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RESEARCH ARTICLE

Hausdorff dimension of sets defined by almost convergent binary expansion sequences

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Abstract

In this paper, we study the Hausdorff dimension of sets defined by almost convergent binary expansion sequences. More precisely, the Hausdorff dimension of the following set

$$\left\{x \in [0,1) : \frac{1}{n}\sum_{k=a}^{a+n-1} x_k \longrightarrow \alpha \text{ uniformly in } a \in \mathbb{N} \text{ as } n \to \infty\right\}$$

is determined for any $\alpha \in [0,1]$. This completes a question considered by Usachev [*Glasg. Math. J.* **64** (2022), 691–697] where only the dimension for rational α is given.

1. Introduction

The concept *almost convergence* was first introduced by Lorentz [6] in 1948 which is used to study the property of divergent sequences, that is what will happen if all the Banach limits of a sequence are equal. In [6], Lorentz defined almost convergence by the equality of all the Banach limits and discovered that this definition is equivalent to the one we give later. Lorentz [6] also studied the relationship between almost convergence, or in other words summation by method F, and matrix methods, then found that most of the commonly used matrix methods contain the method F. One is referred to [6] for details.

Definition 1.1. *A bounded sequence $\{x_k\}_{k=1}^{\infty}$ is called almost convergent to a number $t \in \mathbb{R}$, if*

$$\frac{1}{n}\sum_{k=a}^{a+n-1} x_k \longrightarrow t$$

uniformly in $a \in \mathbb{N}$ as $n \to \infty$. We write this by $\{x_k\}_{k=1}^{\infty} \in \mathbf{AC}(t)$.

It is easy to notice that if take $a = 1$, the summation is exactly Cesàro summation. So a sequence that is almost convergent to t must be Cesàro convergent to t. Borel [2] presented a classical result that the set such that the corresponding sequences with binary expansions of numbers in [0, 1] are Cesàro convergent to $\frac{1}{2}$ has full Lebesgue measure. That is

$$\mathscr{L}\left(\left\{x \in [0,1) : \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} x_k = \frac{1}{2}\right\}\right) = 1,$$

where $\mathscr{L}$ denotes the Lebesgue measure and $x = (x_1x_2 \ldots)_2$ denotes the binary expansion of $x \in [0, 1]$. Besicovitch [1] showed that for all $0 \le \alpha \le 1$, the set

$$F_\alpha = \left\{x \in [0, 1) : \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} x_k = \alpha\right\}$$

has Hausdorff dimension $-\alpha \log_2 \alpha - (1 - \alpha) \log_2(1 - \alpha)$. Besicovitch's result is also extended to other matrix summations [8]. For further research about almost convergence and its applications, one can refer to [4, 7, 8] and the references therein.

Almost convergence can imply Cesàro convergence, and a Cauchy sequence must be an almost convergent sequence. Besides, there are many other sences of convergence. For each one, we can study the Hausdorff dimension of the sets similar to F_α. In this paper, we consider numbers with almost convergent sequences associated with their binary expansions. Connor [3] proved that there is no almost convergent binary expansion sequences for almost all numbers in [0, 1]. In 2022, Usachev [8] showed that for rational α, the Hausdorff dimension of the set

$$\begin{aligned} G_\alpha &:= \left\{x \in [0, 1) : \frac{1}{n} \sum_{k=a}^{a+n-1} x_k \longrightarrow \alpha \text{ uniformly in } a \in \mathbb{N} \text{ as } n \to \infty\right\} \\ &= \left\{x \in [0, 1) : \{x_k\}_{k=1}^{\infty} \in \mathbf{AC}(\alpha)\right\} \end{aligned}$$

is also $-\alpha \log_2 \alpha - (1 - \alpha) \log_2(1 - \alpha)$, but left a problem for the case when α is irrational. Usachev's proof depends highly on the rationality of α, while our method is applicable for any α.

Theorem 1.2. *For all* $0 \le \alpha \le 1$, *we have*

$$\dim_{\mathrm{H}} G_\alpha = -\alpha \log_2 \alpha - (1 - \alpha) \log_2(1 - \alpha).$$

We refer to Lorentz [6] for the definition of strongly regular matrix method.

Definition 1.3. *A matrix method of summation is a mapping on the space of all sequences* $\{x_k\}_{k=1}^{\infty}$ *generated by a matrix* $A = \{a_{nk}\}_{n,k=1}^{\infty}$, *which is*

$$\{x_k\}_{k=1}^{\infty} \longmapsto \left\{\sum_{k=1}^{\infty} a_{nk}x_k\right\}_{n=1}^{\infty}.$$

We call it strongly regular if

$$\sum_{k=1}^{\infty} |a_{nk} - a_{n,k+1}| \longrightarrow 0 \text{ as } n \longrightarrow \infty.$$

Corollary 3.5 in [8] remains true for irrational α. That is

Corollary 1.4. *Let* $A = \{a_{nk}\}_{n,k=1}^{\infty}$ *be a strongly regular matrix method, which is weaker than (or consistent with) the Cesàro method, and let* $0 \le \alpha \le 1$. *The Hausdorff dimension of the set*

$$\left\{x \in [0, 1) : \lim_{n\to\infty} \sum_{k=1}^{\infty} a_{nk}x_k = \alpha\right\},$$

is $-\alpha \log_2 \alpha - (1 - \alpha) \log_2(1 - \alpha)$.

The proof of this corollary is completely same as which in [8].

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2. Preliminaries

In this section, we recall the definition of Hausdorff dimension and mass distribution principle that will be used later.

Definition 2.1. [5] *Given a set $E \subset \mathbb{R}^n$, its s -dimensional Hausdorff measure is*

$$\mathscr{H}^s(E) = \lim_{\delta\to 0} \inf\left\{ \sum_{i=1}^{\infty} |O_i|^s : E \subset \bigcup_{i=1}^{\infty} O_i, 0 \leq |O_i| \leq \delta \right\}$$

where $\{O_i\}_{i=1}^{\infty}$ is an open cover, and $|\cdot|$ denotes the diameter. Besides, the Hausdorff dimension of E is

$$\dim_{\mathrm{H}} E := \inf\{s : \mathscr{H}^s(E) = 0\}.$$

Theorem 2.2. (Mass distribution principle) [5] *Let E be a set, and there is a strictly positive Borel measure μ supported on E. If some $s \geq 0$,*

$$\liminf_{r\to 0} \frac{\log \mu(B(x, r))}{\log r} \geq s$$

holds for all $x \in E$, then

$$\dim_{\mathrm{H}} E \geq s.$$

At the end, we fix a notation. For any $x \in [0, 1)$, let

$$x = \frac{x_1}{2} + \frac{x_2}{2^2} + \ldots$$

be the binary expansion of x. We write $x = (x_1, x_2, \ldots)_2$. For any $m \geq 1$ and a finite block $(\varepsilon_1\varepsilon_2 \ldots \varepsilon_n)$ with $\varepsilon_i \in \{0, 1\}$ for all $1 \leq i \leq n$, we write

$$I_n(\varepsilon_1\varepsilon_2 \ldots \varepsilon_n) = \left\{ x \in [0, 1) : x_i = \varepsilon_i, 1 \leq i \leq n \right\},$$

which is the set of points whose binary expansion begin with the digits $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$. Recall that

$$F_\alpha = \left\{ x \in [0, 1) : \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} x_k = \alpha \right\}$$

is the set which consists of numbers with binary expansion sequences that are Cesàro convergent to α, and

$$G_\alpha = \left\{ x \in [0, 1) : \frac{1}{n} \sum_{k=a}^{a+n-1} x_k \longrightarrow \alpha \text{ uniformly in } a \in \mathbb{N} \text{ as } n \to \infty \right\}.$$

The upper bound of the Hausdorff dimension of G_α is trivial since G_α is a subset of F_α, so by Besicovitch's result [1],

$$\dim_{\mathrm{H}} G_\alpha \leq \dim_{\mathrm{H}} F_\alpha = -\alpha \log_2 \alpha - (1-\alpha) \log_2(1-\alpha).$$

Thus, we need only to focus on the lower bound of the Hausdorff dimension of G_α.

3. Proof of Theorem 1.2

The lower bound of the Hausdorff dimension of G_α is given by classical methods:

(1) Construct a Cantor subset of G_α, denoted by $E_{m,\alpha}$ for each $m \in \mathbb{N}$ large;
(2) Define a probability measure supported on $E_{m,\alpha}$;
(3) Use mass distribution principle to find the lower bound of the Hausdorff dimension of $E_{m,\alpha}$.

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So define the set $E_{m,\alpha}$ as:

$$\text{For every } m \in \mathbb{N},\ E_{m,\alpha} := \left\{ x \in [0, 1) : \sum_{k=(j-1)m+1}^{jm} x_k = [m\alpha] + \xi_j \text{ for all } j \in \mathbb{N} \right\},$$

where $[m\alpha]$ denotes the largest integer less than or equal to $m\alpha$, and $\{\xi_j\}_{j\in\mathbb{N}}$ is defined as follows. Firstly, take ξ_1 to be 0, and ξ_2 to be the integer such that $2[m\alpha] + \xi_1 + \xi_2 = [2m\alpha]$, so ξ_2 could be 0 or 1. Then, secondly, we define ξ_j inductively, that is, we choose ξ_j to be an integer satisfying

$$j[m\alpha] + \xi_1 + \xi_2 + \ldots + \xi_j = [jm\alpha].$$

By a simple calculation, we have

$$\xi_j = \left[jm\alpha\right] - \left[(j-1)m\alpha\right] - \left[m\alpha\right] = \left\{(j-1)m\alpha\right\} + \left\{m\alpha\right\} - \left\{jm\alpha\right\},$$

where $\{m\alpha\} = m\alpha - [m\alpha]$. Therefore, ξ_j could be -1, 0, 1, or 2.

In other words, $E_{m,\alpha}$ is a collection of such numbers, whose binary expansion sequences satisfy that for all $j \in \mathbb{N}$, the first jm digits contain exactly $[jm\alpha]$ many ones, and if we cut the sequences into blocks of length m, the j-th block contains exactly $[m\alpha] + \xi_j$ ones. Except that, there is no request for the position of ones.

We check that $E_{m,\alpha}$ is indeed a subset of G_α when m is sufficiently large.

Lemma 3.1. *Let* $0 < \alpha \le 1$, $m > \left[\frac{100}{\alpha}\right] + 100$, *or* $\alpha = 0$, $m \in \mathbb{N}$. *Then for any* $x \in E_{m,\alpha}$,

$$\frac{1}{n}\sum_{k=a}^{a+n-1} x_k \longrightarrow \alpha \textit{ uniformly in } a \in \mathbb{N} \textit{ as } n \to \infty.$$

Proof. First, for $\alpha = 0$, $E_{m,0}$ contains a single point 0, so it is a subset of G_0. Second, for $0 < \alpha \le 1$, we see that $[m\alpha]$ cannot be 0 for m large. Fix an integer a, for any $x \in E_{m,\alpha}$, let j_1 be the largest j such that $jm < a$, and j_2 be the smallest j such that $jm \ge a + n - 1$. Then, on one hand,

$$\sum_{k=a}^{a+n-1} x_k \le \sum_{k=j_1m+1}^{j_2m} x_k \le \left[j_2m\alpha\right] - \left[j_1m\alpha\right] \le (j_2 - j_1)m\alpha + 1. \tag{3.1}$$

On the other hand,

$$\sum_{k=a}^{a+n-1} x_k \ge \sum_{k=(j_1+1)m+1}^{(j_2-1)m} x_k \ge \left[(j_2-1)m\alpha\right] - \left[(j_1+1)m\alpha\right] \ge (j_2 - j_1)m\alpha - 2m\alpha - 1. \tag{3.2}$$

Furthermore, we have

$$n - 1 \le (j_2 - j_1)m \le n + 2m. \tag{3.3}$$

Combining (3.1) (3.2) (3.3) together, it follows that

$$\frac{(n-1)\alpha - 2m\alpha - 1}{n} \le \frac{1}{n}\sum_{k=a}^{a+n-1} x_k \le \frac{(n+2m)\alpha + 1}{n}. \tag{3.4}$$

Since the left and the right most terms in (3.4) do not depend on a, the convergence is uniform when n tends to infinity. This shows $E_{m,\alpha}$ is a subset of G_α. □

Next, we analyze the Cantor structure of $E_{m,\alpha}$ in detail. Here, we assume that $0 < \alpha \le 1$ and m is sufficiently large. Denote U_j the blocks of length m which contain exactly $[m\alpha] + \xi_j$ ones, $j = 1, 2, \ldots$, that is

$$U_j = \left\{ u = (\varepsilon_1\varepsilon_2 \ldots \varepsilon_m) \in \{0, 1\}^m : \sum_{k=1}^{m} \varepsilon_k = [m\alpha] + \xi_j \right\}.$$

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Then, for each j, the collection U_j contains $D_j = \mathrm{C}_m^{[m\alpha]+\xi_j}$ elements, where C_n^k are binomial coefficients. So the first level of the Cantor structure of $E_{m,\alpha}$ is

$$\mathscr{S}_1 = \{I_m(u_1) : u_1 \in U_1\}, \ S_1 = \bigcup_{u_1 \in U_1} I_m(u_1).$$

The second level is

$$\mathscr{S}_2 = \bigcup_{I_m(u_1)\in \mathscr{S}_1} \Big\{ I_{2m}(u_1u_2) : u_2 \in U_2 \Big\} = \Big\{ I_{2m}(u_1u_2) : u_1 \in U_1, u_2 \in U_2 \Big\},$$

$$S_2 = \bigcup_{u_1\in U_1} \bigcup_{u_2 \in U_2} I_{2m}(u_1u_2),$$

and by induction, the level-j of the Cantor structure of $E_{m,\alpha}$ is

$$\mathscr{S}_j = \{I_{jm}(u_1u_2\ldots u_j) : u_1 \in U_1, u_2 \in U_2, \ldots, u_j \in U_j\},$$

$$S_j = \bigcup_{u_1\in U_1} \bigcup_{u_2 \in U_2} \ldots \bigcup_{u_j \in U_j} I_{jm}(u_1u_2\ldots u_j).$$

Then, we have that

$$\bigcap_{j=1}^{\infty} S_j = \{x = (u_1u_2\ldots)_2 : u_1 \in U_1, u_2 \in U_2 \ldots\} = E_{m,\alpha}.$$

Now we compute the lower bound of the Hausdorff dimension of $E_{m,\alpha}$. We have to deal with some binomial coefficients later, so here we simplify it by using Stirling formula.

Lemma 3.2. *Let $\{d_m\}_{m\geq 1}$ be a sequence of integers with $d_m \leq m$ for all $m \geq 1$ and*

$$\limsup_{m\to\infty} \frac{d_m}{m} = \alpha.$$

Then

$$\limsup_{m\to\infty} \frac{1}{m} \log_2 \mathrm{C}_m^{d_m} = -\alpha \log_2 \alpha - (1-\alpha)\log_2(1-\alpha).$$

Proof. Write $m' = d_m$. By Stirling formula,

$$\begin{aligned}
\mathrm{C}_m^{m'} &= \frac{m!}{m'!(m-m')!} \\
&= \frac{\sqrt{2\pi m}(\frac{m}{\mathrm{e}})^m \mathrm{e}^{O(\frac{1}{m})}}{\sqrt{2\pi m'}(\frac{m'}{\mathrm{e}})^{m'} \mathrm{e}^{O(\frac{1}{m'})}\sqrt{2\pi(m-m')}(\frac{m-m'}{\mathrm{e}})^{m-m'}\mathrm{e}^{O(\frac{1}{m-m'})}} \\
&= \left(\frac{m}{m-m'}\right)^m \left(\frac{m-m'}{m'}\right)^{m'} O\left(m^{-\frac{1}{2}}\right).
\end{aligned}$$

So we have

$$\begin{aligned}
\limsup_{m\to\infty} \frac{1}{m}\log_2 \mathrm{C}_m^{m'} &= \limsup_{m\to\infty}\left(\log_2\frac{1}{1-\frac{m'}{m}} + \frac{m'}{m}\log_2\frac{1-\frac{m'}{m}}{\frac{m'}{m}} + O\left(\frac{\log_2 m}{m}\right)\right) \\
&= \log_2\frac{1}{1-\alpha} + \alpha\log_2\frac{1-\alpha}{\alpha} \\
&= -\alpha\log_2\alpha - (1-\alpha)\log_2(1-\alpha).
\end{aligned}$$

□

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Lemma 3.3. *For any* $0 < \alpha \le 1$ *and* $m > [\frac{100}{\alpha}] + 100$*, we have*

$$\dim_{\mathrm{H}} E_{m,\alpha} \ge \liminf_{j\to\infty} \frac{\log_2 \prod_{k=1}^{j} D_k}{jm}.$$

Proof. Define a measure μ supported on $E_{m,\alpha}$. Let $\mu([0, 1)) = 1$. For all $j > 0$ and every element $I \in \mathscr{S}_j$, we set

$$\mu(I) = \left(\prod_{k=1}^{j} D_k\right)^{-1},$$

and $\mu(E) = 0$ for all $E \cap S_j = \emptyset$. Then, one can see that the set function μ satisfies Kolmogorov's consistency condition, that is, for any $j \ge 1$ and $I \in \mathscr{S}_j$,

$$\sum_{\tilde{I}\in\mathscr{S}_{j+1}, \tilde{I}\subset I} \mu(\tilde{I}) = \mu(I).$$

Thus, it can be extended into a mass distribution supported on $E_{m,\alpha}$ [5]. So for any ball $B(x, r)$ centered at $x \in [0, 1)$ with radius $r < 1$, there is an integer j such that $2^{-(j+1)m} \le r < 2^{-jm}$. It follows that the ball intersects at most 2 elements in $\mathscr{S}_j$, then we have

$$\begin{aligned} \frac{\log_2 \mu(B(x, r))}{\log_2 r} &\ge \frac{\log_2(2 \cdot (\prod_{k=1}^{j} D_k)^{-1})}{\log_2 r} \\ &= \frac{-\log_2 \prod_{k=1}^{j} D_k}{\log_2 r} + O\left(\frac{1}{\log_2 r}\right) \\ &\ge \frac{-\log_2 \prod_{k=1}^{j} D_k}{-(j+1)m} + O\left(\frac{1}{\log_2 r}\right). \end{aligned}$$

Thus

$$\liminf_{r\to 0} \frac{\log \mu(B(x, r))}{\log r} \ge \liminf_{j\to\infty} \frac{\log_2 \prod_{k=1}^{j} D_k}{jm},$$

and the lemma holds by mass distribution principle. □

Remark 1. *For the case* $\alpha = 0$ *and* $m \in \mathbb{N}$*, the result in Lemma 3.3 is trivial since* $E_{m,0}$ *is a set of a single point, and* $D_j = 1$ *for all* $j \in \mathbb{N}$.

Proof. (Proof of Theorem 1.2) As Lemmas 3.1 and 3.3 hold for m sufficiently large, we can take the limit supremum of $\dim_{\mathrm{H}} E_{m,\alpha}$ as $m \to \infty$. Let $\xi_{(m,\alpha)}$ be a number in $\{-1, 0, 1, 2\}$ such that $\mathrm{C}_m^{[m\alpha]+\xi_{(m,\alpha)}}$ is the smallest one among

$$\mathrm{C}_m^{[m\alpha]-1}, \mathrm{C}_m^{[m\alpha]}, \mathrm{C}_m^{[m\alpha]+1}, \mathrm{C}_m^{[m\alpha]+2},$$

and take $m' = [m\alpha] + \xi_{(m,\alpha)}$. Then, we have

$$\begin{aligned} \dim_{\mathrm{H}} G_\alpha &\ge \limsup_{m\to\infty} \dim_{\mathrm{H}} E_{m,\alpha} \\ &= \limsup_{m\to\infty} \liminf_{j\to\infty} \frac{\log_2 \prod_{k=1}^{j} D_k}{jm} \\ &\ge \limsup_{m\to\infty} \liminf_{j\to\infty} \frac{\log_2 \prod_{k=1}^{j} \mathrm{C}_m^{[m\alpha]+\xi_{(m,\alpha)}}}{jm} \end{aligned}$$

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$$
\begin{aligned}
&= \limsup_{m\to\infty} \frac{1}{m} \log_2 \mathrm{C}_m^{m'} \\
&= -\alpha \log_2 \alpha - (1-\alpha)\log_2(1-\alpha)
\end{aligned}
$$

where the last equality follows from Lemma 3.2 since $\limsup_{m\to\infty} \frac{m'}{m} = \alpha$. □

Usachev [8] also posed a potential method to attach the case when α is irrational. Define a sequence of rationals $\{\frac{p_i}{q_i}\}_{i=1}^{\infty}$ which converges to α, and construct a set that consists of numbers with such binary expansions. It requests that among the positions in

$$
\left[\sum_{k=1}^{i-1} q_k m + 1, \sum_{k=1}^{i-1} q_k m + q_i m\right],
$$

the binary expansion sequences have exactly $p_i m$ many ones. Although we do have

$$
\frac{1}{n}\sum_{k=a}^{a+n-1} x_k \longrightarrow \alpha \text{ as } n \to \infty
$$

for every positive integer a, the convergence may not be uniform as mentioned in [8], that in every block of length $q_i m$, all the ones appear at first and then the zeros follow.

So ensure the convergence is uniform, it is necessary to pose suitable restrictions on the distribution of ones, for example, asking zeros and ones appear regularly. Here, we cut the sequence into blocks of the same length, and there is only a little difference about the quantity of ones in different blocks. It avoids the problem caused by q_i going to infinity and ones being separated from zeros.

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