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SOLUTION OF HOMOGENEOUS LINEAR DIFFERENCE EQUATIONS

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Abstract

When the first two elements of a sequence satisfying a second order difference equation are prescribed, the remaining elements are evaluated from a continued fraction and a simple product.

1. Introduction

The sequence u_n satisfies the second order, linear difference equation

$$H(n)\,u_{n+1}+G(n)\,u_n+F(n)\,u_{n-1}=0, \quad n=1,2,\ldots, \tag{1.1}$$

where the $F(n)$, $G(n)$ and $H(n)$ are arbitrary functions of n. We seek the solution for all n when u_0 and u_1 are prescribed.

If one of the complementary solutions of (1.1) can be found by inspection, or if it is prescribed, Brand [1] shows how the second complementary solution is obtained. Let v_n be the known solution, then the second solution w_n is generated by setting $u_n = v_n w_n$ in (1.1). The procedure is analogous to finding the second solution of a differential equation when one solution is known. A linear combination of v_n and w_n is then used to satisfy the prescribed values of u_0 and u_1.

The solution given by Brand was established earlier by Funk [3], but the reference is not so readily available. Funk also solves (1.1) when $u_0 = 0$ and $u_{N+1} = 0$, so that only N equations are involved. His solution for u_1 is expressed as the ratio of two tridiagonal determinants, according to Cramer's rule, and he goes on to show that each tridiagonal determinant is expressible as a continued fraction containing the $F(n)$, $G(n)$ and $H(n)$.

An even earlier solution of (1.1) is given by Perron [4] in terms of a tridiagonal determinant for the case $u_0 = 0$, $u_1 = 1$ and $H(n) = -1$ for all n. He gives

$$u_{n+1} = C_1^n, \quad n = 1, 2, \ldots, \tag{1.2}$$

where C_1^n is defined by (1.4). Perron then points out, like Funk, that C_1^n is equivalent to a continued fraction.

More recently, Brown [2] has given the general solution to (1.1) as a linear combination of u_0 and u_1 according to

$$u_{n+1} = (-)^n\{u_0 F(1)\, C_2^n + u_1\, C_1^n\} \Big/ \prod_{r=1}^{n} H(r) \quad n = 2, 3, \ldots, \tag{1.3}$$

where C_1^n and C_2^n are tridiagonal determinants of orders n and $n-1$ defined by

$$C_m^n = \begin{vmatrix} G(m) & H(m) & 0 & \ldots & 0 & . & . \\ F(m+1) & G(m+1) & H(m+1) & \ldots & 0 & . & . \\ 0 & F(m+2) & G(m+2) & \ldots & H(n-3) & 0 & . \\ . & 0 & F(m+3) & \ldots & G(n-2) & H(n-2) & 0 \\ . & . & . & \ldots & F(n-1) & G(n-1) & H(n-1) \\ . & . & . & \ldots & 0 & F(n) & G(n) \end{vmatrix}. \tag{1.4}$$

This result is the generalization of Perron's solution, and clearly reduces to (1.2) when his conditions are satisfied.

Our motivation is to propose an alternative solution to (1.1), which is as general as Brown's results, but which is simpler and avoids the evaluation of two determinants.

2. Method of solution

The first step in our method of solution is to reduce (1.1) to a first-order equation by the following rearrangement

$$\left[\frac{u_{n+1}}{u_n}\right] + \frac{G(n)}{H(n)} + \frac{\dfrac{F(n)}{H(n)}}{\left[\dfrac{u_n}{u_{n-1}}\right]} = 0. \tag{2.1}$$

This is now a nonlinear, first order difference equation for the ratio u_{n+1}/u_n, which is solved by repeated iteration. The solution is expressed as

$$\frac{u_{n+1}}{u_n} = p_n, \tag{2.2}$$

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where p_n is defined by the continued fraction

$$p_n = -\frac{G(n)}{H(n)} - \cfrac{\frac{F(n)}{H(n)}}{-\frac{G(n-1)}{H(n-1)} - \cfrac{\frac{F(n-1)}{H(n-1)}}{-\frac{G(n-2)}{H(n-2)} - \cdots \cfrac{\cdots}{\cfrac{-\frac{F(1)}{H(1)}}{\frac{u_1}{u_0}}}}}. \tag{2.3}$$

When $n = 0$, we have $p_0 = u_1/u_0$. Thus each p_n as evaluated from (2.3) determines the ratio of successive u_n, that is $u_{n+1} = p_n u_n$.

The second step is to use an identity to express u_{n+1} as a product of ratios of successive u_n

$$u_{n+1} = \frac{u_{n+1}}{u_n} \frac{u_n}{u_{n-1}} \cdots \frac{u_1}{u_0} u_0. \tag{2.4}$$

Substituting from (2.2), the general solution for u_{n+1} is given by

$$u_{n+1} = u_0 \prod_{r=0}^{n} p_r. \tag{2.5}$$

The evaluation of the product is simplified if we observe from (2.2) and (2.3) that successive p_n satisfy

$$p_n = -\frac{G(n)}{H(n)} - \frac{F(n)}{H(n)} \frac{1}{p_{n-1}}, \quad n = 1, 2, \ldots. \tag{2.6}$$

The solution given by (2.5) is formally identical to (1.3), as may be verified, but has the advantage that only one set of calculations is required to evaluate the p_n, whereas two sets of calculations must be employed to determine C_1^n and C_2^n in (1.3).

3. Solutions when u_0 or u_1 vanish

If both u_0 and u_1 are non-zero, u_{n+1} is given by (2.5), and if both u_0 and u_1 vanish, there is only the trivial solution $u_n = 0$ for all n. The two cases of interest occur when either u_0 or u_1 vanishes. When $u_0 = 0$, the identity (2.4) is contracted by removing the dependence on u_0. Consequently, (2.5) is replaced by

$$u_{n+1} = u_1 \prod_{r=1}^{n} p_r, \quad n = 1, 2, \dots. \tag{3.1}$$

Similarly, when $u_1 = 0$, the identity (2.4) is contracted by removing the dependence on both u_0 and u_1. Hence

$$u_{n+1} = u_2 \prod_{r=2}^{n} p_r, \quad n = 2, 3, \dots, \tag{3.2}$$

and u_2 is expressed in terms of u_0 via (1.1)

$$u_2 = -\frac{F(1)}{H(1)} u_0. \tag{3.3}$$

Putting (3.3) into (3.2)

$$u_{n+1} = -\frac{F(1)}{H(1)} u_0 \prod_{r=2}^{n} p_r, \quad n = 2, 3, \dots. \tag{3.4}$$

References

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