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PURE COMPACTIFICATIONS IN QUASI-PRIMAL VARIETIES

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We prove that *if* $\mathfrak{A}$ *is quasi-primal, then every algebra in* **HSP**$\mathfrak{A}$ *has a pure embedding into a product of finite algebras*. For a general theory of varieties $\mathscr{V}$ for which every $\mathfrak{A} \in \mathscr{V}$ can be purely embedded in an equationally compact algebra $\mathfrak{B} \in \mathscr{V}$, and for all notions not explained here, the reader is referred to [**38**; **6**; or **5**]. This theorem was known for Boolean algebras simply as a corollary of the Stone representation theorem and the fact that in the variety of Boolean algebras, *all* embeddings are pure [**2**]. We extend this last result by proving that *if* $\mathfrak{A}$ *is quasi-primal and has no proper subalgebras other than singletons, then all embeddings in* **HSP**$\mathfrak{A}$ *are pure*. Finally, as a corollary of the main theorem, one immediately sees that "Mycielski's problem" [**30**, p. 484] has a positive solution for such varieties: *if* $\mathfrak{A}$ *is quasi-primal, then every equationally compact algebra in* **HSP**$\mathfrak{A}$ *is a retract of a compact topological algebra*.

There are only a few interesting classes K of structures known to have the property that every $\mathfrak{A} \in K$ can be purely embedded in an atomic-compact structure (i.e. for algebras, an equationally compact algebra): Abelian groups [**26**], mono-unary algebras [**40**], Boolean algebras [**38**], G-sets [**4**; **38**, 3.15], multi-unary relational structures (easy), and lattices taken as a class of partially ordered sets (Banaschewski and Nelson—unpublished). This property fails for bi-unary algebras [**38**, 3.17], distributive lattices (R. McKenzie—see [**38**, 3.16]) and semilattices [**31**]—all interesting classes K because they are *residually small*, that is, each $\mathfrak{A} \in K$ can be embedded in an equationally compact algebra. And of course our property fails for any K which is not residually small, of which there are many interesting examples. Again see [**38**; **6**; or **5**] for background. For the status of Mycielski's problem mentioned above see [**40**; or **38**, p. 43] and references given there.

The main theorems mentioned above are in § 1. In § 2 we give a new example of a residually small variety (Stone algebras) in which not every algebra has an equationally compact pure extension. In § 3 we supply an example of a finite algebra $\mathfrak{A}$ with **HSP**$\mathfrak{A}$ not residually small, and in § 4 we answer some questions of Lausch and Nöbauer.

1. Quasi-primal varieties. Quasi-primal algebras were introduced by A. F. Pixley under the name "simple algebraic algebras" in [**32**; see also **33**;

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18; **35**; and **36**]. We will define a finite algebra to be *quasi-primal* if and only if $\mathfrak{A}$ satisfies any (hence all) of the conditions of Theorem 1.1, which we include for information only, since all of our results can be discovered directly by using Q of condition (ii). One may check that an infinite algebra $\mathfrak{A}$ obeying 1.1 (ii) below generates a variety which is not residually small and thus does not satisfy anything like 1.6. But we are unable to decide whether Mycielski's problem holds for this $\mathbf{HSP}\mathfrak{A}$. For B, $C \subseteq A$ and $f : B \to C$, let us say that an *operation* $F : A^k \to A$ *preserves* f if and only if B and C are subuniverses of $\langle A, F\rangle$ and $f : \langle B, F \restriction B^k\rangle \to \langle C, F \restriction C^k\rangle$ is a homomorphism. We will tacitly assume that every polynomial of $\mathfrak{A}$, that is, derived operation (without constants!), is one of the fundamental operations of $\mathfrak{A}$. (We may as well do this since we are studying properties which are invariant under equivalence of varieties.)

THEOREM 1.1 (A. F. Pixley) *For* $\mathfrak{A} = \langle A, F_t\rangle_{t\in T}$ *finite, the following four conditions are equivalent:*

(i) *there exists a family* U *of bijections* $f : B \to C$ *with each* B, $C \subseteq A$, *such that the operations of* $\mathfrak{A}$ *are precisely those operations which preserve each* $f \in U$*;*

(ii) *the quaternary discriminator*

$$Q(a, b, c, d) = \begin{cases} c & \text{if } a = b \\ d & \text{if } a \neq b \end{cases}$$

is an operation of $\mathfrak{A}$*;*

(iii) *the ternary discriminator*

$$T(a, b, c) = \begin{cases} c & \text{if } a = b \\ a & \text{if } a \neq b \end{cases}$$

is an operation of $\mathfrak{A}$*;*

(iv) *every subalgebra of* $\mathfrak{A}$ *is simple and* $\mathbf{HSP}\mathfrak{A}$ *has permutable and distributive congruences.*

The implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) are immediate. For an elegant proof that (iv) $\Rightarrow$ (i), see [**32**, p. 368].

We now fix a quasi-primal $\mathfrak{A} = \langle A; Q, F_1, F_2, \ldots\rangle$, with Q the quaternary discriminator on A. We next review the representation theory for $\mathbf{HSP}\mathfrak{A}$. Parts (ii) and (iii) of the next Proposition were first proved by A. F. Pixley [**32**, Theorem 1.4].

PROPOSITION 1.2 (i) *If* θ *is any congruence on* $\mathfrak{B} \subseteq \mathfrak{A}^I$, *then there exists a filter* G *on* I *such that*

(*) $\theta = \{(\alpha, \beta) \in B^2 : \{i \in I : \alpha(i) = \beta(i)\} \in G\}.$

(ii) $\mathbf{HSP}\mathfrak{A} = \mathbf{ISP}\mathfrak{A}$.

(iii) *Every finite algebra in* $\mathbf{HSP}\mathfrak{A}$ *is a product of subalgebras of* $\mathfrak{A}$.

Sketch of proof. To prove (i), either use Jónsson's Lemma [**21**, Corollary 3.4] or proceed directly by taking

$$G_0 = \{\{i \in I : \alpha(i) = \beta(i)\} : (\alpha, \beta) \in \theta\}$$

and

$$G = \{X \subseteq I : (\exists\, Y \in G_0)\, Y \subseteq X\}.$$

The operation Q quickly shows that G is a filter and that (*) holds. Then (ii) is immediate. To see (iii), note that any finite $\mathfrak{B}$ is $\subseteq \mathfrak{A}^I$ for some finite I; taking $|I|$ as small as possible, one easily gets $\mathfrak{B}$ "rectangular" using Q.

For *injectivity* we refer the reader to [**12**], [**3**], or [**38**, § 2] and references given there. We intend injectivity in the category of non-empty homomorphisms, and so we will not be concerned with "$\emptyset$ – regularity" as in Day [**12**; cf. the remarks in **24**]. The next proposition can be proved directly from Day [**12**], by noticing that $\mathfrak{A}$ must be *self-injective* in Day's terminology, that is, *demi-semi-primal* in Quackenbush's terminology [**35**; see especially Theorem 5.6].

PROPOSITION 1.3. *If $\mathfrak{A}$ has no proper subalgebras other than singletons, then $\mathfrak{A}$ is injective in* **HSP**$\mathfrak{A}$.

N.b. There exist quasi-primal algebras $\mathfrak{A}$ other than those described in 1.3 which are obviously self-injective and hence injective in **HSP**$\mathfrak{A}$, e.g. $\mathfrak{A} = \langle A, Q\rangle$. But some are not injective in **HSP**$\mathfrak{A}$, e.g. $\mathfrak{A} = \langle\{0, 1, 2\}, Q, \wedge\rangle$ (where $\wedge$ denotes g.l.b.). For the reader may easily check that $\{0, 1\}$ is a subuniverse of $\mathfrak{A}$ and that $0 \mapsto 0$, $1 \mapsto 2$ cannot be extended to an endomorphism of $\mathfrak{A}$.

The next proposition is automatic if the conditions of 1.3 hold. Of course $\mathfrak{A}$ is an absolute retract because it is a maximal subdirect irreducible [**38**, 1.8], but this does not automatically give us direct powers of $\mathfrak{A}$ [**39**]. For a stronger fact than 1.4, see Quackenbush [**36**, Theorem 7.5]. The proof here is simpler.

PROPOSITION 1.4. *Every power $\mathfrak{A}^I$ is an absolute retract in* **HSP**$\mathfrak{A}$.

Proof. By 1.2(ii) it is enough to retract any embedding $\varphi : \mathfrak{A}^I \rightarrow \mathfrak{A}^J$. Let π_j denote jth co-ordinate projection $\mathfrak{A}^J \rightarrow \mathfrak{A}$. For each $i \in I$, select $a, b \in \mathfrak{A}^I$ which differ only in the ith place. Clearly there must exist $j = j(i) \in J$ so that $\pi_j\varphi(a) \neq \pi_j\varphi(b)$. It readily follows from 1.2(i) and the simplicity of $\mathfrak{A}$ that $\pi_j \circ \varphi = \alpha_i \circ \pi_i$ for some automorphism α_i of $\mathfrak{A}$. And so

$$\psi(x) = \langle \alpha_i^{-1}(x_{j(i)}) : i \in I\rangle$$

defines a homomorphism $\psi : \mathfrak{A}^J \rightarrow \mathfrak{A}^I$ with $\psi \circ \varphi = 1$.

THEOREM 1.5. *If $\mathfrak{A}$ is quasi-primal, then all embeddings in* **HSP**$\mathfrak{A}$ *are pure embeddings if and only if $\mathfrak{A}$ has no proper subalgebras except (possibly) singletons.*

Proof. If $\mathfrak{B} \subseteq \mathfrak{A}$, $1 < |\mathfrak{B}| < |\mathfrak{A}|$, then there is no retraction of $\mathfrak{A}$ onto $\mathfrak{B}$, since $\mathfrak{A}$ is simple, and hence $\mathfrak{B} \subseteq \mathfrak{A}$ is not a pure embedding.

Conversely, if $\mathfrak{A}$ has no proper non-trivial subalgebras, then every finitely generated algebra in $\mathbf{HSP}\mathfrak{A}$ is a finite power of $\mathfrak{A}$ and hence an absolute retract by 1.4. Thus if $f : \mathfrak{B} \to \mathfrak{C}$ is an embedding, then $f \restriction \mathfrak{B}'$ is retractable, hence pure, whenever $\mathfrak{B}'$ is any finitely generated subalgebra of $\mathfrak{B}$. It follows immediately that f is a pure embedding.

For some information on *absolutely pure* algebras, consult Bacsich [**2**]. In fact, Theorem 1.5 is an immediate corollary of proposition 1.4 and Lemma 4.5 of [**2**].

Note that if $\mathfrak{A}$ is quasi-primal, then for each $n \geqq 1$ there exists a derived $(2n + 2)$-ary operation Q_n of $\mathfrak{A}$ such that

$$Q_n(a_1, \ldots, a_n, b_1, \ldots, b_n, c, d) = \begin{cases} c & \text{if } a_i = b_i \ (1 \leqq i \leqq n) \\ d & \text{otherwise.} \end{cases}$$

For Q_n may be defined recursively via $Q_1 = Q$ and

$$Q_{n+1}(a_1, \ldots, a_{n+1}, b_1, \ldots, b_{n+1}, c, d) = Q(a_{n+1}, b_{n+1}, Q_n(a_1, \ldots, a_n, b_1, \ldots, b_n, c, d), d).$$

THEOREM 1.6. *If $\mathfrak{A}$ is quasi-primal, then every algebra in* $\mathbf{HSP}\mathfrak{A}$ *has a pure embedding into a product of subalgebras of $\mathfrak{A}$.*

Proof. Let $\mathfrak{B} \subseteq \mathfrak{A}^I$; let βI denote the collection of ultrafilters on I; for $\mu \in \beta I$, let $\mathfrak{B}_\mu \subseteq \mathfrak{A}$ denote the image of $\mathfrak{B}$ under the natural map $\pi_\mu : \mathfrak{A}^I \to \mathfrak{A}^I/\mu \cong \mathfrak{A}$. We will be done when we have shown that $\pi : \mathfrak{B} \to \prod \mathfrak{B}_\mu$ is a pure embedding ($\pi(b) = \langle \pi_\mu(b) : \mu \in \beta I \rangle$). To do this, let us take terms $\alpha_1, \beta_1, \ldots, \alpha_k, \beta_k$ in variables $x^1, \ldots, x^n, y^1, \ldots, y^m$ such that, for fixed $b^1, \ldots, b^n \in B$,

$$(1) \quad \mathfrak{B} \models \neg \exists\, y^1 \ldots y^m \bigwedge_{j=1}^{k} (\alpha_j(\vec{b}, \vec{y}) \simeq \beta_j(\vec{b}, \vec{y})).$$

Now define $\mathscr{I}$ to be the family of sets $J \subseteq I$ such that there exist $c^1, \ldots, c^m \in B$ such that

$$(2) \quad \alpha_j(b_i^1, \ldots, b_i^n, c_i^1, \ldots, c_i^m) = \beta_j(b_i^1, \ldots, b_i^n, c_i^1, \ldots, c_i^m) (1 \leqq j \leqq k)$$

whenever $i \in J$. We will show that $\mathscr{I}$ is an ideal of sets; we need only check that if $J, K \in \mathscr{I}$, then $J \cup K \in \mathscr{I}$. Thus let us assume that (2) holds and also that

$$(3) \quad \alpha_j(b_i^1, \ldots, b_i^n, d_i^1, \ldots, d_i^m) = \beta_j(b_i^1, \ldots, b_i^n, d_i^1, \ldots, d_i^m) (1 \leqq j \leqq k)$$

whenever $i \in K$. For $j = 1, \ldots, k$, let $\bar{\alpha}_j, \bar{\beta}_j \in \mathfrak{B}$ be defined by

$$\bar{\alpha}_j = \alpha_j(b^1, \ldots, b^n, c^1, \ldots, c^m)$$
$$\bar{\beta}_j = \beta_j(b^1, \ldots, b^n, c^1, \ldots, c^m).$$

And finally define $e^s (1 \leqq s \leqq m)$ as

$$e^s = Q_k(\bar{\alpha}_1, \ldots, \bar{\alpha}_k, \bar{\beta}_1, \ldots, \bar{\beta}_k, c^s, d^s),$$

where Q_k is the operation defined immediately before this theorem.

To finish showing $J \cup K \in \mathscr{I}$, we will show that

$$(4)\quad \alpha_j(b_i^1, \ldots, b_i^n, e_i^1, \ldots, e_i^m) = \beta_j(b_i^1, \ldots, b_i^n, e_i^1, \ldots, e_i^m)(1 \leqq j \leqq k)$$

whenever $i \in J \cup K$. If $i \in J$, then (2) holds, and so for each j, $\bar{\alpha}_j$ and $\bar{\beta}_j$ are equal in the ith co-ordinate; thus (by the announced property of Q_k) $e_i^s = c_i^s (1 \leqq s \leqq m)$, and so (4) reduces to (2). On the other hand if $i \in K - J$, then for some j, $\bar{\alpha}_j$ and $\bar{\beta}_j$ are unequal in the ith co-ordinate; thus $e_i^s = d_i^s$ $(1 \leqq s \leqq m)$ and so (4) reduces to (3).

We know by (1) that the ideal $\mathscr{I}$ is proper, that is, $I \notin \mathscr{I}$; thus there exists an ultrafilter μ containing no member of $\mathscr{I}$. Finally, we claim that, for this μ,

$$(5)\quad \mathfrak{B}_\mu \models \neg \exists\, y^1 \ldots y^m \bigwedge_{j=1}^{k} [\alpha_j(\pi_\mu \vec{b}, \vec{y}) \simeq \beta_j(\pi_\mu \vec{b}, \vec{y})].$$

For suppose that (5) is false; that is, for some $\pi_\mu(c^1), \ldots, \pi_\mu(c^m)$ we have

$$\alpha_j(\pi_\mu b^1, \ldots, \pi_\mu b^n, \pi_\mu c^1, \ldots, \pi_\mu c^m)$$
$$= \beta_j(\pi_\mu b^1, \ldots, \pi_\mu b^n, \pi_\mu c^1, \ldots, \pi_\mu c^m)\ (1 \leqq j \leqq k).$$

But this says that (2) holds for $i \in I \in \mu$, a contradiction, thus establishing (5). But, as is well known, the validity of (1) $\Rightarrow$ (5) is equivalent to the purity of the embedding $\mathfrak{B} \rightarrow \Pi\mathfrak{B}_\mu$ [**38**, Lemma 3.2].

We remark that the following theorem is an immediate corollary of Theorem 1.6 (and *vice versa*)—for the notion of pure-irreducibility and its connections with this subject see [**38**, § 3; or **6**, § 4]; a more general notion subsuming pure-irreducibility can be found in [**11**].

THEOREM 1.7. *If* $\mathfrak{A}$ *is quasi-primal, then the pure-irreducible algebras in* **HSP**$\mathfrak{A}$ *are precisely the subalgebras of* $\mathfrak{A}$.

We now turn to the possibility of applying Theorem 1.6 to some special algebras which are known to be quasi-primal. We should point out that condition (i) of Theorem 1.1 really gives us a complete catalog of quasi-primal algebras; for each finite A there are only finitely many families U of bijections $f : B \rightarrow C$ with $B, C \subseteq A$, and so there are only finitely many inequivalent varieties **HSP**$\mathfrak{A}$ with $\mathfrak{A}$ quasi-primal of fixed finite power. Nonetheless quasi-primal algebras sometimes arise in a "natural" way, apparently rather different from condition (i) of 1.1.

COROLLARY 1.8. *If* $\mathscr{V}$ *is a variety of commutative rings with unit obeying some law* $x^m = x$, *then every ring in* $\mathscr{V}$ *is a pure subring of a product of finite rings.*

Sketch of proof. The law $x^m = x$ implies the absence of non-zero nilpotent elements, and so, by a theorem of G. Birkhoff [**10**], every subdirectly irreducible ring $R \in \mathscr{V}$ is a field—in which the equation $x^m = x$ can have at most m roots;

thus $|R| \leqq m$. Consider first the case that all fields $R \in \mathscr{V}$ have the same characteristic, p. Now finitely many fields of characteristic p can always be embedded in a single field $GF(p^k)$ for some k. (*N.b.* But $GF(p^k)$ may itself fail to be in $\mathscr{V}$—note Banaschewski's example [**3**] of $x^{22} = x$—this $\mathscr{V}$ contains $GF(4)$ and $GF(8)$, but not $GF(64)$.) In this case we have $\mathscr{V} \subseteq \mathbf{HSP}GF(p^k)$, and $GF(p^k)$ is known [**32**] to be quasi-primal, for it is not hard to express the ternary discriminator in $GF(p^k)$. Finally, if $\mathscr{V}$ has fields of various characteristics $p_1, \ldots, p_s$, then choose integers $a_1, \ldots, a_s$ so that

$$\sum_{i=1}^{s} a_i p_1 \ldots \hat{p}_i \ldots p_s = 1$$

(where ^ indicates a deletion) and let $t(x_1, \ldots, x_s)$ be the ring-theoretic term

$$\sum_{i=1}^{s} a_i p_1 \ldots \hat{p}_i \ldots p_s x_i.$$

It is clear that if $\mathscr{V}_i$ is the subvariety of $\mathscr{V}$ generated by fields in $\mathscr{V}$ of characteristic p_i, then

$$\mathscr{V}_i \models t(x_1, \ldots, x_s) \doteq x_i \quad (1 \leqq i \leqq s),$$

and also

$$\mathscr{V} = \mathbf{HSP}(\mathscr{V}_1 \cup \ldots \cup \mathscr{V}_s).$$

Thus the theory of *independent varieties* [**17**; **19**; **13**; **9**] tells us that every ring $R \in \mathscr{V}$ is a product, $R \cong R_1 \times \ldots \times R_s$, with $R_i \in \mathscr{V}_i$ $(1 \leqq i \leqq s)$, and so we may apply the result for single characteristics. (Note that if $\mathfrak{A} \subseteq \mathfrak{C}$ is pure and $\mathfrak{B} \subseteq \mathfrak{D}$ is pure, then $\mathfrak{A} \times \mathfrak{B} \subseteq \mathfrak{C} \times \mathfrak{D}$ is pure.)

There have been many definitions of "Post algebras," not all equivalent, and certainly not all equational. But the following definition includes a portion of the known theory. Let

$$\mathfrak{A}_n = \langle \{0, \ldots, n-1\}, \wedge, \vee, ', 0, \ldots, n-1 \rangle,$$

where $\wedge$ and $\vee$ are the binary operations of minimum and maximum, $x' = x + 1 \pmod{n}$, and $0, \ldots, n-1$ are constants. Following Ash [**1**] we define the *variety of Post algebras of order n* to be $\mathbf{HSP}\mathfrak{A}_n$. For another definition of Post algebras as a variety, see Traczyk [**42**]; for many other definitions of Post algebras, see references given in [**7**; **8**; **42**; or **43**]. E. L. Post showed in 1921 [**34**] that $\mathfrak{A}_n$ is functionally complete, and hence primal, and so the following corollary is immediate (either from 1.5 or 1.6).

COROLLARY 1.9. *Every Post algebra of order n is a pure subalgebra of a power of* $\mathfrak{A}_n$.

There are some interesting quasi-primal reducts of $\mathfrak{A}_n$ to which we can apply 1.6 but not 1.5, namely the *double Heyting algebras*

$$\mathfrak{L}_n = \{\langle 0, e_2, \ldots, e_{n-1}, 1 \rangle, \wedge, \vee, *, +\},$$

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where $\wedge$ and $\vee$ are meet and join operators for the linear ordering $0 < e_2 < \ldots < e_{n-1} < 1$, $*$ is the binary operation of *relative pseudocomplement*, that is, $x \leqq a * b$ if and only if $x \wedge a \leqq b$, and $+$ is defined dually to $*$. One may check that the quaternary discriminator is defined on $\mathfrak{L}_n$ by taking

$$Q(x, y, z, w) = [(p(x, y) * q(x, y)) \wedge z] \vee [(q(x, y) + p(x, y)) \wedge w],$$

where

$$p(x, y) = (x \vee y) * ((x \wedge y) + (x \vee y))$$

and

$$q(x, y) = (x \wedge y) + ((x \vee y) * (x \wedge y)).$$

(These are only special instances of a wide class of double Heyting algebras considered by Katriňák [**23**].)

COROLLARY 1.10. *For fixed n, every double Heyting algebra in* $\mathbf{HSP}\mathfrak{L}_n$ *is a pure subalgebra of a product of finite algebras.*

We will omit the proof that $\mathfrak{L}_3$ is equivalent to $\langle\{0, e, 1\}, \wedge, \vee, {}^*, {}^+\rangle$ where * and $^+$ are pseudocomplement $x^* = x * 0$ and dual pseudocomplement $x^+ = x + 1$; this latter algebra generates a subvariety of the variety of *double Stone algebras* known as *trivalent Lukasiewicz algebras* (see [**43**] and references given there). Hence the next corollary is immediate; the dual pseudocomplement $^+$ is essential—the corresponding statement for the $(\wedge, \vee, {}^*)$-reduct is false; see § 2 below.

COROLLARY 1.11. *Every trivalent Łukasiewicz algebra is a pure subalgebra of a product of finite algebras.*

Finally, in connection with Mycielski's problem mentioned in the introduction, we have the next corollary. The result concerning Post algebras was previously proved by Beazer [**7**].

COROLLARY 1.12. *If $\mathfrak{A}$ is quasi-primal, then every equationally compact algebra $\mathfrak{B}$ in* $\mathbf{HSP}\mathfrak{A}$ *is a retract of a compact topological algebra (in fact, a product of finite algebras). The conclusion holds, in particular, for $\mathfrak{B}$ a commutative ring obeying some law $x^m = x$, a Post algebra of order n, a double Heyting algebra in* $\mathbf{HSP}\mathfrak{L}_n$ *or a trivalent Lukasiewicz algebra.*

One further example of varieties to which 1.6 applies is any variety $\mathscr{K}_n (n < \omega)$ of *monadic algebras* (see Monk [**29**] or Quackenbush [**36**, § 10]).

2. A Stone algebra with no equationally compact hull. An algebra $\mathfrak{A} = \langle A; \wedge, \vee, 0, 1, {}^*\rangle$ is a *Stone algebra* if and only if $\langle A; \wedge, \vee, 0, 1\rangle$ is a distributive lattice with 0 and 1, * is a unary operation of pseudocomplementation, that is, $x \wedge a = 0 \leftrightarrow x \leqq a^*$, and * satisfies the *Stone identity*

$$x^* \vee x^{**} \doteq 1.$$

Equivalently, the class of Stone algebras is the variety $\mathscr{S}$ given by the equations defining distributive lattices with 0 and 1 together with these equations:

$$x \simeq x \wedge x^{**}$$

$$x \wedge x^{*} \simeq 0 \quad x^{*} \vee x^{**} \simeq 1$$

$$(x \wedge y)^{*} \simeq x^{*} \vee y^{*}$$

$$(x \vee y)^{*} \simeq x^{*} \wedge y^{*}.$$

The only subdirectly irreducible Stone algebras are the two-element Boolean algebra and the three-element algebra $\mathfrak{C}_3 = \langle \{0, e, 1\}, \wedge, \vee, 0, 1, {}^*\rangle$ where $0 < e < 1$, and $0^* = 1$, $e^* = 1^* = 0$. Thus $\mathscr{S} = \mathbf{HSP}\mathfrak{C}_3$ and so $\mathscr{S}$ is residually small [**38**; **6**]; moreover $\mathscr{S}$ has enough injectives. For these and related facts, see [**16**, § 14] and references given there.

THEOREM 2.1. *There exists a Stone algebra which is not a pure subalgebra of any equationally compact algebra.*

Proof. R. McKenzie proved (see [**38**, p. 50]) that there exists a distributive lattice L which is not a pure subalgebra of any equationally compact algebra (In fact, the example is presented in [**38**, p. 50] as an example of *pure-irreducibility*; but our statement here is a corollary—see [**38**, § 3; or **6**]. It is easy to see that the large pure-irreducible algebra given there has a greatest and a least element. If it did not, one could always adjoin them, yielding a pure extension.) Take L to be a family of subsets of a set P which is closed under $\cap$ and $\cup$ and contains $\emptyset$ and P. Our Stone algebra will be a certain subalgebra of $\mathfrak{C}_3{}^P$. For each $\lambda \in L$ define $F(\lambda) \in \mathfrak{C}_3{}^P$ via

$$F(\lambda)(x) = \begin{cases} 1 & \text{if } x \in \lambda \\ e & \text{if } x \notin \lambda. \end{cases}$$

Now let $A = \{(0, 0, \ldots)\} \cup \{F(\lambda) : \lambda \in L\}$. It is easy to check that A is a subuniverse of $\mathfrak{C}_3{}^P$ and so defines a Stone algebra $\mathfrak{A}$. We first claim that $F[L] = \{f(\lambda) : \lambda \in L\}$ is a *pure* sublattice of $\langle A, \wedge, \vee\rangle$—in fact it is obviously a retract of $\langle A, \wedge, \vee\rangle$ by mapping $(0, 0, 0, \ldots)$ onto $(e, e, e, \ldots)$. Now if $\mathfrak{A} \subseteq \mathfrak{B} = \langle B; \wedge, \vee, 0, 1, {}^*\rangle$ were any pure embedding of $\mathfrak{A}$ in an equationally compact Stone algebra then we would have $L \cong F(L) \subseteq \langle A; \wedge, \vee\rangle \subseteq \langle B; \wedge, \vee\rangle$ with both embeddings pure and $\langle B; \wedge, \vee\rangle$ equationally compact—a contradiction.

This theorem should be compared with Theorem 1.11 above about $\mathbf{HSP}$ $\langle\{0, e, 1\}, \wedge, \vee, 0, 1, {}^*, {}^+\rangle$. We are unable to decide whether $\mathbf{HSP}\mathfrak{H}_3$ admits pure compactifications, where $\mathfrak{H}_3$ is a 3-element Heyting algebra (cf. 1.10 above). And of course Theorem 2.1 does not settle Mycielski's question for Stone algebras (which is also open for distributive lattices): *is every equationally compact Stone algebra a retract of some compact topological algebra?*

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3. A finite $\mathfrak{A}$ with HSP $\mathfrak{A}$ not residually small. Recall [**38**] that a variety $\mathscr{V}$ is residually small if and only if $\mathscr{V}$ contains only a *set* of subdirectly irreducible algebras—equivalently, if and only if every algebra in $\mathscr{V}$ can be embedded in an equationally compact algebra. In [**38**] we remarked that if $\mathfrak{A}$ is finite and **HSP**$\mathfrak{A}$ has distributive congruences, then **HSP**$\mathfrak{A}$ is residually small (even residually finite) as follows readily from Jónsson's Lemma [**21**], but we were unable to state whether there exists any finite $\mathfrak{A}$ with **HSP**$\mathfrak{A}$ not residually small. We thank R. W. Quackenbush for pointing out that J. A. Gerhard had in effect already found such $\mathfrak{A}$ of power 3. We will see that 3 is best possible.

THEOREM 3.1. *There exists a 3-element idempotent semigroup $\mathfrak{A}$ with* **HSP**$\mathfrak{A}$ *not residually small.*

Proof. Let $\mathfrak{A} = \langle\{0, 1, 2\}, \cdot\,\rangle$ with this multiplication table:

$$\begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 2 & 0 & 1 & 2 \end{array}.$$

(Equivalently take the concrete semigroup consisting of the identity function and two constant functions on any set of more than one element.) It follows from work of J. A. Gerhard [**14**] that **HSP**$\mathfrak{A}$ is defined by the laws

$$x(yz) \doteq (xy)z$$

$$xx \doteq x$$

$$xyx \doteq xy.$$

We next note that in [**15**] Gerhard has given an example of a (countably) infinite subdirectly irreducible semigroup in this variety, but his construction really applies to any cardinality. In fact, let X be any set, $0, 1 \in X$, and for each $x \in X$, define mappings $a_x, b_x, c_x : X \to X$ as follows

$$a_x(y) = x$$

$$b_x(y) = \begin{cases} y & \text{if } y = x \\ 0 & \text{if } y \neq x \end{cases}$$

$$c_x(y) = \begin{cases} y & \text{if } y = x \\ 1 & \text{if } y \neq x. \end{cases}$$

One may check that $\{a_0, a_1\} \cup \{a_x, b_x, c_x : x \in X, x \neq 0, 1\}$ is closed under composition and defines a semigroup S in this variety. Now if θ is any congruence with $a_x \,\theta\, a_y (x \neq y)$, then

$$a_0 = b_x a_y \,\theta\, b_x a_x = a_x = c_x a_x \,\theta\, c_x a_y = a_1,$$

and so $a_0 \,\theta\, a_1$. Thus if θ is a maximal congruence separating a_0 and a_1, then S/θ is subdirectly irreducible and $|S/\theta| = |X|$.

Notice that the semigroup $\mathfrak{A}$ of Theorem 3.1 is isomorphic to $\langle(0, e, 1), \cdot\rangle$ under the correspondence

$$0 \leftrightarrow 2$$
$$e \leftrightarrow 1$$
$$1 \leftrightarrow 0,$$

where $\cdot$ is defined in the 3-element Stone algebra via

$$x \cdot y = x \vee (x^* \wedge y).$$

Thus $\mathbf{HSP}\mathfrak{A}$ is (within equivalence) a reduct of the variety of Stone algebras.

J. Baldwin and J. Berman have supplied another example of a three-element algebra which generates a variety which is not residually small: a *three-element pseudocomplemented semilattice* [**20**; **37**]. If $\mathfrak{A}$ is such, then $\mathfrak{A}$ is not a Boolean algebra; by Jones [**20**], the only proper subvariety of pseudocomplemented semilattices is the variety of Boolean algebras; hence $\mathbf{HSP}\mathfrak{A}$ = the variety of pseudocomplemented semilattices, which is known to be not residually small. One easily sees that these two examples are not equivalent.

THEOREM 3.2. *If $\mathfrak{A}$ is any 2-element algebra, then $\mathbf{HSP}\mathfrak{A}$ is residually small, in fact residually of power 2.*

Sketch of proof. It is enough to check through equivalence classes of two-element algebras as enumerated by Post in 1941. We will use the reformulation by Lyndon in 1951 [**27**]. Systems I are all familiar (Boolean rings, algebras, groups, 3-groups, etc.). Systems II have implication algebras as reducts, and these are congruence-distributive [**28**; see also **22** for an explicit representation]. Systems III are explicitly represented in [**27**], and Systems IV possess a "median" operator, and so are well known to have distributive congruences.

4. Some problems of Lausch and Nöbauer.

With quasi-primal algebras we can solve three open problems of Lausch and Nöbauer [**25**]. The first, on page 42, asks whether there exists *a variety $\mathscr{V}$ without constants which is semi-degenerate*, that is, no algebra of power > 1 in $\mathscr{V}$ has a one-element subalgebra. Clearly if $\mathfrak{A} = (\{0, 1\}, Q, p)$, where $p(0) = 1$, $p(1) = 0$, then $\mathscr{V} = \mathbf{HSP}\mathfrak{A}$ is as desired (in [**41**] $\mathscr{V}$ is called the variety of "Boolean 3-algebras").

Problem (a) on page 70 asks, "*if $\mathfrak{A} \subseteq \mathfrak{B} \in \mathscr{V}$, and Σ is a set of equations with constants from A which is satisfiable in some $\mathfrak{C} \supseteq \mathfrak{A}$, $\mathfrak{C} \in \mathscr{V}$, then must Σ be satisfiable in some $\mathfrak{D} \supseteq \mathfrak{B}$, $\mathfrak{D} \in \mathscr{V}$?*" To see the negative answer, we let $\mathscr{V}$ be the variety of commutative rings with unit obeying the law $x^{22} \simeq x$, $\mathfrak{A} = GF(2)$, $\mathfrak{B} = GF(8)$, $\Sigma = \{x^2 \simeq x + 1\}$. Clearly Σ is satisfiable in $\mathfrak{C} = GF(4) \supseteq \mathfrak{A}$; but Σ is clearly not satisfiable in $\mathfrak{B}$, and hence not in any $\mathscr{V}$-extension of $\mathfrak{B}$, since $\mathfrak{B}$ is a $\mathscr{V}$-maximal subdirect irreducible, and hence an absolute retract in $\mathscr{V}$ (see [**38**], especially 2.7). (This example is essentially due to B. Banaschewski. An isomorphic example, in the language of quasi-primals, was given by R. W. Quackenbush [**36**, 8.2].)

Problem (b) on page 71 asks, "*if* $\mathfrak{A} \in \mathscr{V}$ *and* Σ *is a set of equations with constants from A which is satisfiable in some* $\mathfrak{B} \supseteq \mathfrak{A}$, $\mathfrak{B} \in \mathscr{V}$, *and which has at most one solution in any* $\mathfrak{B} \supseteq \mathfrak{A}$, $\mathfrak{B} \in \mathscr{V}$, *then must* Σ *be satisfiable in* $\mathfrak{A}$?" To see the negative answer, we let $\mathfrak{B} = \langle\{0, 1, 2\}, T, 0, 1\rangle$ (where 0, 1 are constants and T is the ternary discriminator as in 1.1 (iii) above), and take $\mathscr{V} = \mathbf{HSP}\mathfrak{B}$ and $\mathfrak{A} = \langle\{0, 1\}, T, 0, 1\rangle \subseteq \mathfrak{B}$. We take

$$\Sigma = \begin{cases} T(0, x, 1) = 0 \\ T(1, x, 0) = 1. \end{cases}$$

One easily checks that Σ is satisfiable in $\mathfrak{B}$ (by $x = 2$), but not in $\mathfrak{A}$; to finish, we need to see the uniqueness of a solution of Σ in any $\mathfrak{C} \supseteq \mathfrak{A}$, $\mathfrak{C} \in \mathscr{V}$. By 1.2 (ii), we need only see the uniqueness of a solution in any power $\mathfrak{B}^I$; but obviously the only solution in $\mathfrak{B}^I$ is $x = (2, 2, \ldots, 2)$.

Added in Proof. For further information on quasi-primal varieties, consult Keimel and Werner [**47**].

Bulman-Fleming and Werner [**45**] have proved that in a quasi-primal variety the equationally compact algebras are precisely (finite) products of extensions by complete Boolean algebras of subalgebras of the quasi-primal generator, and that the topologically compact algebras are all products of finite algebras. Also see Banaschewski and Nelson [**44**].

The author thanks S. O. Macdonald and J. Groves for pointing out that the 8-element quaternion group generates a variety which is not residually small. The proof is a natural generalization of [**49**, Example 51.33, p. 147]. This variety is given by the laws $[x^2, y] = 1$ and $x^4 = 1$ (together with laws for group theory) [**48**, Theorem 3.2]. This group is as small as possible, for the 6-element non-Abelian group generates a residually small variety (see e.g. [**50**]).

Problem (a) of Lausch and Nöbauer has also been solved by Hule and Müller [**46**].

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