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THE TENSOR PRODUCT FORMULA FOR REFLEXIVE SUBSPACE LATTICES

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ABSTRACT. We give a characterisation of $\mathcal{L}_1 \otimes \mathcal{L}_2$ where $\mathcal{L}_1$ and $\mathcal{L}_2$ are subspace lattices with $\mathcal{L}_1$ commutative and either completely distributive or complemented. We use it to show that $\operatorname{Lat}(\mathcal{A}_1 \otimes \mathcal{A}_2) = \operatorname{Lat}\mathcal{A}_1 \otimes \operatorname{Lat}\mathcal{A}_2$ if $\mathcal{A}_1$ is a CSL algebra with a completely distributive or complemented lattice and $\mathcal{A}_2$ is any operator algebra.

1. **Introduction.**

The algebra tensor product formula (ATPF):

(ATPF) $$\operatorname{Alg}(\mathcal{L}_1 \otimes \mathcal{L}_2) = \operatorname{Alg}\mathcal{L}_1 \bar{\otimes} \operatorname{Alg}\mathcal{L}_2.$$

for reflexive operator algebra has been studied in a series of papers [3], [5], [6], [7], [8], and [9]. Although not universally valid [10], the ATPF has been shown to hold in various circumstances. If $\mathcal{L}_1$ and $\mathcal{L}_2$ are both orthocomplemented then $\operatorname{Alg}\mathcal{L}_1$ and $\operatorname{Alg}\mathcal{L}_2$ are von Neumann algebras, and in these circumstances the ATPF is a formulation of Tomita's commutation theorem. The formula also holds if one of the subspace lattices $\mathcal{L}_1$ or $\mathcal{L}_2$ is commutative and completely distributive [9].

The dual equation is the lattice tensor product formula for reflexive subspace lattices (LTPF):

(LTPF) $$\operatorname{Lat}(\mathcal{A}_1 \bar{\otimes} \mathcal{A}_2) = \operatorname{Lat}A_1 \otimes \operatorname{Lat}A_2.$$

The validity of the LTPF has been established only in special cases. It holds, for example, if $\mathcal{A}_1$ and $\mathcal{A}_2$ are both CSL algebras and $\operatorname{Lat}\mathcal{A}_1$ is completely distributive [9], or if $\mathcal{A}_1$ and $\mathcal{A}_2$ are both approximately finite-dimensional von Neumann algebras [4], or if $\mathcal{A}_1$ is a CSL algebra and $\operatorname{Lat}\mathcal{A}_1$ is a nest or is totally atomic and $\mathcal{A}_2$ consists of just scalar multiples of the identity [4]. The main obstacle preventing more general results is the difficulty in determining $\mathcal{L}_1 \otimes \mathcal{L}_2$, except in a few special cases. In this paper we obtain a tractable description of $\mathcal{L}_1 \otimes \mathcal{L}_2$ when $\mathcal{L}_1$ is completely distributive and commutative, and use it to extend the known validity of the LTPF.

We consider only separable Hilbert spaces, bounded linear operators and orthogonal projections. For any set $\mathcal{A}$ of operators on a Hilbert space $\mathcal{H}$, $\operatorname{Lat}\mathcal{A}$ denotes the set of all projections left invariant by each $A \in \mathcal{A}$. Each $\operatorname{Lat}\mathcal{A}$ is a subspace lattice, *i.e.* a strongly

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closed, complete sublattice of $\mathrm{Proj}(\mathcal{H})$, the lattice of all projections on $\mathcal{H}$. For any set $\mathcal{L}$ of projections on $\mathcal{H}$, $\mathrm{Alg}\,\mathcal{L}$ denotes the set of all operators which leave invariant each $P \in \mathcal{L}$. Each $\mathrm{Alg}\,\mathcal{L}$ is an operator algebra, *i.e.* a weakly closed subalgebra of $\mathcal{B}(\mathcal{H})$. We say that $\mathcal{A}$ is reflexive if $\mathcal{A} = \mathrm{Alg}\,\mathrm{Lat}\,\mathcal{A}$, and that $\mathcal{L}$ is reflexive if $\mathcal{L} = \mathrm{Lat}\,\mathrm{Alg}\,\mathcal{L}$.

Suppose that $\mathcal{A}_i$ is an operator algebra and $\mathcal{L}_i$ is a subspace lattice on a Hilbert space $\mathcal{H}_i$, for $i = 1$ and 2. The tensor product $\mathcal{A}_1 \otimes \mathcal{A}_2$ is the operator algebra on $\mathcal{H}_1 \otimes \mathcal{H}_2$ generated by all elementary tensors $A_1 \otimes A_2$, where $A_i \in \mathcal{A}_i$. Similarly, $\mathcal{L}_1 \otimes \mathcal{L}_2$ is the smallest subspace lattice on $\mathcal{H}_1 \otimes \mathcal{H}_2$ which contains all elementary tensors $P_1 \otimes P_2$, where $P_i \in \mathcal{L}_i$. The lattice $\mathcal{L}_1 \otimes \mathcal{L}_2$ is, in general, difficult to determine. However a useful description can be given if one of the factors is completely distributive and commutative. This description is based upon Arveson's representation of commutative subspace lattices [1], which we now briefly outline. A more complete account also appears in [2] (Chapter 22).

Let μ be a regular measure on a compact metric space X, and let $\leq$ be a standard pre-order on X, *i.e.* for all $x, y \in X$, $x \leq y$ if and only if $f_n(x) \leq f_n(y)$ for all n, where $f_1, f_2, \ldots$ is a countable family of continuous real-valued functions on X. If E is a Borel subset of X, P_E will denote the corresponding projection on $L^2(X,\mu)$; *i.e.* P_E is multiplication by χ_E, the characteristic function of E. A subset E of X is increasing if $x \in X$ and $x \leq y$ implies $y \in E$. Let $\mathcal{L}(X,\mu,\leq) = \{P_E : E \text{ is an increasing Borel set}\}$. Then $\mathcal{L}(X,\mu,\leq)$ is a commutative subspace lattice (CSL), and every CSL is unitarily equivalent to one of the form $\mathcal{L}(X,\mu,\leq)$.

Arveson established the reflexivity of $\mathcal{L} = \mathcal{L}(X,\mu,\leq)$ by introducing $\mathcal{A}_{\min}$, the minimal algebra corresponding to $(X,\mu,\leq)$, and showing that $\mathcal{L} = \mathrm{Lat}\,\mathcal{A}_{\min}$. Of all ultraweakly closed algebra of operators $\mathcal{A}$ on $L^2(X,\mu)$ for which $\mathrm{Lat}\,\mathcal{A} = \mathcal{L}$ and $\mathcal{A} \cap \mathcal{A}^* = \mathcal{L}'$, $\mathcal{A}_{\min}$ is the smallest. The largest of such algebras is $\mathrm{Alg}\,\mathcal{L}$, and $\mathcal{L}$ is said to be *synthetic* if $\mathcal{A}_{\min} = \mathrm{Alg}\,\mathcal{L}$.

2. **The lattice $\mathcal{L}(X,\mu,\leq,\mathcal{P})$.** We shall study lattice tensor products $\mathcal{L}_1 \otimes \mathcal{L}_2$, acting on spaces $\mathcal{H}_1 \otimes \mathcal{H}_2$, in which the first factor, $\mathcal{L}_1$, is a CSL. We shall assume that $\mathcal{H}_1 = L^2(X,\mu)$ and $\mathcal{H}_2 = \mathcal{H}$, and that $\mathcal{H}_1 \otimes \mathcal{H}_2$ is identified, via a unitary equivalence, with $L^2(X,\mu,\mathcal{H})$, the Hilbert space of weakly-measurable, square-integrable, $\mathcal{H}$-valued functions on X. Under this identification, $L^\infty(X,\mu) \otimes \mathcal{B}(\mathcal{H}) = L^\infty\big(X,\mu,\mathcal{B}(\mathcal{H})\big)$, the space of measurable, essentially bounded $\mathcal{B}(\mathcal{H})$-valued functions defined on X. In particular, if χ_E is the characteristic function of a Borel subset E of X, and if P is a projection on $\mathcal{H}$, then $\chi_E \otimes P = \chi_E P$.

For any subset $\mathcal{A}$ of $\mathcal{B}(\mathcal{H})$, let $L^\infty(X,\mu,\mathcal{A})$ denote the space of essentially bounded, $\mathcal{A}$-valued functions on X. Let $\mathcal{B}(\mathcal{H})_+$ denote the positive cone of $\mathcal{B}(\mathcal{H})$. We say that $\phi \in L^\infty\big(X,\mu,\mathcal{B}(\mathcal{H})_+\big)$ is increasing if $\phi(x) \leq \phi(y)$ whenever $x \leq y$. Let $L^\infty(X,\mu,\leq)$ denote the space of essentially bounded, positive, increasing functions on X, and for each $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})_+$ let

$$L^\infty(X,\mu,\leq,\mathcal{A}) = L^\infty(X,\mu,\mathcal{A}) \cap L^\infty(X,\mu,\leq).$$

For each $\phi \in L^\infty\big(X,\mu,\mathcal{B}(\mathcal{H})\big)$, the multiplication operator M_ϕ is defined on $L^2(X,\mu,\mathcal{H})$ by $M_\phi f(x) = \phi(x)f(x)$ for each $f \in L^2(X,\mu,\mathcal{H})$. For any subspace lattice $\mathcal{P}$ on $\mathcal{H}$, let

$$\mathcal{L}(X,\mu,\mathcal{P}) = \{M_\phi : \phi \in L^\infty(X,\mu,\mathcal{P})\}.$$

Each $M_\phi \in \mathcal{L}(X,\mu,\mathcal{P})$ is a projection, and $\mathcal{L}(X,\mu,\mathcal{P})$ is, in fact, a subspace lattice. The lattice operations in $\mathcal{L}(X,\mu,\mathcal{P})$ are performed pointwise, and $M_{\phi^{(\alpha)}} \longrightarrow M_\psi$ strongly if and only if $\phi^{(\alpha)}(x) \longrightarrow \psi(x)$ strongly a.e.

We also define

$$\mathcal{L}(X,\mu,\leq,\mathcal{P}) = \{M_\phi : \phi \in L^\infty(X,\mu,\leq,P)\}.$$

Since the partial order $\leq$ is preserved under arbitrary joins and intersections, $\mathcal{L}(X,\mu,\leq,\mathcal{P})$ is a strongly closed sublattice of $\mathcal{L}(x,\mu,\mathcal{P})$.

The tensor product $\mathcal{L}(X,\mu,\leq)\otimes\mathcal{P}$ is generated by projections of the form $P_E \otimes Q$, where P_E is multiplication by the characteristic function χ_E of an increasing subset E of X, and $Q \in \mathcal{P}$. But $P_E \otimes Q = M_\phi$ where ϕ is the increasing function $\chi_E Q$. So $\mathcal{L}(X,\mu,\leq)\otimes\mathcal{P} \subseteq \mathcal{L}(X,\mu,\leq,\mathcal{P})$. We shall show that $\mathcal{L}(X,\mu,\leq)\otimes\mathcal{P} = \mathcal{L}(X,\mu,\leq,\mathcal{P})$ for certain types of CSLs $\mathcal{L}(X,\mu,\leq)$. But first we establish some properties of $\mathcal{L}(X,\mu,\leq,\mathcal{P})$.

Arveson introduced the lattice $\mathcal{L}\big(X,\mu,\leq,\mathrm{Proj}(\mathcal{H})\big)$ in [1], and established its reflexivity by showing that

$$\mathcal{L}\big(X,\mu,\leq,\mathrm{Proj}(\mathcal{H})\big) = \mathrm{Lat}(\mathcal{A}_{\min}\otimes 1). \tag{1}$$

The next theorem is a simple generalisation.

THEOREM 1. $\mathcal{L}(X,\mu,\leq,\mathrm{Lat}\,\mathcal{B}) = \mathrm{Lat}(\mathcal{A}_{\min}\otimes\mathcal{B})$ *for any operator algebra* $\mathcal{B} \subseteq \mathcal{B}(\mathcal{H})$.

PROOF. Clearly $\mathcal{L}(X,\mu,\leq,\mathrm{Lat}\,\mathcal{B}) \subseteq L\big(X,\mu,\leq,\mathrm{Proj}(\mathcal{H})\big)$, and $\mathrm{Lat}(\mathcal{A}_{\min}\otimes\mathcal{B}) \subseteq \mathrm{Lat}(\mathcal{A}_{\min}\otimes 1) = \mathcal{L}\big(X,\mu,\leq,\mathrm{Proj}(\mathcal{H})\big)$ by (1). So suppose that $P = M_\phi \in \mathcal{L}\big(X,\mu,\leq,\mathrm{Proj}(\mathcal{H})\big)$. It is enough to show that $P \in \mathrm{Lat}(1\otimes\mathcal{B})$ if and only if $\phi(x) \in \mathrm{Lat}\,\mathcal{B}$ a.e.

Now $1\otimes B = M_B$ for each $B \in \mathcal{B}$, where $M_B f(x) = Bf(x)$, for all $f \in L^2(X,\mu,\mathcal{H})$. So $P^\perp(1\otimes B)P = M_{\phi^\perp}M_B M_\phi = M_{\phi^\perp B\phi} = 0$ if and only if $\phi(x) \in \mathrm{Lat}\,B$ a.e. It follows that if $\phi(x) \in \mathrm{Lat}\,\mathcal{B}$ a.e. then $P^\perp(1\otimes B)P = 0$, and hence $P \in \mathrm{Lat}(1\otimes\mathcal{B})$ since B is arbitrary in $\mathcal{B}$. On the other hand, if $P \in \mathrm{Lat}(1\otimes\mathcal{B})$, then for each $B \in \mathcal{B}$, $\phi^\perp(x)B\phi(x) = 0$ a.e. Since the unit ball of $\mathcal{B}$ is weakly separable, there is a null set N, such that $\phi^\perp(x)B\phi(x) = 0$ for all $x \in X\setminus N$ and for all $B \in \mathcal{B}$. So $\phi(x) \in \mathrm{Lat}\,\mathcal{B}$ for all $x \in X\setminus N$.

COROLLARY 2. *If $\mathcal{P}$ is reflexive then $\mathcal{L}(x,\mu,\leq,\mathcal{P})$ is reflexive.*

PROOF. If $\mathcal{P}$ is reflexive, then $\mathcal{P} = \mathrm{Lat\,Alg}\,\mathcal{P}$, and so by Theorem 1

$$\mathcal{L}(X,\mu,\leq,\mathcal{P}) = \mathcal{L}(X,\mu,\leq,\mathrm{Lat\,Alg}\,\mathcal{P}) = \mathrm{Lat}(\mathcal{A}_{\min}\otimes\mathrm{Alg}\,\mathcal{P}).$$

Hence $\mathcal{L}(X,\mu,\leq,\mathcal{P})$ is reflexive.

The following lemma will be used to establish the lattice tensor product formula for certain types of operator algebras.

LEMMA 3. *If $\mathcal{L}(X,\mu,\leq)\otimes\mathcal{P}=\mathcal{L}(X,\mu,\leq,\mathcal{P})$ for arbitrary subspace lattices $\mathcal{P}$, and if $\mathcal{L}(X,\mu,\leq)$ is synthetic, then the* LTPF *holds if $\mathcal{A}=\operatorname{Alg}\mathcal{L}(X,\mu,\leq)$ and $\mathcal{B}$ is any operator algebra.*

PROOF. The hypotheses and Theorem 1 imply that

$$\operatorname{Lat}(\mathcal{A}\otimes\mathcal{B})=\operatorname{Lat}(\mathcal{A}_{\min}\otimes\mathcal{B})=\mathcal{L}(X,\mu,\leq,\operatorname{Lat}\mathcal{B})=\mathcal{L}(X,\mu,\leq)\otimes\operatorname{Lat}\mathcal{B}.$$

The reflexivity of $\mathcal{L}(x,\mu,\leq)$ completes the proof.

3. **Boolean algebras.** If $\mathcal{L}(X,\mu,\leq)$ is complemented, we may assume that $\leq$ is trivial, *i.e.* $x\leq y$ if and only if $x=y$. Hence $\mathcal{L}(X,\mu,\leq)=\mathcal{L}(X,\mu)=\{P_E : E \text{ is a Borel set}\}$, and $\mathcal{L}(X,\mu,\leq,\mathcal{P})=\mathcal{L}(X,\mu,\mathcal{P})$.

THEOREM 4. *$\mathcal{L}(X,\mu)\otimes\mathcal{P}=\mathcal{L}(X,\mu,\mathcal{P})$ for any subspace lattice $\mathcal{P}$.*

PROOF. We must show that $\mathcal{L}(X,\mu,\mathcal{P})\subseteq\mathcal{L}(X,\mu)\otimes\mathcal{P}$. So suppose that $M_\phi\in\mathcal{L}(X,\mu,\mathcal{P})$. Since the weak and strong closures of any set of projections contain the same projections, and since $\mathcal{L}(X,\mu)\otimes\mathcal{P}$ is strongly closed, it is enough to show that every weak neighbourhood of M_ϕ contains a projection $M_\psi\in\mathcal{L}(X,\mu)\otimes\mathcal{P}$.

Suppose that $f_1,g_1,f_2,g_2,\ldots,f_n,g_n$ are vectors in $\mathcal{H}$, and $\varepsilon>0$. Let $\Phi: X\longrightarrow \mathbb{C}^n$ be defined by

$$\Phi(x)=\big(\langle\phi(x)f_1,g_1\rangle,\langle\phi(x)f_2,g_2\rangle,\ldots,\langle\phi(x)f_n,g_n\rangle\big).$$

Since Φ is bounded, its range can be covered by open subsets $U_1,U_2,\ldots,U_r$ of $\mathbb{C}^n$, each of diameter less than ε. Define disjoint subsets $X_1,X_2,\ldots,X_r$ inductively by $X_1=\Phi^{-1}(U_1)$ and $X_j=\Phi^{-1}(U_i)\setminus(X_1\cup X_2\cup\cdots\cup X_{j-1})$ for $j=2,3,\ldots,r$. Let $\psi=\sum_{j=1}^r\phi(x_j)\chi_j$, where for each j, χ_j is the characteristic function of X_j, and $x_j\in X_j$. (If X_j is empty, set $\phi(x_j)\chi_j=0$.) Then $M_\psi\in\mathcal{L}(X,\mu)\otimes\mathcal{P}$. Furthermore, if $F_i=\chi_Y\otimes f_i$, and $G_i=\chi_Y\otimes g_i$, where Y is a Borel subset of X, then for each i,

$$\begin{aligned}|\langle(M_\phi-M_\psi)F_i,G_i\rangle| &= \left|\int_Y\big\langle\big(\phi(x)-\psi(x)\big)f_i,g_i\big\rangle\mu\,dx\right|,\\ &\leq\sum_{j=1}^r\int_{X_j\cap Y}\left|\big\langle\big(\phi(x)-\phi(x_j)\big)f_i,g_i\big\rangle\right|\mu\,dx,\\ &\leq\varepsilon\cdot\sum_{j=1}^r\mu(X_j\cap Y)\leq\varepsilon\cdot\mu(Y).\end{aligned}$$

Suppose that $F_1,G_1,F_2,G_2,\ldots,F_n,G_n$ are step functions in $L^2(X,\mu,\mathcal{H})$. Then there exist disjoint, measurable sets $Y_1,Y_2,\ldots,Y_m$, and vectors f_{ij} and g_{ij} in $\mathcal{H}$, such that

$$F_i=\sum_{j=1}^m\chi_j\otimes f_{ij},\quad\text{and}\quad G_i=\sum_{j=1}^m\chi_j\otimes g_{ij},$$

for each i, where χ_j is the characteristic function of Y_j. For each j choose a step function ψ_j such that $M_{\psi_j}\in\mathcal{L}(X,\mu)\otimes\mathcal{P}$, and such that for each i and for each j,

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$|\langle (M_\phi - M_{\psi_j})(\chi_j \otimes f_{ij}), (\chi_j \otimes g_{ij})| \le \varepsilon \cdot \mu(Y_j)$. Now let $\psi = \chi_1\psi_1 + \chi_2\psi_2 + \cdots + \chi_m\psi_m$. Then $M_\psi \in \mathcal{L}(X,\mu) \otimes \mathcal{P}$, and for each i,

$$|\langle (M_\phi - M_\psi)F_i, G_i\rangle| = \sum_{j=1}^{m} |\langle (M_\phi - M_{\psi_j})(\chi_j \otimes f_{ij}), (\chi_j \otimes g_{ij})\rangle|,$$
$$\le \varepsilon \sum_{j=1}^{m} \mu(Y_j) \le \varepsilon \cdot \mu(X).$$

Since step functions are norm-dense in $L^2(X,\mu,\mathcal{H})$, it follows that M_ϕ is in the weak closure of $\mathcal{L}(X,\mu) \otimes \mathcal{P}$, as required.

COROLLARY 5. *If $\mathcal{A}$ is a von Neumann algebra with an abelian commutant, and if $\mathcal{B}$ is any operator algebra, then* $\mathrm{Lat}(\mathcal{A} \otimes \mathcal{B}) = \mathrm{Lat}\,\mathcal{A} \otimes \mathrm{Lat}\,\mathcal{B}$.

PROOF. The conditions on $\mathcal{A}$ ensure that $\mathcal{A} = \mathrm{Alg}\,\mathcal{L}(X,\mu)$ for some complemented CSL $\mathcal{L}(X,\mu)$. Such subspace lattices are synthetic [2] (Corollary 22.20), and so the result follows from Lemma 3 and Theorem 4.

4. **Complete distributivity.** Complete distributivity is an infinite version of ordinary distributivity for lattices. A complete lattice $\mathcal{L}$ is completely distributive if the identity:

$$\bigwedge_{\alpha\in I}\Big(\bigvee_{\beta\in J} x_{\alpha,\beta}\Big) = \bigvee_{\psi\in J^I}\Big(\bigwedge_{\alpha\in I} x_{\alpha,\psi(\alpha)}\Big), \tag{2}$$

and its lattice dual

$$\bigvee_{\alpha\in I}\Big(\bigwedge_{\beta\in J} x_{\alpha,\beta}\Big) = \bigwedge_{\psi\in J^I}\Big(\bigvee_{\alpha\in I} x_{\alpha,\psi(\alpha)}\Big). \tag{3}$$

hold, where I and J are arbitrary indexing sets, J^I is the set of functions from I into J, and where $x_{\alpha,\beta} \in \mathcal{L}$ for each $\alpha \in I$ and each $\beta \in J$.

Other characterizations of complete distributivity have been obtained [11], [12] and [13]. In particular Raney has shown that (2) and (3) are equivalent [14]. We shall use the following splitting property which was shown by Raney [14] to be equivalent to complete distributivity.

THEOREM 6 [14]. *A complete lattice $\mathcal{L}$ is completely distributive if and only if, whenever $v, w \in \mathcal{L}$, $v \not\le w$, there exist $a, b \in \mathcal{L}$, such that $a \not\le w$ and $v \not\le b$, and either $a \le c$ or $c \le b$ for each $c \in \mathcal{L}$.*

Any lattice of commuting projections is distributive. However it may not be completely distributive. The following measure-theoretic characterisation of complete distributivity for the commutative subspace lattice $\mathcal{L}(X,\mu,\le)$ is due to Hopenwasser, Laurie and Moore [6]:

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THEOREM 7. *The lattice $\mathcal{L}(X, \leq, \mu)$ is completely distributive if and only if for every Borel set A with $\mu(A) > 0$, $(\mu \times \mu)\big(A \times A \cap G(\leq)\big) > 0$, where $G(\leq) = \{(x,y) : x \leq y\}$ is the graph of $\leq$.*

We shall give a variation of Theorem 7 based upon interval subsets of X. For $x, y \in X$, define $[x,\infty] = \{z \in X : x \leq z\}$, $[-\infty, y] = \{z \in X : z \leq x\}$, and $[x,y] = \{z \in X : x \leq z \leq y\}$. Let $P_x = P_{[x,\infty]}$ and $Q_y = P_{[-\infty,y]}$. The intervals $[x,\infty]$, $[-\infty,y]$ and $[x,y]$ are closed, $[x,\infty]$ is increasing and $[-\infty,y]$ is decreasing. So P_x and $Q_y^{\perp}$ are in $\mathcal{L}(X, \leq, \mu)$, and $P_x Q_y = P_{[x,y]}$.

If $\leq$ is trivial and if μ has no atoms, then for each x, $\mu[x,\infty] = \mu[-\infty, x] = \mu\{x\} = 0$, and so $P_x = Q_x = 0$. In this case $\mathcal{L}(X, \leq, \mu)$ is a non-atomic Boolean algebra and is not completely distributive. We show that the projections P_x and Q_x are more substantial if $\mathcal{L}(X, \leq, \mu)$ is completely distributive.

LEMMA 8. *The lattice $\mathcal{L}(X, \leq, \mu)$ is completely distributive if and only if for each Borel set A with $\mu(A) > 0$, $\exists x, y \in A$ such that $\mu([x,y] \cap A) > 0$.*

PROOF. First suppose that $\mathcal{L}(X, \leq, \mu)$ is completely distributive, and that $\mu(A) > 0$. Then by Theorem 7 and Fubini's theorem,

$$(\mu \times \mu)\big(A \times A \cap G(\leq)\big) = \int_A \mu([x,\infty] \cap A)\mu\, dx > 0.$$

So $\mu([x,\infty]) \cap A) > 0$ for some $x \in A$.

Let $B = [x,\infty] \cap A$. Then by Theorem 7 and Fubini's theorem again,

$$(\mu \times \mu)\big(B \times B \cap G(\leq)\big) = \int_B \mu\big([-\infty, y) \cap B\big)\mu\, dy > 0,$$

and so $\mu([-\infty, y] \cap B) > 0$ for some $y \in B$. But $[-\infty, y] \cap B = [x,y] \cap A$, and so $x, y \in A$ and $\mu([x,y] \cap A) > 0$.

For the converse, assume that $\mu(A) > 0$ for some Borel set A, and let $B = \{x \in A : \mu([x,\infty] \cap A) = 0\}$. Now B is a Borel set, and if $\mu(B) > 0$ it follows from the hypothesis that $\mu([u,v] \cap B) > 0$ for some $u, v \in B$. Now $[u,v] \subseteq [u,\infty]$ and $B \subseteq A$, and so $\mu([u,\infty] \cap A) > 0$. But this is impossible since $u \in B$, and so we conclude that $\mu(B) = 0$. Therefore $\mu\big([x,\infty) \cap A\big) > 0$ for almost all $x \in A$, and so $(\mu \times \mu)\big(A \times A \cap G(\leq)\big) > 0$, by Fubini's theorem. So $\mathcal{L}(X, \leq, \mu)$ is completely distributive by Theorem 7.

COROLLARY 9. *The lattice $\mathcal{L}(X, \leq, \mu)$ is completely distributive if and only if for each Borel set A and each $\varepsilon > 0$, $\exists x_i, y_i \in A$, $i = 1, 2, \ldots, n$, such that $\mu(A \setminus \bigcup_{i=1}^{n} [x_i, y_i]) < \varepsilon$.*

PROOF. First suppose that $\mathcal{L} = \mathcal{L}(X, \leq, \mu)$ is completely distributive, and that A is a Borel subset of X. Let $\sigma = \sup_{\mathcal{F} \in \Omega} \mu(\bigcup_{[x,y] \in \mathcal{F}} [x,y] \cap A)$, where Ω consists of all finite sets of intervals $[x,y]$, where $x, y \in A$. Then there exists countably many intervals $[x_i, y_i]$: $i = 1, 2, 3, \ldots$, with $x_i, y_i \in A$ for each i, such that $\mu(\bigcup_{i=1}^{\infty} [x_i, y_i] \cap A) = \sigma$. Let $A^{(\infty)} = \bigcup_{i=1}^{\infty} [x_i, y_i] \cap A$, and let $B = A \setminus A^{(\infty)}$. Assume that $\mu(B) > 0$. Then $\mu([x', y'] \cap B) > 0$ for some $x', y' \in B$, by Lemma 8.

Choose n such that $\mu(A^{(n)}) > \sigma - \delta/2$, where $A^{(n)} = \bigcup_{i=1}^{n}[x_i, y_i] \cap A$, and $\delta = \mu([x', y'] \cap B)$. Let $A' = A^{(n)} \cup ([x', y'] \cap A)$. Then

$$\mu(A') = \mu(A' \cap A^{(\infty)}) + \mu(A' \cap B) \geq \mu(A^{(n)}) + \mu([x', y'] \cap B) > \sigma + \delta/2.$$

But this is a contradiction, since $\{[x', y'], [x_1, y_1], \ldots, [x_n, y_n]\} \in \Omega$. So we conclude that $\mu(B) = 0$, and hence $\mu(A \setminus \bigcup_{i=1}^{n}[x_i, y_i]) < \varepsilon$ for sufficiently large n.

The converse is an easy application of Lemma 8.

Corollary 9 can be expressed in terms of projections.

COROLLARY 10. *The lattice $\mathcal{L}(X, \leq, \mu)$ is completely distributive if and only if $P_A \leq \bigvee_{x,y \in A} P_x Q_y$ for each Borel set A.*

PROOF. The measure condition in Lemma 9 is equivalent to the statement that $\bigvee_{[x,y] \in \mathcal{F}} P_x Q_y P_A^{\perp} \longrightarrow 0$ strongly as $\mathcal{F}$ increases along the net Ω. But $\bigvee_{[x,y] \in \mathcal{F}} P_x Q_y \longrightarrow \bigvee_{x,y \in A} P_x Q_y$. So the condition in Lemma 9 is equivalent to $\bigvee_{x,y \in A} P_x Q_y P_A^{\perp} = 0$, *i.e.* $P_A \leq \bigvee_{x,y \in A} P_x Q_y$.

COROLLARY 11. *If $\mathcal{L}(X, \leq, \mu)$ is completely distributive, then $P_A = \bigvee_{x \in A} P_x$ for each increasing set A.*

PROOF. Suppose that $P_A \in \mathcal{L}(X, \leq, \mu)$, with $\mu(A) > 0$. Now $P_A \leq (\bigvee_{x,y \in A} P_x Q_y)$ by Corollary 10. But $P_x Q_y \leq P_x \leq P_A$ for each $x, y \in A$, since A is increasing. So $P_A = \bigvee_{x,y \in A} P_x Q_y = \bigvee_{x \in A} P_x$.

Corollary 11 will be used to show that $\mathcal{L}(X, \mu, \leq \mathcal{P})$ is a tensor product if $\mathcal{L}(X, \mu, \leq)$ is completely distributive.

THEOREM 12. *If $\mathcal{L}(X, \mu, \leq)$ is completely distributive, and if $\mathcal{P}$ is any subspace lattice, then $\mathcal{L}(X, \mu, \leq) \otimes \mathcal{P} = \mathcal{L}(X, \mu, \leq, \mathcal{P})$.*

PROOF. It is enough to show that $\mathcal{L}(X, \mu, \leq, \mathcal{P}) \subseteq \mathcal{L}(X, \mu, \leq) \otimes \mathcal{P}$. Suppose that $M_\phi \in \mathcal{L}(X, \mu, \leq, \mathcal{P})$, and let $M_\psi = \bigvee_{x \in X} P_x \otimes \phi(x)$. For each $x \in X$, $P_x \otimes \phi(x) \in \mathcal{L}(X, \mu, \leq) \otimes \mathcal{P}$, and since $\mathcal{L}(X, \mu, \leq) \otimes \mathcal{P}$ is complete, it follows that $M_\psi \in \mathcal{L}(X, \mu, \leq) \otimes \mathcal{P}$. Furthermore, $P_x \otimes \phi(x) \leq M_\phi$ since ϕ is increasing, and so $M_\psi \leq M_\phi$. We show that $M_\psi = M_\phi$.

Choose $f \in \mathcal{H}$ and $t \geq 0$, and let $C = \{x \in X : \langle \phi(x) f, f \rangle \geq t\}$. Since ϕ is increasing, C_t is an increasing subset of X. By Corollary 11, $P_C = \bigvee_{x \in C} P_x$, and since the unit ball of $\mathcal{B}\big(L^2(X, \mu)\big)$ is strongly metrizable, $P_C = \bigvee_{i=1}^{\infty} P_{x_i}$ for some countable set of points $x_i \in C$. Now for each i, $P_{x_i} \otimes \phi(x_i) \leq M_\psi$, and so

$$t \leq \langle \phi(x_i) f, f \rangle \leq \langle \psi(x) f, f \rangle \quad \text{a.e. on } [x_i, \infty].$$

It follows that $\langle \psi(x) f, f \rangle \geq t$ a.e. on C, and since $t \geq 0$ is arbitrary, $\langle \psi(x) f, f \rangle \geq \langle \phi(x) f, f \rangle$ a.e. Therefore, since $\mathcal{H}$ is separable, there is a null set N such that for all $f \in \mathcal{H}$ and all $x \in X \setminus N$, $\langle \psi(x) f, f \rangle \geq \langle \phi(x) f, f \rangle$. Therefore $\psi(x) \geq \phi(x)$ a.e. and so $M_\psi \geq M_\phi$. Therefore $M_\phi = M_\psi$, as required.

COROLLARY 13. *If $\mathcal{L}(X,\mu,\leq)$ is completely distributive and $\mathcal{P}$ is any subspace lattice, then every projection in $\mathcal{L}(X,\mu,\leq,\mathcal{P})$ is the join of elementary projections.*

PROOF. The proof of Theorem 12 shows that $M_\phi = \bigvee_{x\in X} P_x \otimes \phi(x)$ for each $M_\phi \in \mathcal{L}(X,\mu,\leq,\mathcal{P})$.

Corollary 13 is not true for arbitrary CSLs. If, for example, $\mathcal{L}(X,\mu,\leq)$ is a Boolean algebra without atoms, then $P_x \otimes \phi(x) = 0$ for each $x \in X$.

COROLLARY 14. *If $\mathcal{A} = \mathrm{Alg}\big(\mathcal{L}(X,\mu,\leq)\big)$, where $\mathcal{L}(X,\mu,\leq)$ is completely distributive and if $\mathcal{B}$ is any operator algebra, then $\mathrm{Lat}(\mathcal{A}\otimes\mathcal{B}) = \mathrm{Lat}\,\mathcal{A}\otimes\mathrm{Lat}\,\mathcal{B}$.*

PROOF. This follows from Lemma 3, Theorem 12 and the fact that completely distributive commutative subspace lattices are synthetic [6] (Corollary 9).

As a second application of Theorem 12, we show that the tensor product of two completely distributive subspace lattices is also completely distributive if one of the factors is also commutative. This generalises a result in [6], where it is shown that the tensor product of two completely distributive subspace lattices is also completely distributive if both factors are commutative.

THEOREM 15. *If $\mathcal{L}_1$ and $\mathcal{L}_2$ are completely distributive subspace lattices and if $\mathcal{L}_1$ is commutative, then $\mathcal{L}_1 \otimes \mathcal{L}_2$ is completely distributive.*

PROOF. We may suppose that $\mathcal{L}_1 = \mathcal{L}(X,\mu,\leq)$ and that $\mathcal{L}_2 = \mathcal{P}$. Then $\mathcal{L}_1 \otimes \mathcal{L}_2 = \mathcal{L}(X,\mu,\leq,\mathcal{P})$ by Theorem 12. We shall use Theorem 6 to show that $\mathcal{L}(X,\mu,\leq,\mathcal{P})$ is completely distributive.

Suppose that M_ϕ and M_ψ are projections in $\mathcal{L}(X,\mu,\leq,\mathcal{P})$, and $M_\phi \not\leq M_\psi$. Since $M_\phi = \bigvee_{x\in X} P_x \otimes \phi(x)$ by Corollary 13, $P_u \otimes \phi(u) \not\leq M_\psi$ for some $u \in X$. Let $Z = \{x \in X : u \leq x$ and $\phi(u) \not\leq \psi(x)\}$. Then $\mu(Z) > 0$, and so by Lemma 8, $\mu([v,w]\cap Z) > 0$ for some v, w in Z. Since $Z \subseteq [u,\infty]$, we have $u \leq v \leq w$ and $\phi(u) \not\leq \psi(w)$. Furthermore, $\mu([v,w]) \geq \mu([v,w]\cap Z) > 0$, and since ϕ and ψ are increasing, $\phi(v) \not\leq \psi(x)$ and $\phi(x) \not\leq \psi(w)$ for all $x \in [v,w]$.

Now P_v and $Q_w^\perp \in \mathcal{L}(X,\mu,\leq)$, and since $\mu([v,w]) > 0$, $P_v \not\leq Q_w^\perp$. Since $\mathcal{L}(X,\mu,\leq)$ is completely distributive, there are increasing subsets A and B in $\mathcal{L}(X,\mu,\leq)$ such that $P_v \not\leq P_B$ and $P_A \not\leq Q_w^\perp$, and such that either $P_A \leq P_C$ or $P_C \leq P_B$ for each increasing subset C of X. Furthermore, $\mathcal{P}$ is completely distributive, and hence contains projections α and β such that $\alpha \not\leq \psi(w)$ and $\phi(v) \not\leq \beta$, and either $\alpha \leq \gamma$ or $\gamma \leq \beta$ for each $\gamma \in \mathcal{P}$. Now define

$$\Phi(x) = \begin{cases} \alpha & \text{for } x \in A, \\ 0 & \text{for } x \notin A, \end{cases} \quad \text{and} \quad \Psi(x) = \begin{cases} 1 & \text{for } x \in B, \\ \beta & \text{for } x \notin B. \end{cases}$$

We show that M_Φ and M_Ψ split $\mathcal{L}(X,\mu,\leq,\mathcal{P})$.

If $M_\Phi \leq M_\psi$, then $\alpha \leq \psi(x)$ a.e. on A. On the other hand, $\alpha \not\leq \psi(w)$, and since ψ is increasing, $\alpha \not\leq \psi(x)$ for all $x \leq w$. So $\mu([-\infty,w]\cap A) = 0$. But this is a contradiction since $P_A \not\leq Q_w^\perp$. So we conclude that $M_\Phi \not\leq M_\psi$. Similarly, if $M_\phi \leq M_\Psi$, then $\phi(x) \leq \beta$

a.e. on $B^c = X\backslash B$. But $\phi(v) \not\leq \beta$ and so $\phi(x) \not\leq \beta$ for all $x \geq v$. Therefore $\mu([v, \infty] \cap B^c) = 0$, and since this contradicts $P_v \not\leq P_B$, we have $M_\phi \not\leq M_\Psi$.

Suppose that $M_\theta \in \mathcal{L}(X, \mu, \leq, \mathcal{P})$, let $C = \{x : \alpha \leq \theta(x)\}$ and let $D = \{x : \theta(x) \leq \beta\}$. Then $C \cup D = X$. Furthermore, $P_C \in \mathcal{L}(X, \mu, \leq)$, and so either $P_A \leq P_C$ or $P_C \leq P_B$. If $P_A \leq P_C$ then $\alpha \leq \theta(x)$ a.e. on A, and hence $M_\Phi \leq M_\theta$. On the other hand, if $P_C \leq P_B$ then $P_B^\perp \leq P_C^\perp \leq P_D$. Now $P_B^\perp = P_{B^c}$, and so $\theta(x) \leq \beta$ a.e. on B^c. Hence $M_\theta \leq M_\Psi$.

So $\mathcal{L}(X, \mu, \leq, \mathcal{P})$ splits, and so by Theorem 6, it is completely distributive.

COROLLARY 16. *The tensor product of a finite number of completely distributive, commutative subspace lattices is completely distributive.*

PROOF. The tensor product of commutative subspace lattices is commutative, since it is generated by commuting projections. So Theorem 15 can be used inductively to establish the result.

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