate ring of each $SL_2(\mathbb{C})$ component of the product space $SL_2(\mathbb{C})^{E(\Gamma)}$ comes with a copy of the valuation $v\colon \mathbb{C}[SL_2(\mathbb{C})] \to \mathbb{Z} \cup \{-\infty\}$ constructed in Section 4. We let v_e be the copy of v associated with the the $SL_2(\mathbb{C})$ component assigned to the edge $e \in E(\Gamma)$. By Lemma 3.2, the $\mathbb{R}_{\geq 0}$ combinations of the v_e form a cone of valuations C_Γ on the coordinate ring $\mathbb{C}[SL_2(\mathbb{C})^{E(\Gamma)}]$, and therefore $\mathbb{C}[M_\Gamma(SL_2(\mathbb{C}))]$. Let $v_\ell = \sum_{e\in E(\Gamma)} \ell(e) v_e$ be the valuation corresponding to an assignment $\ell\colon E(\Gamma) \to \mathbb{R}_{\geq 0}$. The following proposition allows us to compute the associated graded algebra of an interior valuation from this cone.

Proposition 5.1 *For an interior point $\ell \in C_\Gamma$, the associated graded algebra of $\mathbb{C}[M_\Gamma(SL_2(\mathbb{C}))]$ with respect to v_ℓ is isomorphic to $\mathbb{C}[M_\Gamma(SL_2(\mathbb{C})^c)]$.*

Proof This follows from Lemmas 3.2 and 3.1 and Proposition 4.1. ∎

5.2 The Graph Functor

We show that the construction $M_\Gamma(SL_2(\mathbb{C}))$ is functorial with respect to finite cellular graph maps; in particular, any marking $\phi\colon \Gamma_g \to \Gamma$, induces an isomorphism $\phi^* : M_\Gamma(SL_2(\mathbb{C})) \cong \mathcal{X}(F_g, SL_2(\mathbb{C})) = M_{\Gamma_g}(SL_2(\mathbb{C}))$. We show that if two finite cellular homotopy isomorphisms $\phi_1, \phi_2\colon \Gamma_1 \to \Gamma_2$ induce the same map on homotopy, $\pi_1(\phi_1) = \pi_1(\phi_2)$, then the maps on varieties also agree, $\phi_1^* = \phi_2^*$. This means that each equivalence class of markings $[\phi]\colon \Gamma_g \to \Gamma$ defines a unique associated isomorphism $\phi^* : M_\Gamma(SL_2(\mathbb{C})) \cong \mathcal{X}(F_g, SL_2(\mathbb{C}))$.

A finite cellular map $\phi\colon \Gamma_1 \to \Gamma_2$ takes an edge $e \in E(\Gamma_1)$ to a finite path of edges $\phi(e) = f(1)\cdots f(k)$ in Γ_2. These paths determine a regular map of affine varieties

$$\phi^s\colon SL_2(\mathbb{C})^{E(\Gamma_2)} \longrightarrow SL_2(\mathbb{C})^{E(\Gamma_1)}$$

by sending $(\dots, h_f, \dots) \in SL_2(\mathbb{C})^{E(\Gamma_2)}$ to $(\dots, \prod h_{f(i)}, \dots) \in SL_2(\mathbb{C})^{E(\Gamma_1)}$. In particular, if ϕ collapses the edge e, then the associated component is assigned the identity Id. Notice that the value of the e-th component of $\phi^s(\dots, h_f, \dots)$ only depends on the reduction of the path $\phi(e)$, as the $SL_2(\mathbb{C})$ elements in any backtracks cancel each other out. In particular, if e is a closed loop in Γ_1, then the e-th component of ϕ^s depends only on the minimal length loop in the free homotopy class of $\phi(e)$ (see Proposition 2.2).

Let $\phi^a\colon SL_2(\mathbb{C})^{V(\Gamma_2)} \to SL_2(\mathbb{C})^{V(\Gamma_1)}$ send $(\dots, g_w, \dots)$ to the tuple defined by the property that $g_u = g_w$ for all $u \in \phi^{-1}(w)$. It is clear that ϕ^a is a map of reductive groups.

Proposition 5.2 *The map ϕ^s descends to define a map of affine varieties on the GIT quotients $\phi^*\colon M_{\Gamma_2}(SL_2(\mathbb{C})) \to M_{\Gamma_1}(SL_2(\mathbb{C}))$. Furthermore, the construction $M_\Gamma(SL_2(\mathbb{C}))$ and the maps ϕ^* define a functor from the category of finite graphs with finite cellular maps to the category of complex affine varieties.*

Proof If we act on $(\dots, h_f, \dots) \in SL_2(\mathbb{C})^{E(\Gamma_2)}$ with $(\dots, g_w, \dots) \in SL_2(\mathbb{C})^{V(\Gamma_2)}$ and pass the result through ϕ^s, by the definition of the action the resulting element is $g_v(\prod h_{f(i)})g_u^{-1}$, where $\delta(e) = (v, u)$. This calculation implies that ϕ^s intertwines the actions of $SL_2(\mathbb{C})^{V(\Gamma_2)}$ and $SL_2(\mathbb{C})^{V(\Gamma_1)}$ as related through ϕ^a. This implies that ϕ^s descends to a map $\phi^*: M_{\Gamma_2}(SL_2(\mathbb{C})) \to M_{\Gamma_1}(SL_2(\mathbb{C}))$. For the second part of the proposition, note that by definition, $(\phi_2 \circ \phi_1)^s = \phi_1^s \circ \phi_2^s$ and $(\phi_2 \circ \phi_1)^a = \phi_1^a \circ \phi_2^a$; as a consequence, $(\phi_2 \circ \phi_1)^* = \phi_1^* \circ \phi_2^*$. ∎

Example 5.3 Let $\phi: \Gamma \to \Gamma$ be an isomorphism of graphs, which reverses the orientation of one edge $e \in E(\Gamma)$ with $\delta(e) = (u, v)$. This induces an automorphism of $M_\Gamma(SL_2(\mathbb{C}))$ by $(-)^{-1}: SL_2(\mathbb{C})^e \to SL_2(\mathbb{C})^e \subset SL_2(\mathbb{C})^{E(\Gamma)}$. This takes the edge element $g_u h_e g_v^{-1}$ to $g_v h_e^{-1} g_u^{-1}$, and therefore intertwines the $SL_2(\mathbb{C})^{V(\Gamma)}$ action, giving an isomorphism of varieties.

Example 5.4 Let $\theta: \Gamma_g \to \Gamma_g$ be induced from an outer automorphism of F_g, namely an assignment $e_i \to W_i$ of g generating words to the edges of Γ_g. Then $\theta^s: SL_2(\mathbb{C})^{E(\Gamma_g)} \to SL_2(\mathbb{C})^{E(\Gamma_g)}$ is the regular map defined by sending

$$\vec{h} = (h_1, \dots, h_g) \longrightarrow \big(W_1(\vec{h}), \dots, W_g(\vec{h})\big).$$

The inverse of this map is likewise defined by the inverse of θ. These automorphisms define the action of $\mathrm{Out}(F_g)$ on $\mathcal{X}(F_g, SL_2(\mathbb{C})) = M_{\Gamma_g}(SL_2(\mathbb{C}))$.

Example 5.5 Let $\phi_{\mathcal{T}}: \Gamma \to \Gamma_g$ and $\psi_{V,\mathcal{T}}: \Gamma_g \to \Gamma$ be the pair of maps induced by a choice of spanning tree $\mathcal{T} \subset \Gamma$.

The map $\phi_{\mathcal{T}}^s$ sends $(h_1, \dots, h_g) \in SL_2(\mathbb{C})^{E(\Gamma_g)}$ to the element in $SL_2(\mathbb{C})^{E(\Gamma)}$ obtained by assigning Id to all edges in $\mathcal{T}$, and h_i to the edge in $E(\Gamma) \setminus E(\mathcal{T})$ corresponding to the i-th edge of Γ_g. As $\phi_{\mathcal{T}}$ maps each vertex in Γ down to the unique vertex of Γ_g, $\phi_{\mathcal{T}}^a: SL_2(\mathbb{C}) \to SL_2(\mathbb{C})^{V(\Gamma)}$ is the diagonal map of reductive groups.

The map $\psi_{V,\mathcal{T}}^s$ sends an element $(\dots, h_e, \dots) \in SL_2(\mathbb{C})^{E(\Gamma)}$ to an assignment of words $w_i(\dots, h_e, \dots)$ on the edges of Γ_g. Let $e_i \in E(\Gamma)$ be the preimage of the i-th edge of Γ_g under $\phi_{\mathcal{T}}$, with $\delta(e_i) = (u_i, v_i)$. The word $w_i(\dots, h_e, \dots)$ is computed by concatenating the unique path $V \to u_i$ in $\mathcal{T}$ with e_i and the unique path $v_i \to V$ in $\mathcal{T}$, and recording h_e or h_e^{-1} for each traversed edge, where the sign depends on the orientation of e with respect to the direction of the path.

Lemma 5.6 *For any distinguished map $\psi_{V,\mathcal{T}}: \Gamma_g \to \Gamma$, the induced map $\psi_{V,\mathcal{T}}^*$ is an isomorphism of affine schemes.*

Proof We must check that $\psi_{V,\mathcal{T}}^*$ and $\phi_{\mathcal{T}}^*$ discussed in Example 5.5 are inverse to each other. The discussion in Example 5.5 immediately implies that $\psi_{V,\mathcal{T}}^* \circ \phi_{\mathcal{T}}^*$ is the identity on $M_{\Gamma_g}(SL_2(\mathbb{C}))$.

Consider any edge $e \in \mathcal{T} \subset \Gamma$ with $\delta(e) = (u, v)$, $v \neq V$. Using the action of $SL_2(\mathbb{C})^{\{v\}} \subset SL_2(\mathbb{C})^{V(\Gamma)}$ we can move $(\dots, h_e, \dots)$ to $(\dots, 1, \dots)$, by introducing h_e or h_e^{-1} on the other incident edges of v according to their orientations. The W_i do not change under this operation. The same construction can be applied to $u \neq V$ as

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above. It follows that we can move any point in $SL_2(\mathbb{C})^{E(\Gamma)}$ into the image of $\phi^* \circ \psi^*$ using the action of $SL_2(\mathbb{C})^{V(\Gamma)}$. ∎

Notice that Lemma 5.6 implies that $\psi^*_{V,\mathcal{T}}$ does not depend on the choice $V \in V(\Gamma)$. Indeed, the image $\psi^*_{V,\mathcal{T}}(\dots, h_e, \dots) = (\dots, W_i(\dots, h_e, \dots), \dots)$ differs under a change of basepoint by including or taking away backtracks in the words W_i, a modification that does not change the value of $W_i(\dots, h_e, \dots) \in SL_2(\mathbb{C})$. Next we prove a generalization of this fact.

Proposition 5.7 *Let $\phi_1, \phi_2 \colon \Gamma_g \to \Gamma$ be markings that satisfy $\pi_1(\phi_1) = \pi_1(\phi_2)$; then $\phi_1^* = \phi_2^*$.*

Proof For any edge $e \in E(\Gamma_g)$, the loops $\phi_1(e), \phi_2(e) \subset \Gamma$ are freely homotopic. It follows that both differ from a common minimal length cellular class γ by backtracks. Therefore, we must have $W_i^1(\dots, h_e, \dots) = W_i^2(\dots, h_e, \dots)$ for any

$$(\dots, h_e, \dots) \in M_\Gamma\big(SL_2(\mathbb{C})\big). \qquad ∎$$

Corollary 5.8 *Let $\phi_1, \phi_2 \colon \Gamma_1 \to \Gamma_2$ satisfy $\pi_1(\phi_1) = \pi_1(\phi_2)$; then $\phi_1^* = \phi_2^*$.*

Proof We can choose a spanning tree in Γ_1, and a corresponding distinguished map $\theta \colon \Gamma_g \to \Gamma_1$. This defines two cellular maps $\theta_i = \phi_i \circ \theta \colon \Gamma_g \to \Gamma_2$, which are the same map under π_1. It follows from the proof of Proposition 5.7 that θ_1^*, θ_2^* and therefore ϕ_1^*, ϕ_2^* define the same maps on varieties; see Figure 4. ∎

From Proposition 5.7 and Lemma 5.6, it follows that any equivalence class of markings $[\phi] \colon \Gamma_g \to \Gamma$ corresponds to a well-defined isomorphism of varieties

$$\phi^* : M_\Gamma(SL_2(\mathbb{C})) \cong \mathcal{X}(F_g, SL_2(\mathbb{C})),$$

and an isomorphism of coordinate rings:

$$\widehat{\phi} \colon \mathbb{C}\big[\mathcal{X}(F_g, SL_2(\mathbb{C}))\big] \longrightarrow \mathbb{C}\big[M_\Gamma(SL_2(\mathbb{C}))\big].$$

As a consequence, there is a cone of valuations $C_{\Gamma,\phi} \subset \mathcal{X}(F_g, SL_2(\mathbb{C}))^{an}$ obtained by pulling back C_Γ along $\widehat{\phi}$.

6 The Sets $\mathcal{R}(\Gamma, \phi), \mathcal{S}(\Gamma, \phi)$ and Length Functions

We discuss two spanning sets $\mathcal{R}(\Gamma, \phi), \mathcal{S}(\Gamma, \phi) \subset \mathbb{C}[\mathcal{X}(F_g, SL_2(\mathbb{C}))]$ associated with a marking $\phi \colon \Gamma_g \to \Gamma$. Elements of $S(\Gamma, \phi)$ correspond to arrangements of closed loops in Γ, and the elements of $R(\Gamma, \phi)$ are in bijection with the so-called spin diagrams with topology Γ. We relate these sets to each other, showing that any element of $\mathcal{S}(\Gamma, \phi)$ has a lower-triangular expansion into spin diagrams in $\mathcal{R}(\Gamma, \phi)$ with respect to a natural partial ordering. We then show how to evaluate elements from both of these sets in the valuations $v_{\Gamma,\ell,\phi} \in C_{\Gamma,\phi}$. We finish the section by showing that the set $\mathcal{S}(\Gamma, \phi)$ does not depend on the marking ϕ or the graph Γ, and we use this to give an embedding of $\widehat{O}(g)$ into $\mathcal{X}(F_g, SL_2(\mathbb{C}))^{an}$.

$$\begin{tikzcd}
M_{\Gamma_1}(SL_2(\mathbb{C})) \arrow[rr, shift left, "\phi_2^*"] \arrow[rr, shift right, "\phi_1^*"] \arrow[dr, shift right, "\theta_2^*"'] \arrow[dr, shift left, "\theta_1^*"] & & M_{\Gamma_1}(SL_2(\mathbb{C})) \arrow[dl, "\theta^*"] \\
& \mathcal{X}(F_g, SL_2(\mathbb{C})) &
\end{tikzcd}$$

Figure 4

6.1 Spin Diagrams

We fix a marking of a trivalent graph $\phi: \Gamma_g \to \Gamma$ and use the induced isomorphism $\phi^*: M_\Gamma(SL_2(\mathbb{C})) \to \mathcal{X}(F_g, SL_2(\mathbb{C}))$ to define a basis in $\mathbb{C}[\mathcal{X}(F_g, SL_2(\mathbb{C}))]$ with invariant theory. Observe that the coordinate ring of the space $SL_2(\mathbb{C})^{E(\Gamma)}$ has the following isotypical decomposition under the action of $SL_2(\mathbb{C})^{E(\Gamma)} \times SL_2(\mathbb{C})^{E(\Gamma)}$.

$$\mathbb{C}[SL_2(\mathbb{C})^{E(\Gamma)}] = \bigoplus_{a:E(\Gamma)\to\mathbb{Z}_{\geq 0}} \bigotimes_{e\in E(\Gamma)} V(a(e)) \otimes V(a(e)).$$

Passing to $SL_2(\mathbb{C})^{V(\Gamma)}$ invariants gives a direct sum decomposition of the coordinate ring of $M_\Gamma(SL_2(\mathbb{C}))$:

$$\mathbb{C}\big[M_\Gamma(SL_2(\mathbb{C}))\big] = \bigoplus_{a:E(\Gamma)\to\mathbb{Z}_{\geq 0}} \Big[\bigotimes_{e\in E(\Gamma)} V(a(e)) \otimes V(a(e))\Big]^{SL_2(\mathbb{C})^{V(\Gamma)}}.$$

Recall that there is a copy of $SL_2(\mathbb{C})$ for each vertex $v \in V(\Gamma)$ acting on the tensor product $V(a(e)) \otimes V(a(f)) \otimes V(a(g))$, where $v \in \delta(e), \delta(f), \delta(g)$. The Clebsch–Gordon rule implies that the invariant subspace of such a tensor product is at most one dimensional, and this space is nontrivial if and only if the numbers $a(e), a(f), a(g)$ satisfy two conditions:

(a) $a(e) + a(f) + a(g) \in 2\mathbb{Z}$

(b) $a(e), a(f), a(g)$ are the sides of a triangle: $|a(e) - a(g)| \leq a(f) \leq a(e) + a(g)$.

Definition 6.1 We let $\mathcal{P}_\Gamma$ be the polyhedral cone of $a: E(\Gamma) \to \mathbb{R}_{\geq 0}$ that satisfies condition (b) above, and we let $L_\Gamma \subset \mathbb{Z}^{E(\Gamma)}$ be the lattice defined by condition (a) above. Finally, $P_\Gamma = \mathcal{P}_\Gamma \cap L_\Gamma$ is defined to be the associated affine semigroup.

For $a: E(\Gamma) \to \mathbb{Z}_{\geq 0}$ the invariant spaces $[V(a(e)) \otimes V(a(f)) \otimes V(a(g))]^{SL_2(\mathbb{C})}$ are multiplicity-free, it follows that each space $[\bigotimes_{e\in E(\Gamma)} V(a(e)) \otimes V(a(e))]^{SL_2(\mathbb{C})^{V(\Gamma)}}$ is multiplicity-free as well, with dimension 1 occurring precisely when $a \in P_\Gamma$. We fix a non-zero element $\Phi_a \in [\bigotimes_{e\in E(\Gamma)} V(a(e)) \otimes V(a(e))]^{SL_2(\mathbb{C})^{V(\Gamma)}}$. This choice defines a direct sum decomposition:

$$\mathbb{C}\big[M_\Gamma(SL_2(\mathbb{C}))\big] = \bigoplus_{a \in P_\Gamma} \mathbb{C}\Phi_a.$$

Let $R(\Gamma) \subset \mathbb{C}[M_\Gamma(SL_2(\mathbb{C}))]$ be the set composed of the Φ_a; these functions are called spin diagrams. For any valuation $v_\ell \in C_\Gamma$, the following formula holds by definition (see Figure 5):

$$v_\ell(\Phi_a) = \sum_{e \in E(\Gamma)} \ell(e)a(e).$$

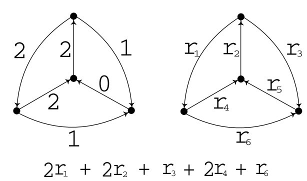

Figure 5: Evaluating a spin diagram element on the left in a valuation on the right.

Proposition 6.2 *For any $\ell \in C_\Gamma$, the space $v_\ell^{\leq r}$ is equal to the direct sum*

$$\bigoplus_{a \mid v_\ell(\Phi_a) \leq r} \mathbb{C}\Phi_a.$$

Furthermore, for any $f \in \mathbb{C}[M_\Gamma(SL_2(\mathbb{C}))]$ with $f = \sum C_a \Phi_a$ we have:

$$v_\ell(f) = \mathrm{MAX}\{v_\ell(\Phi_a) | C_a \neq 0\}.$$

Proof This is a consequence of the definition of the space $\mathbb{C}\Phi_a$ and Lemmas 3.2 and 3.1. ∎

We place a partial ordering on the lattice points in P_Γ, where $a \leq a'$ if $a' - a \in C_\Gamma$.

Proposition 6.3 *A product $\Phi_a \Phi_{a'}$ is a linear combination of Φ_t with $t \leq a + a'$; furthermore, the term $\Phi_{a+a'}$ always appears with a non-zero coefficient.*

Proof The first statement is a consequence of the corresponding fact for $\mathbb{C}[SL_2(\mathbb{C})]$. For the second statement, we choose a valuation v_ℓ with ℓ in the interior of C_Γ, and consider $v_\ell(\Phi_a\Phi_{a'}) = v_\ell(\Phi_a) + v_\ell(\Phi_{a'})$. If $\Phi_{a+a'}$ did not appear in $\Phi_a\Phi_{a'}$, then this element would be a linear combination of Φ_t with $t(e) < a(e) + a'(e)$ for some $e \in E(\Gamma)$. For each of these terms we must have $v_\ell(\Phi_t) < v_\ell(\Phi_a) + v_\ell(\Phi_{a'})$ by Formula 6.1, which contradicts that v_ℓ is a valuation. ∎

For any $f \in \mathbb{C}[M_\Gamma(SL_2(\mathbb{C}))]$ we can write $f = \sum_{a \in P_\Gamma} C_a \Phi_a$. If there is a $a \in P_\Gamma$ with $C_a \neq 0$ and $a' < a$ for all $C_{a'} \neq 0$, we say that Φ_a is the initial term of f, $\mathrm{in}(f) = C_a\Phi_a$. This notion can also be defined in the coordinate ring $\mathbb{C}[SL_2(\mathbb{C})^{E(\Gamma)}]$, where

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in(f) is taken to be the contribution from the isotypical space $\otimes_{e\in E(\Gamma)} V(a(e)) \otimes V(a(e))$ with a maximal under $<$. We note that the $SL_2(\mathbb{C})^{V(\Gamma)}$-stability of this decomposition implies that the initial term of an invariant $f \in \mathbb{C}[M_\Gamma(SL_2(\mathbb{C}))] \subset \mathbb{C}[SL_2(\mathbb{C})^{E(\Gamma)}]$ can be computed in either ring, with the same result.

The coordinate ring $\mathbb{C}[M_\Gamma(SL_2(\mathbb{C})^c)]$ has a decomposition into invariant spaces identical to the decomposition of $\mathbb{C}[M_\Gamma(SL_2(\mathbb{C}))]$ discussed above. Each of these spaces is of course one-dimensional, and the multiplication operation on $\mathbb{C}[SL_2(\mathbb{C})^c]$ is graded by the highest weights in the isotypical decomposition. The following is an immediate consequence of these observations.

Proposition 6.4 *The space $M_\Gamma(SL_2(\mathbb{C})^c)$ is the affine toric variety* $\mathrm{Spec}(\mathbb{C}[P_\Gamma])$.

6.2 An Alternative Construction of $M_\Gamma(SL_2(\mathbb{C}))$.

We give a different construction of the space $M_\Gamma(SL_2(\mathbb{C}))$, which is useful for describing the second spanning set $S(\Gamma) \subset \mathbb{C}[M_\Gamma(SL_2(\mathbb{C}))]$. Consider the following left diagonal GIT quotient:

$$M_k(SL_2(\mathbb{C})) = SL_2(\mathbb{C})\backslash\backslash SL_2(\mathbb{C})^k.$$

This space retains a right action by $SL_2(\mathbb{C})^k$.

For an oriented graph Γ, we form the product variety $\prod_{v\in V(\Gamma)} M_{\epsilon(v)}(SL_2(\mathbb{C}))$. This product is taken over vertices so each edge $e \in E(\Gamma)$ is represented twice, and each corresponding copy of $SL_2(\mathbb{C})$ has a right action by $SL_2(\mathbb{C})$ as above. Consequently, this product space comes with a residual action by $SL_2(\mathbb{C})^{E(\Gamma)} \times SL_2(\mathbb{C})^{E(\Gamma)}$. A right-hand side quotient by a diagonally embedded $SL_2(\mathbb{C})^{E(\Gamma)} \subset SL_2(\mathbb{C})^{2E(\Gamma)}$ gives the following identity:

$$\begin{aligned}
&\Big[\prod_{v\in V(\Gamma)} M_{\epsilon(v)}(SL_2(\mathbb{C}))\Big] /\!/ SL_2(\mathbb{C})^{E(\Gamma)}\\
&\quad\cong SL_2(\mathbb{C})^{V(\Gamma)}\backslash\backslash\Big[\prod_{e\in E(\Gamma)} ([SL_2(\mathbb{C})^2] /\!/ SL_2(\mathbb{C}))^{E(\Gamma)}\Big]\\
&\quad= M_\Gamma(SL_2(\mathbb{C})).
\end{aligned}$$

To see this, we view the group $SL_2(\mathbb{C})$ as an affine GIT quotient

$$\big[SL_2(\mathbb{C}) \times SL_2(\mathbb{C})\big] /\!/ SL_2(\mathbb{C}),$$

where $SL_2(\mathbb{C})$ acts on the right-hand side of both components. The equivariant $SL_2(\mathbb{C}) \times SL_2(\mathbb{C})$ isomorphism between these spaces is provided by the maps $(g,h) \to h^{-1}g$ and $g \to (g,1)$.

For an oriented tree $\mathcal{T}$, we define a space $M_{\mathcal{T}}(SL_2(\mathbb{C}))$ using the same quotient recipe:

$$M_{\mathcal{T}}(SL_2(\mathbb{C})) = \Big[\prod_{v\in V(\mathcal{T})} M_{\epsilon(v)}(SL_2(\mathbb{C}))\Big] /\!/ SL_2(\mathbb{C})^{E(\mathcal{T})}.$$

Lemma 6.5 *For a tree $\mathcal{T}$ with leaf set $\mathcal{L}(\mathcal{T})$, there is an isomorphism:*

$$\Phi_{\mathcal{T}}: M_{|\mathcal{L}(\mathcal{T})|}\big(SL_2(\mathbb{C})\big) \longrightarrow M_{\mathcal{T}}\big(SL_2(\mathbb{C})\big).$$

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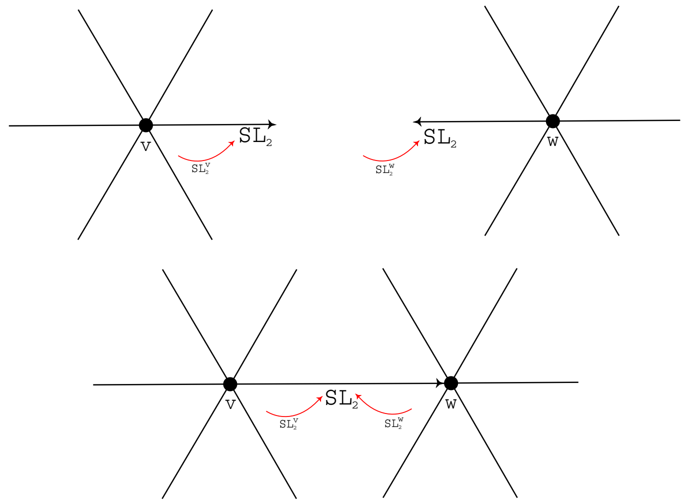

Figure 6: The isomorphism takes $SL_2(\mathbb{C})\backslash\backslash[SL_2(\mathbb{C}) \times SL_2(\mathbb{C})]$ with its two right hand actions to $SL_2(\mathbb{C})$ with its left and right actions.

Proof By repeatedly applying the isomorphism

$$\left[SL_2(\mathbb{C}) \times SL_2(\mathbb{C})\right] /\!/ SL_2(\mathbb{C}) \cong SL_2(\mathbb{C}),$$

we find that $M_{\mathcal{T}}(SL_2(\mathbb{C}))$ is isomorphic to the quotient space:

$$SL_2(\mathbb{C})^{V(\mathcal{T})}\backslash\backslash\left[SL_2(\mathbb{C})^{E(\mathcal{T})\cup\mathcal{L}(\mathcal{T})}\right].$$

For ease of notation we set $n = |\mathcal{L}(\mathcal{T})|$, and we let

$$\Phi_{\mathcal{T}}\colon M_{\mathcal{L}(\mathcal{T})}\big(SL_2(\mathbb{C})\big) \to M_{\mathcal{T}}\big(SL_2(\mathbb{C})\big)$$

be the map that assigns $(g_1, \dots, g_n)$ to the leaves of $\mathcal{T}$ and $\mathrm{Id} \in SL_2(\mathbb{C})$ to all edges, similar to the map ϕ_Γ from Section 5.

We choose a base point $V \in V(\mathcal{T})$ and define the map

$$\Psi_{V,\mathcal{T}}\colon M_{\mathcal{T}}(SL_2(\mathbb{C})) \to M_{\mathcal{L}(\mathcal{T})}(SL_2(\mathbb{C})).$$

This map sends $(\dots, g_e, \dots)$ to $(W_1(\dots, g_e, \dots), \dots, W_n(\dots, g_e, \dots))$, where W_i is the oriented word obtained by following the unique path from V to the i-th leaf of $\mathcal{T}$. Now the proof of Lemma 5.6 shows that $\Phi_{\mathcal{T}}$ and $\Psi_{V,\mathcal{T}}$ are inverses. ∎

For a graph Γ, and a spanning tree $\mathcal{T}$, we let $T(\mathcal{T}, \Gamma)$ be the tree obtained by splitting each edge in $E(\Gamma) \setminus E(\mathcal{T})$; see Figure 7. By once again making use of the isomorphism $[SL_2(\mathbb{C}) \times SL_2(\mathbb{C})]/\!/SL_2(\mathbb{C}) \cong SL_2(\mathbb{C})$, we observe that there is a natural quotient map associated with gluing the two ends of a split edge back together:

$$\pi_{\mathcal{T},\Gamma}\colon M_{T(\mathcal{T},\Gamma)}\big(SL_2(\mathbb{C})\big) \to M_\Gamma\big(SL_2(\mathbb{C})\big) = M_{T(\mathcal{T},\Gamma)}\big(SL_2(\mathbb{C})\big) /\!/ SL_2(\mathbb{C})^{E(\Gamma)\setminus E(\mathcal{T})}.$$

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We let π_0 be this quotient map for the special case $\Gamma = \Gamma_g$:

$$\pi_0 \colon M_{2g}\big(SL_2(\mathbb{C})\big) \longrightarrow M_{\Gamma_g}\big(SL_2(\mathbb{C})\big).$$

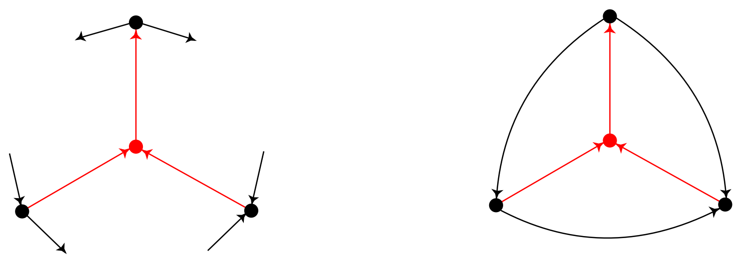

Figure 7: A graph Γ on the right, with spanning tree $\mathcal{T}$ in red, and the tree $T(\mathcal{T},\Gamma)$ on the left.

The following proposition allows us to relate our constructions on the graph Γ to those on the tree $T(\mathcal{T},\Gamma)$; it is a consequence of the argument used in Lemmas 5.6 and 6.5.

Proposition 6.6 *The maps $\Phi_{T(\mathcal{T},\Gamma)}$ and $\Psi_{T(\mathcal{T},\Gamma)}$ are isomorphisms. Furthermore, the following diagram commutes:*

$$\begin{tikzcd}
M_{T(\mathcal{T},\Gamma)}(SL_2(\mathbb{C})) \arrow[r, "\pi_{\mathcal{T},\Gamma}"] \arrow[d, shift right, "\Psi_{V,\mathcal{T}}"'] & M_\Gamma(SL_2(\mathbb{C})) \arrow[d, shift right, "\psi_{V,\Gamma}"'] \\
M_{2g}(SL_2(\mathbb{C})) \arrow[r, "\pi_0"] \arrow[u, shift right, "\Phi_{\mathcal{T}}"'] & M_{\Gamma_g}(SL_2(\mathbb{C})). \arrow[u, shift right, "\phi_\Gamma"']
\end{tikzcd}$$

Now we give a presentation of the coordinate ring $\mathbb{C}[M_n(SL_2(\mathbb{C}))]$ (this also appears in [21, Section 8]). The space $SL_2(\mathbb{C})^n$ is an affine subspace of the space of $2\times 2n$ matrices, cut out by n determinant equations. Below, $C_{i,j}$ is a column of an element of $M_{2,2n}(\mathbb{C})$:

$$SL_2(\mathbb{C})^n = \big\{[C_{1,1}, C_{1,2}, \dots, C_{n,1}, C_{n,2}] \in M_{2,2n}(\mathbb{C}) \mid \det(C_{i,1}, C_{i,2}) = 1\big\}.$$

We can realize $M_n(SL_2(\mathbb{C}))$ as a subvariety of $SL_2(\mathbb{C})\backslash\backslash M_{2,2n}(SL_2(\mathbb{C}))$. Following Weyl's First Fundamental Theorem of Invariant Theory, the latter is cut out of $\mathbb{C}^{\binom{2n}{2}}$ by Plücker equations. The invariant ring $\mathbb{C}[M_{2,2n}(SL_2(\mathbb{C}))]^{SL_2(\mathbb{C})}$ is generated by the forms $p_{(i,a),(j,b)} = \det(C_{i,a}, C_{j,b})$. The space $M_n(SL_2(\mathbb{C}))$ is then the intersection of the hypersurfaces in this quotient determined by the equations $p_{(i,1),(j,2)} = 1$.

Proposition 6.7 *Order the indices (i,a) lexicographically. The coordinate ring of $M_n(SL_2(\mathbb{C}))$ is generated by the $\binom{2n}{2}$ generators $p_{(i,a)}$, subject to the following equations, where $(i,a) < (j,b) < (k,c) < (l,d)$:*

$$p_{(i,a),(j,b)}p_{(k,c),(l,d)} + p_{(i,a),(l,d)}p_{(j,b),(k,c)} = p_{(i,a),(k,c)}p_{(j,b),(l,d)}, \quad p_{(i,1),(i,2)} = 1$$

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The algebra $\mathbb{C}[M_n(SL_2(\mathbb{C}))]$ controls the structure of $M_\Gamma(SL_2(\mathbb{C}))$ "at a vertex of valence n." Each index i is associated with an incident edge, and the ordering of the indices i above is tantamount to a choice of a cyclic ordering of the edges incident on such a vertex. For a graph Γ such a choice is known as a ribbon structure on Γ.

We will also make use of the "opposite" Plücker generators $-p_{(i,a)(j,b)} = p_{(j,b)(i,a)}$ when direction is important. Of course, $M_n(SL_2(\mathbb{C})) = SL_2(\mathbb{C})^{n-1}$, so this is perhaps an overly complicated presentation; however, the combinatorics of the Plücker generators and equations play an important role in describing the set $S(\Gamma) \subset \mathbb{C}[M_\Gamma(SL_2(\mathbb{C}))]$. Throughout we will use "direction" to mean the ordering of indices, as in $p_{(i,a),(j,b)}$ goes from i to j, and "sign" to mean the data a, b.

6.3 The Space $M_3(SL_2(\mathbb{C}))$

Let v_1, v_2, v_3 be three copies of $v: \mathbb{C}[SL_2(\mathbb{C})] \to \mathbb{Z} \cup \{-\infty\}$ defined on

$$\mathbb{C}\big[M_3(SL_2(\mathbb{C}))\big] \subset \mathbb{C}\big[SL_2(\mathbb{C})^3\big].$$

We apply Lemmas 3.2 and 3.1 to show that the associated graded algebra of an interior element of $C_3 = \mathbb{R}_{\geq 0}\{v_1, v_2, v_3\}$ is $\mathbb{C}[SL_2(\mathbb{C})^c \times SL_2(\mathbb{C})^c \times SL_2(\mathbb{C})^c]^{SL_2(\mathbb{C})} = \mathbb{C}[M_3(SL_2(\mathbb{C})^c)]$, and we show that this is an affine semigroup algebra.

Definition 6.8 The cone $\mathcal{P}_3$ is defined to be the set of labelings of the following diagram by non-negative real numbers that satisfy $x_1 + y_1 = a, x_2 + y_2 = b, x_3 + y_3 = c$, such that a, b, c form the sides of a triangle, $|a - c| \leq b \leq a + c$. The lattice L_3 is the set of integer labelings of this diagram with $a + b + c \in 2\mathbb{Z}$. We let P_3 be the affine semigroup $\mathcal{P}_3 \cap L_3$, see Figure 8.

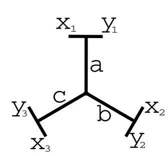

Figure 8: Defining diagram of P_3.

Proposition 6.9 *The algebra $\mathbb{C}[M_3(SL_2(\mathbb{C})^c)]$ is isomorphic to the affine semigroup algebra $\mathbb{C}[P_3]$.*

Proof Following Section 4, $M_3(SL_2(\mathbb{C})^c)$ can be rewritten as the GIT quotient

$$SL_2(\mathbb{C})\backslash\!\backslash\big[(SL_2(\mathbb{C})/\!/U_+)^3\big] \times \big[(U_-\backslash\!\backslash SL_2(\mathbb{C}))^3\big]/\!/(\mathbb{C}^*)^3.$$

We compute this as a reduction in steps. The quotient $SL_2(\mathbb{C})\backslash\!\backslash[(SL_2(\mathbb{C})/\!/U_+)^3] = SL_2(\mathbb{C})\backslash\!\backslash[(\mathbb{C}^2)^3]$ is isomorphic to the affine space $\bigwedge^2(\mathbb{C}^3)$ by the First Fundamental Theorem of Invariant Theory. It can be viewed as the variety associated with the affine semigroup algebra $\mathbb{C}[Q_3]$, where Q_3 is the set of triples a, b, c as in the definition of

P_3 above. Here a, b, c are the highest weights of the components

$$\big(V(a) \otimes V(b) \otimes V(c)\big)^{SL_2(\mathbb{C})} \subset SL_2(\mathbb{C})\backslash\backslash\big[(\mathbb{C}^2)^3\big].$$

Recall that the character spaces of the diagonal action of $\mathbb{C}^*$ on $SL_2(\mathbb{C})//U = \mathbb{C}^2$ are precisely the $V(a) \subset \mathbb{C}[\mathbb{C}^2]$. It follows that $\mathbb{C}[M_3(SL_2(\mathbb{C}))]$ is graded by the subspaces $((V(a)\otimes V(b)\otimes V(c))^{SL_2(\mathbb{C})} \otimes V(a)\otimes V(b)\otimes V(c)$. Since $\mathbb{C}[U_-\backslash\backslash SL_2(\mathbb{C})] = \mathbb{C}[\mathbb{C}^2]$ is itself toric, we can further grade this algebra by viewing each $V(a)$ as a direct sum of the monomial spaces $\mathbb{C}X^{x_1}Y^{y_1}$, where $x_1 + y_1 = a$. A similar decomposition holds for $V(b), V(c)$. ∎

The affine semigroup P_3 is generated by the $3 \times 4 = 12$ weightings with one of a, b, c equal to 0 and the other two entries equal to 1. We label these generators $X_{(i,s),(j,t)}$, indicating a path from i to j with sign markings on the s, t ends of these paths, respectively. Now we recall the forms $p_{(i,s)(j,t)} \in \mathbb{C}[M_3(SL_2(\mathbb{C}))]$. If we view $M_3(SL_2(\mathbb{C}))$ as the space of 2×6 matrices $[C_{1,1}C_{1,2}C_{2,1}, C_{2,2}, C_{3,1}C_{3,2}]$ that satisfy $\det(C_{i,1}C_{i,2}) = 1$, then $p_{(i,s),(j,t)}$ is the function $\det(C_{i,s}C_{j,t})$. We let the initial form $\mathrm{in}(f) \in \mathbb{C}[M_3(SL_2(\mathbb{C})^c)]$ (when it exists) for $f \in \mathbb{C}[M_3(SL_2(\mathbb{C}))]$ be as in Subsection 6.1.

Proposition 6.10 *The initial forms* $\mathrm{in}(p_{(i,s),(j,t)})$ *give a generating set of* P_3.

Proof Recall that the algebra $\mathbb{C}[SL_2(\mathbb{C})^c]$ is isomorphic to $\mathbb{C}[\mathbb{C}^2 \times \mathbb{C}^2]^{\mathbb{C}^*}$, and the isomorphism does the following to generators: $A \to X \otimes X$, $B \to X \otimes Y$, $C \to Y \otimes X$, $D \to Y \otimes Y$. We consider the image of the tensor $p_{(1,1),(2,1)} \in \mathbb{C}[M_3(SL_2(\mathbb{C})^c]$, given by the determinant

$$A_1C_2 - A_2C_1 = (X_1 \otimes X_1)(Y_2 \otimes X_2) - (X_2 \otimes X_2)(Y_1 \otimes X_1) = (X_1Y_2 - X_2Y_1) \otimes X_1X_2.$$

We obtain the Plücker invariant

$$X_1Y_2 - X_2Y_1 \in \mathbb{C}[\mathbb{C}^2 \times \mathbb{C}^2 \times \mathbb{C}^2]^{SL_2(\mathbb{C})} = \mathbb{C}[M_3(SL_2(\mathbb{C})//U_-^3]$$

tensored with the leaf data X_1X_2; this is precisely the element $X_{(1,1),(2,1)} \in P_3$. This computation works for all generators $X_{(i,a)(j,b)}$. ∎

Note that if generators $p_{(i_1,a_1),(j_1,b_1)}, \dots, p_{(i_n,a_n),(j_n,b_n)}$ are multiplied, the x_i, y_i components of the initial term $\mathrm{in}(\prod p_{(i_s,a_s),(j_s,b_s)})$ records the number of each type of sign that appears; see Figure 9.

6.4 Γ-tensors

For a graph Γ, we introduce a combinatorial object called an abstract Γ-tensor $V(S, \phi)$ (see Figure 10); this is the following information (see also [21, Section 8]):

(a) A set of reduced directed paths S_v in each link $\Gamma_v \subset \Gamma$.
(b) For each edge $e \in E(\Gamma)$ with $\delta(e) = (v, u)$, a direction preserving bijection $\phi_e : S_{v,e} \to S_{u,e}$ between the paths through e in S_v and S_u.

A marking A of an abstract Γ-tensor is an assignment $A: S_v \to \{1, 2\}^2$ to the end points of each path such that the label on meeting endpoints of two paths connected by one of the maps ϕ_e are different. When a path $p_{ij} \in S_v$ is marked $A(i) = a, A(j) = b$,

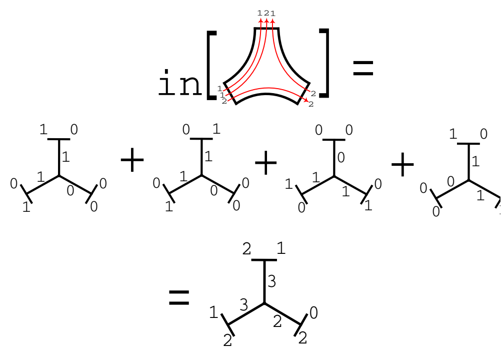

Figure 9: Multiplying initial terms in P_3.

it can be viewed as a Plücker generator $p_{(i,a),(j,b)} \in \mathbb{C}[M_{\epsilon(v)}(SL_2(\mathbb{C}))]$. In this way an abstract Γ tensor $V(S,\phi)$ with A defines a monomial

$$\mathcal{V}(S,\phi,A) \in \mathbb{C}\Big[\prod_{v\in V(\Gamma)} M_{\epsilon(v)}(SL_2(\mathbb{C}))\Big].$$

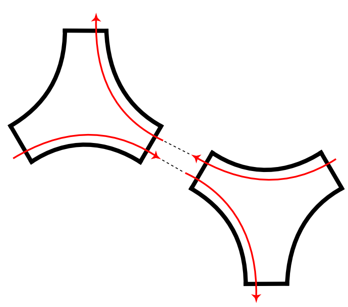

Figure 10: An abstract Γ-tensor

By [21, Proposition 8.5] (also see Proposition 6.15), we obtain an invariant form $\mathcal{V}(S,\phi) \in \mathbb{C}[M_\Gamma(SL_2(\mathbb{C}))] \subset \mathbb{C}[\prod_{v\in V(\Gamma)} M_{\epsilon(v)}(SL_2(\mathbb{C}))]$ by taking a signed sum over all $2^{E(\Gamma)}$ possible markings.

$$\mathcal{V}(S,\phi) = \sum_A (-1)^{\rho(A)}\mathcal{V}(S,\phi,A).$$

Here $(-1)^{\rho(A)}$ is a product of signs defined by ϕ_e and the marking A at an edge. For each identification of endpoints $\phi_e(i) = j$, there is a contribution $(-1)^{\rho(\phi_e,i,j)}$, which is positive if $a(i) = 1, a(j) = 2$ and i is outgoing, and negative if $a(i) = 1, a(j) = 2$ and i is incoming. This assignment is illustrated on an example in Figure 11.

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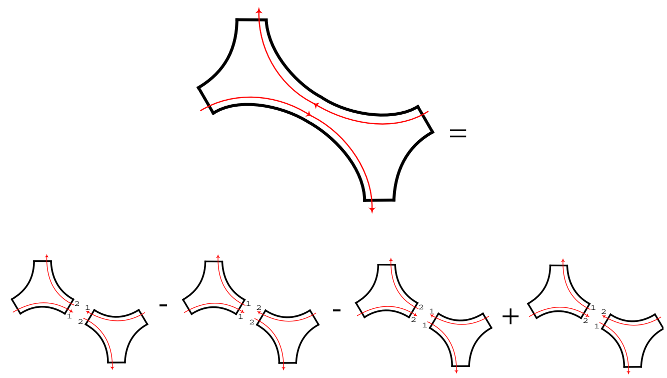

Figure 11: A Γ tensor obtained as a signed sum over sign assignments to an abstract Γ-tensor

We let $\mathcal{S}(\Gamma) \subset \mathbb{C}[M_\Gamma(SL_2(\mathbb{C}))]$ be the set of all the forms $\mathcal{V}(S,\phi)$. Each abstract Γ-tensor $V(S,\phi)$ corresponds naturally to a weighting $a(S,\phi)\colon E(\Gamma) \to \mathbb{Z}_{\geq 0}$, where $a(S,\phi)(e)$ is the number of paths passing through e. The next proposition relates $\mathcal{V}(S,\phi)$ to the spin element $\Phi_{a(S,\phi)} \in \mathbb{C}[M_\Gamma(SL_2(\mathbb{C}))]$ when Γ is trivalent.

Proposition 6.11 *Let Γ be trivalent; then the initial form* $\mathrm{in}(\mathcal{V}(S,\phi))$ *of a Γ-tensor is a non-zero multiple of $\Phi_{a(S,\phi)}$.*

Proof We observe that the initial form $\mathrm{in}(\mathcal{V}(S,\phi)) \in \mathbb{C}[M_\Gamma(SL_2(\mathbb{C})^c)]$ can be computed in the algebra $\mathbb{C}[M_3(SL_2(\mathbb{C}))^{V(\Gamma)}]$; the $SL_2(\mathbb{C})^{E(\Gamma)}$ stability of the filtrations involved guarantees that the initial form of an invariant in $\mathbb{C}[M_3(SL_2(\mathbb{C}))^{V(\Gamma)}]$ is also invariant. The initial form of $\mathrm{in}(\mathcal{V}(S,\phi,A))$ is a tensor product (over $v \in V(\Gamma)$) of initial forms of the monomials defined by S in $\mathbb{C}[M_3(SL_2(\mathbb{C}))]$. Each of the $\mathrm{in}(\mathcal{V}(S,\phi,A))$ lies in the $a(S,\phi)$ isotypical component of $\mathbb{C}[M_3(SL_2(\mathbb{C}))^{V(\Gamma)}]$, so it follows that $\mathrm{in}(\mathcal{V}(S,\phi))$ equals the sum of these terms. Initial forms with different markings on a path at an edge are linearly independent, as this is the case for the associated elements of $\mathbb{C}[P_3^{V(\Gamma)}]$. Since $\mathcal{V}(S,\phi)$ always has exactly one term with all outgoing assignments 1, this sum cannot vanish. ∎

Corollary 6.12 *A valuation $v_\ell \in C_\Gamma$ is computed on a Γ-tensor*

$$\mathcal{V}(S,\phi) \in \mathbb{C}[\mathcal{X}(F_g, SL_2(\mathbb{C}))]$$

as follows:

$$v_\ell(\mathcal{V}(S,\phi)) = \sum_{e \in E(\Gamma)} \ell(e) a(S,\phi)(e).$$

We have stated this corollary for trivalent Γ, but the general case can be recovered by considering a weighting on Γ as a weighting with 0 entries on a trivalent cover $\widetilde{\Gamma} \to \Gamma$.

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6.5 Γ-tensors and F_g

For a marking $\phi: \Gamma_g \to \Gamma$, let $S(\Gamma, \phi) \subset \mathbb{C}[\mathcal{X}(F_g, SL_2(\mathbb{C})]$ be the pullback of $S(\Gamma) \subset \mathbb{C}[M_\Gamma(SL_2(\mathbb{C}))]$. We show that the sets $S(\Gamma, \phi) \subset \mathbb{C}[\mathcal{X}(F_g, SL_2(\mathbb{C}))]$ all coincide and how to compute them as regular functions. From each connected abstract Γ tensor $V(S, \phi)$ we obtain a reduced word $\omega(S, \phi) \in F_g$, this defines a bijection between abstract Γ-tensors and elements of $\langle F_g \rangle$. The Γ-tensor $\mathcal{V}(S, \phi)$ is then shown to coincide with the trace-word function $\tau_{\omega(S,\phi)}$ (see Figure 13). The fact that these two sets of regular functions are the same is a consequence of classical results in invariant theory (*i.e.*, the classical results of Procesi, [28]); we give a presentation that emphasizes the combinatorial features of these functions. We conclude by showing that for a valuation $v_{\Gamma,\ell,\phi} \in C_{\Gamma,\phi}$, $v_{\Gamma,\ell,\phi}(\tau_{\omega(S,P)})$ is the evaluation of the length function $d_{\omega(S,P)}$ on the metric graph $(\Gamma, \ell, \phi) \in \widehat{O}(g)$.

In order to identify the $\mathcal{V}(S, \phi)$ as regular functions, we consider similar forms defined on trees. A $\mathcal{T}$-tensor $\mathcal{V}(S, \phi, B) \in \mathbb{C}[M_\mathcal{T}(SL_2(\mathbb{C}))]$ is defined in the same way as a Γ-tensor, with the addition of sign information B specified at the leaves. By construction, any $\mathcal{V}(S, \phi, B)$ factors as a monomial in simple paths $\mathcal{V}(S_\gamma, \phi_\gamma, B_\gamma)$; the next proposition shows that each of these path tensors can be considered to be a Plücker generator from Subsection 6.4.

Proposition 6.13 *The set of $\mathcal{T}$-tensors $\mathcal{S}(\mathcal{T}) \subset \mathbb{C}[M_\mathcal{T}(SL_2(\mathbb{C}))]$ coincides with the set of Plücker monomials in $\mathbb{C}[M_{\mathcal{L}(\mathcal{T})}(SL_2(\mathbb{C}))]$ under the isomorphisms $\Phi_\mathcal{T}$ and $\Psi_\mathcal{T}$.*

Proof This straightforward calculation is also covered in [21, Proposition 8.4]. ∎

We use this proposition to prove that $\mathcal{S}(\Gamma, \phi)$ coincides with the set

$$\mathcal{S}(\Gamma_g) \subset \mathbb{C}\big[\mathcal{X}(F_g, SL_2(\mathbb{C}))\big]$$

for $\psi_{V,\mathcal{T}}: \Gamma_g \to \Gamma$ and $\phi_\mathcal{T}: \Gamma \to \Gamma_g$ a distinguished pair of maps corresponding to a spanning tree $\mathcal{T} \subset \Gamma$. Recall the maps $\Phi_\mathcal{T}$, $\Psi_\mathcal{T}$, and $\pi_\mathcal{T}$ from Proposition 6.6.

Proposition 6.14 *The set of Γ-tensors $\mathcal{S}(\Gamma) \subset \mathbb{C}[M_\Gamma(SL_2(\mathbb{C}))]$ coincides with the set of Γ_g-tensors in $\mathbb{C}[M_{\Gamma_g}(SL_2(\mathbb{C}))]$ under the isomorphisms $\Phi_\mathcal{T}$ and $\Psi_\mathcal{T}$.*

Proof This follows from the observation (see [21, Proposition 8.5]) that any Γ (respectively Γ_g-tensor) can be decomposed into a sum of $T(\mathcal{T}, \Gamma)$-tensors, using the map $\pi_\mathcal{T}$.

$$\mathcal{V}_\Gamma(S, \phi) = \sum_B \mathcal{V}_{T(\mathcal{T},\Gamma)}(S_{T(\mathcal{T},\Gamma)}, \phi_{T(\mathcal{T},\Gamma)}, B)(-1)^{\rho(B)}.$$

Each of the components $\mathcal{V}_{T(\mathcal{T},\Gamma)}(S_{T(\mathcal{T},\Gamma)}, \phi_{T(\mathcal{T},\Gamma)}, B)$ is a monomial in Plücker generators by Proposition 6.13. This sum therefore amounts to an element of $\mathcal{S}(\Gamma_g)$. This argument can then be run in reverse to give the other inclusion. ∎

Next we show that the set $\mathcal{S}(\Gamma_g) \subset \mathbb{C}[M_{\Gamma_g}(SL_2(\mathbb{C}))]$ coincides with another set of distinguished regular functions on $M_{\Gamma_g}(SL_2(\mathbb{C})) = \mathcal{X}(F_g, SL_2(\mathbb{C}))$. The trace-word function τ_ω associated with a word $\omega \in F_g$ is the regular function that takes $(A_1, \dots, A_g) \in \mathcal{X}(F_g, SL_2(\mathbb{C}))$ to the complex number $\mathrm{tr}(\omega(A_1, \dots, A_g))$. Let $\mathcal{W}_g \subset$

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$\mathbb{C}[\mathcal{X}(F_g, SL_2(\mathbb{C}))]$ be the set of monomials in these functions, we will prove the following proposition with a series of lemmas.

Proposition 6.15 *The set $\mathcal{S}(\Gamma_g)$ coincides with $\mathcal{W}_g$.*

Notice that $\mathcal{W}_g$ is naturally closed under the action of the outer automorphism group $\mathrm{Out}(F_g)$ on $\mathbb{C}[\mathcal{X}(F_g, SL_2(\mathbb{C}))]$, so Proposition 6.15 implies that $\mathcal{S}(\Gamma, \phi) = \mathcal{S}(\Gamma_g) \subset \mathbb{C}[\mathcal{X}(F_g, SL_2(\mathbb{C}))]$ for ϕ any composition of an outer automorphism with a distinguished graph map. Proposition 5.7 then implies that this is the case for any marking $\phi: \Gamma_g \to \Gamma$.

First we show that these sets are bijective combinatorially. Let $\mathcal{S}_{\mathrm{con}}(\Gamma)$ be the set of connected abstract Γ-tensors. Fix a connected abstract Γ_g-tensor $V(S, \phi) \in \mathcal{S}_{\mathrm{con}}(\Gamma_g)$. We let $T(\Gamma_g, v)$ be the $2g$ tree obtained from the "spanning tree" $\{v\} \subset \Gamma_g$. We label the leaves of $T(\Gamma_g, v)$ with the $2g$ indices $2i-1, 2i$ $1 \le i \le g$, where the $2i-1$ and $2i$ leaves are identified in the quotient map $T(\Gamma_g, v) \to \Gamma_g$. We let $p_{(i_1, j_1)} \in S$ be a path in $V(S, \phi)$. The endpoint j_1 is connected with the starting point of the next path $p_{(i_2, j_2)}$, so $j_1 = 2a_1 - 1, i_2 = 2a_1$ or $j_1 = 2a_1, i_2 = 2a_1 - 1$. Continuing this way, we obtain a word $\omega(V(S, \phi), p_{(i_1, j_1)}) = x_{a_1}^{\epsilon_1} x_{a_2}^{\epsilon_2} \cdots x_{a_m}^{\epsilon_m}$, where the sign ϵ_i is determined by the rule $j_1 = 2a_1 - 1, i_2 = 2a_1 = 2 \Rightarrow \epsilon_i = 1$, $j_1 = 2a_1, i_2 = 2a_1 - 1 \Rightarrow \epsilon_i = -1$. By construction, this word is reduced, as Γ_g tensors do not have back-tracks by definition. Given a word ω, we may reverse this recipe to obtain a Γ_g-tensor $V(S_\omega, \phi_\omega)$. All paths in $T(\Gamma_g, v)$ are determined by their endpoints, so we have a $1-1$ map from the set of connected Γ_g-tensors $V(S, \phi)$ with a choice of initial path $p_{(i_1, j_1)} \in S$ to the set of reduced words in F_g. Changing the initial path amounts to changing the word $\omega(V(S, \phi), p_{(i_1, j_1)})$ by a cyclic permutation, so we have proved the following lemma.

Lemma 6.16 *The set $\mathcal{S}_{\mathrm{con}}(\Gamma_g)$ is in bijection with cyclic equivalence classes of reduced words in F_g.*

By Proposition 6.14, the same is true for the connected Γ tensors $\mathcal{S}_{\mathrm{con}}(\Gamma)$ of any graph Γ. The recipe to compute $\omega(V(S, \phi), p_1) \in F_g$ is almost identical in the general case, except one replaces a simple path $p_{(i_s, j_s)}$ with the shortest path p_s in the chosen spanning tree $\mathcal{T} \subset \Gamma$.

Now we show that any connected Γ_g-tensor can be "untangled," in the sense that it is realized as the pullback of a fixed $\Gamma_{g'}$ tensor by a map $\phi_*: M_{\Gamma_{g'}}(SL_2(\mathbb{C})) \to M_{\Gamma_g}(SL_2(\mathbb{C}))$, where $\Gamma_{g'}$ has a different genus.

Definition 6.17 We let $\mathcal{V}_g \in \mathbb{C}[M_{\Gamma_g}(SL_2(\mathbb{C}))] = \mathbb{C}[SL_2(\mathbb{C}) \backslash\backslash SL_2(\mathbb{C})^{2g} // SL_2(\mathbb{C})^g]$ be the Γ_g-tensor defined by joining the paths $p_{(2g,1)} \to p_{(2,3)} \to \cdots \to p_{(2g-2, 2g-1)}$, as in Figure 12.

We fix a reduced word $\omega \in F_g$, and we let $n = |\omega|$. There is a natural map of $\phi_\omega: \mathcal{X}(F_g, SL_2(\mathbb{C})) \to \mathcal{X}(F_n, SL_2(\mathbb{C}))$ defined by sending $(A_1, \dots, A_g)$ to the entries of ω in order, $(\omega_1(\vec{A}), \dots, \omega_n(\vec{A}))$. Recall that $\mathcal{X}(F_g, SL_2(\mathbb{C}))$ is the quotient

$$SL_2(\mathbb{C}) \backslash\backslash SL_2(\mathbb{C})^{2g} // SL_2(\mathbb{C})^g = M_{2g}(SL_2(\mathbb{C})) // SL_2(\mathbb{C})^g,$$

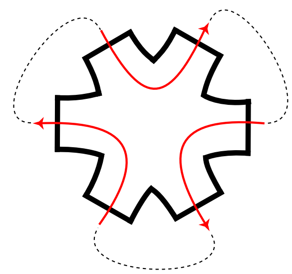

Figure 12: The tensor $\mathcal{V}_3$

by way of the map $\pi_{0,g}(k_1, h_1, \ldots, k_g, h_g) = (k_1h_1^{-1}, \ldots, k_gh_g^{-1})$. We define a map ϕ'_ω by sending the tuple $(k_1, h_1, \ldots, k_g, h_g)$ to $(\omega_1(\vec{k}), \omega_1(\vec{h}), \ldots, \omega_n(\vec{k}), \omega_n(\vec{h}))$, this commutes with ϕ_ω under the isomorphisms $\pi_{0,g}, \pi_{0,n}$. We will see where $\mathcal{V}_g$ goes under ϕ^*_ω.

Lemma 6.18 $\phi^*_\omega(\mathcal{V}_n) = \mathcal{V}(S_\omega, \phi_\omega)$

Proof The Plücker element $p_{(2i,a),(2i+1,b)}$, evaluated on $\phi'_\omega(k_1, h_1, \ldots, k_g, h_g)$, gives the determinant of the a column of $\omega_i(\vec{h})$ and the b column of $\omega_{i+1}(\vec{k})$. The pullback $\phi^*_\omega(\mathcal{V}_n)$ therefore coincides with $\mathcal{V}(S_\omega, \phi_\omega)$ by definition. ∎

We now introduce the model trace-word $\tau_g = \mathrm{tr}(A_1A_2\ldots A_g)$. By the previous lemma, if we can show that $\tau_n = \mathcal{V}_n \in \mathbb{C}[\mathcal{X}(F_n, SL_2(\mathbb{C}))]$, we would have $\mathcal{V}(S_\omega, \phi_\omega) = \tau_\omega = \mathrm{tr}(\omega(\vec{A}))$, and Proposition 6.15 would follow.

Lemma 6.19 $\tau_n = \mathcal{V}_n \in \mathbb{C}[\mathcal{X}(F_n, SL_2(\mathbb{C}))]$

Proof This can be found in [28, Theorem 1.2]. ∎

For each metric graph (Γ, ℓ), with a marking $\phi\colon \Gamma_g \to \Gamma$, with corresponding valuation $v_{\Gamma,\ell,\phi}\colon \mathbb{C}[\mathcal{X}(F_g, SL_2(\mathbb{C}))] \to \mathbb{Z} \cup \{-\infty\}$, the following holds by Corollary 6.12:

$$v_{\Gamma,\ell,\phi}(\tau_\omega) = v_{\Gamma,\ell,\phi}\big(\mathcal{V}(S_\omega, \phi_\omega)\big) = d_\omega(\Gamma, \ell, \phi).$$

6.6 Proof of Theorem 1.1

Now we finish the proof of Theorem 1.1 by showing that $\Sigma\colon \widehat{O}(g) \to \mathcal{X}(F_g, SL_2(\mathbb{C}))^{an}$, $\Sigma(\Gamma, \ell, \phi) = v_{\Gamma,\ell,\phi}$ gives an embedding. Outer space $O(g)$ is the closed subset of $\widehat{O}(g)$ of marked metric graphs (Γ, ℓ, ϕ) with volume 1; this also gives an embedding of outer space into $\mathcal{X}(F_g, SL_2(\mathbb{C}))^{an}$.

The topology on $\mathcal{X}(F_g, SL_2(\mathbb{C}))^{an}$ is generated by the the sets $ev_f^{-1}(U)$ where the $U \subset \mathbb{R}$ are open intervals. This is similar to the topology on $\widehat{O}(g)$, which is generated by the sets $d_w^{-1}(U)$ ([6]). Each length function on $\widehat{O}(g)$ extends to a continuous function on $\mathcal{X}(F_g, SL_2(\mathbb{C}))^{an}$ by Equation 6.5, so it follows that any open set in $\widehat{O}(g)$ is

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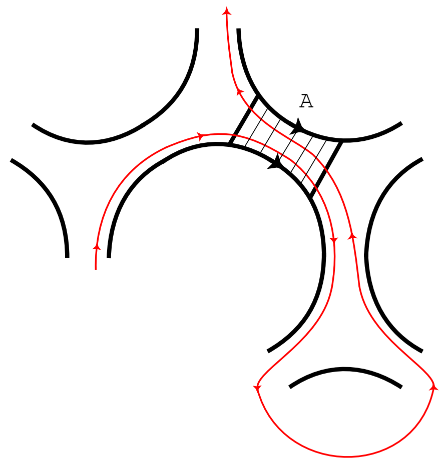

Figure 13: A path representing $\mathrm{tr}(\dots, A, \dots, A^{-1}, \dots)$

induced from its inclusion into $\mathcal{X}(F_g, SL_2(\mathbb{C}))^{an}$. It remains to check that any evaluation function is continuous on $\widehat{O}(g)$, as this would imply that no new open sets are induced from $f \in \mathbb{C}[\mathcal{X}(F_g, SL_2(\mathbb{C}))]$, $f \neq \tau_\omega$.

Note that f can be expanded in a unique way into (Γ, ϕ) spin diagrams $\Phi_a \in \mathcal{R}(\Gamma)$, and that the value $v_{\Gamma,\ell,\phi}(f)$ is obtained on one or more of these diagrams. Now we can replace each spin diagram function Φ_a with a lower-triangular expansion into trace-word functions $\Phi_a = \sum \tau_{w_{a,i}}$, with $v_{\Gamma,\ell,\phi}(\Phi_a) = v_{\Gamma,\ell,\phi}(\tau_{w_{a,1}})$, the value of the leading term. The equivalence class of Φ_a equals that of $\tau_{w_{a,1}}$ in the associated graded algebra $\mathbb{C}[P_\Gamma]$ of any interior point of $C_{\Gamma,\phi}$; it follows that

$$v_{\Gamma,\ell,\phi}(f) = \max_{C_a \neq 0}\{\dots, v_{\Gamma,\ell,\phi}(\Phi_a), \dots\} = \max_{C_a \neq 0}\{\dots, v_{\Gamma,\ell,\phi}(\tau_{w_{a,1}}), \dots\}$$

for any $\ell \in C_{\Gamma,\phi}$. This shows that ev_f is the max of a finite number of continuous functions over the simplicial cone $C_{\Gamma,\phi}$. As the function ev_f is continuous when restricted to any $C_{\Gamma,\phi}$, it must be continuous on $\widehat{O}(g)$ by the Pasting Lemma. This proves Theorem 1.1.

6.7 The Tropical Variety $\mathbb{T}(I_{2,g})$ and Υ_g

We refer the reader to the book of Maclagan and Sturmfels [24] for background on tropical geometry. We will need the following result (see [27]) relating the analytification X^{an} of an affine variety to the tropical variety $\mathbb{T}(I)$ of an ideal that cuts X out of affine space.

Theorem 6.20 (Payne) *Let $\pi\colon k[x_1, \dots, x_n] \to k[X]$ be a presentation of the coordinate ring of X with $I = \mathrm{Ker}(\pi)$; then the image of the map $(ev_{\pi(x_1)}, \dots, ev_{\pi(x_n)})\colon X^{an} \to \mathbb{R}^n$ is the tropical variety $\mathbb{T}(I)$.*

We fix a set of generators $x_1, \dots, x_g \in F_g$ and consider the set $S_{2,g}$ of trace word functions where no x_i appears more than twice, including inverses and multiplicity. For a parameter set $Y_{2,g}$ in bijection with $S_{2,g}$ we let $I_{2,g} \subset \mathbb{C}[Y_{2,g}]$ be the ideal of forms which vanishes on the associated map $\mathbb{C}[Y_{2,g}] \to \mathbb{C}[\mathcal{X}(F_g, SL_2(\mathbb{C}))]$. The choice of generators $x_1, \dots, x_g$ determines an embedding of the space of spanned metric graphs Υ_g into $\widehat{O}_g$ (Subsection 2.3). We finish this section with a proof of the following theorem.

Theorem 6.21 *The map $(\dots, ev_{\tau_w}, \dots)_{w \in S_{2,g}} \colon X(F_g, SL_2(\mathbb{C}))^{an} \to \mathbb{T}(I_{2,g})$ is $1-1$ on $\Upsilon_g \subset \widehat{O}_g$.*

The general principle at work in this statement is encapsulated in the following lemma.

Lemma 6.22 *Let $G \subset \mathbb{C}[\mathcal{X}(F_g, SL_2(\mathbb{C}))]$ be any set so that the equivalence classes of the elements $f \in X$ generate $gr_{v_{\Gamma,\mathcal{T},\ell}}(\mathbb{C}[\mathcal{X}(F_g, SL_2(\mathbb{C}))]) \cong \mathbb{C}[P_\Gamma]$ for all $(\Gamma, \mathcal{T}, \ell) \in \Upsilon_g$; then the map $(\dots, ev_f, \dots)_{f \in G}$ is $1-1$ on Υ_g.*

Proof A choice $(\Gamma, \mathcal{T})$ determines a basis $\mathcal{R}(\Gamma, \mathcal{T}) \subset \mathbb{C}[\mathcal{X}(F_g, SL_2(\mathbb{C}))]$ of spin diagrams. Each $f \in G$ has an expansion $f = \sum C_a \Phi_a$ in these diagrams, and the equivalence class of f in $gr_{v_\ell}(\mathbb{C}[\mathcal{X}(F_g, SL_2(\mathbb{C}))])$ for ℓ an interior point of $C_{\Gamma,\mathcal{T}} \subset \Upsilon_g$ is the initial form of f with respect to v_ℓ. By assumption the a of these initial forms generate P_Γ, and by Proposition 6.2 the values on these generators determine all outputs of the valuation v_ℓ. It follows that if $v_, v_2 \in \Upsilon_g$ have the same values on every element of G, then they must be the same valuation. ∎

Theorem 6.21 is a consequence of Lemma 6.22 and the following proposition.

Proposition 6.23 ([22, Theorem 7.2]) *The affine semigroup P_Γ is generated by those $a \colon E(\Gamma) \to \mathbb{Z}_{\geq 0}$ with $a(e) \leq 2$.*

The set $S_{2,g}$ is said to be a Khovanskii basis of $\mathbb{C}[\mathcal{X}(F_g, SL_2(\mathbb{C}))]$ with respect to each valuation v_ℓ with $\ell \in \Upsilon_g$ because the equivalence classes of these functions generate each associated graded algebra of each v_ℓ. This terminology is used at the suggestion of Bernd Sturmfels in honor of the contributions of Askold Khovanskii to the intersection of combinatorics and algebraic geometry. Results from [22] also imply that $S_{2,g}$ is a Khovanskii basis for a collection of maximal rank (*i.e.*, $\dim(\mathcal{X}(F_g, SL_2(\mathbb{C})))$) valuations $v_{\Gamma,<}$ (see Section 8.4).

7 The Integrable Hamiltonian System in $\mathcal{X}(F_g, SL_2(\mathbb{C}))$ Associated with Γ

In this section we construct the spaces $M_\Gamma(SL_2(\mathbb{C}))$ and $M_\Gamma(SL_2(\mathbb{C})^c)$ as symplectic reductions. We show that each distinguished map $\phi_\mathcal{T} \colon \Gamma \to \Gamma_g$ associated with a spanning tree $\mathcal{T}$ gives a symplectomorphism $\phi_\mathcal{T}^* \colon \mathcal{X}(F_g, SL_2(\mathbb{C})) \to M_\Gamma(SL_2(\mathbb{C}))$. Then we construct a surjective, continuous contraction map $\Xi_\Gamma \colon M_\Gamma(SL_2(\mathbb{C})) \to M_\Gamma(SL_2(\mathbb{C})^c)$, and we show that this map is a symplectomorphism on a dense,

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open subspace $M^o_\Gamma(SL_2(\mathbb{C})) \subset M_\Gamma(SL_2(\mathbb{C}))$. The compact part $\mathbb{T}_\Gamma$ of the defining torus T_Γ of the toric variety $M_\Gamma(SL_2(\mathbb{C})^c)$ is used to define an integrable system on $M^o_\Gamma(SL_2(\mathbb{C}))$ with momentum map $\mu_{\mathbb{T}_\Gamma} \circ \Xi_\Gamma$. We show that the fibers of the contraction map Ξ_Γ are all connected, and that the fiber over the origin is the compact character variety $\mathcal{X}(F_g, SU(2))$.

7.1 $M_\Gamma(SL_2(\mathbb{C}))$ and $M_\Gamma(SL_2(\mathbb{C})^c)$ as Symplectic Reductions

The product spaces $SL_2(\mathbb{C})^{E(\Gamma)}$ and $[SL_2(\mathbb{C})^c]^{E(\Gamma)}$ have natural Kähler structures as subspaces of $M_{2\times 2}(\mathbb{C})^{E(\Gamma)}$, and the actions of $SU(2)^{V(\Gamma)}$ (as a subgroup of $SL_2(\mathbb{C})^{V(\Gamma)}$) on these spaces are Hamiltonian. The work of Kirwan [14] and Kempf and Ness [16] shows that the reductions $SL_2(\mathbb{C})^{E(\Gamma)}/_0 SU(2)^{V(\Gamma)}$, $[SL_2(\mathbb{C})^c]^{E(\Gamma)}/_0 SU(2)^{V(\Gamma)}$ carry stratified Kähler structures, and can be identified with the spaces $M_\Gamma(SL_2(\mathbb{C}))$ and $M_\Gamma(SL_2(\mathbb{C})^c)$, respectively. We show that the symplectic structure on $M_\Gamma(SL_2(\mathbb{C}))$ obtained by this reduction procedure does not depend on Γ.

Let $\mathcal{T}$ be a directed tree with one internal edge $e \in E(\mathcal{T})$ and endpoints $\delta(e) = (u, v)$, and let $\mathcal{T}'$ be the tree obtained from $\mathcal{T}$ by collapsing e to a single vertex w. We let $S_1 \cup S_2 = \mathcal{L}(\mathcal{T})$ be the partition of the leaves of $\mathcal{T}$ defined by e.

Lemma 7.1 *There are symplectomorphisms*

$$\phi: M_{\mathcal{T}}\big(SL_2(\mathbb{C})\big) \longrightarrow M_{\mathcal{T}'}\big(SL_2(\mathbb{C})\big), \quad \psi: M_{\mathcal{T}'}\big(SL_2(\mathbb{C})\big) \longrightarrow M_{\mathcal{T}}\big(SL_2(\mathbb{C})\big).$$

Furthermore, both of these maps are isomorphisms of $SU(2)^{\mathcal{L}(\mathcal{T})}$ Hamiltonian spaces.

Proof Recall that $SL_2(\mathbb{C})$ can be identified with $T^*(SU(2))$ as a Hamiltonian $SU(2) \times SU(2)$ space. For any Hamiltonian $SU(2)$ space X there are isomorphisms $\overline{\phi}: X \to SU(2)\backslash_0[SL_2(\mathbb{C}) \times X]$, $\overline{\psi}: SU(2)\backslash_0[SL_2(\mathbb{C}) \times X] \to X$ computed as follows ([9, proof of Lemma 4.8]):

$$\overline{\phi}(x) = \big([\mathrm{Id}, -\mu(x)], x\big) \quad \overline{\psi}\big([k, v], x\big) = k^{-1}x$$

Furthermore, these maps intertwine the left $SU(2)$ action on $SL_2(\mathbb{C}) \times X$ with the $SU(2)$ action on X, and if X carries a Hamiltonian K action for some compact Lie group K, this action is carried through $\overline{\phi}$.

The space $M_{\mathcal{T}'}(SL_2(\mathbb{C}))$ is an $SU(2) \times SU(2)$ reduction of $SL_2(\mathbb{C})^{S_1} \times SL_2(\mathbb{C}) \times SL_2(\mathbb{C})^{S_2}$. By setting $X = SL_2(\mathbb{C})^{S_2}$, $\overline{\phi}$ and $\overline{\psi}$ identify $SU(2)\backslash_0[SL_2(\mathbb{C}) \times SL_2(\mathbb{C})^{S_2}]$ with $SL_2(\mathbb{C})^{S_2}$. Under this isomorphism the right $SU(2)$ action on $SL_2(\mathbb{C})$ is identified with the left diagonal action on $SL_2(\mathbb{C})^{S_2}$. After taking the product with $SL_2(\mathbb{C})^{S_1}$ and reducing by $SU(2)$, $\overline{\phi}$ and $\overline{\psi}$ descend to give isomorphisms ϕ and ψ between $M_{\mathcal{T}'}(SL_2(\mathbb{C}))$ and the diagonal left $SU(2)$ reduction of $SL_2(\mathbb{C})^{\mathcal{L}(\mathcal{T})} = SL_2(\mathbb{C})^{S_1 \cup S_2}$. This latter space is $M_{\mathcal{T}}(SL_2(\mathbb{C}))$. ∎

Proposition 7.2 *Let $\mathcal{T} \subset \Gamma$ be a spanning tree, with corresponding markings $\phi_{\mathcal{T},V}: \Gamma_g \to \Gamma$, $\psi_{\mathcal{T}}: \Gamma \to \Gamma_g$. This induces maps*

$$\Phi_{\mathcal{T}}: M_\Gamma\big(SL_2(\mathbb{C})\big) \longrightarrow M_{\Gamma_g}\big(SL_2(\mathbb{C})\big), \quad \Psi_{\mathcal{T}}: M_{\Gamma_g}\big(SL_2(\mathbb{C})\big) \longrightarrow M_\Gamma\big(SL_2(\mathbb{C})\big),$$

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which are symplectomorphisms.

Proof This follows by repeatedly applying Lemma 7.1 to the edges of $\mathcal{T}$. ∎

7.2 The Hamiltonian $\mathbb{T}_\Gamma$-space $M_\Gamma(SL_2(\mathbb{C})^c)$

By Proposition 6.4, the space $M_\Gamma(SL_2(\mathbb{C})^c)$ is the affine toric variety associated with the convex polyhedral cone $\mathcal{P}_\Gamma$. We describe a Hamiltonian action by a torus $\mathbb{T}_\Gamma$ on this space with momentum image equal to $\mathcal{P}_\Gamma$. The space $SL_2(\mathbb{C})^c$ is an S^1 reduction of $SL_2(\mathbb{C})/\!/U_- \times U_+\backslash\!\backslash SL_2(\mathbb{C})$, considered as a product of imploded cotangent bundles of $SU(2)$, and the momentum map $\mu\colon SL_2(\mathbb{C})^c \to \mathbb{R}$ of the residual S^1 action takes a class $([k,w][h,v])$ to $|w| = |v|$. The $SU(2)\times SU(2)$ and S^1 actions commute on $SL_2(\mathbb{C})^c$, so it follows that we can realize $M_\Gamma(SL_2(\mathbb{C})^c)$ as an $SU(2)^{V(\Gamma)} \times (S^1)^{E(\Gamma)}$ reduction of $[SL_2(\mathbb{C})/\!/U_- \times U_+\backslash\!\backslash SL_2(\mathbb{C})]^{E(\Gamma)}$.

By first reducing with respect to $SU(2)^{V(\Gamma)}$, we are led to consider the spaces $SU(2)_0\backslash[SL_2(\mathbb{C})/\!/U]^3$. The momentum level 0 set for the action of $SU(2)$ on $[SL_2(\mathbb{C})/\!/U]^3$ is the space $\mu_{SU(2)}^{-1}(0)$ of imploded triples $[k_1,r_1\rho][k_2,r_2\rho][k_3,r_3\rho]$ such that

$$r_1k_1\circ\rho + r_2k_2\circ\rho + r_3k_3\rho = 0.$$

The quotient space $SU(2)\backslash\mu_{SU(2)}^{-1}(0)$ is known as the space of spin-framed triangles in $\mathbb{R}^3 \cong su(2)^*$, see [12]. It is known that $SU(2)\backslash\mu_{SU(2)}^{-1}(0)$ is symplectomorphic to the affine toric manifold $\wedge^2(\mathbb{C}^3)$. Notice that the action of $(S^1)^3$ on this space has momentum map

$$\mu_{(S^1)^3}\big([k_1,r_1\rho][k_2,r_2\rho][k_3,r_3\rho]\big) = (r_1,r_2,r_3).$$

The momentum image is therefore the polyhedral cone Δ_3 of non-negative real triples (r_1,r_2,r_3) that can be the sides of a triangle.

Now we can realize $M_\Gamma(SL_2(\mathbb{C})^c)$ as an $(S^1)^{E(\Gamma)}$ reduction of the affine toric symplectic manifold $\prod_{v\in V(\Gamma)} SU(2)\backslash\mu_{SU(2)}^{-1}(0)$. The torus $(S^1)^{2E(\Gamma)}$ acts on the product $\prod_{v\in V(\Gamma)} SU(2)_0\backslash[SL_2(\mathbb{C})/\!/U]^3$, where there is a copy of $(S^1)^2$ for each edge $e \in E(\Gamma)$. The torus $(S^1)^{E(\Gamma)}$ is a product of the anti-diagonally embedded copies of the circle: $S^1 \subset (S^1)^2$, $t \to (t,t^{-1})$. Following the definition of $SL_2(\mathbb{C})^c$, the momentum map of the copy of S^1 associated with an edge $e \in E(\Gamma)$ is the difference in lengths of the two imploded coordinates associated with e in this product. We let $\mathbb{T}_\Gamma \subset (S^1)^{2E(\Gamma)}$ be the product of circles $S^1 \subset (S^1)^2$ embedded by $t \to (t,1)$; as remarked above, the component associated with a single edge $e \in E(\Gamma)$ has momentum map $\mu\colon SL_2(\mathbb{C})^c \to \mathbb{R}$.

Proposition 7.3 *The space $M_\Gamma(SL_2(\mathbb{C})^c)$ is a Hamiltonian toric space, and the momentum image of the torus $\mathbb{T}_\Gamma$ is the polyhedral cone $\mathcal{P}_\Gamma$.*

Proof The space $\prod_{v\in V(\Gamma)} SU(2)\backslash\mu_{SU(2)}^{-1}(0)$ has a Hamiltonian action of $(S^1)^{2E(\Gamma)}$, with momentum image $\prod_{v\in V(\Gamma)}\Delta_3$. In reducing by $(S^1)^{E(\Gamma)}$, we pass to the subquotient with residual momentum image the subcone of $\prod_{v\in V(\Gamma)}\Delta_3$ where two components corresponding to the same edge $e \in E(\Gamma)$ have the same value; this is $\mathcal{P}_\Gamma$ by

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definition. Furthermore, this is the momentum map of the residual action by the torus $\mathbb{T}_\Gamma$. ■

7.3 The Γ-collapsing map

Recall that the collapsing map $\Xi: SL_2(\mathbb{C}) \to SL_2(\mathbb{C})^c$ is a surjective, continuous map of Hamiltonian $SU(2) \times SU(2)$ spaces, and that it intertwines the left and right $SU(2)$ momentum mappings on these spaces. We let $\Xi^{E(\Gamma)}: SL_2(\mathbb{C})^{E(\Gamma)} \to [SL_2(\mathbb{C})^c]^{E(\Gamma)}$ be the product map, which is likewise a map of Hamiltonian $SU(2)^{V(\Gamma)}$ spaces; see Figures 14 and 15. It follows that $\Xi^{E(\Gamma)}$ restricts to a surjective, continuous map on the 0-level sets of the map(s) $\mu_{SU(2)^{V(\Gamma)}}$. We let Ξ_Γ be the map on the quotient by $SU(2)^{V(\Gamma)}$:

$$\Xi_\Gamma : \mathcal{X}\big(F_g, SL_2(\mathbb{C})\big) \cong M_\Gamma\big(SL_2(\mathbb{C})\big) \to M_\Gamma\big(SL_2(\mathbb{C})^c\big)$$

$$\begin{tikzcd} SL_2(\mathbb{C}) \arrow[rr, "\Xi"] \arrow[dr, "\mu_L \times \mu_R"'] & & SL_2(\mathbb{C})^c \arrow[dl, "\mu_L \times \mu_R"] \\ & su(2)^* \times su(2)^* & \end{tikzcd}$$

Figure 14: Ξ intertwines the momentum maps.

$$\begin{tikzcd} SL_2(\mathbb{C})^{E(\Gamma)} \arrow[rr, "\Xi^{E(\Gamma)}"] \arrow[dr, "\mu_{SU(2)^{V(\Gamma)}}"'] & & {[SL_2(\mathbb{C})^c]^{E(\Gamma)}} \arrow[dl, "\mu_{SU(2)^{V(\Gamma)}}"] \\ & {[su(2)^*]^{V(\Gamma)}} & \end{tikzcd}$$

Figure 15: $\Xi^{E(\Gamma)}$ also intertwines the momentum maps.

Following Subsection 4.3, we consider the family $\det^\Gamma: M_{2\times 2}(\mathbb{C})^{E(\Gamma)} \to \mathbb{C}^{E(\Gamma)}$. This family is the Vinberg enveloping monoid of the group $SL_2(\mathbb{C})^{E(\Gamma)}$. By [13, Section 6], for any assignment of dominant $SL_2(\mathbb{C})$ coweights $a: E(\Gamma) \to \mathbb{Z}_{>0}$ (namely a choice of a point in C_Γ) there is an $SU(2)^{2E(\Gamma)}$-equivariant gradient Hamiltonian flow $V_{\pi,a}$ on the flat $\mathbb{C}$-family E_a obtained by base change under the map $\phi_a: \mathbb{C} \to \mathbb{C}^{E(\Gamma)}$, $z \to (\dots, z^{a(e)}, \dots)$:

$$\begin{tikzcd} E_a = M_{2\times 2}(\mathbb{C}) \times_{\phi_a, \det^\Gamma} \mathbb{C} \arrow[r] \arrow[d] & M_{2\times 2}(\mathbb{C})^{E(\Gamma)} \arrow[d, "\det^\Gamma"'] \\ \mathbb{C} \arrow[r, "\phi_a"] & \mathbb{C}^{E(\Gamma)}. \end{tikzcd}$$

By [13, Section 6], this flow is completed by the continuous map $\Xi^{E(\Gamma)}: SL_2(\mathbb{C})^{E(\Gamma)} \to [SL_2(\mathbb{C})^c]^{E(\Gamma)}$. As this map is $SU(2)^{V(\Gamma)}$-equivariant, it descends to the symplectic reductions, where it must agree with Ξ_Γ.

7.4 The Integrable System in $\mathfrak{X}(F_g, SL_2(\mathbb{C}))$

By Proposition 4.2, $\Xi^{E(\Gamma)}: SL_2(\mathbb{C})^{E(\Gamma)} \to [SL_2(\mathbb{C})^c]^{E(\Gamma)}$ is a symplectomorphism on $SL_2(\mathbb{C})_o^{E(\Gamma)}$; the latter being the set of points $(\dots, [k_e, v_e], \dots)$ with $v_e \neq 0$ for all $e \in E(\Gamma)$. The product $SL_2(\mathbb{C})_o^{E(\Gamma)}$ is preserved by the $SU(2)^{V(\Gamma)}$ action; in particular, it inherits an $SU(2)^{V(\Gamma)}$ Hamiltonian structure from $SL_2(\mathbb{C})^{E(\Gamma)}$, and $\Xi^{E(\Gamma)}$ restricts to give an isomorphism of Hamiltonian $SU(2)^{V(\Gamma)}$ spaces onto the image in $[SL_2(\mathbb{C})^c]^{E(\Gamma)}$.

Definition 7.4 The subspace $M_\Gamma^o(SL_2(\mathbb{C})) \subset M_\Gamma(, SL_2(\mathbb{C}))$ is the space of those points $(\dots, [k_e, v_e], \dots)$ with $v_e \neq 0$.

Proposition 7.5 *The map $\Xi_\Gamma: M_\Gamma(SL_2(\mathbb{C})) \to M_\Gamma(SL_2(\mathbb{C})^c)$ restricts to a symplectomorphism from $M_\Gamma^o(SL_2(\mathbb{C}))$ onto its image in $M_\Gamma(SL_2(\mathbb{C})^c)$.*

Proof This follows from the discussion above and the fact that $M_\Gamma^o(SL_2(\mathbb{C}))$ is the reduction at level 0 of $SL_2(\mathbb{C})_o^{E(\Gamma)}$ by $SU(2)^{V(\Gamma)}$. ∎

This completes the proof of Theorem 1.4. The integrable $\mathbb{T}_\Gamma = [S^1]^{E(\Gamma)}$ action on $M_\Gamma(SL_2(\mathbb{C})^c)$ also defines a Hamiltonian action on the image of $M_\Gamma^o(SL_2(\mathbb{C}))$. The momentum map for this action is computed as

$$\mu_{\mathbb{T}_\Gamma} \circ \Xi_\Gamma: \mathfrak{X}\big(F_g, SL_2(\mathbb{C})\big) \to \mathcal{P}_\Gamma \subset \mathbb{R}^{E(\Gamma)}.$$

In particular, the Hamiltonians on $M_\Gamma^o(SL_2(\mathbb{C}))$ can be extended continuously to $\mathfrak{X}(F_g, SL_2(\mathbb{C}))$.

We let $\Xi_\Gamma^{-1}: M_\Gamma^o(SL_2(\mathbb{C})^c) \to M_\Gamma^o(SL_2(\mathbb{C}))$ be the inverse map to the restriction of Ξ_Γ to $M_\Gamma^o(SL_2(\mathbb{C}))$. The Hamiltonian action of $\mathbb{T}_\Gamma$ is computed through Ξ_Γ^{-1} as follows. Let $(t_e) \in \mathbb{T}_\Gamma$ and let $(\dots, [k_e, v_e], \dots) \in M_\Gamma(SL_2(\mathbb{C})) = \mathfrak{X}(F_g, SL_2(\mathbb{C}))$. Then

$$\begin{aligned} (t_e) \star [k_e, v_e] &= \Xi_\Gamma^{-1} \circ (t_e) \star \Xi_\Gamma[k_e, v_e] = \Xi_\Gamma^{-1} \circ \big([k_e h_e^{-1}, w_e][t_e h_e, v_e]\big) \\ &= [k_e h_e^{-1} t_e h_e, v_e]. \end{aligned}$$

Recall that h_e is an element chosen so that $h_e \circ v_e = w_e = r_e\rho_0$, for some $r_e \in \mathbb{R}_{\geq 0}$. The action by (t_e) is well-defined on any point (k_e, v_e) with $|v_e| = |w_e| = r_e > 0$ by Proposition 4.2.

7.5 Induced Γ-decomposition

By Proposition 4.2, the map Ξ_Γ is not $1-1$ on the subspace

$$M_\Gamma\big(SL_2(\mathbb{C})\big) \smallsetminus M^o_\Gamma\big(SL_2(\mathbb{C})\big).$$

These are the points $(\dots,[k_e, v_e],\dots)$ with $|v_e| = 0$ for some subset of edges $S \subset E(\Gamma)$. For any $S \subset E(\Gamma)$, define $\Gamma_S \subset \Gamma$ to be the graph induced by the complement of S, and Γ^S to be the graph induced by S itself (see Figure 16). The next lemma addresses the structure of the possible Γ_S.

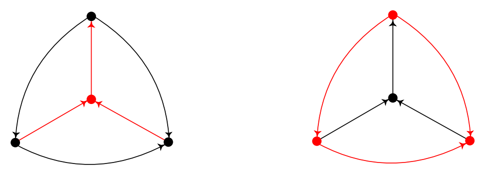

Figure 16: The graphs Γ^S (left) and Γ_S (right)

Lemma 7.6 Let $S \subset E(\Gamma)$ be a set of edges for which some $(\dots,[k_e, v_e],\dots)$ has $v_e = 0$ if and only if $e \in S$; then $\Gamma_S \subset \Gamma$ has no leaves. For any subgraph $\Gamma' \subset \Gamma$ with no leaves, there exists a $(\dots,[k_e, v_e],\dots) \in \mu^{-1}_{SU(2)^{V(\Gamma)}}(0)$ with $v_e = 0$ for precisely $e \in E(\Gamma) \smallsetminus E(\Gamma')$.

Proof Let $w \in V(\Gamma)$ and let $\epsilon(w) = \{e_1,\dots,e_m\} \cup \{f_1,\dots,f_k\}$ with the e_i incoming and f_j outgoing. For $(k_e, v_e) \in \mu^{-1}_{SU(2)^{V(\Gamma)}}(0)$ the momentum 0 condition for the copy of $SU(2)$ associated with w is the following:

$$k_{e_1} \circ v_{e_1} + \cdots + k_{e_m} \circ v_{e_m} - (v_{f_1} + \cdots + v_{f_k}) = 0$$

If $\Gamma_S \subset \Gamma$ has a leaf, then for some $w \in V(\Gamma)$ all of the v_{e_i}, v_{f_j} above are 0 save one. This contradicts the equation.

For the second part, observe that if we can assign non-zero vectors to the edges of any graph that satisfy the momentum 0 condition above, we can take such an assignment to Γ_S and extend it to Γ by placing 0 on all edges of S. For this reason, it suffices to show that any graph supports a non-zero vector assignment.

We fix the genus g and consider the graph Γ_g with one vertex. Any assignment of vectors $v_1,\dots,v_g$ to the edges of Γ_g satisfies the momentum 0 condition, because $\sum \mathrm{Id} \circ v_1 + \cdots + \mathrm{Id} \circ v_g - (v_1 + \cdots + v_g) = 0$. We take this as the base case of an induction on the number of edges in a graph Γ. We let Γ' be obtained from Γ by collapsing an edge $f \in E(\Gamma)$, $\delta(f) = (x, y)$ to a vertex $w \in V(\Gamma')$, and we suppose there is some assignment of $(\dots,[k_e, v_e],\dots)$ to the edges of Γ' that satisfies the momentum

0 condition, with all $v_e \neq 0$. The edges incident on w are partitioned into two sets $\epsilon(w) = X \cup Y$ by f, with those in X incident on x and those Y incident on y, with associated sums

$$\sigma = -\Big[\sum_{i\in X} k_i \circ v_i - \sum_{j\in X} v_j\Big] = \Big[\sum_{i\in Y} h_i \circ z_i - \sum_{j\in Y} z_j\Big].$$

There are two cases, $\sigma \neq 0$ and $\sigma = 0$. If $\sigma \neq 0$, we place (Id, σ) on the edge f. As f is incoming on y, we have $-\sigma + [\sum_{i\in Y} h_i \circ z_i - \sum_{j\in Y} z_j] = -\sigma + \sigma = 0$. We also have $\sigma + [\sum_{i\in X} k_i \circ v_i - \sum_{j\in X} v_j] = \sigma - \sigma = 0$, so this assignment satisfies the requisite conditions for Γ.

If $\sigma = 0$, we have two more cases. If f is a separating edge, we rotate all $[k_i, v_i]$, $i \in Y$ by some $g \in SU(2)$ to make the difference $\neq 0$, and reduce to the previous case. If f is not separating, we take it to be part of a simple loop $\gamma = (f, e_2, \dots, e_k)$ in Γ. Each edge of this loop is directed in some way. If two consecutive edges share a direction, we add a q or $-q$ to both (correcting as necessary with k_i^{-1}. If two consecutive edges switch directions, we switch signs. This can always be done, because there must always be an even number of switch vertices. To see this last point, note that if we have an odd number of switch vertices, then the edge orientation must switch an odd number of times as we go around γ, but this implies that f has both possible orientations, a contradiction. ∎

Corollary 7.7 *The polyhedral cone $\mathcal{P}_\Gamma \subset \mathbb{R}^{E(\Gamma)}$ intersects the coordinate subspace $\mathbb{R}^K \subset \mathbb{R}^{E(\Gamma)}$ if and only if $K = E(\Gamma) \smallsetminus E(\Gamma')$, for $\Gamma' \subset \Gamma$ a subgraph with no leaves.*

We let $M_{\Gamma,S}(SL_2(\mathbb{C})) \subset M_\Gamma(SL_2(\mathbb{C}))$ and $M_{\Gamma,S}(SL_2(\mathbb{C})^c) \subset M_\Gamma(SL_2(\mathbb{C})^c)$ be the subspaces defined by the condition $|v_e| = 0$ if and only if $e \in S$. Clearly, Ξ_Γ restricts to a surjective map $\Xi_\Gamma\colon M_{\Gamma,S}(SL_2(\mathbb{C})) \to M_{\Gamma,S}(SL_2(\mathbb{C})^c)$. The condition $|v_e| = 0$ if and only if $e \in S$ defines an open face of $\mathcal{P}_\Gamma$, which is the image of $M_{\Gamma,S}(SL_2(\mathbb{C}))$ and $M_{\Gamma,S}(SL_2(\mathbb{C})^c)$ under the maps $\mu_{\mathbb{T}} \circ \Xi_\Gamma$ and $\mu_{\mathbb{T}}$, respectively. Furthermore, $M_{\Gamma,S}(SL_2(\mathbb{C})^c)$ is the $\mathbb{T}^{\mathbb{C}}$ orbit corresponding to this face. For the next proposition, we allow a graph Γ_S that may have bivalent vertices, the definition of $M_{\Gamma_S}(\,\cdot\,)$ is identical.

Proposition 7.8 *The closure of $M_{\Gamma,S}(SL_2(\mathbb{C})^c)$ is isomorphic to $M_{\Gamma_S}(SL_2(\mathbb{C})^c)$.*

Proof By definition $M_{\Gamma,S}(SL_2(\mathbb{C})^c)$ is covered by the space of $(\dots, [k_e, v_e], \dots)$ satisfying the momentum 0 condition $SU(2)^{V(\Gamma)}$ with $v_e = 0, k_e = \mathrm{Id}$ for precisely those $e \in E(\Gamma_S)$. The closure of this space admits points where $v_e = 0$, and therefore k_e is identified to Id. This is the intersection of the subspace $(SL_2(\mathbb{C})^c)^{E(\Gamma_S)} \subset (SL_2(\mathbb{C})^c)^{E(\Gamma)}$ with the momentum level 0 space. As all $e \in S$ are assigned 0 lengths, this subspace is precisely the momentum level 0 set for the action of $SU(2)^{V(\Gamma_S)}$, and the condition $k_e = \mathrm{Id}$ implied by the definition of the contraction $SL_2(\mathbb{C})^c$ implies the action of $SU(2)^{V(\Gamma)}$ reduces to that of $SU(2)^{V(\Gamma_S)}$. ∎

Let $\overline{\Gamma}_S$ be the graph obtained from Γ_S by replacing any pair of edges connected to a bivalent vertex with a single edge. We leave it to the reader to check that $M_{\Gamma_S}(SL_2(\mathbb{C})^c) \cong M_{\overline{\Gamma}_S}(SL_2(\mathbb{C})^c)$.

The space $M_{\Gamma,S}(SL_2(\mathbb{C}))$ is an $SU(2)^{V(\Gamma)}$ quotient of the product space $SU(2)^S \times \mu^{-1}_{SU(2)^{V(\Gamma_S)}}(0)$. Here,

$$\mu^{-1}_{SU(2)^{V(\Gamma_S)}}(0) \subset SL_2(\mathbb{C})^{E(\Gamma)\setminus S}$$

,is the level 0 subset defined by the $SU(2)^{V(\Gamma_S)}$ action in the definition of $M_{\Gamma_S}(SL_2(\mathbb{C}))$. In particular, if we take $S = E(\Gamma)$, the fiber over the unique point in $M_\Gamma(SL_2(\mathbb{C})^c)$ defined by the condition $|v_e| = 0$ for all $e \in E(\Gamma)$ is the character variety $M_\Gamma(SU(2)) = \mathcal{X}(F_g, SU(2)) \subset \mathcal{X}(F_g, SL_2(\mathbb{C}))$ associated with the group $SU(2)$.

Proposition 7.9 *The map $\Xi_\Gamma: M_{\Gamma,S}(SL_2(\mathbb{C})) \to M_{\Gamma,S}(SL_2(\mathbb{C})^c)$ has connected fibers.*

Proof The fiber over an equivalence class $(\dots,[k_e,v_e],\dots) \in M_{\Gamma,S}(SL_2(\mathbb{C})^c)$ is $SU(2)^S \times (\dots,[k_e,v_e],\dots)$, which is connected because the subspace of points in the equivalence class $(\dots,[k_e,v_e],\dots)$ is connected. ■

Now Proposition 7.9 and Lemma 7.6 complete a proof of Theorem 1.3. ■

8 Compactifications of $\mathcal{X}(F_g, SL_2(\mathbb{C}))$

In this section we show that the cone $C_{\Gamma,\phi}$ of valuations on $\mathbb{C}[\mathcal{X}(F_g, SL_2(\mathbb{C}))]$ defined by a marking $\phi: \Gamma_g \to \Gamma$ is induced from a compactification $M_\Gamma(SL_2(\mathbb{C})) \subset M_\Gamma(X)$. Here $X \subset \mathbb{P}^5$ is the projective $SL_2(\mathbb{C}) \times SL_2(\mathbb{C})$ scheme from Section 4 obtained by taking Proj of the algebra $\overline{R} = \mathbb{C}[a,b,c,d,t]/\langle ad - bc - t^2\rangle$. We construct $M_\Gamma(X)$ as a GIT quotient as in Section 5, using the $SL_2(\mathbb{C}) \times SL_2(\mathbb{C})$-linearized line bundle $\mathcal{O}(1)$ on $\mathbb{P}^4$.

8.1 The GIT Quotient $M_\Gamma(X)$

As X is an $SL_2(\mathbb{C}) \times SL_2(\mathbb{C})$ variety, $X^{E(\Gamma)}$ carries an action of the group $SL_2(\mathbb{C})^{V(\Gamma)}$. For each edge $e \in E(\Gamma)$ there is a corresponding divisor $\overline{D}_e = D \times \prod_{e \neq f \in E(\Gamma)} X$. Furthermore, there is a subspace $D_S = \cap_{e\in S} D_e = \prod_{e\in S} D \times \prod_{f\in E(\Gamma)\setminus S} X$ for every $S \subset E(\Gamma)$. Each subspace D_S is $SL_2(\mathbb{C})^{E(\Gamma)}$ stable and smooth, making $\widehat{D}_\Gamma = \cup D_e$ a normal crossings divisor on $X^{E(\Gamma)}$.

We let $\mathcal{L} = i^*(\mathcal{O}(1))$ under $i: X \subset \mathbb{P}^4$, this bundle is $SL_2(\mathbb{C}) \times SL_2(\mathbb{C})$-linearized and the section ring of $\mathcal{L}$ is then $\overline{R}$. Similarly we let $\mathcal{M} = j^*(O(1))$ for $j: D \subset \mathbb{P}^4$ the inclusion of the divisor $D \subset X$; the section ring of $\mathcal{M}$ is $\overline{R}/\langle t\rangle$. We place $\mathcal{L}^{\boxtimes E(\Gamma)}$ on $X^{E(\Gamma)}$, which then comes with an $SL_2(\mathbb{C})^{V(\Gamma)}$ action. We define $M_\Gamma(X)$ to be the following GIT quotient:

$$M_\Gamma(X) = SL_2(\mathbb{C})^{V(\Gamma)} \backslash\!\backslash_{\mathcal{L}^{\boxtimes E(\Gamma)}} X^{E(\Gamma)}.$$

The projective coordinate ring $\overline{R}_\Gamma$ associated with the line bundle $\mathcal{L}^{\boxtimes E(\Gamma)}$ is a direct sum of the following spaces.

$$\overline{R}_\Gamma(m) = \bigoplus_{a:E(\Gamma)\to[0,m]} \bigotimes_{e\in E(\Gamma)} V(a(e)) \otimes V(a(e)) t^m.$$

Once again, t^m serves as a placeholder in this algebra, and multiplication by $t \in V(0,0)^{\otimes E(\Gamma)}t \subset \overline{R}_\Gamma(1)$ shifts a summand in $\overline{R}_\Gamma(m)$ to its isomorphic image in $\overline{R}_\Gamma(m+1)$. The projective coordinate ring $R_\Gamma = [\overline{R}_\Gamma]^{SL_2(\mathbb{C})^{V(\Gamma)}}$ of $M_\Gamma(X)$ has a similar description to the affine coordinate ring of $\mathbb{C}[M_\Gamma(SL_2(\mathbb{C}))]$; it is a direct sum of the following spaces:

$$R_\Gamma(m) = \bigoplus_{a:E(\Gamma)\to[0,m]} [\bigotimes_{e\in E(\Gamma)} V(a(e)) \otimes V(a(e))]^{SL_2(\mathbb{C})^{V(\Gamma)}} t^m.$$

Each of the spaces in this sum is spanned by a single spin diagram Φ_a, with $a(e) \le m$ for all $e \in E(\Gamma)$. Multiplication on R_Γ is induced from the inclusions $R_\Gamma(m) \subset \mathbb{C}[M_\Gamma(SL_2(\mathbb{C}))]$.

Proposition 8.1 *The space $M_\Gamma(SL_2(\mathbb{C}))$ is a dense, open subset of $M_\Gamma(X)$.*

Proof The subspaces $R_\Gamma(m)$ filter the algebra $\mathbb{C}[M_\Gamma(SL_2(\mathbb{C}))]$, so the algebra $\frac{1}{t}R_\Gamma$ is isomorphic to $\mathbb{C}[M_\Gamma(SL_2(\mathbb{C}))] \otimes \mathbb{C}[t, \frac{1}{t}]$ by standard properties of Rees algebras. This implies that the complement of the $t = 0$ hypersurface is $M_\Gamma(SL_2(\mathbb{C}))$. ∎

8.2 The Boundary Divisor D_Γ

The divisor $D_\Gamma \subset M_\Gamma(X)$ is defined by the equation $t = 0$. The m-th graded piece of the ideal $\langle t \rangle$ is the following subspace of R_Γ:

$$\langle t \rangle \cap R_\Gamma(m) = \bigoplus_{a:E(\Gamma)\to[0,m-1]} \mathbb{C}\Phi_a t^m.$$

The ideal $\langle t \rangle$ is the intersection of the following subspaces I_e, $e \in E(\Gamma)$:

$$I_e(m) = \bigoplus_{a:E(\Gamma)\to[0,m],a(e)<m} \mathbb{C}\Phi_a t^m.$$

Proposition 8.2 *The subspaces I_e are prime ideals, and therefore define the irreducible components of D_Γ. The locus D_e of I_e is the image in $M_\Gamma(X)$ of $\overline{D}_e \subset X^{E(\Gamma)}$, and can be identified with $SL_2(\mathbb{C})^{V(\Gamma)} \backslash\backslash_{\mathcal{M}\boxtimes\prod_{e'\in E(\Gamma)\setminus\{e\}}\mathcal{L}} [D \times \prod_{e'\in E(\Gamma)\setminus\{e\}} X]$.*

Proof We let $\overline{I}_e \subset \overline{R}_\Gamma$ be the sum of the spaces:

$$\overline{I}_e(m) = \bigoplus_{a:E(\Gamma)\to[0,m],a(e)<m} \bigotimes_{e'\in E(\Gamma)} V(a(e')) \otimes V(a(e'))t^m \subset \overline{R}_\Gamma(m),$$

so that $I_e = \overline{I}_e \cap R_\Gamma$ and $I_e = \overline{I}_e^{SL_2(\mathbb{C})^{V(\Gamma)}}$. By definition, the locus of $\overline{I}_e$ is $\overline{D}_e = D \times \prod_{e'\in E(\Gamma)\setminus\{e\}} X \subset X^{E(\Gamma)}$. ∎

The space $SL_2(\mathbb{C})^{E(\Gamma)}$ is compactified by $X^{E(\Gamma)}$, and the cone C_Γ of valuations on $\mathbb{C}[SL_2(\mathbb{C})^{E(\Gamma)}]$ are the divisorial valuations associated with the components $\overline{D}_e \subset \overline{D}_\Gamma$. These valuations are induced on $\mathbb{C}[M_\Gamma(SL_2(\mathbb{C}))]$ by the inclusion $\mathbb{C}[M_\Gamma(SL_2(\mathbb{C}))] \subset \mathbb{C}[SL_2(\mathbb{C})^{E(\Gamma)}]$; this allows us to characterize the cone C_Γ.

Proposition 8.3 *The divisorial valuations on $\mathbb{C}[M_\Gamma(SL_2(\mathbb{C}))]$ attached to the components D_e are points on the extremal rays of C_Γ.*

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Proof For an edge $e \in E(\Gamma)$, let v_e be the valuation on $\mathbb{C}[M_\Gamma(SL_2(\mathbb{C}))]$ corresponding to the metric that assigns e a 1 and every other edge a 0. By definition, v_e is the valuation induced on $\mathbb{C}[M_\Gamma(SL_2(\mathbb{C}))] \subset \mathbb{C}[SL_2(\mathbb{C})^{E(\Gamma)}]$ by taking degree along $\overline{D}_e$. Let $\overline{\eta}_e$ be the generic point of $\overline{D}_e \subset X^{E(\Gamma)}$, and let η_e be the generic point of $D_e \subset M_\Gamma(X)$. We have the inclusion of local rings

$$\mathcal{O}_{\eta_e} = \mathcal{O}_{\overline{\eta}_e}^{SL_2(\mathbb{C})^{E(\Gamma)}} \subset \mathcal{O}_{\overline{\eta}_e}.$$

Furthermore, the generator t_e of the maximal ideal of $\mathcal{O}_{\overline{\eta}_e}$ is an invariant, it follows that this same element generates the maximal ideal of $\mathcal{O}_{\eta_e}$. ∎

For a set $S \subset E(\Gamma)$, the ideal $I_S = \sum_{e \in S} I_e$ cuts out $D_S = \bigcap_{e \in S} D_e$:

$$I_S(m) = \bigoplus_{a: E(\Gamma) \to [0,m], \exists e \in E(\Gamma), a(e) < m} \mathbb{C}\Phi_a t^m.$$

Proposition 8.4 *Each $I_S \subset R_\Gamma$ is prime, and* $\operatorname{codim}(D_S) = |S|$.

Proof If $f = \sum C_i \Phi_{a_i} \in R_\Gamma(m)$, $g = \sum K_i \Phi_{b_i} \in R_\Gamma(n)$, $f, g \notin I_S$, then $a_i(e) = m$ and $b_i(e) = n$ for all $e \in S$. Let $v_S = \sum_{e \in S} v_e : \mathbb{C}[M_\Gamma(SL_2(\mathbb{C}))] \to \mathbb{Z} \cup \{\infty\}$ be the sum of the extremal ray generators of C_Γ corresponding to the edges in S. We have $v_S(fg) = v_S(f) + v_S(g) = |S|(n+m)$, so it follows that some component Φ_c of $fg \in R_\Gamma(n+m)$ must have $c(e) = n+m$ for all $e \in S$.

For the second part, we show that for any $S \subset S'$ with $|S' \setminus S| = 1$ there is an element $\Phi_a \in I_{S'}, \notin I_S$. This implies that the height of I_S is $|S|$, as it can placed in a strict chain of prime ideals of length $|E(\Gamma)| = \dim(M_\Gamma(SL_2(\mathbb{C})))$. The element Φ_a must have an $a: E(\Gamma) \to [0,m]$ with $a(e) = m$ for all $e \in S$ and $a(f) < m$ for $f \in S' \setminus S$, for some $m \in \mathbb{Z}$. We let $m = 4$, and observe that each of the trinodes in Figure 17, and all of their permutations, can be part of a spin diagram. These can be combined without limitations to produce a weighting with the desired properties for any Γ. ∎

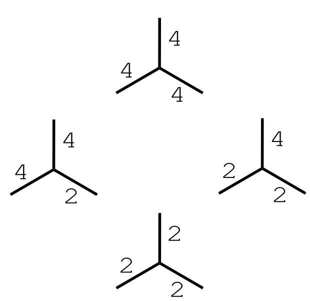

Figure 17: Building blocks of the element $a \in I_{S'}, \notin I_S$

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8.3 The Induced Toric Degeneration of $M_\Gamma(X)$

The projective coordinate ring R_Γ is a subring of $\mathbb{C}[M_\Gamma(SL_2(\mathbb{C}))]\otimes\mathbb{C}[t]$ and inherits the basis of spin diagrams $\Phi_a t^m$. Lemma 3.2 implies that the valuations $v_{\Gamma,\ell} \in C_\Gamma$ all extend to $\mathbb{C}[M_\Gamma(SL_2(\mathbb{C})]] \otimes \mathbb{C}[t]$ and R_Γ. We use an interior element of C_Γ to induce a toric degeneration on $M_\Gamma(X)$. The following scheme is the special fiber of this degeneration, in analogy with $M_\Gamma(SL_2(\mathbb{C})^c)$. Recall the flat degeneration $i : X_0 \subset \mathbb{P}^4$ of X from Section 4 and let $\mathcal{L}_0 = i^*(O(1))$.

Definition 8.5 Let $M_\Gamma(X_0)$ be the GIT quotient of $X_0^{E(\Gamma)}$ with respect to the $SL_2(\mathbb{C})^{V(\Gamma)}$-linearized line bundle $\mathcal{L}_0^{\boxtimes E(\Gamma)}$.

Proposition 8.6 *An interior point $v_{\Gamma,\ell} \in C_\Gamma$ defines a flat degeneration $M_\Gamma(X) \Rightarrow M_\Gamma(X_0)$.*

Proof This argument is identical to the proof of Proposition 5.1. ∎

We let $\overline{T}_\Gamma$ be the graded coordinate ring of $X_0^{E(\Gamma)}$ with respect to the line bundle $\mathcal{L}_0^{\boxtimes E(\Gamma)}$. The invariant subring $T_\Gamma \subset \overline{T}_\Gamma$ has an identical description to R_Γ as a direct sum of spaces associated with spin diagrams $a: E(\Gamma) \to Z$:

$$T_\Gamma(m) = \bigoplus_{a:E(\Gamma)\to[0,m]} \Big[\bigotimes_{e\in E(\Gamma)} V(a(e)) \otimes V(a(e))\Big]^{SL_2(\mathbb{C})^{V(\Gamma)}} t^m.$$

The multiplication operation on this algebra is induced from $\overline{T}_\Gamma$, so it follows that the product of elements from components labeled by $a, b: E(\Gamma) \to \mathbb{Z}$ is an element in the component labeled by $a+b: E(\Gamma) \to \mathbb{Z}$. As each of these components is multiplicity free, it follows that T_Γ is a graded (by the exponent of t) affine semigroup algebra.

Definition 8.7 We define the polytope $\mathcal{Q}_\Gamma$ to be the subset of $a \in \mathcal{P}_\Gamma$ satisfying $a(e) \le 1$.

Proposition 8.8 *The algebra T_Γ is isomorphic to the graded affine semigroup algebra defined by $\mathcal{Q}_\Gamma$ with respect to the lattice L_Γ.*

Proof The description of $T_\Gamma(m)$ above implies that the basis of this space defined by the direct sum decomposition is in bijection with the lattice points of $m\mathcal{Q}_\Gamma$. Multiplication in this algebra is computed by addition on the direct sum labels $a: E(\Gamma) \to \mathbb{Z}, m \in \mathbb{Z}$. ∎

Example 8.9 (Graphs of genus 2) The character variety $\mathcal{X}(F_2, SL_2(\mathbb{C}))$ is isomorphic to $\mathbb{C}^3$. We depict the compactification polytopes $\mathcal{Q}_{\Gamma_1}, \mathcal{Q}_{\Gamma_2}$ for the two graphs of genus 2 in Figure 18. We have colored the intersection of $\mathcal{Q}_{\Gamma_1}, \mathcal{Q}_{\Gamma_2}$ with the coordinate hyperplanes in red. The fibers of Ξ_{Γ_i} for points in these intersections are products of 3-spheres $\mathbb{S}^3 \cong SU(2)$, with the exception of the fiber over the origin, which is $\mathcal{X}(F_2, SU(2))$.

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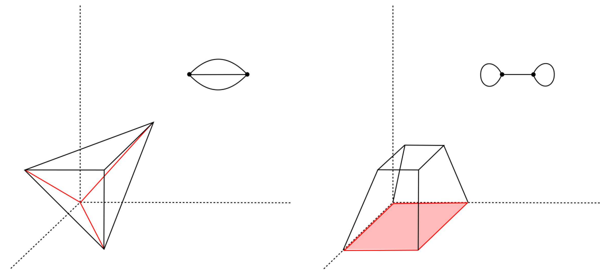

Figure 18: The polytopes $\mathcal{Q}_{\Gamma_1}, \mathcal{Q}_{\Gamma_2}$ for graphs of genus 2

The conditions that define $I_S \subset R_\Gamma$ likewise define ideals $J_S \subset T_\Gamma$. We let $K_S = \cap K_e$, for K_e the divisor defined by J_e.

Proposition 8.10

(i) *The ideal J_S cuts out the toric subspace $K_S \subset M_\Gamma(X_0)$ defined by the face $F_S = \{w \in \mathcal{Q}_\Gamma, e \in S | w(e) = 1\}$.*
(ii) $\operatorname{codim}(F_S) = \operatorname{codim}(K_S) = |S|$.

Proof The first part follows from the observation that $S_\Gamma^{SL_2(\mathbb{C})^{V(\Gamma)}}(m)/J_S(m)$ has a basis in bijection with the lattice points of mF_S. The second part is identical to the proof of Proposition 8.4. ∎

Finally we relate D_S and K_S.

Proposition 8.11 *The space K_S is a toric degeneration of D_S.*

Proof The graded component $R_\Gamma(m)/I_S(m)$ of the coordinate ring has a basis $[\Phi_a]$ for $[a] \in mF_S \subset m\mathcal{Q}_\Gamma$. For $[\Phi_a] \in R_\Gamma(m)/I_S(m)$, $[\Phi_b] \in R_\Gamma(n)/I_S(n)$, multiplication $[\Phi_a][\Phi_b]$ gives a sum $C_i[\Phi_{c_i}]$ for $[c_i] \in (n+m)F_S$, and $c_1 = a+b$. This expansion is obtained from the one in R_Γ by eliminating those Φ_{c_i} with $c_i \in I_S$. This implies that we can use any filtration induced by a v_ℓ in the interior of C_Γ to obtain the toric degeneration to K_S. ∎

This concludes the proof of Theorem 1.5. ∎

8.4 Newton-Okounkov Bodies of $\mathcal{X}(F_g, SL_2(\mathbb{C}))$

We fix a graph Γ, an isomorphism $\gamma : \pi_1(\Gamma) \cong F_g$, and we let $v_{\Gamma,\gamma,\bar{1}}$ be the valuation on $\mathbb{C}[M_\Gamma(SL_2(\mathbb{C}))]$ associated with the metric that assigns every edges $e \in E(\Gamma)$ length 1. The toric degeneration $\mathbb{C}[P_\Gamma]$ matches the associated graded ring of a Newton-Okounkov body construction explored in [22]. We show that that the maximal rank valuation used in this construction can be built from a flag of subvarieties in the boundary divisor $D_\Gamma \subset M_\Gamma(X)$.

Let X be a projective variety of dimension d, and let $\vec{F}$ be a full flag of irreducible subvarieties, with $F_1 = \{pt\}$ a smooth point of X. Following [15], one obtains a maximal rank valuation $\mathfrak{v}_{\vec{F}}$ on the rational functions $K(X)$ as follows. For $f \in K(X)$, one takes the degree $v_{F_d}(f)$ along the divisor F_d. For $y_d \in \mathcal{O}_{pt}$ a local equation which defining F_d, one then repeats this process on $y_d^{-v_{F_d}(f)} f$, thought of as a regular function on F_d, with respect to the flag $F_1 \subset \cdots \subset F_{d-1}$. The resulting function from $K(X)$ to $\mathbb{Z}^d$ (considered with the lexicographic ordering) defines the associated valuation on $K(X)$.

A total ordering $\prec$ on $E(\Gamma)$ defines an associated lexicographic ordering on the spin diagrams Φ_a ([22, Theorem 1.1]). We say $a < b$ if this is the case in the lexicographic ordering defined on the entries $a(e), b(e)$, $e \in E(\Gamma)$ with respect to the lexicographic ordering defined by $\prec$. This construction defines a filtration $\mathfrak{v}_{\Gamma,\prec}$ on $\mathbb{C}[M_\Gamma(SL_2(\mathbb{C}))]$ that yields the Newton–Okounkov body construction in [22]. The image of $\mathcal{R}(\Gamma,\phi)$ under $\mathfrak{v}_{\Gamma,\prec}$ is shown to give a basis, and it follows by Proposition 6.11 that the same is true for $\mathcal{S}(\Gamma,\gamma)$. There is an associated ordering on the irreducible components $D_e \subset D_\Gamma$, we let D_i be the intersection of the first i of these components under $\prec$, this defines a complete flag $\vec{D}_\prec$ of subspaces of $M_\Gamma(X)$.

Theorem 8.12 *Let $\prec$ be an ordering on the edges $E(\Gamma)$ and let $\mathfrak{v}_{\Gamma,\prec}$ be the associated maximal rank valuation. Then $\mathfrak{v}_{\Gamma,\prec}$ is the maximal rank valuation defined by the flag $\vec{D}_\prec$.*

Proof Everything is $SL_2(\mathbb{C})^{V(\Gamma)}$ invariant, so we carry out our analysis in the scheme $X^{E(\Gamma)}$. Note that the scheme D_i is the image of the subspace $D^i \times X^{|E(\Gamma)|-i} \subset X^{E(\Gamma)}$.

Let $\tau_w \in \mathcal{S}(\Gamma,\gamma)$ be the trace-word function for a reduced word $w \in F_g$ with associated Γ tensor $\mathcal{V}(P,\phi)$. By the description of X in Subsection 4.5, τ_w can be written as a polynomial in $\mathbb{C}[SL_2(\mathbb{C})]$ generators A_e, B_e, C_e, D_e for $e \in E(\Gamma)$. We let these be represented by $a_e/t_e, b_e/t_e, c_e/t_e$, and d_e/t_e for t_e the form that cuts out $D \subset X$. Note that the t_e are algebraically independent. We claim that some monomial in this polynomial has t_e power equal to $-a(P,\phi)(e)$ for all $e \in E(\Gamma)$. It follows from the definition of τ_w as a regular function on $M_\Gamma(SL_2(\mathbb{C}))$ that each monomial has t_e power $\geq -a(P,\phi)(e)$. Furthermore, if no monomial has all minimal possible t_e powers, then

$$v_{\Gamma,\gamma,\bar{1}}(\tau_w) < \sum_{e\in E(\Gamma)} a(P,\phi)(e),$$

where $v_{\Gamma,\gamma,\bar{1}}$ is considered as a valuation on the rational functions of $X^{E(\Gamma)}$.

Following the Newton–Okounkov body recipe, we record the value $a(P,\phi)(e_1)$ and consider the product $t_{e_1}^{a(P,\phi)(e_1)}\tau_w$ as a regular function on $D \times X^{|E(\Gamma)|-1}$. But by the above argument, the leading term of this product has t_{e_2} degree $\geq -a(P,\phi)(e_2)$, and this value is achieved on one of the same monomials with t_{e_1} power $t_{e_1}^{-a(P,\phi)(e_1)}$. Furthermore, this pattern must continue as we proceed deeper into the flag. We conclude that $\mathfrak{v}_{\Gamma,\prec}(\tau_w) = (a(P,\phi)(e_1),\dots,a(P,\phi)(e_{|E(\Gamma)|}))$, where the ordering is determined by $\prec$. ∎

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