Canad. Math. Bull. Vol. **61** (2), 2018 pp. 328–345
http://dx.doi.org/10.4153/CMB-2017-030-4

Moduli of Space Sheaves with Hilbert Polynomial $4m + 1$

Mario Maican

Abstract. We investigate the moduli space of sheaves supported on space curves of degree 4 and having Euler characteristic 1. We give an elementary proof of the fact that this moduli space consists of three irreducible components.

1 Introduction and Preliminaries

Let $\mathrm{M}_{\mathbb{P}^n}(rm+\chi)$ be the moduli space of Gieseker semi-stable sheaves on the complex projective space $\mathbb{P}^n$ having Hilbert polynomial $P(m) = rm + \chi$. Le Potier [11] showed that $\mathrm{M}_{\mathbb{P}^2}(rm+\chi)$ is irreducible and, if r and χ are coprime, smooth. For low multiplicity, the homology of $\mathrm{M}_{\mathbb{P}^2}(rm+\chi)$ has been studied in [3, 4] using the wall-crossing method and in [6, 13, 14] using the Białynicki–Birula method. When $n > 2$, the moduli space is no longer irreducible. Thus, according to [8], $\mathrm{M}_{\mathbb{P}^3}(3m+1)$ has two irreducible components meeting transversally. The focus of this paper is the moduli space $\mathbf{M} = \mathrm{M}_{\mathbb{P}^3}(4m+1)$ of stable sheaves on $\mathbb{P}^3$ with Hilbert polynomial $4m+1$. This was investigated in [5] using wall-crossing, by relating $\mathbf{M}$ to $\mathrm{Hilb}_{\mathbb{P}^3}(4m+1)$. The main result of [5] states that $\mathbf{M}$ consists of three irreducible components, denoted $\overline{\mathbf{R}}$, $\overline{\mathbf{E}}$, $\mathbf{P}$, of dimensions 16, 17, and 20, respectively. The generic sheaves in $\overline{\mathbf{R}}$ are structure sheaves of rational quartic curves. The generic sheaves in $\overline{\mathbf{E}}$ are of the form $\mathcal{O}_E(P)$, where E is an elliptic quartic curve and P is a point on E. The third irreducible component parametrizes the planar sheaves.

The purpose of this paper is to reprove the decomposition of $\mathbf{M}$ into irreducible components without using the wall-crossing method; see Theorem 4.3. We achieve this as follows. Using the decomposition of $\mathrm{Hilb}_{\mathbb{P}^3}(4m+1)$ into irreducible components, found in [2], we show that the subset of $\mathbf{M}$ of sheaves generated by a global section is irreducible; see Proposition 2.4. This provides our first irreducible component. We then describe the sheaves whose support is an elliptic quartic curve; see Section 3. To show that the set of such sheaves $\mathcal{F}$ is irreducible we use results from [17] regarding the geometry of $\mathrm{Hilb}_{\mathbb{P}^3}(4m)$. Given $\mathcal{F}$, we construct at Proposition 4.2 a variety $\mathbf{W}$ together with a map $\sigma\colon \mathbf{W} \to \Gamma$, the support map, where $\Gamma \subset \mathrm{Hilb}_{\mathbb{P}^3}(4m)$ is an irreducible quasi-projective curve, such that $\mathcal{F} \in \sigma^{-1}(x)$ for a point $x \in \Gamma$ and such that $\Gamma \smallsetminus \{x\}$ consists only of smooth curves. Moreover, the fibers of σ are irreducible, hence $\mathbf{W}$ is irreducible, and hence $\mathcal{F}$ is contained in the closure of the set of sheaves with support smooth elliptic curves. Thus, we obtain the second irreducible

Received by the editors February 22, 2017.
Published electronically July 12, 2017.
AMS subject classification: **14D20**, 14D22.
Keywords: moduli of sheaves, semi-stable sheaves.

component. The set $\mathbf{P}$ of planar sheaves is irreducible because it is a bundle over the Grassmannian of planes in $\mathbb{P}^3$ with fiber $\mathrm{M}_{\mathbb{P}^2}(4m+1)$, which is, as mentioned above, irreducible.

We also rely on the cohomological classification of sheaves in $\mathbf{M}$ found at [5, Theorem 6.1], which does not use the wall-crossing method (it uses the Beilinson spectral sequence). We fix a 4-dimensional vector space V over $\mathbb{C}$ and we identify $\mathbb{P}^3$ with $\mathbb{P}(V)$. We fix a basis $\{X, Y, Z, W\}$ of V^*.

Theorem 1.1 ([5, Theorem 6.1]) *Let $\mathcal{F}$ give a point in $\mathrm{M}_{\mathbb{P}^3}(4m+1)$. Then $\mathcal{F}$ satisfies one of the following cohomological conditions:*

(i) $\mathrm{h}^0(\mathcal{F}\otimes\Omega^2(2)) = 0$, $\mathrm{h}^0(\mathcal{F}\otimes\Omega^1(1)) = 0$, $\mathrm{h}^0(\mathcal{F}) = 1$;
(ii) $\mathrm{h}^0(\mathcal{F}\otimes\Omega^2(2)) = 0$, $\mathrm{h}^0(\mathcal{F}\otimes\Omega^1(1)) = 1$, $\mathrm{h}^0(\mathcal{F}) = 1$;
(iii) $\mathrm{h}^0(\mathcal{F}\otimes\Omega^2(2)) = 1$, $\mathrm{h}^0(\mathcal{F}\otimes\Omega^1(1)) = 3$, $\mathrm{h}^0(\mathcal{F}) = 2$.

Let $\mathbf{M}_0, \mathbf{M}_1, \mathbf{M}_2 \subset \mathbf{M}$ be the subsets of sheaves satisfying conditions (i), (ii), and (iii), respectively. We will call them *strata*. Clearly, $\mathbf{M}_0$ is open, $\mathbf{M}_1$ is locally closed, and $\mathbf{M}_2$ is closed. We also quote the classification of the sheaves in each stratum in terms of locally free resolutions, which was carried out at [5, Theorem 6.1]. The sheaves in $\mathbf{M}_0$ are precisely the sheaves having a resolution of the form

$$0 \longrightarrow 3\mathcal{O}(-3) \xrightarrow{\psi} 5\mathcal{O}(-2) \xrightarrow{\varphi} \mathcal{O}(-1)\oplus\mathcal{O} \longrightarrow \mathcal{F} \longrightarrow 0 \tag{1.1}$$
$$\varphi = \begin{bmatrix} X & Y & Z & W & 0 \\ q_1 & q_2 & q_3 & q_4 & q_5 \end{bmatrix}$$

or a resolution of the form

$$0 \longrightarrow 3\mathcal{O}(-3) \xrightarrow{\psi} 5\mathcal{O}(-2) \xrightarrow{\varphi} \mathcal{O}(-1)\oplus\mathcal{O} \longrightarrow \mathcal{F} \longrightarrow 0 \tag{1.2}$$
$$\varphi = \begin{bmatrix} l_1 & l_2 & l_3 & 0 & 0 \\ q_1 & q_2 & q_3 & q_4 & q_5 \end{bmatrix},$$

where l_1, l_2, l_3 are linearly independent. Let $\mathbf{R}, \mathbf{E} \subset \mathbf{M}_0$ be the subsets of sheaves having resolution (1.1) (resp. (1.2)). Clearly, $\mathbf{R}$ is an open subset of $\mathbf{M}$ and consists of structure sheaves of rational quartic curves. The set $\mathbf{E}$ contains all extensions of $\mathbb{C}_P$ by $\mathcal{O}_E$, where E is an elliptic quartic curve and P is a point on E. The sheaves in $\mathbf{M}_1$ are precisely the sheaves having a resolution of the form

$$0 \longrightarrow 3\mathcal{O}(-3) \xrightarrow{\psi} 5\mathcal{O}(-2)\oplus\mathcal{O}(-1) \xrightarrow{\varphi} 2\mathcal{O}(-1)\oplus\mathcal{O} \longrightarrow \mathcal{F} \longrightarrow 0, \tag{1.3}$$

where $\varphi_{12} = 0$ and $\varphi_{11}\colon 5\mathcal{O}(-2) \to 2\mathcal{O}(-1)$ is not equivalent to a morphism represented by a matrix of the form

$$\begin{bmatrix} \star & \star & 0 & 0 & 0 \\ \star & \star & \star & \star & \star \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} \star & \star & \star & \star & 0 \\ \star & \star & \star & \star & 0 \end{bmatrix}.$$

The sheaves in $\mathbf{M}_2$ are precisely the sheaves of the form $\mathcal{O}_C(-P)(1)$, where $\mathcal{O}_C(-P)$ in $\mathcal{O}_C$ denotes the ideal sheaf of a closed point P in a planar quartic curve C.

Assume now that $\mathcal{F}$ has resolution (1.1). Let $S \subset \mathbb{P}^3$ be the quadric surface given by the equation $q_5 = 0$. From the snake lemma we get the resolution

$$0 \longrightarrow 3\mathcal{O}(-3) \longrightarrow \Omega^1(-1) \longrightarrow \mathcal{O}_S \longrightarrow \mathcal{F} \longrightarrow 0.$$

We consider first the case when S is smooth. The semi-stable sheaves on a smooth quadric surface with Hilbert polynomial $4m+1$ have been investigated in [1]. We cite below the main result of [1]:

Proposition 1.2 *Let $\mathcal{F}$ be a coherent sheaf on $\mathbb{P}^1 \times \mathbb{P}^1$ that is semi-stable relative to the polarization $\mathcal{O}(1,1)$ and such that $P_{\mathcal{F}}(m) = 4m+1$. Then precisely one of the following is true:*

(i) *$\mathcal{F}$ is the structure sheaf of a curve of type $(1,3)$;*
(ii) *$\mathcal{F}$ is the structure sheaf of a curve of type $(3,1)$;*
(iii) *$\mathcal{F}$ is a non-split extension $0 \to \mathcal{O}_E \to \mathcal{F} \to \mathbb{C}_P \to 0$ for a curve E in $\mathbb{P}^1 \times \mathbb{P}^1$ of type $(2,2)$ and a point $P \in E$. Such an extension is unique up to isomorphism and satisfies the condition $\mathrm{H}^1(\mathcal{F}) = 0$.*

Thus, $\mathrm{M}_{\mathbb{P}^1\times\mathbb{P}^1}(4m+1)$ has three connected components. Two of these, $\mathbb{P}(\mathrm{H}^0(\mathcal{O}(1,3)))$ and $\mathbb{P}(\mathrm{H}^0(\mathcal{O}(3,1)))$, are isomorphic to $\mathbb{P}^7$. The third one is smooth, has dimension 9, and is isomorphic to the universal elliptic curve in $\mathbb{P}(\mathrm{H}^0(\mathcal{O}(2,2))) \times (\mathbb{P}^1 \times \mathbb{P}^1)$. The sheaves at (iii) *are precisely the sheaves having a resolution of the form*

$$0 \longrightarrow \mathcal{O}(-2,-1) \oplus \mathcal{O}(-1,-2) \xrightarrow{\varphi} \mathcal{O}(-1,-1) \oplus \mathcal{O} \longrightarrow \mathcal{F} \longrightarrow 0$$

with $\varphi_{11} \neq 0$, $\varphi_{12} \neq 0$.

The following well-known lemma provides one of our main technical tools.

Lemma 1.3 *Let X be a projective scheme and let Y be a subscheme. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module and let $\mathcal{G}$ be a coherent $\mathcal{O}_Y$-module. Then there is an exact sequence of vector spaces*

$$\begin{aligned}(1.4)\quad 0 \longrightarrow \mathrm{Ext}^1_{\mathcal{O}_Y}(\mathcal{F}_{|Y}, \mathcal{G}) \longrightarrow \mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \longrightarrow \mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{T}or_1^{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_Y), \mathcal{G})\\ \longrightarrow \mathrm{Ext}^2_{\mathcal{O}_Y}(\mathcal{F}_{|Y}, \mathcal{G}) \longrightarrow \mathrm{Ext}^2_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}).\end{aligned}$$

In particular, if $\mathcal{F}$ is an $\mathcal{O}_Y$-module, then the above exact sequence takes the form

$$\begin{aligned}(1.5)\quad 0 \longrightarrow \mathrm{Ext}^1_{\mathcal{O}_Y}(\mathcal{F}, \mathcal{G}) \longrightarrow \mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \longrightarrow \mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{I}_Y, \mathcal{G})\\ \longrightarrow \mathrm{Ext}^2_{\mathcal{O}_Y}(\mathcal{F}, \mathcal{G}) \longrightarrow \mathrm{Ext}^2_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}).\end{aligned}$$

2 Sheaves Supported on Rational Quartic Curves

Let $\mathbf{R}_0 \subset \mathbf{R}$ be the set of isomorphism classes of structure sheaves $\mathcal{O}_R$ of curves $R \subset S$ of type $(1,3)$ or $(3,1)$ on smooth quadrics $S \subset \mathbb{P}^3$. A curve of type $(1,3)$ on S can be deformed inside $\mathbb{P}^3$ to a curve of type $(3,1)$, hence $\mathbf{R}_0$ is irreducible of dimension 16. Let $\mathbf{E}_0 \subset \mathbf{E}$ be the set of isomorphism classes of non-split extensions of $\mathbb{C}_P$ by $\mathcal{O}_E$ for $E \subset S$ a curve of type $(2,2)$ on a smooth quadric $S \subset \mathbb{P}^3$ and P a closed point on E.

From (1.5) and Proposition 1.2(iii) we have the exact sequence

$$0 \longrightarrow \operatorname{Ext}^1_{\mathcal{O}_S}(\mathbb{C}_P, \mathcal{O}_E) \simeq \mathbb{C} \longrightarrow \operatorname{Ext}^1_{\mathcal{O}_{\mathbb{P}^3}}(\mathbb{C}_P, \mathcal{O}_E) \longrightarrow \operatorname{Hom}_{\mathcal{O}_S}(\mathbb{C}_P, \mathcal{O}_E) = 0.$$

We denote by $\mathcal{O}_E(P)$ the unique non-split extension of $\mathbb{C}_P$ by $\mathcal{O}_E$. Clearly, $\mathbf{E}_0$ is irreducible of dimension 17. Let $\mathbf{E}_{\text{free}} \subset \mathbf{E}_0$ denote the open subset of sheaves that are locally free on their schematic support, which is equivalent to saying that $P \in \operatorname{reg}(E)$. Let $\mathbf{P} \subset \mathrm{M}_{\mathbb{P}^3}(4m+1)$ be the closed set of planar sheaves. It has dimension 20. Let $\mathbf{P}_{\text{free}} \subset \mathbf{P}$ be the open subset of sheaves that are locally free on their support. According to [10], $\mathbf{P} \smallsetminus \mathbf{P}_{\text{free}}$ has codimension 2 in $\mathbf{P}$.

Proposition 2.1 *The closed sets $\overline{\mathbf{R}}_0$, $\overline{\mathbf{E}}_0$, and $\mathbf{P}$ are irreducible components of $\mathrm{M}_{\mathbb{P}^3}(4m+1)$. Moreover, $\mathbf{R}_0$, $\mathbf{E}_{\text{free}}$ and $\mathbf{P}_{\text{free}}$ are smooth open subsets of the moduli space.*

Proof Let $\mathcal{F} = \mathcal{O}_R$ give a point in $\mathbf{R}_0$, where $R \subset S$ is a curve of, say, type $(1,3)$. From Serre duality we have

$$\operatorname{Ext}^2_{\mathcal{O}_S}(\mathcal{F},\mathcal{F}) \simeq \operatorname{Hom}_{\mathcal{O}_S}(\mathcal{F},\mathcal{F}(-2,-2))^* = 0.$$

From the exact sequence (1.5) we get the relation

$$\operatorname{ext}^1_{\mathcal{O}_{\mathbb{P}^3}}(\mathcal{F},\mathcal{F}) = \operatorname{ext}^1_{\mathcal{O}_S}(\mathcal{F},\mathcal{F}) + \hom_{\mathcal{O}_S}(\mathcal{F}(-2),\mathcal{F}) = 7 + \mathrm{h}^0(\mathcal{O}_R(2,2)) = 16.$$

This shows that $\overline{\mathbf{R}}_0$ is an irreducible component of $\mathbf{M}$ and that $\mathbf{R}_0$ is smooth.

Next, consider $\mathcal{F} = \mathcal{O}_E(P)$ giving a point in $\mathbf{E}_0$. As above, we have the relation

$$\operatorname{ext}^1_{\mathcal{O}_{\mathbb{P}^3}}(\mathcal{F},\mathcal{F}) = \operatorname{ext}^1_{\mathcal{O}_S}(\mathcal{F},\mathcal{F}) + \hom_{\mathcal{O}_S}\big(\mathcal{F}(-2),\mathcal{F}\big) = 9 + \hom_{\mathcal{O}_S}\big(\mathcal{F},\mathcal{F}(2,2)\big).$$

Assume, in addition, that $\mathcal{F}$ is locally free on E. Its rank must be 1, because E is a curve of multiplicity 4. Thus,

$$\operatorname{Hom}_{\mathcal{O}_S}(\mathcal{F},\mathcal{F}(2,2)) \simeq \mathrm{H}^0(\mathcal{O}_E(2,2)) \simeq \mathbb{C}^8,$$

hence $\operatorname{ext}^1_{\mathcal{O}_{\mathbb{P}^3}}(\mathcal{F},\mathcal{F}) = 17$. This shows that $\overline{\mathbf{E}}_0$ is an irreducible component of $\mathbf{M}$ and that $\mathbf{E}_{\text{free}}$ is smooth.

Assume now that $\mathcal{F}$ is supported on a planar quartic curve $C \subset H$. Using Serre duality and (1.5) we get the relation

$$\operatorname{ext}^1_{\mathcal{O}_{\mathbb{P}^3}}(\mathcal{F},\mathcal{F}) = \operatorname{ext}^1_{\mathcal{O}_H}(\mathcal{F},\mathcal{F}) + \hom_{\mathcal{O}_H}\big(\mathcal{F}(-1),\mathcal{F}\big) = 17 + \hom_{\mathcal{O}_H}\big(\mathcal{F},\mathcal{F}(1)\big).$$

Assume, in addition, that $\mathcal{F}$ is locally free on C, so a line bundle. Thus,

$$\operatorname{Hom}_{\mathcal{O}_H}(\mathcal{F},\mathcal{F}(1)) \simeq \mathrm{H}^0(\mathcal{O}_C(1)) \simeq \mathbb{C}^3,$$

hence $\operatorname{ext}^1_{\mathcal{O}_{\mathbb{P}^3}}(\mathcal{F},\mathcal{F}) = 20$. This shows that $\mathbf{P}$ is an irreducible component of $\mathbf{M}$ and that $\mathbf{P}_{\text{free}}$ is smooth. ∎

Remark 2.2 Let $\mathcal{F}$ be a one-dimensional sheaf on $\mathbb{P}^3$ without zero-dimensional torsion. Let $\mathcal{F}'$ be a planar subsheaf such that $\mathcal{F}/\mathcal{F}'$ has dimension zero. Then $\mathcal{F}$ is planar. Indeed, say that $\mathcal{F}'$ is an $\mathcal{O}_H$-module for a plane $H \subset \mathbb{P}^3$. From (1.4) we have the exact sequence

$$0 \to \operatorname{Ext}^1_{\mathcal{O}_H}\big((\mathcal{F}/\mathcal{F}')_{|H},\mathcal{F}'\big) \to \operatorname{Ext}^1_{\mathcal{O}_{\mathbb{P}^3}}(\mathcal{F}/\mathcal{F}',\mathcal{F}') \to \operatorname{Hom}_{\mathcal{O}_H}(\mathcal{T}or_1^{\mathcal{O}_{\mathbb{P}^3}}(\mathcal{F}/\mathcal{F}',\mathcal{O}_H),\mathcal{F}').$$

https://doi.org/10.4153/CMB-2017-030-4 Published online by Cambridge University Press

The group on the right vanishes, because $\mathcal{T}or_1^{\mathcal{O}_{\mathbb{P}^3}}(\mathcal{F}/\mathcal{F}', \mathcal{O}_H)$ is supported on finitely many points, yet $\mathcal{F}'$ has no zero-dimensional torsion. Thus, $\mathcal{F} \in \operatorname{Ext}^1_{\mathcal{O}_H}((\mathcal{F}/\mathcal{F}')_{|H}, \mathcal{F}')$, so $\mathcal{F}$ is an $\mathcal{O}_H$-module.

Proposition 2.3 *The non-planar sheaves in* $\mathrm{M}_{\mathbb{P}^3}(4m+1)$ *having resolution* (1.3) *are precisely the non-split extensions of the form*

$$0 \longrightarrow \mathcal{O}_C \longrightarrow \mathcal{F} \longrightarrow \mathcal{O}_L \longrightarrow 0, \tag{2.1}$$

where C is a planar cubic curve and L is a line meeting C with multiplicity 1*. For such a sheaf,* $\mathrm{H}^0(\mathcal{F})$ *generates* $\mathcal{O}_C$*. The set* $\mathbf{R}$ *consists precisely of the sheaves generated by a global section. The set* $\mathbf{E}$ *consists precisely of the sheaves $\mathcal{F}$ such that* $\mathrm{H}^0(\mathcal{F})$ *generates a subsheaf with Hilbert polynomial* $4m$*.*

Proof Let φ be a morphism as at (1.3). Denote $\mathcal{G} = \mathcal{C}oker(\varphi_{11})$ and let $H \subset \mathbb{P}^3$ be the plane given by the equation $\varphi_{22} = 0$. From the snake lemma we have the exact sequence

$$\mathcal{O}_H \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow 0.$$

We examine first the case when

$$\varphi_{11} \nsim \begin{bmatrix} 0 & 0 & \star & \star & \star \\ \star & \star & \star & \star & \star \end{bmatrix}.$$

Thus, we can write

$$\varphi_{11} = \begin{bmatrix} X & Y & Z & W & 0 \\ 0 & l_1 & l_2 & l_3 & l_4 \end{bmatrix}.$$

If l_4 is a multiple of X, then $P_{\mathcal{G}} = 3$ (see the proof of [5, Theorem 6.1(iii)]); hence, by Remark 2.2, $\mathcal{F}$ is planar. Assume now that l_4 is not a multiple of X and let $L \subset \mathbb{P}^3$ be the line given by the equations $X = 0$, $l_4 = 0$. Then $\mathcal{G}$ is a proper quotient sheaf of $\mathcal{O}_L(-1)$, hence it has support of dimension zero, and hence, by Remark 2.2, $\mathcal{F}$ is planar. It remains to examine the case when

$$\varphi_{11} = \begin{bmatrix} u_1 & u_2 & u_3 & 0 & 0 \\ 0 & v_1 & v_2 & v_3 & v_4 \end{bmatrix}.$$

Let P be the point given by the ideal (u_1, u_2, u_3) and let L be the line given by the equations $v_3 = 0$, $v_4 = 0$. We have an exact sequence

$$\mathcal{O}_L(-1) \longrightarrow \mathcal{G} \longrightarrow \mathbb{C}_P \longrightarrow 0.$$

If the first morphism is not injective, then $\mathcal{G}$ has dimension zero, hence $\mathcal{F}$ is planar. If $\mathcal{G}$ is an extension of $\mathbb{C}_P$ by $\mathcal{O}_L(-1)$, then this extension does not split; otherwise, $\mathcal{O}_L(-1)$ would be a destabilizing quotient sheaf of $\mathcal{F}$. Thus, $\mathcal{G} \simeq \mathcal{O}_L$, and we have an exact sequence

$$0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{F} \longrightarrow \mathcal{O}_L \longrightarrow 0,$$

where $\mathcal{E}$ gives a point in $\mathrm{M}_H(3m)$ and is generated by a global section. Thus, $\mathcal{E}$ is the structure sheaf of a cubic curve $C \subset H$. If $L \subset H$, then from (1.5) we would have the exact sequence

$$0 \longrightarrow \operatorname{Ext}^1_{\mathcal{O}_H}(\mathcal{O}_L, \mathcal{O}_C) \longrightarrow \operatorname{Ext}^1_{\mathcal{O}_{\mathbb{P}^3}}(\mathcal{O}_L, \mathcal{O}_C) \longrightarrow \operatorname{Hom}_{\mathcal{O}_H}\big(\mathcal{O}_L(-1), \mathcal{O}_C\big).$$

The group on the right vanishes, because $\mathcal{O}_C$ is stable. We deduce that $\mathcal{F}$ lies in $\mathrm{Ext}^1_{\mathcal{O}_H}(\mathcal{O}_L, \mathcal{O}_C)$, hence $\mathcal{F}$ is planar.

Thus far we have shown that if $\mathcal{F}$ is non-planar and has resolution (1.3), then $\mathcal{F}$ is an extension as in the proposition. Conversely, given a non-split extension (2.1), $\mathcal{F}$ is semi-stable, because $\mathcal{O}_C$ and $\mathcal{O}_L$ are stable. In view of Theorem 1.1, since $\mathcal{F}$ is non-planar, we have $\mathrm{h}^0(\mathcal{F}) = 1$. Thus, $\mathrm{H}^0(\mathcal{F})$ generates $\mathcal{O}_C$. It follows that $\mathcal{F}$ cannot have resolutions (1.1) or (1.2); otherwise, $\mathrm{H}^0(\mathcal{F})$ would generate $\mathcal{F}$ or would generate a subsheaf with Hilbert polynomial $4m$. We conclude that $\mathcal{F}$ has resolution (1.3).

The rest of the proposition follows from Theorem 1.1 and from the fact, proved in [7], that for a planar sheaf $\mathcal{F}$ having resolution (1.3), the space of global sections generates a subsheaf with Hilbert polynomial $4m - 2$ or it generates the structure sheaf of a cubic curve. ∎

Proposition 2.4 *The set* $\mathbf{R}$ *of sheaves in* $\mathrm{M}_{\mathbb{P}^3}(4m+1)$ *generated by a global section is irreducible.*

Proof Let $\mathrm{Hilb}_{\mathbb{P}^3}(4m+1)^{\mathrm{s}} \subset \mathrm{Hilb}_{\mathbb{P}^3}(4m+1)$ be the open subset of semi-stable quotients. The image of the canonical map $\mathrm{Hilb}_{\mathbb{P}^3}(4m+1)^{\mathrm{s}} \to \mathrm{M}_{\mathbb{P}^3}(4m+1)$ is $\mathbf{R}$. According to [2, Theorem 4.9], $\mathrm{Hilb}_{\mathbb{P}^3}(4m+1)$ has four irreducible components, denoted H_1, H_2, H_3, H_4. The generic point in H_1 is a rational quartic curve. The generic curve in H_2 is the disjoint union of a planar cubic and a line. The generic member of H_3 is the disjoint union of a point and an elliptic quartic curve. The generic member of H_4 is the disjoint union of a planar quartic curve and three distinct points. Thus, $H_2 \cup H_3 \cup H_4$ lies in the closed subset

$$H = \left\{ [\mathcal{O} \twoheadrightarrow \mathcal{S}] \mid \mathrm{h}^0(\mathcal{S}) \geq 2 \right\} \subset \mathrm{Hilb}_{\mathbb{P}^3}(4m+1).$$

According to Theorem 1.1, $H^{\mathrm{s}} = \emptyset$. Indeed, any sheaf in $\mathbf{M}_2$ cannot be generated by a single global section. Thus, $\mathrm{Hilb}_{\mathbb{P}^3}(4m+1)^{\mathrm{s}}$ is an open subset of H_1, hence it is irreducible, and hence $\mathbf{R}$ is irreducible. ∎

3 Sheaves Supported on Elliptic Quartic Curves

We will next examine the sheaves $\mathcal{F}$ having resolution (1.2). Let P be the point given by the ideal (l_1, l_2, l_3). Notice that the subsheaf of $\mathcal{F}$ generated by $\mathrm{H}^0(\mathcal{F})$ is the kernel of the canonical map $\mathcal{F} \to \mathbb{C}_P$. This shows that $\mathcal{F}$ is non-planar, because, according to [7], the global sections of a sheaf in $\mathrm{M}_{\mathbb{P}^2}(4m+1)$ whose first cohomology vanishes generate a subsheaf with Hilbert polynomial $4m-2$ or the structure sheaf of a planar cubic curve, which is not the case here. We consider first the case when q_4 and q_5 have no common factor, so they define a curve E. Applying the snake lemma to the diagram in Figure 1 we see that $\mathcal{F}$ is an extension of $\mathbb{C}_P$ by $\mathcal{O}_E$. From Serre duality, we have

$$\mathrm{Ext}^1_{\mathcal{O}_{\mathbb{P}^3}}(\mathbb{C}_P, \mathcal{O}_E) \simeq \mathrm{Ext}^2_{\mathcal{O}_{\mathbb{P}^3}}(\mathcal{O}_E, \mathbb{C}_P)^* \simeq \mathbb{C}.$$

The group in the middle can be determined by applying $\mathrm{Hom}(\,\cdot\,, \mathbb{C}_P)$ to the first row of the diagram. We may write $\mathcal{F} = \mathcal{O}_E(P)$.

$$
\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & \mathcal{O}(-4) \arrow[r, "\begin{bmatrix} q_5 \\ q_4 \end{bmatrix}"] & 2\mathcal{O}(-2) \arrow[r, "{[-q_4\ q_5]}"] \arrow[d] & \mathcal{O} \arrow[r] \arrow[d] & \mathcal{O}_E \arrow[r] & 0 \\
0 \arrow[r] & 3\mathcal{O}(-3) \arrow[r] & 5\mathcal{O}(-2) \arrow[r, "\varphi"] \arrow[d] & \mathcal{O}(-1) \oplus \mathcal{O} \arrow[r] \arrow[d] & \mathcal{F} \arrow[r] & 0 \\
0 \arrow[r] & \mathcal{K} \arrow[r] & 3\mathcal{O}(-2) \arrow[r, "{[l_1\ l_2\ l_3]}"] \arrow[d] & \mathcal{O}(-1) \arrow[r] \arrow[d] & \mathbb{C}_P \arrow[r] & 0 \\
 & & 0 & 0 & &
\end{tikzcd}
$$

Figure 1

Proposition 3.1 *The sheaf $\mathcal{O}_E(P)$ is stable.*

Proof We will show that $\mathcal{O}_E$ is stable, forcing $\mathcal{O}_E(P)$ to be stable. To prove that $\mathcal{O}_E$ is stable, we must show that it does not contain a stable subsheaf $\mathcal{E}$ having one of the following Hilbert polynomials: m, $m+1$ (*i.e.*, the structure sheaf of a line), $2m$, $2m+1$ (*i.e.*, the structure sheaf of a conic curve), $3m$, $3m+1$. The structure sheaf of a line contains subsheaves having Hilbert polynomial m and the structure sheaf of a conic curve contains subsheaves having Hilbert polynomial $2m$. Thus, it is enough to consider only the Hilbert polynomials m, $2m$, $3m+1$, $3m$. In the first case, we have a commutative diagram

$$
\begin{tikzcd}
0 \arrow[r] & \mathcal{O}(-3) \arrow[r] \arrow[d, "\gamma"] & 2\mathcal{O}(-2) \arrow[r] \arrow[d, "\beta"] & \mathcal{O}(-1) \arrow[r] \arrow[d, "\alpha"] & \mathcal{E} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{O}(-4) \arrow[r] & 2\mathcal{O}(-2) \arrow[r] & \mathcal{O} \arrow[r] & \mathcal{O}_E \arrow[r] & 0
\end{tikzcd}
$$

in which $\alpha \neq 0$. It follows that $\mathcal{O}(-3) \simeq \mathcal{K}er(\gamma) \simeq \mathcal{K}er(\beta)$, which is absurd. In the second case, we get a commutative diagram

$$
\begin{tikzcd}
0 \arrow[r] & 2\mathcal{O}(-3) \arrow[r] \arrow[d, "\gamma"] & 4\mathcal{O}(-2) \arrow[r] \arrow[d, "\beta"] & 2\mathcal{O}(-1) \arrow[r] \arrow[d, "\alpha"] & \mathcal{E} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{O}(-4) \arrow[r] & 2\mathcal{O}(-2) \arrow[r] & \mathcal{O} \arrow[r] & \mathcal{O}_E \arrow[r] & 0
\end{tikzcd}
$$

in which $\alpha \neq 0$, hence $\mathcal{K}er(\alpha) \simeq \mathcal{O}(-1)$ or $\mathcal{O}(-2)$. From the exact sequence

$$0 \longrightarrow 2\mathcal{O}(-3) \simeq \mathcal{K}er(\gamma) \longrightarrow \mathcal{K}er(\beta) \longrightarrow \mathcal{K}er(\alpha) \longrightarrow \mathcal{C}oker(\gamma) \simeq \mathcal{O}(-4),$$

https://doi.org/10.4153/CMB-2017-030-4 Published online by Cambridge University Press

we see that $\mathcal{K}er(\beta) \simeq 3\mathcal{O}(-2)$, and we get the exact sequence

$$0 \longrightarrow 2\mathcal{O}(-3) \longrightarrow 3\mathcal{O}(-2) \longrightarrow \mathcal{K}er(\alpha) \longrightarrow 0.$$

Such an exact sequence cannot exist. In the third case, we use the resolution of $\mathcal{E}$ given at [8, Theorem 1.1]. We obtain a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & 2\mathcal{O}(-3) \arrow[r] \arrow[d, "\gamma"] & 3\mathcal{O}(-2) \oplus \mathcal{O}(-1) \arrow[r] \arrow[d, "\beta"] & \mathcal{O}(-1) \oplus \mathcal{O} \arrow[r] \arrow[d, "\alpha"] & \mathcal{E} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathcal{O}(-4) \arrow[r] & 2\mathcal{O}(-2) \arrow[r] & \mathcal{O} \arrow[r] & \mathcal{O}_E \arrow[r] & 0 \end{tikzcd}$$

in which α is non-zero on global sections, hence $\mathcal{K}er(\alpha) \simeq \mathcal{O}(-1)$. We obtain a contradiction from the exact sequence

$$0 \longrightarrow 2\mathcal{O}(-3) \simeq \mathcal{K}er(\gamma) \longrightarrow \mathcal{K}er(\beta_{11}) \oplus \mathcal{O}(-1) \longrightarrow \mathcal{K}er(\alpha) \longrightarrow 0.$$

Assume, finally, that $\mathcal{E}$ gives a stable point in $\mathrm{M}_{\mathbb{P}^3}(3m)$. If $\mathrm{H}^0(\mathcal{E}) \neq 0$, then it is easy to see that $\mathcal{E}$ is the structure sheaf of a planar cubic curve, hence we get a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{O}(-4) \arrow[r] \arrow[d, "\gamma"] & \mathcal{O}(-3) \oplus \mathcal{O}(-1) \arrow[r] \arrow[d, "\beta"] & \mathcal{O} \arrow[r] \arrow[d, "\alpha"] & \mathcal{E} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathcal{O}(-4) \arrow[r] & 2\mathcal{O}(-2) \arrow[r] & \mathcal{O} \arrow[r] & \mathcal{O}_E \arrow[r] & 0 \end{tikzcd}$$

in which α is injective. We get a contradiction from the fact that $\mathcal{O}(-1)$ is a subsheaf of $\mathcal{K}er(\beta) \simeq \mathcal{K}er(\gamma)$. If $\mathrm{H}^0(\mathcal{E}) = 0$, then we get a commutative diagram of the form

$$\begin{tikzcd} 0 \arrow[r] & 3\mathcal{O}(-3) \arrow[r] \arrow[d, "\gamma"] & 6\mathcal{O}(-2) \arrow[r] \arrow[d, "\beta"] & 3\mathcal{O}(-1) \arrow[r] \arrow[d, "\alpha"] & \mathcal{E} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathcal{O}(-4) \arrow[r] & 2\mathcal{O}(-2) \arrow[r] & \mathcal{O} \arrow[r] & \mathcal{O}_E \arrow[r] & 0. \end{tikzcd}$$

It is easy to see that $\alpha(1)$ is injective on global sections, hence $\mathcal{C}oker(\alpha)$ is isomorphic to the structure sheaf of a point and $\mathcal{C}oker(\beta) \simeq \mathcal{O}(-2)$. We get a contradiction from the exact sequence

$$\mathcal{O}(-4) \simeq \mathcal{C}oker(\gamma) \longrightarrow \mathcal{C}oker(\beta) \longrightarrow \mathcal{C}oker(\alpha). \qquad \blacksquare$$

To finish the discussion about sheaves at Theorem 1.1(i), we need to examine the case when $q_4 = uv_1$ and $q_5 = uv_2$ with linearly independent $v_1, v_2 \in V^*$. Let H be the plane given by the equation $u = 0$ and let L be the line given by the equations $v_1 = 0$, $v_2 = 0$. We apply the snake lemma to the diagram in Figure 2. The kernel of the canonical map $\mathcal{G} \to \mathbb{C}_P$ is an $\mathcal{O}_H$-module. This shows that $\mathcal{F}$ is not isomorphic to $\mathcal{G}$; otherwise, in view of Remark 2.2, $\mathcal{F}$ would be planar. Thus, $\mathcal{O}_L(-1) \to \mathcal{F}$ is non-zero, hence it is injective. We get a non-split extension

$$0 \longrightarrow \mathcal{O}_L(-1) \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow 0, \tag{3.1}$$

and it becomes clear that $P \in H$ and that $\mathcal{G}$ gives a point in $\mathrm{M}_{\mathbb{P}^3}(3m+1)$. From Remark 2.2 we see that $\mathcal{G}$ gives a point in $\mathrm{M}_H(3m+1)$. Thus, $\mathcal{G}$ is the unique non-split

https://doi.org/10.4153/CMB-2017-030-4 Published online by Cambridge University Press

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & \mathcal{O}(-3) \arrow[r] & 2\mathcal{O}(-2) \arrow[r, "{[v_1\ v_2]}"] \arrow[d] & \mathcal{O}(-1) \arrow[r] \arrow[d, "{\left[\begin{smallmatrix}0\\u\end{smallmatrix}\right]}"] & \mathcal{O}_L(-1) \arrow[r] & 0 \\
0 \arrow[r] & 3\mathcal{O}(-3) \arrow[r] & 5\mathcal{O}(-2) \arrow[r, "\varphi"] \arrow[d] & \mathcal{O}(-1)\oplus\mathcal{O} \arrow[r] \arrow[d] & \mathcal{F} \arrow[r] & 0 \\
0 \arrow[r] & \mathcal{K} \arrow[r] & 3\mathcal{O}(-2) \arrow[r, "{\left[\begin{smallmatrix}l_1 & l_2 & l_3\\ \star & \star & \star\end{smallmatrix}\right]}"] \arrow[d] & \mathcal{O}(-1)\oplus\mathcal{O}_H \arrow[r] \arrow[d] & \mathcal{G} \arrow[r] & 0 \\
 & & 0 & 0 & &
\end{tikzcd}$$

Figure 2

extension of $\mathbb{C}_P$ by $\mathcal{O}_C$ for a cubic curve $C \subset H$ containing P. We write $\mathcal{G} = \mathcal{O}_C(P)$. Let $\mathbf{D} \subset \mathrm{M}_{\mathbb{P}^3}(4m+1)$ be the set of non-split extension sheaves as in (3.1) that are non-planar (we allow the possibility that $L \subset H$, in which case the support of $\mathcal{F}$ is contained in the double plane $2H$).

We examine first the case when $L \not\subset H$; that is, L meets C with multiplicity 1, at a point P'. According to [8, Theorem 1.1] there is a resolution

$$0 \longrightarrow 2\mathcal{O}(-3) \xrightarrow{\delta} 3\mathcal{O}(-2)\oplus\mathcal{O}(-1) \xrightarrow{\gamma} \mathcal{O}(-1)\oplus\mathcal{O} \longrightarrow \mathcal{G} \longrightarrow 0$$

$$\delta = \begin{bmatrix} u & 0 \\ 0 & u \\ -u_1 & -u_2 \\ -g_1 & -g_2 \end{bmatrix}, \qquad \gamma = \begin{bmatrix} u_1 & u_2 & u & 0 \\ g_1 & g_2 & 0 & u \end{bmatrix},$$

where $\mathrm{span}\{u_1, u_2, u\} = \mathrm{span}\{l_1, l_2, l_3\}$ and C has equation $u_1g_2 - u_2g_1 = 0$ in H. Note that $\mathcal{G}_{|L} \simeq \mathbb{C}_{P'}$ unless $\gamma(P') = 0$, in which case $\mathcal{G}_{|L} \simeq \mathbb{C}_{P'} \oplus \mathbb{C}_{P'}$. But $\gamma(P') = 0$ if and only if $P' = P \in \mathrm{sing}(C)$. From (1.4) we have the exact sequence

$$0 \to \mathrm{Ext}^1_{\mathcal{O}_L}(\mathcal{G}_{|L}, \mathcal{O}_L(-1)) \to \mathrm{Ext}^1_{\mathcal{O}_{\mathbb{P}^3}}(\mathcal{G}, \mathcal{O}_L(-1)) \to \mathrm{Hom}_{\mathcal{O}_L}(\mathcal{T}or_1^{\mathcal{O}_{\mathbb{P}^3}}(\mathcal{G}, \mathcal{O}_L), \mathcal{O}_L(-1)).$$

The group on the right vanishes, because $\mathcal{O}_L(-1)$ has no zero-dimensional torsion. It follows that

$$\mathrm{Ext}^1_{\mathcal{O}_{\mathbb{P}^3}}(\mathcal{G}, \mathcal{O}_L(-1)) \simeq \begin{cases} \mathbb{C} & \text{if } P \neq P' \text{ or if } P = P' \in \mathrm{reg}(C), \\ \mathbb{C}^2 & \text{if } P = P' \in \mathrm{sing}(C). \end{cases}$$

Let $\mathbf{D}_0 \subset \mathbf{D}$ be the open subset given by the conditions that $L \not\subset H$ and either $P \neq P'$ or $P = P' \in \mathrm{reg}(C)$. The map

$$\mathbf{D}_0 \longrightarrow \mathrm{Hilb}_{\mathbb{P}^3}(m+1) \times \mathrm{M}_{\mathbb{P}^3}(3m+1), \qquad [\mathcal{F}] \longmapsto (L, [\mathcal{G}])$$

is injective and has irreducible image. We deduce that $\mathbf{D}_0$ is irreducible and has dimension 16.

Let $\mathbf{D}' \subset \mathrm{M}_{\mathbb{P}^3}(4m+1)$ be the subset of non-split extensions (2.1). Denote $P = L \cap C$. From (1.4) we have the exact sequence

$$0 \to \mathbb{C} \simeq \mathrm{Ext}^1_{\mathcal{O}_H}(\mathbb{C}_P, \mathcal{O}_C) \to \mathrm{Ext}^1_{\mathcal{O}_{\mathbb{P}^3}}(\mathcal{O}_L, \mathcal{O}_C) \to \mathrm{Hom}_{\mathcal{O}_H}(\mathcal{T}or_1^{\mathcal{O}_{\mathbb{P}^3}}(\mathcal{O}_L, \mathcal{O}_H), \mathcal{O}_C) = 0.$$

We deduce that, given L and C, there is a unique non-split extension of $\mathcal{O}_L$ by $\mathcal{O}_C$. The map

$$\mathbf{D}' \longrightarrow \mathrm{Hilb}_{\mathbb{P}^3}(m+1) \times \mathrm{Hilb}_{\mathbb{P}^3}(3m)$$

sending $\mathcal{F}$ to (L, C) is injective and has irreducible image. We deduce that $\mathbf{D}'$ is irreducible and has dimension 15. Tensoring (2.1) with $\mathcal{O}_H$, we get the exact sequence

$$0 = \mathcal{T}or_1^{\mathcal{O}_{\mathbb{P}^3}}(\mathcal{O}_L, \mathcal{O}_H) \longrightarrow \mathcal{O}_C \longrightarrow \mathcal{F}_{|H} \longrightarrow \mathbb{C}_P \longrightarrow 0$$

from which we see that $\mathcal{F}_{|H} \simeq \mathcal{O}_C(P)$. We obtain the extension

$$0 \longrightarrow \mathcal{O}_L(-1) \longrightarrow \mathcal{F} \longrightarrow \mathcal{O}_C(P) \longrightarrow 0.$$

We deduce that $[\mathcal{F}] \in \mathbf{D}$. Thus, $\mathbf{D}' \subset \mathbf{D}$. Moreover, $\mathbf{D}' \cap \mathbf{D}_0$ is open and non-empty in $\mathbf{D}'$, because it consists precisely of extensions as above for which $P \in \mathrm{reg}(C)$. Thus, $\mathbf{D}' \subset \overline{\mathbf{D}}_0$.

Remark 3.2 Note that $\mathbf{D}_0 \setminus \mathbf{D}'$ is the open subset of $\mathbf{D}$ given by the conditions $L \not\subset H$ and $P \neq P'$. We claim that $\mathbf{D}_0 \setminus \mathbf{D}'$ is the set of sheaves of the form $\mathcal{O}_D(P)$, where $D = L \cup C$ is the union of a line and a planar cubic curve having intersection of multiplicity 1 and $P \in C \setminus L$. First we show that the notation $\mathcal{O}_D(P)$ is justified. From (1.4) we have the exact sequence

$$\begin{aligned}0 \longrightarrow \mathbb{C} \simeq \mathrm{Ext}^1_{\mathcal{O}_L}\big(\mathbb{C}_{P'}, \mathcal{O}_L(-1)\big) &\longrightarrow \mathrm{Ext}^1_{\mathcal{O}_{\mathbb{P}^3}}\big(\mathcal{O}_C, \mathcal{O}_L(-1)\big) \\ &\longrightarrow \mathrm{Hom}\big(\mathcal{T}or_1^{\mathcal{O}_{\mathbb{P}^3}}(\mathcal{O}_C, \mathcal{O}_L), \mathcal{O}_L(-1)\big) = 0,\end{aligned}$$

which shows that $\mathcal{O}_D$ is the unique non-split extension of $\mathcal{O}_C$ by $\mathcal{O}_L(-1)$. The long exact sequence of groups

$$\begin{aligned}0 = \mathrm{Ext}^1_{\mathcal{O}_{\mathbb{P}^3}}\big(\mathbb{C}_P, \mathcal{O}_L(-1)\big) \longrightarrow \mathrm{Ext}^1_{\mathcal{O}_{\mathbb{P}^3}}(\mathbb{C}_P, \mathcal{O}_D) &\longrightarrow \mathrm{Ext}^1_{\mathcal{O}_{\mathbb{P}^3}}(\mathbb{C}_P, \mathcal{O}_C) \simeq \mathbb{C} \\ &\longrightarrow \mathrm{Ext}^2_{\mathcal{O}_{\mathbb{P}^3}}\big(\mathbb{C}_P, \mathcal{O}_L(-1)\big) = 0\end{aligned}$$

shows that there is a unique non-split extension of $\mathbb{C}_P$ by $\mathcal{O}_D$, which we denote by $\mathcal{O}_D(P)$. Given $\mathcal{F} \in \mathbf{D}_0 \setminus \mathbf{D}'$, the pull-back of $\mathcal{O}_C$ in $\mathcal{F}$, denoted $\mathcal{F}'$, is a non-split extension of $\mathcal{O}_C$ by $\mathcal{O}_L(-1)$. Indeed, if $\mathcal{F}'$ were a split extension, then $\mathcal{O}_C \subset \mathcal{F}$ and $\mathcal{F}/\mathcal{O}_C \simeq \mathcal{O}_L(-1) \oplus \mathbb{C}_P$, so $\mathcal{O}_L(-1)$ would be a destabilising quotient sheaf of $\mathcal{F}$. Thus, $\mathcal{F}' \simeq \mathcal{O}_D$ and $\mathcal{F} \simeq \mathcal{O}_D(P)$. Conversely, $\mathcal{O}_D(P)/O_L(-1)$ is an extension of $\mathbb{C}_P$ by $\mathcal{O}_C$, hence $\mathcal{O}_D(P)/O_L(-1) \simeq \mathcal{O}_C(P)$.

Remark 3.3 If $L \cap C = \{P\}$ is a regular point of C, and $D = L \cup C$, then there are no semi-stable extensions of the form

$$0 \longrightarrow \mathcal{O}_D \longrightarrow \mathcal{F} \longrightarrow \mathbb{C}_P \longrightarrow 0.$$

https://doi.org/10.4153/CMB-2017-030-4 Published online by Cambridge University Press

Indeed, if $\mathcal{F}$ were such a semi-stable extension, then we would also have an extension

$$0 \longrightarrow \mathcal{O}_L(-1) \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow 0,$$

where $\mathcal{G}$ is an extension of $\mathbb{C}_P$ by $\mathcal{O}_C$. Note that $\mathcal{G}$ is a non-split extension; otherwise, $\mathcal{O}_C$ would be a destabilizing quotient sheaf of $\mathcal{F}$. Thus, $\mathcal{F}$ is the unique non-split extension of $\mathcal{O}_C(P)$ by $\mathcal{O}_L(-1)$, so it is also the unique non-split extension of $\mathcal{O}_L$ by $\mathcal{O}_C$. Thus, $\mathrm{H}^0(\mathcal{F})$ generates $\mathcal{O}_C$, hence $\mathcal{O}_D$ is a subsheaf of $\mathcal{O}_C$, which is absurd.

Remark 3.4 The set $\mathbf{S} \subset \mathrm{M}_{\mathbb{P}^2}(3m) \times \mathrm{M}_{\mathbb{P}^2}(3m+1)$ of pairs $([\mathcal{E}],[\mathcal{G}])$ such that $\mathrm{H}^0(\mathcal{E}) = 0$ and $\mathcal{E}$ is a subsheaf of $\mathcal{G}$ is irreducible. By duality, this is equivalent to saying that the set $\mathbf{S}^{\mathrm{D}} \subset \mathrm{M}_{\mathbb{P}^2}(3m-1) \times \mathrm{M}_{\mathbb{P}^2}(3m)$ of pairs $([\mathcal{G}],[\mathcal{E}])$ such that $\mathrm{H}^0(\mathcal{E}) = 0$ and $\mathcal{G}$ is a subsheaf of $\mathcal{E}$ is irreducible. Given an exact sequence

$$0 \longrightarrow \mathcal{G} \longrightarrow \mathcal{E} \longrightarrow \mathbb{C}_{P'} \longrightarrow 0,$$

we can combine the resolutions of sheaves on $\mathbb{P}^2$

$$0 \longrightarrow \mathcal{O}(-3) \oplus \mathcal{O}(-2) \xrightarrow{\left[\begin{smallmatrix} q_1 & u_1 \\ q_2 & u_2 \end{smallmatrix}\right]} 2\mathcal{O}(-1) \longrightarrow \mathcal{G} \longrightarrow 0$$

and

$$0 \longrightarrow \mathcal{O}(-3) \longrightarrow 2\mathcal{O}(-2) \xrightarrow{[\, v_1 \;\; v_2 \,]} \mathcal{O}(-1) \longrightarrow \mathbb{C}_{P'} \longrightarrow 0$$

to form the resolution

$$0 \longrightarrow \mathcal{O}(-3) \xrightarrow{\psi} \mathcal{O}(-3) \oplus 3\mathcal{O}(-2) \xrightarrow{\varphi} 3\mathcal{O}(-1) \longrightarrow \mathcal{E} \longrightarrow 0,$$

$$\varphi = \begin{bmatrix} q_1 & u_1 & l_{11} & l_{12} \\ q_2 & u_2 & l_{21} & l_{22} \\ 0 & 0 & v_1 & v_2 \end{bmatrix}.$$

We indicate by the index i the maximal minor of a matrix obtained by deleting column i. The condition $\mathrm{H}^0(\mathcal{E}) = 0$ is equivalent to the condition $\psi_{11} \neq 0$, which is equivalent to the following conditions: $\varphi_1 \neq 0$ and φ_1 divides $\varphi_2, \varphi_3, \varphi_4$. As φ_1 divides both $(q_1u_2 - u_1q_2)v_1$ and $(q_1u_2 - u_1q_2)v_2$, we see that φ_1 is a multiple of $q_1u_2 - u_1q_2$. It follows that φ is equivalent to the matrix

$$v = \begin{bmatrix} l_{11}v_2 - l_{12}v_1 & u_1 & l_{11} & l_{12} \\ l_{21}v_2 - l_{22}v_1 & u_2 & l_{21} & l_{22} \\ 0 & 0 & v_1 & v_2 \end{bmatrix}.$$

Let $U \subset \mathrm{Hom}(\mathcal{O}(-3) \oplus 3\mathcal{O}(-2), 3\mathcal{O}(-1))$ be the set of morphisms represented by matrices v as above satisfying the following conditions: $v_1 \neq 0$, u_1 and u_2 are linearly independent, v_1 and v_2 are linearly independent. Clearly, U is irreducible. Let $v' \in \mathrm{Hom}(\mathcal{O}(-3) \oplus \mathcal{O}(-2), 2\mathcal{O}(-1))$ be the morphism represented by the matrix

$$\begin{bmatrix} l_{11}v_2 - l_{12}v_1 & u_1 \\ l_{21}v_2 - l_{22}v_1 & u_2 \end{bmatrix}.$$

The above discussion shows that the map $\pi\colon U \to \mathbf{S}^{\mathrm{D}}$, $v \mapsto ([\mathcal{C}oker(v')], [\mathcal{C}oker(v)])$ is surjective. Thus, $\mathbf{S}^{\mathrm{D}}$ is irreducible. The open subset $\mathbf{S}_{\mathrm{irr}} \subset \mathbf{S}$, given by the condition that the schematic support of $\mathcal{G}$ be irreducible, is irreducible.

Let $\mathbf{D}_1 \subset \mathbf{D}$ be the locally closed subset given by the conditions $L \not\subset H$ and $P = P' \in \operatorname{sing}(C)$. Since $\dim \operatorname{Ext}^1_{\mathcal{O}_{\mathbb{P}^3}}(\mathcal{G}, \mathcal{O}_L(-1)) = 2$, we see that $\dim \mathbf{D}_1 = 14$. The set of cubic curves in $\mathbb{P}^2$ that are singular at a fixed point is irreducible. It follows that $\mathbf{D}_1$ is irreducible as well.

Proposition 3.5 *The set* $\mathbf{D}_1$ *is contained in the closure of* $\mathbf{D}_0$.

Proof Consider $[\mathcal{F}] \in \mathbf{D}_0 \cup \mathbf{D}_1$. Consider extension (3.1) in which $\mathcal{G} = \mathcal{O}_C(P)$ and $L \cap H = \{P'\}$. Dualizing, we get the extension

$$0 \longrightarrow \mathcal{O}_C(-P) \longrightarrow \mathcal{F}^{\mathrm{D}} \longrightarrow \mathcal{O}_L(-1) \longrightarrow 0.$$

Tensoring with $\mathcal{O}_H$, we get the exact sequence

$$0 = \mathcal{T}or_1^{\mathcal{O}_{\mathbb{P}^3}}(\mathcal{O}_L(-1), \mathcal{O}_H) \longrightarrow \mathcal{O}_C(-P) \longrightarrow (\mathcal{F}^{\mathrm{D}})_{|H} \longrightarrow \mathbb{C}_{P'} \longrightarrow 0.$$

This short exact sequence does not split. Indeed, by [12], $\mathcal{F}^{\mathrm{D}}$ is stable and has slope $-1/4$, hence $\mathcal{O}_C(-P)$, which has slope $-1/3$, cannot be a quotient sheaf of $\mathcal{F}^{\mathrm{D}}$. Since $\mathcal{O}_C(-P)$ is stable, it is easy to see that $(\mathcal{F}^{\mathrm{D}})_{|H}$ gives a sheaf in $\mathrm{M}_H(3m)$ supported on C. The kernel of the map $\mathcal{F}^{\mathrm{D}} \to (\mathcal{F}^{\mathrm{D}})_{|H}$ is supported on L and has no zero-dimensional torsion, hence it is isomorphic to $\mathcal{O}_L(-2)$. Denote $\mathcal{E} = ((\mathcal{F}^{\mathrm{D}})_{|H})^{\mathrm{D}}$. Dualizing the exact sequence

$$0 \longrightarrow \mathcal{O}_L(-2) \longrightarrow \mathcal{F}^{\mathrm{D}} \longrightarrow (\mathcal{F}^{\mathrm{D}})_{|H} \longrightarrow 0,$$

we obtain the extension

$$0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{F} \longrightarrow \mathcal{O}_L \longrightarrow 0.$$

Tensoring with $\mathcal{O}_H$, and taking into account the fact that $\mathcal{T}or_1^{\mathcal{O}_{\mathbb{P}^3}}(\mathcal{O}_L, \mathcal{O}_H) = 0$, we get the exact sequence

$$0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}_C(P) \longrightarrow \mathbb{C}_{P'} \longrightarrow 0.$$

From (1.4) we have the exact sequence

$$0 \longrightarrow \operatorname{Ext}^1_{\mathcal{O}_H}(\mathbb{C}_{P'}, \mathcal{E}) \xrightarrow{\epsilon} \operatorname{Ext}^1_{\mathcal{O}_{\mathbb{P}^3}}(\mathcal{O}_L, \mathcal{E}) \longrightarrow \operatorname{Hom}(\mathcal{T}or_1^{\mathcal{O}_{\mathbb{P}^3}}(\mathcal{O}_L, \mathcal{O}_H), \mathcal{E}) = 0.$$

It is clear now that the isomorphism class of $\mathcal{F}$ corresponds to the isomorphism class of $\mathcal{O}_C(P)$ under the bijective map ϵ. Let $\mathbf{D}'' \subset (\mathbf{D}_0 \cup \mathbf{D}_1) \smallsetminus \mathbf{D}'$ be the subset given by the condition that C be irreducible. Note that $\mathbf{D}''$ is an open subset of $\mathbf{D}$ and contains an open subset of $\mathbf{D}_1$. We will prove below that $\mathbf{D}''$ is irreducible. Since $\mathbf{D}_1$ is irreducible, we arrive at the conclusion of the proposition

$$\mathbf{D}_1 \subset \overline{\mathbf{D}'' \cap \mathbf{D}_1} \subset \overline{\mathbf{D}''} = \overline{\mathbf{D}'' \cap \mathbf{D}_0} \subset \overline{\mathbf{D}}_0.$$

Consider the subset

$$\mathbf{S}'' \subset \operatorname{Hilb}_{\mathbb{P}^3}(m+1) \times \mathrm{M}_{\mathbb{P}^3}(3m) \times \mathrm{M}_{\mathbb{P}^3}(3m+1)$$

of triples $(L, [\mathcal{E}], [\mathcal{G}])$ satisfying the following conditions: $\mathcal{E}$ and $\mathcal{G}$ are supported on a planar irreducible cubic curve C, $\mathrm{H}^0(\mathcal{E}) = 0$, $\mathcal{E}$ is a subsheaf of $\mathcal{G}$, and $L \cap C = \{P'\}$, where $\mathbb{C}_{P'} \simeq \mathcal{G}/\mathcal{E}$. Note that the projection $\mathbf{S}'' \to \mathrm{M}_{\mathbb{P}^3}(3m) \times \mathrm{M}_{\mathbb{P}^3}(3m+1)$ has fibers affine planes and has image the irreducible variety $\mathbf{S}_{\mathrm{irr}}$ from Remark 3.4. It follows that $\mathbf{S}''$ is irreducible. To prove that $\mathbf{D}''$ is irreducible, we will show that the morphism

$$\eta\colon \mathbf{D}'' \longrightarrow \mathbf{S}'', \qquad \eta([\mathcal{F}]) = \big(L, [((\mathcal{F}^{\mathrm{D}})_{|H})^{\mathrm{D}}], [\mathcal{F}_{|H}]\big)$$

is bijective. We first verify surjectivity. Given an extension

$$0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{G} \longrightarrow \mathbb{C}_{P'} \longrightarrow 0$$

we let $\mathcal{F} \in \mathrm{Ext}^1_{\mathcal{O}_{\mathbb{P}^3}}(\mathcal{O}_L, \mathcal{E})$ be the image of $\mathcal{G}$ under ϵ. Since $\mathcal{G}$ does not split, neither does $\mathcal{F}$. By hypothesis $\mathcal{E}$ has irreducible support, hence $\mathcal{E}$ is stable, and, a fortiori, $\mathcal{F}$ is stable. Applying the snake lemma to the diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{E} \arrow[r] \arrow[d, equal] & \mathcal{F} \arrow[r] \arrow[d] & \mathcal{O}_L \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{E} \arrow[r] & \mathcal{G} \arrow[r] & \mathbb{C}_{P'} \arrow[r] & 0,
\end{tikzcd}$$

we get the extension

$$0 \longrightarrow \mathcal{O}_L(-1) \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow 0.$$

Thus, $[\mathcal{F}] \in \mathbf{D}_0 \cup \mathbf{D}_1$ and $\mathcal{F}_{|H} \simeq \mathcal{G}$, where H is the plane containing C. Dualizing the first row of the above diagram we see that $(\mathcal{F}^{\mathrm{D}})_{|H} \simeq \mathcal{E}^{\mathrm{D}}$. By hypothesis $\mathcal{E}$ is not isomorphic to $\mathcal{O}_C$, hence $[\mathcal{F}] \notin \mathbf{D}'$. Thus, $[\mathcal{F}] \in \mathbf{D}''$ and $\eta([\mathcal{F}]) = (L, [\mathcal{E}], [\mathcal{G}])$. This proves that η is surjective. Since $[\mathcal{F}] = \epsilon([\mathcal{G}])$, we see that η is also injective. ∎

We will next examine the sheaves in $\mathbf{D}$ for which $L \subset H$. From (1.5) we have the exact sequence

$$\begin{aligned}
0 \longrightarrow \mathrm{Ext}^1_{\mathcal{O}_H}(\mathcal{O}_C(P), \mathcal{O}_L(-1)) &\longrightarrow \mathrm{Ext}^1_{\mathcal{O}_{\mathbb{P}^3}}(\mathcal{O}_C(P), \mathcal{O}_L(-1)) \\
&\longrightarrow \mathrm{Hom}(\mathcal{O}_C(P)(-1), \mathcal{O}_L(-1)) \\
&\longrightarrow \mathrm{Ext}^2_{\mathcal{O}_H}(\mathcal{O}_C(P), \mathcal{O}_L(-1)) \simeq \mathrm{Hom}_{\mathcal{O}_H}(\mathcal{O}_L(-1), \mathcal{O}_C(P)(-3))^* = 0.
\end{aligned}$$

Thus, we have non-planar sheaves precisely if $\mathrm{Hom}(\mathcal{O}_C(P), \mathcal{O}_L) \neq 0$. Any non-zero morphism $\alpha\colon \mathcal{O}_C(P) \to \mathcal{O}_L$ fits into a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & 2\mathcal{O}_H(-2) \arrow[r, "v"] \arrow[d, "\gamma"] & \mathcal{O}_H(-1) \oplus \mathcal{O}_H \arrow[r] \arrow[d, "\beta"] & \mathcal{O}_C(P) \arrow[r] \arrow[d, "\alpha"] & 0 \\
0 \arrow[r] & \mathcal{O}_H(-1) \arrow[r, "l"] & \mathcal{O}_H \arrow[r] & \mathcal{O}_L \arrow[r] & 0
\end{tikzcd}$$

$$\beta = \begin{bmatrix} v & c \end{bmatrix}, \qquad \gamma = \begin{bmatrix} v_1 & v_2 \end{bmatrix}, \qquad v = \begin{bmatrix} u_1 & u_2 \\ g_1 & g_2 \end{bmatrix}$$

with $\beta \neq 0$. Note that $c \neq 0$; otherwise, $\mathcal{C}oker(\beta)$ would be the structure sheaf of a line, and we would have the relation $(vu_1, vu_2) = (lv_1, lv_2)$. Thus, v_1 and v_2 would be linearly independent, hence $\mathcal{C}oker(\gamma)$ would be zero-dimensional, and hence $\mathcal{C}oker(\beta)$ would be zero-dimensional, which is absurd. Replacing, possibly, v with an equivalent matrix, we can assume that g_1 and g_2 are divisible by l. Conversely, if $\mathcal{O}_C(P)$ is the cokernel of the morphism

$$v = \begin{bmatrix} u_1 & u_2 \\ lv_1 & lv_2 \end{bmatrix}, \qquad \text{then, denoting} \qquad v' = \begin{bmatrix} u_1 & u_2 \\ v_1 & v_2 \end{bmatrix},$$

we can apply the snake lemma to the commutative diagram

$$\begin{tikzcd} & 2\mathcal{O}_H(-2) \arrow[r, equal] \arrow[d, "v'"] & 2\mathcal{O}_H(-2) \arrow[d, "v"] & & \\ 0 \arrow[r] & 2\mathcal{O}_H(-1) \arrow[r, "1\oplus l"] & \mathcal{O}_H(-1)\oplus\mathcal{O}_H \arrow[r] & \mathcal{O}_L \arrow[r] & 0 \end{tikzcd}$$

to get a surjective map $\mathcal{O}_C(P) \to \mathcal{O}_L$. This discussion shows that $\mathrm{Hom}(\mathcal{O}_C(P), \mathcal{O}_L)$ does not vanish precisely if $C = L \cup C'$ for a conic curve $C' \subset H$ and for $P \in C'$. In this case we have a commutative diagram

$$\begin{tikzcd} \mathrm{Hom}(\mathcal{O}_C, \mathcal{O}_L(-1)) = 0 \arrow[d] & & \\ \mathrm{Ext}^1_{\mathcal{O}_H}(\mathbb{C}_P, \mathcal{O}_L(-1)) \arrow[d] & & \mathrm{Hom}(\mathbb{C}_P, \mathcal{O}_L) = 0 \arrow[d] \\ \mathrm{Ext}^1_{\mathcal{O}_H}(\mathcal{O}_C(P), \mathcal{O}_L(-1)) \arrow[r, hook] \arrow[d] & \mathrm{Ext}^1_{\mathcal{O}_{\mathbb{P}^3}}(\mathcal{O}_C(P), \mathcal{O}_L(-1)) \arrow[r, two heads] \arrow[d, "\delta"] & \mathrm{Hom}(\mathcal{O}_C(P), \mathcal{O}_L) \arrow[d, "\simeq"] \\ \mathrm{Ext}^1_{\mathcal{O}_H}(\mathcal{O}_C, \mathcal{O}_L(-1)) \arrow[r, hook] \arrow[d] & \mathrm{Ext}^1_{\mathcal{O}_{\mathbb{P}^3}}(\mathcal{O}_C, \mathcal{O}_L(-1)) \arrow[r, two heads] & \mathrm{Hom}(\mathcal{O}_C, \mathcal{O}_L) \simeq \mathbb{C} \\ \mathrm{Ext}^2_{\mathcal{O}_H}(\mathbb{C}_P, \mathcal{O}_L(-1)) \arrow[r, "\simeq"] \arrow[d] & \mathrm{Hom}_{\mathcal{O}_H}(\mathcal{O}_L, \mathbb{C}_P)^* & \\ \mathrm{Ext}^2_{\mathcal{O}_H}(\mathcal{O}_C(P), \mathcal{O}_L(-1)) = 0 & & \end{tikzcd}$$

Here $\delta(\mathcal{F})$ is the pull-back of $\mathcal{O}_C$ in $\mathcal{F}$. If $P \notin L$, then δ is an isomorphism. If $P \in L$, then we have an exact sequence

$$0 \longrightarrow \mathbb{C} \longrightarrow \mathrm{Ext}^1_{\mathcal{O}_{\mathbb{P}^3}}\big(\mathcal{O}_C(P), \mathcal{O}_L(-1)\big) \xrightarrow{\delta} \mathrm{Ext}^1_{\mathcal{O}_{\mathbb{P}^3}}(\mathcal{O}_C, \mathcal{O}_L(-1)) \longrightarrow \mathbb{C} \longrightarrow 0.$$

If $\mathcal{F}$ is non-planar, then $\delta(\mathcal{F})$ is generated by a global section. Indeed, in view of Proposition 2.3, $\mathcal{F}$ cannot have resolution (1.3), so it has resolution (1.1) or (1.2). Also, $\mathcal{F}$ is not generated by a global section, because $\mathcal{O}_C(P)$ is not generated by a global section. It follows that $P_{\mathcal{F}'}(m) = 4m$, where $\mathcal{F}' \subset \mathcal{F}$ is the subsheaf generated by $\mathrm{H}^0(\mathcal{F})$. But $\mathcal{F}'$ maps to $\mathcal{O}_C$, hence $\delta(\mathcal{F}) \subset \mathcal{F}'$. These two sheaves have the same Hilbert polynomial, so they coincide. We conclude that $\delta(\mathcal{F})$ is the structure sheaf $\mathcal{O}_D$ of a quartic curve D. If $P \notin L$, then $\mathcal{F} \simeq \mathcal{O}_D(P)$.

Assume now that $P \in L$. The preimage of $[\mathcal{O}_D]$ under the induced map

$$\mathbb{P}\big(\,\mathrm{Ext}^1_{\mathcal{O}_{\mathbb{P}^3}}(\mathcal{O}_C(P), \mathcal{O}_L(-1))\big) \smallsetminus \mathbb{P}(\mathbb{C}) \longrightarrow \mathbb{P}\big(\,\mathrm{Ext}^1_{\mathcal{O}_{\mathbb{P}^3}}(\mathcal{O}_C, \mathcal{O}_L(-1))\big)$$

https://doi.org/10.4153/CMB-2017-030-4 Published online by Cambridge University Press

is an affine line that maps to a curve in $\mathrm{M}_{\mathbb{P}^3}(4m+1)$. The exact sequence

$$0 = \mathrm{Hom}(\mathbb{C}_P, \mathcal{O}_C) \longrightarrow \mathrm{Ext}^1_{\mathcal{O}_{\mathbb{P}^3}}(\mathbb{C}_P, \mathcal{O}_L(-1)) \simeq \mathbb{C} \longrightarrow \mathrm{Ext}^1_{\mathcal{O}_{\mathbb{P}^3}}(\mathbb{C}_P, \mathcal{O}_D) \\ \longrightarrow \mathrm{Ext}^1_{\mathcal{O}_{\mathbb{P}^3}}(\mathbb{C}_P, \mathcal{O}_C) \simeq \mathbb{C}$$

shows that $\mathrm{Ext}^1_{\mathcal{O}_{\mathbb{P}^3}}(\mathbb{C}_P, \mathcal{O}_D)$ has dimension 2. Indeed, if this vector space had dimension 1, then its image in $\mathrm{M}_{\mathbb{P}^3}(4m+1)$ would be a point. This, as we saw above, is not the case.

Let $\mathbf{D}_2 \subset \mathbf{D}$ be the closed subset given by the condition $L \subset H$. Equivalently, $\mathbf{D}_2$ is given by the condition $C = L \cup C'$ and $P \in C'$ for a conic curve C'. According to [5, Proposition 4.10], the set $\mathbf{D}_2$ is irreducible of dimension 14. Indeed, let

$$\mathbf{S} \subset \mathrm{Hilb}_{\mathbb{P}^2}(m+1) \times \mathrm{M}_{\mathbb{P}^2}(3m+1)$$

be the locally closed subset of pairs $(L, [\mathcal{O}_C(P)])$ for which $C = L \cup C'$ and $P \in C'$, for a conic curve $C' \subset \mathbb{P}^2$. According to [5, Lemma 4.9], $\mathbf{S}$ is irreducible. The canonical map $\mathbf{D}_2 \to \mathbf{S}$ is surjective and its fibers are irreducible of dimension 3.

4 The Irreducible Components

Let

$$\mathbf{W}_0 \subset \mathrm{Hom}\big(3\mathcal{O}(-3), 5\mathcal{O}(-2)\big) \times \mathrm{Hom}\big(5\mathcal{O}(-2), \mathcal{O}(-1) \oplus \mathcal{O}\big)$$

be the subset of pairs of morphisms equivalent to pairs (ψ, φ) occurring in resolutions (1.1) and (1.2). We claim that $\mathbf{W}_0$ is locally closed. To see this, consider first the locally closed subset $\mathbb{W}$ given by the following conditions: ψ is injective, φ is generically surjective, $\varphi \circ \psi = 0$. We have the universal sequence

$$3\mathcal{O}_{\mathbb{W}\times\mathbb{P}^3}(-3) \xrightarrow{\Psi} 5\mathcal{O}_{\mathbb{W}\times\mathbb{P}^3}(-2) \xrightarrow{\Phi} \mathcal{O}_{\mathbb{W}\times\mathbb{P}^3}(-1) \oplus \mathcal{O}_{\mathbb{W}\times\mathbb{P}^3}.$$

Denote $\widetilde{\mathcal{F}} = \mathcal{C}oker(\Phi)$. Corresponding to the polynomial $P(m) = 4m + 1$ we have the locally closed subset

$$\mathbb{W}_P = \{x \in \mathbb{W},\ P_{\widetilde{\mathcal{F}}_x} = P\} \subset \mathbb{W}$$

constructed when we flatten $\widetilde{\mathcal{F}}$, see [9, Theorem 2.1.5]. Now $\mathbf{W}_0 \subset \mathbb{W}_P$ is the subset given by the condition that $\widetilde{\mathcal{F}}_x$ be semi-stable, which is an open condition, because $\widetilde{\mathcal{F}}_{|\mathbb{W}_P\times\mathbb{P}^3}$ is flat over $\mathbb{W}_P$. We endow $\mathbf{W}_0$ with the induced reduced structure. Consider the map

$$\rho_0 \colon \mathbf{W}_0 \longrightarrow \mathbf{M}_0, \qquad (\psi, \varphi) \longmapsto [\mathcal{C}oker(\varphi)].$$

On $\mathbf{W}_0$ we have the canonical action of the linear algebraic group

$$\mathbf{G}_0 = \big(\,\mathrm{Aut}(3\mathcal{O}(-3)) \times \mathrm{Aut}(5\mathcal{O}(-2)) \times \mathrm{Aut}(\mathcal{O}(-1) \oplus \mathcal{O})\big)/\mathbb{C}^*$$

where $\mathbb{C}^*$ is identified with the subgroup $\{(t\cdot\mathrm{id}, t\cdot\mathrm{id}, t\cdot\mathrm{id}),\ t \in \mathbb{C}^*\}$. It is easy to check that the fibers of ρ_0 are precisely the $\mathbf{G}_0$-orbits. Let

$$\mathbf{W}_1 \subset \mathrm{Hom}(3\mathcal{O}(-3), 5\mathcal{O}(-2) \oplus \mathcal{O}(-1)) \times \mathrm{Hom}(5\mathcal{O}(-2) \oplus \mathcal{O}(-1), 2\mathcal{O}(-1) \oplus \mathcal{O})$$

https://doi.org/10.4153/CMB-2017-030-4 Published online by Cambridge University Press

be the locally closed subset of pairs of morphisms equivalent to pairs (ψ,φ) occurring in resolution (1.3) and let

$$\mathbf{W}_2 \subset \operatorname{Hom}(\mathcal{O}(-4)\oplus\mathcal{O}(-2), \mathcal{O}(-3)\oplus 3\mathcal{O}(-1)) \times \operatorname{Hom}(\mathcal{O}(-3)\oplus 3\mathcal{O}(-1), 2\mathcal{O})$$

be the set of pairs given at [5, Theorem 6.1(iii)]. The groups $\mathbf{G}_1$, $\mathbf{G}_2$ are defined by analogy with the definition of $\mathbf{G}_0$. As before, for $i = 1, 2$, the fibers of the canonical quotient map $\rho_i: \mathbf{W}_i \to \mathbf{M}_i$ are precisely the $\mathbf{G}_i$-orbits.

Proposition 4.1 *For $i = 0, 1$, $\mathbf{M}_i$ is the categorical quotient of $\mathbf{W}_i$ modulo $\mathbf{G}_i$. The subvariety $\mathbf{M}_2$ is the geometric quotient of $\mathbf{W}_2$ modulo $\mathbf{G}_2$.*

Proof The argument at [7, Theorem 3.1.6] shows that ρ_0, ρ_1, ρ_2 are categorical quotient maps. Since $\mathbf{M}_2$ is normal (being smooth), we can apply [15, Theorem 4.2] to conclude that ρ_2 is a geometric quotient map. ∎

Consider the closed subset $\mathbf{W}_{\mathrm{ell}} = \rho_0^{-1}(\mathbf{E}) \subset \mathbf{W}_0$. Consider the restriction to the second direct summand of the map

$$\mathcal{O}_{\mathbf{W}_{\mathrm{ell}}\times\mathbb{P}^3}(-1) \oplus \mathcal{O}_{\mathbf{W}_{\mathrm{ell}}\times\mathbb{P}^3} \longrightarrow \widetilde{\mathcal{F}}_{|\mathbf{W}_{\mathrm{ell}}\times\mathbb{P}^3}$$

and denote its image by $\widetilde{\mathcal{F}}'$. The quotient $[\mathcal{O}_{\mathbf{W}_{\mathrm{ell}}\times\mathbb{P}^3} \twoheadrightarrow \widetilde{\mathcal{F}}']$ induces a morphism

$$\sigma: \mathbf{W}_{\mathrm{ell}} \longrightarrow \operatorname{Hilb}_{\mathbb{P}^3}(4m).$$

According to [2, Examples 2.8 and 4.8], $\operatorname{Hilb}_{\mathbb{P}^3}(4m)$ has two irreducible components, denoted H_1, H_2. The generic member of H_1 is a smooth elliptic quartic curve. The generic member of H_2 is the disjoint union of a planar quartic curve and two isolated points. Note that H_2 lies in the closed subset

$$H = \{[\mathcal{O} \twoheadrightarrow \mathcal{S}] \mid \mathrm{h}^0(\mathcal{S}) \geq 3\} \subset \operatorname{Hilb}_{\mathbb{P}^3}(4m).$$

Since σ factors through the complement of H, we deduce that σ factors through H_1. By an abuse of notation, we denote the corestriction by $\sigma: \mathbf{W}_{\mathrm{ell}} \to H_1$.

Proposition 4.2 *The sets $\mathbf{D}_0$, $\mathbf{D}_1$, $\mathbf{D}_2$, $\mathbf{D}$, and $\mathbf{E}$ are contained in the closure of $\mathbf{E}_0$. The set $\mathbf{D}$ is irreducible and $\mathbf{D}_0$ is dense in $\mathbf{D}$. Moreover,*

$$\overline{\mathbf{E}} \smallsetminus \mathbf{P} = \mathbf{E} \cup \mathbf{D} = \mathbf{E} \cup \mathbf{D}', \qquad \overline{\mathbf{R}} \smallsetminus (\overline{\mathbf{E}} \cup \mathbf{P}) = \mathbf{R}.$$

Proof Let $\mathbf{E}_{\mathrm{reg}} \subset \mathbf{E}_0$ be the open subset of sheaves with smooth support. Let $H_{10} \subset H_1$ be the open subset consisting of smooth elliptic quartic curves. For any $x \in H_1 \smallsetminus H_{10}$ there is an irreducible quasi-projective curve $\Gamma \subset H_1$ such that $x \in \Gamma$ and $\Gamma \smallsetminus \{x\} \subset H_{10}$. To produce Γ proceed as follows. Embed H_1 into a projective space. Intersect with a suitable linear subspace passing through x to obtain a subscheme of dimension 1 all of whose irreducible components meet H_{10}. Retain one of these irreducible components and remove the points, other than x, that lie outside H_{10}.

Notice that if $y = [\mathcal{O} \twoheadrightarrow \mathcal{O}_E]$ is a point in H_{10}, then $\sigma^{-1}\{y\}$ is irreducible of dimension $1 + \dim \mathbf{G}_0$. Indeed,

$$\sigma^{-1}\{y\} = \rho_0^{-1}\{[\mathcal{O}_E(P)],\ P \in E\}.$$

https://doi.org/10.4153/CMB-2017-030-4 Published online by Cambridge University Press

Assume now that $x = [\mathcal{O} \twoheadrightarrow \mathcal{O}_E]$ where E is the schematic support of a sheaf in $\mathbf{E} \setminus \mathbf{D}$. We denote its irreducible components by $Z_0, \dots, Z_m$. Denote by $(\mathbf{E} \setminus \mathbf{D})^0$ the open subset of sheaves of the form $\mathcal{O}_{E'}(P')$ with P' lying outside $Z_1 \cup \dots \cup Z_m$ and let $\mathbf{W}^0$ be its preimage under ρ_0. Denote by σ_0 the restriction of σ to $\mathbf{W}^0$. Clearly, $\sigma_0^{-1}\{y\}$ is irreducible of dimension $1 + \dim \mathbf{G}_0$, and the same is true for $\sigma_0^{-1}\{x\}$. Thus, the fibers of the map $\sigma_0^{-1}(\Gamma) \to \Gamma$ are all irreducible of the same dimension. By [16, Theorem 8, p. 77] we deduce that $\sigma_0^{-1}(\Gamma)$ is irreducible. Thus, $\rho_0(\sigma^{-1}(\Gamma))$ is irreducible, hence any sheaf of the form $\mathcal{O}_E(P)$, $P \in Z_0 \setminus (Z_1 \cup \cdots \cup Z_m)$, is the limit of sheaves in $\mathbf{E}_{\mathrm{reg}}$. The same argument applies to $\mathcal{O}_E(P)$ for P belonging to exactly one of the components of E. A fortiori, $\mathcal{O}_E(P)$ lies in the Zariski closure of $\mathbf{E}_{\mathrm{reg}}$ for all $P \in E$. We conclude that $\mathbf{E} \setminus \mathbf{D} \subset \overline{\mathbf{E}}_0$.

Let D be the union of a line L and a planar irreducible cubic curve C, where L and C meet precisely at a regular point of C. Take $x = [\mathcal{O} \twoheadrightarrow \mathcal{O}_D]$. Then

$$\sigma^{-1}\{x\} = \rho_0^{-1}\{[\mathcal{O}_D(P)],\ P \in C \setminus L\}$$

is irreducible of dimension $1 + \dim \mathbf{G}_0$. We deduce as above that any sheaf of the form $\mathcal{O}_D(P)$, $P \in C \setminus L$, is the limit of sheaves in $\mathbf{E}_{\mathrm{reg}}$. The set of sheaves of the form $\mathcal{O}_D(P)$ is dense in $\mathbf{D}_0$. We conclude that $\mathbf{D}_0 \subset \overline{\mathbf{E}}_0$.

Let $\mathbf{D}^o \subset \mathbf{D} \cap \mathbf{E} = \mathbf{D} \setminus \mathbf{D}'$ be the open subset given by the condition that $P \notin L$. Let $\sigma^o \colon \mathbf{D}^o \to H_1$ denote the restriction of σ. According to [17, Theorem 5.2 (4)], there is an irreducible closed subset $\widehat{\mathbf{B}} \subset H_1$ whose generic member is the union of a planar cubic curve and an incident line. Let D be the schematic support of a sheaf in $\mathbf{D}_2$. According to [17, Theorem 5.2 (5)], the point $x = [\mathcal{O} \twoheadrightarrow \mathcal{O}_D]$ belongs to $\widehat{\mathbf{B}}$. By the same argument as above, there is an irreducible quasi-projective curve $\Gamma \subset \widehat{\mathbf{B}}$ containing x such that the points $y \in \Gamma \setminus \{x\}$ are of the form $[\mathcal{O} \twoheadrightarrow \mathcal{O}_{L\cup C}]$, where C is a planar irreducible cubic curve and L is an incident line. Notice that

$$(\sigma^o)^{-1}\{y\} = \rho_0^{-1}\big\{[\mathcal{O}_{L\cup C}(P)],\ P \in C \setminus L\big\}$$

is irreducible of dimension $1 + \dim \mathbf{G}_0$. Assume, in addition, that D is the union of an irreducible plane conic curve C' and a double line supported on L'. Then

$$(\sigma^o)^{-1}\{x\} = \rho_0^{-1}\big\{[\mathcal{O}_D(P)],\ P \in C' \setminus L'\big\}$$

is irreducible of dimension $1 + \dim \mathbf{G}_0$. We deduce, as above, that $(\sigma^o)^{-1}(\Gamma)$ is irreducible, hence $\rho_0((\sigma^o)^{-1}(\Gamma))$ is irreducible, and hence any sheaf of the form $\mathcal{O}_D(P)$, $P \in C' \setminus L'$, is the limit of sheaves in $\mathbf{D}_0$. But $\mathbf{D}_2$ is irreducible, hence the set of sheaves $\mathcal{O}_D(P)$ as above is dense in $\mathbf{D}_2$. We deduce that $\mathbf{D}_2 \subset \overline{\mathbf{D}}_0$. Thus, $\mathbf{D}_2 \subset \overline{\mathbf{E}}_0$.

Recall from Proposition 3.5 that $\mathbf{D}_1 \subset \overline{\mathbf{D}}_0$. Since $\mathbf{D} = \mathbf{D}_0 \cup \mathbf{D}_1 \cup \mathbf{D}_2$, we see that $\mathbf{D} \subset \overline{\mathbf{D}}_0 \subset \overline{\mathbf{E}}_0$.

The inclusion $\overline{\mathbf{E}} \setminus \mathbf{P} \subset \mathbf{E} \cup \mathbf{D}'$ follows from Theorem 1.1 and Proposition 2.3. Indeed, $\mathbf{E}$ is closed in $\mathbf{M}_0$. The reverse inclusion was proved above. Finally,

$$\overline{\mathbf{R}} \setminus (\overline{\mathbf{E}} \cup \mathbf{P}) = \overline{\mathbf{R}} \setminus (\mathbf{E} \cup \mathbf{D}' \cup \mathbf{P}) \subset \mathbf{M} \setminus (\mathbf{E} \cup \mathbf{D}' \cup \mathbf{P}) = \mathbf{M}_0 \setminus \mathbf{E} = \mathbf{R}.$$

The reverse inclusion is obvious because by definition $\mathbf{R}$ is disjoint from $\mathbf{E}$, $\mathbf{D}'$, $\mathbf{P}$. ∎

From Proposition 4.2 we obtain the decomposition of $\mathrm{M}_{\mathbb{P}^3}(4m+1)$ into irreducible components.

Theorem 4.3 *The moduli space* $\mathrm{M}_{\mathbb{P}^3}(4m+1)$ *consists of three irreducible components* $\overline{\mathbf{R}}$, $\overline{\mathbf{E}}$, *and* $\mathbf{P}$.

The intersections $\overline{\mathbf{R}} \cap \mathbf{P}$, $\overline{\mathbf{E}} \cap \mathbf{P}$, $\overline{\mathbf{R}} \cap \overline{\mathbf{E}}$ were described generically in [5]. They are irreducible and have dimension 14, 16, and 15, respectively. The generic member of $\overline{\mathbf{R}} \cap \mathbf{P}$ has the form $[\mathcal{O}_C(P_1 + P_2 + P_3)]$, where C is a planar quartic curve and P_1, P_2, P_3 are three distinct nodes. The generic point in $\overline{\mathbf{E}} \cap \mathbf{P}$ has the form $[\mathcal{O}_C(P_1 + P_2 + P)]$, where C is a planar quartic curve, P_1 and P_2 are distinct nodes, and P is a third point on C. The generic sheaves in $\overline{\mathbf{R}} \cap \overline{\mathbf{E}}$ have the form $\mathcal{O}_E(P)$, where E is a singular $(2,2)$-curve on a smooth quadric surface and $P \in \mathrm{sing}(E)$.

References

[1] E. Ballico and S. Huh, Stable sheaves on a smooth quadric surface with linear Hilbert bipolynomials. The Scientific World Journal (2014), Article ID 346126.

[2] D. Chen and S. Nollet, *Detaching embedded points.* Algebra Number Theory **6**(2012), 731–756. http://dx.doi.org/10.2140/ant.2012.6.731

[3] J. Choi and K. Chung, *The geometry of the moduli space of one-dimensional sheaves.* Sci. China Math. **58**(2015), 487–500. http://dx.doi.org/10.1007/s11425-014-4889-9

[4] ———, *Moduli spaces of α-stable pairs and wall-crossing on* $\mathbb{P}^2$. J. Math. Soc. Japan **68**(2016), 685–709. http://dx.doi.org/10.2969/jmsj/06820685

[5] J. Choi, K. Chung, and M. Maican, *Moduli of sheaves supported on quartic space curves.* Michigan Math. J. **65**(2016), 637–671. http://dx.doi.org/10.1307/mmj/1472066152

[6] J. Choi and M. Maican, *Torus action on the moduli spaces of torsion plane sheaves of multiplicity four.* J. Geom. Phy. **83**(2014), 18–35. http://dx.doi.org/10.1016/j.geomphys.2014.05.005

[7] J.-M. Drézet and M. Maican, *On the geometry of the moduli spaces of semi-stable sheaves supported on plane quartics.* Geom. Dedicata **152**(2011), 17–49. http://dx.doi.org/10.1007/s10711-010-9544-1

[8] H.-G. Freiermuth and G. Trautmann, *On the moduli scheme of stable sheaves supported on cubic space curves.* Amer. J. Math. **126**(2004), 363–393. http://dx.doi.org/10.1353/ajm.2004.0013

[9] D. Huybrechts and M. Lehn, *The geometry of moduli spaces of sheaves.* Aspects of Mathematics, E31, Vieweg, Braunschweig, 1997. http://dx.doi.org/10.1007/978-3-663-11624-0

[10] O. Iena, *On the singular* 1*-dimensional planar sheaves supported on quartics.* Rend. Istit. Mat. Univ. Trieste **48**(2016), 565–586.

[11] J. Le Potier, *Faisceaux semi-stables de dimension* 1 *sur le plan projectif.* Rev. Roumaine Math. Pures Appl. **38**(1993), 635–678.

[12] M. Maican, *A duality result for moduli spaces of semistable sheaves supported on projective curves.* Rend. Semin. Mat. Univ. Padova **123**(2010), 55–68. http://dx.doi.org/10.4171/RSMUP/123-3

[13] ———, *The homology groups of certain moduli spaces of plane sheaves.* Internat. J. Math. **24**(2013), Article ID 1350098. http://dx.doi.org/10.1142/S0129167X13500985

[14] ———, *On the homology of the moduli space of plane sheaves with Hilbert polynomial* $5m + 3$. Bull. Sci. Math. **139**(2015), 1–32. http://dx.doi.org/10.1016/j.bulsci.2014.08.001

[15] V. Popov and E. Vinberg, *Invariant theory*. In: Algebraic Geometry IV, Encyclopedia of Mathematical Sciences, 55, Springer Verlag, Berlin, 1994. http://dx.doi.org/10.1007/978-3-662-03073-8

[16] I. Shafarevich, *Basic algebraic geometry. I.* Second ed., Springer-Verlag, Berlin, Heidelberg, 1994.

[17] I. Vainsencher and D. Avritzer, *Compactifying the space of elliptic quartic curves.* In: Complex projective geometry, (Trieste, 1989), London Math. Soc. Lecture Note Ser., 179, Cambridge University Press, Cambridge, 47–58.

Institute of Mathematics of the Romanian Academy, Calea Grivitei 21, Bucharest 010702, Romania
e-mail: maican@imar.ro