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SPACE-LIKE HYPERSURFACES IN LOCALLY SYMMETRIC LORENTZ SPACE

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Abstract

Let M be an n-dimensional space-like hypersurface in a locally symmetric Lorentz space, with $n(n-1)R = \kappa H (\kappa > 0)$ and satisfying certain additional conditions on the sectional curvature. Denote by S and H the squared norm of the second fundamental form and the mean curvature of M, respectively. We show that if the mean curvature is nonnegative and attains its maximum on M, then:

(1) if $H^2 < 4(n-1)c/n^2$, M is totally umbilical;
(2) if $H^2 = 4(n-1)c/n^2$, M is totally umbilical or is an isoparametric hypersurface;
(3) if $H^2 > 4(n-1)c/n^2$ and S satisfies some pinching conditions, M is totally umbilical or is an isoparametric hypersurface.

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1. Introduction

Let M_s^m be an m-dimensional connected semi-Riemannian manifold of index $s (s \geq 0)$; it is called a semi-definite space of index s. In particular, M_1^m is called a Lorentz space. When the Lorentz space M_1^m is of constant curvature c, it is called a Lorentz space form and denoted by $M_1^m(c)$. A hypersurface M of a Lorentz space M_1^m is said to be space-like if the metric on M induced by that of the Lorentz space M_1^m is positive definite.

It is well-known that space-like hypersurfaces with constant mean curvature in arbitrary space–time are of interest in relativity theory (see [10] and [15]). Therefore, space-like hypersurfaces in a Lorentz space form have recently been investigated by many differential geometers from both the physical and the mathematical points of view; see, for example, [1, 4, 5, 7, 8, 11, 13] and [14]. Goddard [8] conjectured that a complete space-like hypersurface in de Sitter space M_1^{n+1} with constant mean

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curvature H must be totally umbilical. Akutagawa [1] and Ramanathan [13] proved independently that the conjecture is true if $H^2 \leq 1$ when $n = 2$ and $n^2H^2 < 4(n-1)$ when $n \geq 3$.

We note that the investigation of space-like hypersurfaces for which the scalar curvature $n(n-1)R$ and the mean curvature H are linearly related is also a very important and interesting problem. Cheng [4] and Li [9] obtained some characteristic theorems, in terms of the sectional curvature, on space-like hypersurfaces where the scalar curvature $n(n-1)R$ and the mean curvature H are linearly related. Recently, the author proved a characteristic theorem concerning such hypersurfaces in terms of the mean curvature H [14].

All of the above results were obtained under the assumption that the ambient manifolds possess very nice symmetry properties. Many researchers have recently begun to study ambient manifolds which do not have symmetry in general: for example, the general Lorentz space or locally symmetric Lorentz space; see [3] and [16], for instance. In [3], Baek*et al.* obtained some important results on complete space-like hypersurfaces in locally symmetric Lorentz space with constant mean curvature.

In this paper, we consider $(n+1)$-dimensional Lorentz space M_1^{n+1} of index 1. We denote by $\bar{\nabla}$, $\bar{K}$ and $\bar{R}$ the semi-Riemannian connection, sectional curvature and curvature tensor on M_1^{n+1}, respectively. If the Lorentz space M_1^{n+1} satisfies the following conditions:

1. for any space-like vector u and any time-like vector v, $\bar{K}(u, v) = -c_1/n$ where c_1 is a constant;
2. for any space-like vectors u and v,

$$\bar{K}(u, v) \geq c_2, \tag{1.1}$$

   where c_2 is a constant;

then we shall say that M_1^{n+1} is a locally symmetric Lorentz space satisfying condition (*).

REMARK 1. The Lorentz space form $M_1^{n+1}(c)$ satisfies condition (*), with $-c_1/n = c_2 = c$.

In what follows, we shall investigate space-like hypersurfaces, with the scalar curvature $n(n-1)R$ and the mean curvature H being linearly related, in a locally symmetric Lorentz space satisfying condition (*). We shall prove the following results.

THEOREM 1.1. *Let M be an n-dimensional space-like hypersurface with $n(n-1)R = \kappa H$, where κ is a positive constant, in a locally symmetric Lorentz space M_1^{n+1} that satisfies condition (*). Suppose that the mean curvature H is nonnegative and attains its maximum on M; then the following properties hold.*

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(1) *If* $H^2 < 4(n-1)c/n^2$, *then* M *is totally umbilical.*
(2) *If* $H^2 = 4(n-1)c/n^2$, *then* M *is totally umbilical or is an isoparametric hypersurface with two distinct principal curvatures, one of which is simple.*
(3) *If* $H^2 > 4(n-1)c/n^2$ *and the squared norm* S *of the second fundamental form satisfies* $S \leq nH^2 + (B^-_{H,n,c})^2$ *or* $S \geq nH^2 + (B^+_{H,n,c})^2$, *then* M *is totally umbilical or is an isoparametric hypersurface with two distinct principal curvatures, one of which is simple, with* $c = 2c_2 + (c_1/n)$ $(c_2 > 0)$ *and*

$$B^{\pm}_{H,n,c} = \sqrt{\frac{n}{4(n-1)}}\Big[(n-2)H \pm \sqrt{n^2H^2 - 4(n-1)c}\,\Big].$$

REMARK 2. Note that if M_1^{n+1} is the de Sitter space $M_1^{n+1}(c)$, where $c = -c_1/n = c_2 > 0$, then Theorem 1.1 reduces to [14, Theorem 1.2].

2. Preliminaries

Let M be an n-dimensional space-like hypersurface in Lorentz space M_1^{n+1}. Let $\{e_1, e_2, \ldots, e_n, e_{n+1}\}$ be a local frame of orthonormal vector fields in M_1^{n+1} such that, restricted to M, the vectors $\{e_1, e_2, \ldots, e_n\}$ are tangent to M, and the vector e_{n+1} is normal to M. Let $\{\omega_1, \omega_2, \ldots, \omega_n, \omega_{n+1}\}$ be the dual frame field. We shall use the following convention on the ranges of indices:

$$1 \leq i, j, k, \ldots \leq n, \quad 1 \leq A, B, C, \ldots \leq n+1.$$

We write $\varepsilon_i = 1$ and $\varepsilon_{n+1} = -1$; then M_1^{n+1} satisfies the structure equations

$$\begin{aligned} d\omega_A &= -\sum_B \varepsilon_B \omega_{AB} \wedge \omega_B, \quad \omega_{AB} + \omega_{BA} = 0, \\ d\omega_{AB} &= -\sum_C \varepsilon_C \omega_{AC} \wedge \omega_{CB} - \frac{1}{2}\sum_{C,D} \varepsilon_C \varepsilon_D \bar{R}_{ABCD}\omega_C \wedge \omega_D, \end{aligned} \tag{2.1}$$

where $\bar{R}_{ABCD}$ denotes the components of the Riemannian curvature tensor of M_1^{n+1}. We denote by $\bar{R}_{CD}$ and $\bar{R}$ the Ricci tensor and the scalar curvature of M_1^{n+1}, respectively; then

$$\bar{R}_{CD} = \sum_B \varepsilon_B \bar{R}_{BCDB}, \quad \bar{R} = \sum_A \varepsilon_A \bar{R}_{AA}.$$

Now, let us write $\bar{R}_{ABCD;E}$ for the covariant derivative of $\bar{R}_{ABCD}$. Then the components $\bar{R}_{ABCD;E}$ are defined by

$$\begin{aligned} &\sum_E \varepsilon_E \bar{R}_{ABCD;E}\omega_E \\ &\quad = d\bar{R}_{ABCD} - \sum_E \varepsilon_E(\bar{R}_{EBCD}\omega_{EA} + \bar{R}_{AECD}\omega_{EB} + \bar{R}_{ABED}\omega_{EC} + \bar{R}_{ABCE}\omega_{ED}). \end{aligned}$$

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Since M_1^{n+1} is a locally symmetric manifold,

$$\bar{R}_{ABCD;E} = 0. \tag{2.2}$$

We have, for M, that

$$d\omega_i = -\sum_j \omega_{ij} \wedge \omega_j, \quad \omega_{ij} + \omega_{ji} = 0,$$
$$d\omega_{ij} = -\sum_k \omega_{ik} \wedge \omega_{kj} - \frac{1}{2}\sum_{k,l} R_{ijkl}\omega_k \wedge \omega_l.$$

The Gauss equation is given by

$$R_{ijkl} = \bar{R}_{ijkl} - (h_{il}h_{jk} - h_{ik}h_{jl}), \tag{2.3}$$
$$n(n-1)R = \sum_{i,j} \bar{R}_{ijji} - n^2H^2 + S, \tag{2.4}$$

where $S = \sum_{i,j}(h_{ij})^2$, $H = (1/n)\sum_i h_{ii}$ and R denote, respectively, the squared norm of the second fundamental form, the mean curvature and the normalized scalar curvature of M.

Let $\{h_{ijk}\}$ and $\{h_{ijkl}\}$ be the covariant derivatives of $\{h_{ij}\}$ and $\{h_{ijk}\}$, respectively. Then the Codazzi equation and Ricci identities are

$$h_{ijk} - h_{ikj} = \bar{R}_{n+1ijk}, \tag{2.5}$$
$$h_{ijkl} - h_{ijlk} = \sum_m h_{im}R_{mjkl} + \sum_m h_{mj}R_{mikl}. \tag{2.6}$$

Upon restricting $\bar{R}_{ABCD;E}$ to M, $\bar{R}_{n+1ijk;l}$ is given by

$$\bar{R}_{n+1ijk;l} = \bar{R}_{n+1ijkl} + h_{jl}\bar{R}_{n+1in+1k} + h_{kl}\bar{R}_{n+1ijn+1} + \sum_m h_{ml}\bar{R}_{mijk}, \tag{2.7}$$

where the $\bar{R}_{n+1ijkl}$ are defined by

$$\sum_l \bar{R}_{n+1ijkl}\omega_l = d\bar{R}_{n+1ijk} - \sum_l \bar{R}_{n+1ljk}\omega_{li} - \sum_l \bar{R}_{n+1ilk}\omega_{lj} - \sum_l \bar{R}_{n+1ijl}\omega_{lk}.$$

Let f be a smooth function on M. The first and second covariant derivatives f_i, f_{ij} and the Laplacian of f are defined by

$$df = \sum_i f_i\theta_i, \quad \sum_j f_{ij}\theta_j = df_i + \sum_j f_j\theta_{ji}, \quad \Delta f = \sum_i f_{ii}.$$

We introduce an operator $\square$ due to Cheng and Yau [6]:

$$\square f = \sum_{i,j}(nH\delta_{ij} - h_{ij})f_{ij}. \tag{2.8}$$

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Setting $f = nH$ in (2.8), from (2.2) and (2.4) we obtain

$$\begin{aligned}\Box(nH) &= \sum_{i,j}(nH\delta_{ij} - h_{ij})(nH)_{ij} \\ &= \sum_{i}(nH)(nH)_{ii} - \sum_{i,j} h_{ij}(nH)_{ij} \\ &= \frac{1}{2}\Delta(nH)^2 - \sum_{i}(nH_i)^2 - \sum_{i,j} h_{ij}(nH)_{ij} \\ &= -\frac{1}{2}n(n-1)\Delta R + \frac{1}{2}\Delta S - n^2|\nabla H|^2 - \sum_{i,j} h_{ij}(nH)_{ij}. \end{aligned} \tag{2.9}$$

The Laplacian Δh_{ij} of the second fundamental form h of M is defined by $\Delta h_{ij} = \sum_{k=1}^{n} h_{ijkk}$. From (2.5) and (2.6) it follows that

$$\begin{aligned}\Delta h_{ij} &= \sum_{k} h_{ikjk} + \sum_{k} \bar{R}_{n+1ijkk} \\ &= \sum_{k}\left\{h_{kikj} - \sum_{l}(h_{kl}R_{lijk} + h_{il}R_{lkjk}) + \sum_{k}\bar{R}_{n+1ijkk}\right\}.\end{aligned}$$

Using $h_{kikj} = h_{kkij} + \bar{R}_{n+1kikj}$, we find that

$$\Delta h_{ij} = \sum_{k} h_{kkij} + \sum_{k}\left(\bar{R}_{n+1ijkk} + \bar{R}_{n+1kikj}\right) - \sum_{k,l}(h_{kl}R_{lijk} + h_{il}R_{lkjk}); \tag{2.10}$$

and from (2.3), (2.7) and (2.10), we obtain

$$\begin{aligned}\Delta h_{ij} = &\sum_{k} h_{kkij} + \sum_{k}\left(\bar{R}_{n+1ijk;k} + \bar{R}_{n+1kik;j}\right) \\ &- \sum_{k}\left(h_{ij}\bar{R}_{n+1kn+1k} - h_{kk}\bar{R}_{n+1in+1j}\right) \\ &- \sum_{k,l}\left(2h_{kl}\bar{R}_{lijk} + h_{jl}\bar{R}_{lkik} + h_{il}\bar{R}_{lkjk}\right) - nH\sum_{l} h_{il}h_{lj} + Sh_{ij}.\end{aligned}$$

Since M_1^{n+1} is locally symmetric, from (2.2) we have $\bar{R}_{n+1ijk;k} = 0$ and $\bar{R}_{n+1kik;j} = 0$. Choose a local frame of orthonormal vector fields $\{e_i\}$ such that, at an arbitrary point of M,

$$h_{ij} = \lambda_i\delta_{ij}. \tag{2.11}$$

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Then

$$\begin{aligned}\frac{1}{2}\Delta S &= \sum_{i,j,k} h_{ijk}^2 + \sum_{i,j} h_{ij}\Delta h_{ij} \\ &= \sum_{i,j,k} h_{ijk}^2 + \sum_{i,j} h_{ij}(nH)_{ij} - \Big(S\sum_k \bar{R}_{n+1kn+1k} - \sum_{i,j} nHh_{ij}\bar{R}_{n+1jn+1i}\Big) \\ &\quad - \sum_{i,j,k,l} 2\big(h_{ij}h_{kl}\bar{R}_{lijk} + h_{ij}h_{li}\bar{R}_{lkjk}\big) - nH\sum_j \lambda_j^3 + S^2. \end{aligned} \tag{2.12}$$

By (2.9) and (2.12),

$$\begin{aligned}\Box(nH) &= \sum_{i,j,k} h_{ijk}^2 - n^2|\nabla H|^2 - \frac{1}{2}n(n-1)\Delta R \\ &\quad - \Big(S\sum_k \bar{R}_{n+1kn+1k} - \sum_{i,j} nHh_{ij}\bar{R}_{n+1jn+1i}\Big) \\ &\quad - \sum_{i,j,k,l} 2\big(h_{ij}h_{kl}\bar{R}_{lijk} + h_{ij}h_{li}\bar{R}_{lkjk}\big) - nH\sum_j \lambda_j^3 + S^2. \end{aligned} \tag{2.13}$$

The following result, due to Okumura [12] and Alencar and do Carmo [2], will be very important for our purposes.

LEMMA 2.1 ([2, 12]). *Let $\mu_1, \mu_2, \ldots, \mu_n$ be real numbers such that $\sum_i \mu_i = 0$ and $\sum_i \mu_i^2 = \beta^2$, where β is a nonnegative constant. Then*

$$-\frac{n-2}{\sqrt{n(n-1)}}\beta^3 \le \sum_i \mu_i^3 \le \frac{n-2}{\sqrt{n(n-1)}}\beta^3,$$

with equality if and only if $(n-1)$ of the numbers μ_i are equal to $\beta/\sqrt{n(n-1)}$ or $(n-1)$ of the numbers μ_i are equal to $-\beta/\sqrt{n(n-1)}$.

3. Proof of Theorem 1.1

Let $|\Phi|^2$ be a nonnegative C^2 function defined by

$$|\Phi|^2 = S - nH^2; \tag{3.1}$$

then M is totally umbilical if and only if $|\Phi|^2 = 0$.

By (2.11) and condition (*),

$$\begin{aligned} &-\Big(S\sum_k \bar{R}_{n+1kn+1k} - \sum_{i,j} nHh_{ij}\bar{R}_{n+1jn+1i}\Big) \\ &\quad = -\Big(S\sum_k \bar{R}_{n+1kn+1k} - \sum_k nH\lambda_k\bar{R}_{n+1kn+1k}\Big) \\ &\quad = -\sum_k (S - nH\lambda_k)\bar{R}_{n+1kn+1k} = \sum_k (S - nH\lambda_k)\frac{c_1}{n} = c_1|\Phi|^2 \end{aligned} \tag{3.2}$$

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and

$$-\sum_{i,j,k,l}\big(h_{ij}h_{kl}\bar{R}_{lijk}+h_{ij}h_{li}\bar{R}_{lkjk}\big)=-\sum_{j,k}\big(\lambda_j\lambda_k\bar{R}_{kjjk}-\lambda_k^2\bar{R}_{kjjk}\big)$$
$$=-\sum_{j,k}\big(\lambda_j\lambda_k-\lambda_k^2\big)\bar{R}_{kjjk}\geq nc_2\big(S-nH^2\big)$$
$$=nc_2|\Phi|^2. \tag{3.3}$$

Since $\sum_i(H-\lambda_i)=0$ and $\sum_i(H-\lambda_i)^2=S-nH^2=|\Phi|^2$, it follows from Lemma 2.1 that

$$\left|\sum(H-\lambda_i)^3\right|\leq\frac{n-2}{\sqrt{n(n-1)}}|\Phi|^3.$$

Hence

$$-nH\sum_i\lambda_i^3=-3nH^2S+2n^2H^4+nH\sum_i(H-\lambda_i)^3$$
$$\geq-3nH^2\big(|\Phi|^2+nH^2\big)+2n^2H^4-n|H|\frac{n-2}{\sqrt{n(n-1)}}|\Phi|^3$$
$$=-3nH^2|\Phi|^2-n^2H^4-n|H|\frac{n-2}{\sqrt{n(n-1)}}|\Phi|^3. \tag{3.4}$$

From (2.13) and (3.1)–(3.4), we obtain

$$\Box(nH)\geq\sum_{i,j,k}h_{ijk}^2-n^2|\nabla H|^2-\frac{1}{2}n(n-1)\Delta R$$
$$+|\Phi|^2\left\{nc-nH^2-\frac{(n-2)}{\sqrt{n(n-1)}}n|H||\Phi|+|\Phi|^2\right\}, \tag{3.5}$$

where $c=2c_2+c_1/n$.

In order to prove our theorems, we introduce an important operator

$$L=\Box+(\kappa/2n)\Delta.$$

We can now establish the following propositions.

PROPOSITION 3.1. *Let M be an n-dimensional space-like hypersurface with nonnegative mean curvature in a locally symmetric Lorentz space that satisfies condition (*). If $n(n-1)R=\kappa H(\kappa>0)$ and $c_2>0$, then $L=\Box+(\kappa/2n)\Delta$ is elliptic and $R>0$, $H>0$.*

PROOF. Since the mean curvature of M is nonnegative, we have the scalar curvature $n(n-1)R\geq 0$. Choose a local frame of orthonormal vector fields $\{e_i\}$ such that, at an arbitrary point of M, (2.11) holds. Then

$$n(n-1)R=\sum_{i,j}\bar{R}_{ijji}-n^2H^2+\sum_j\lambda_j^2, \tag{3.6}$$
$$\sum_j\lambda_j^2=\kappa H-\sum_{i,j}\bar{R}_{ijji}+n^2H^2.$$

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Therefore, $R > 0$. In fact, if there exists a point x such that $R = 0$, then $H = 0$ at this point; however, from (1.1) and (3.6),

$$0 = \sum_{i,j} \bar{R}_{ijji} + \sum_j \lambda_j^2 \geq n(n-1)c_2$$

at this point, which is impossible since we have assumed $c_2 > 0$. Thus, we obtain $R > 0$ and $H > 0$.

By (3.6) and (1.1), for any i,

$$\begin{aligned}
\left(nH - \lambda_i + \frac{\kappa}{2n}\right) &= \sum_j \lambda_j - \lambda_i + \frac{1}{2}\frac{\sum_j \lambda_j^2 - n^2H^2 + \sum_{i,j} \bar{R}_{ijji}}{nH} \\
&\geq \left[\left(\sum_j \lambda_j\right)^2 - \lambda_i \sum_j \lambda_j - \frac{1}{2}\sum_{l\neq j} \lambda_l\lambda_j + \frac{1}{2}n(n-1)c_2\right](nH)^{-1} \\
&= \left[\sum_j \lambda_j^2 + \frac{1}{2}\sum_{l\neq j} \lambda_l\lambda_j - \lambda_i \sum_j \lambda_j + \frac{1}{2}n(n-1)c_2\right](nH)^{-1} \\
&= \left[\sum_{j\neq i} \lambda_j^2 + \frac{1}{2}\sum_{\substack{l\neq j \\ l,j\neq i}} \lambda_l\lambda_j + \frac{1}{2}n(n-1)c_2\right](nH)^{-1} \\
&= \frac{1}{2}\left[\sum_{j\neq i} \lambda_j^2 + \left(\sum_{j\neq i} \lambda_j\right)^2 + n(n-1)c_2\right](nH)^{-1} > 0.
\end{aligned}$$

Thus L is an elliptic operator. This completes the proof of Proposition 3.1. □

PROPOSITION 3.2. *Let M be an n-dimensional space-like hypersurface in a locally symmetric Lorentz space that satisfies condition (*). If $n(n-1)R = \kappa H$ ($\kappa > 0$) and $c_2 > 0$, then $\sum_{i,j,k} h_{ijk}^2 \geq n^2|\nabla H|^2$.*

PROOF. We choose an orthonormal frame field as in the proof of Proposition 3.1; then we have $S = \sum_{i,j} h_{ij}^2 \neq 0$. In fact, if $S = \sum_i \lambda_i^2 = 0$ at a point x of M, then $\lambda_i = 0$ (for $i = 1, 2, \ldots, n$) at this point. Therefore $H = 0$ and $R = 0$ at this point. But, from (3.6) and (1.1), we have $0 = \sum_{i,j} \bar{R}_{ijji} \geq n(n-1)c_2$, which is impossible since we have assumed $c_2 > 0$.

Since $\bar{R}_{ABCD;E} = 0$, from (2.4) and $n(n-1)R = \kappa H$ it follows that

$$\kappa \nabla_i H = -2n^2 H \nabla_i H + 2\sum_{j,k} h_{kj}h_{kji},$$

$$\left(\frac{\kappa}{2} + n^2 H\right)^2 |\nabla H|^2 = \sum_i \left(\sum_{j,k} h_{kj}h_{kji}\right)^2 \leq \sum_{i,j} h_{ij}^2 \sum_{i,j,k} h_{ijk}^2 = S\sum_{i,j,k} h_{ijk}^2$$

and

$$\begin{aligned}
\sum_{i,j,k} h_{ijk}^2 - n^2|\nabla H|^2 &\geq \left[\left(\frac{\kappa}{2} + n^2 H\right)^2 - n^2 S\right]|\nabla H|^2 \frac{1}{S} \\
&= \left[\frac{\kappa^2}{4} + n^2\big(\kappa H + n^2 H^2 - S\big)\right]|\nabla H|^2 \frac{1}{S} \\
&= \left(\frac{\kappa^2}{4} + n^2 \sum_{i,j} \bar{R}_{ijji}\right)|\nabla H|^2 \frac{1}{S} \\
&\geq \left(\frac{\kappa^2}{4} + n^3(n-1)c_2\right)|\nabla H|^2 \frac{1}{S} \geq 0.
\end{aligned}$$

This completes the proof of Proposition 3.2. □

PROOF OF THEOREM 1.1. From (3.5) and Proposition 3.2,

$$\begin{aligned}
L(nH) &= \square(nH) + (\kappa/2n)\Delta(nH) \\
&= \square(nH) + \frac{1}{2}n(n-1)\Delta R \\
&\geq |\Phi|^2\left\{nc - nH^2 - \frac{(n-2)}{\sqrt{n(n-1)}}n|H||\Phi| + |\Phi|^2\right\} = |\Phi|^2 P_{H,n,c}(|\Phi|) \quad (3.7)
\end{aligned}$$

where

$$P_{H,n,c}(|\Phi|) = nc - nH^2 - \frac{(n-2)}{\sqrt{n(n-1)}}n|H||\Phi| + |\Phi|^2.$$

The discriminant of $P_{H,n,c}(|\Phi|)$ is $(n/(n-1))(n^2H^2 - 4(n-1)c)$.

(1) If $H^2 < 4(n-1)c/n^2$ on M, then $P_H(|g|) > 0$ on M and the right-hand side of (3.7) is nonnegative. Since the operator L is elliptic and H attains its maximum on M, from (3.7) we know that H is constant on M. From (3.7) again, we get that $|\Phi|^2 P_{H,n,c}(|\Phi|) = 0$, so $|\Phi|^2 = 0$ and M is totally umbilical.

(2) If $H^2 = 4(n-1)c/n^2$ on M, then $P_{H,n,c}(|\Phi|) = (|\Phi| - (n-2)\sqrt{c}/\sqrt{n})^2 \geq 0$ on M. Similarly to the proof of (1), from (3.7) we deduce that H is constant on M and $|\Phi|^2 P_{H,n,c}(|\Phi|) = 0$. Hence, either $|\Phi|^2 = 0$ and M is totally umbilical, or $P_{H,n,c}(|\Phi|) = 0$.

If $P_{H,n,c}(|\Phi|) = 0$, then $|\Phi| = (n-2)\sqrt{c}/\sqrt{n}$. By (3.7), equality holds in Lemma 2.1. Therefore we know that $(n-1)$ of the numbers $H - \lambda_i$ are either equal to

$$\frac{|\Phi|}{\sqrt{n(n-1)}} = \frac{n-2}{n\sqrt{n-1}}\sqrt{c}$$

or equal to the negative of the above expression. This implies that M has $(n-1)$ principal curvatures which are equal and constant. As H is constant, the remaining

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principal curvature is constant as well; so M is an isoparametric hypersurface with two distinct principal curvatures, one of which is simple.

(3) If $H^2 > 4(n-1)c/n^2$ on M, we consider two cases (a) and (b).

(a) If $H^2 < c$, then $P_{H,n,c}(|\Phi|)$ has two real roots $B^-_{H,n,c}$ and $B^+_{H,n,c}$, given by

$$B^{\pm}_{H,n,c} = \sqrt{\frac{n}{4(n-1)}}\Big[(n-2)H \pm \sqrt{n^2H^2 - 4(n-1)c}\,\Big].$$

Clearly, we have $B^+_{H,n,c} > 0$, $B^-_{H,n,c} > 0$ and $B^-_{H,n,c} < B^+_{H,n,c}$. Since we are supposing that $S \le nH^2 + (B^-_{H,n,c})^2$ or $S \ge nH^2 + (B^+_{H,n,c})^2$ on M, which means that $|\Phi| \le B^-_{H,n,c}$ or $|\Phi| \ge B^+_{H,n,c}$ on M, we know that $P_{H,n,c}(|\Phi|) \ge 0$ on M. Because L is elliptic and H attains its maximum on M, we know that H is constant on M from (3.7). Thus we obtain that $|\Phi|^2 P_{H,n,c}(|\Phi|) = 0$ so that $|\Phi|^2 = 0$ and M is totally umbilical, or that $P_{H,n,c}(|\Phi|) = 0$. If $P_{H,n,c}(|\Phi|) = 0$, then $|\Phi| = B^-_{H,n,c}$ or $|\Phi| = B^+_{H,n,c}$ on M. If $|\Phi| = B^-_{H,n,c}(> 0)$, then by (3.7) equality holds in Lemma 2.1. By making use of the same assertion as in the proof of (2) above, we infer that M is an isoparametric hypersurface with two distinct principal curvatures, one of which is simple. If $|\Phi| = B^+_{H,n,c}(> 0)$, we also have that M is an isoparametric hypersurface with two distinct principal curvatures, one of which is simple.

(b) If $H^2 \ge c$, then $B^+_{H,n,c} > 0$ and $B^-_{H,n,c} \le 0$. By making use of the same assertion as in the proof of case (a) above, we get that $|\Phi|^2 P_{H,n,c}(|\Phi|) = 0$ so that $|\Phi|^2 = 0$ and M is totally umbilical, or that $P_{H,n,c}(|\Phi|) = 0$. If $P_{H,n,c}(|\Phi|) = 0$, then $|\Phi| = B^-_{H,n,c}$ or $|\Phi| = B^+_{H,n,c}$ on M. If $|\Phi| = B^-_{H,n,c}(\le 0)$, then $|\Phi| = 0$ and M is totally umbilical. If $|\Phi| = B^+_{H,n,c}(> 0)$, then by (3.7) equality holds in Lemma 2.1. As in the proof of (2), we also have that M is an isoparametric hypersurface with two distinct principal curvatures, one of which is simple. This completes the proof of Theorem 1.1. □

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