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INEQUALITIES WITH WEIGHTS FOR DISCRETE HILBERT TRANSFORMS†

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1. **Introduction.** Let $Z(Z^+)$ denote the set of all (positive) integers and let T denote the *discrete Hilbert transform* defined for suitable sequences $a=\{a_k\}_{k\in Z}$ by

$$(Ta)_n=\frac{1}{\pi}\sum_{-\infty}^{\infty}{}' \frac{a_k}{k-n}, \quad (n\in Z)$$

where as usual the prime denotes omission of the term $k=n$. When $a=\{a_k\}$ is an odd sequence, $\{(Ta)_n\}$ is even and given by the *discrete odd Hilbert transform* T_0*:*

$$(T_0a)_n=\frac{2}{\pi}\sum_{k=1}^{\infty}{}' \frac{ka_k}{k^2-n^2}+\frac{a_n}{2\pi n}, \quad (n\in Z^+)$$

while for $a=\{a_k\}$ even with $a_0=0$, $\{(Ta)_n\}$ is odd and is given by the *discrete even Hilbert transform* T_e:

$$(T_ea)_n=\frac{2}{\pi}\sum_{k=1}^{\infty}{}' \frac{na_k}{k^2-n^2}-\frac{a_n}{2\pi n}, \quad (n\in Z^+).$$

Norm inequalities of the form

$$\text{(1.1)} \qquad \sum_{-\infty}^{\infty}|(Ta)_n|^p\, w_n\le C_p^p\sum_{-\infty}^{\infty}|a_k|^p\, w_k, \quad (1<p<\infty)$$

and weak type inequalities of the form

$$\text{(1.2)} \qquad \sum_{\{n\in Z:|(Ta)_n|>\alpha\}} w_n\le (C_p/\alpha)^p\sum_{-\infty}^{\infty}|a_k|^p\, w_k, \quad (1\le p<\infty, \alpha>0)$$

where $w=\{w_k\}$ is a fixed non-negative sequence and C_p is a constant independent of $a=\{a_k\}$ have been widely studied. In [4] Hunt, Muckenhoupt, and Wheeden have shown that for $1<p<\infty$ inequalities (1.1) and (1.2) are each equivalent to the A_p condition for $w=\{w_k\}$, namely:

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There exists a constant K such that

$$\left(\sum_{k=m}^{n} w_k\right)\left(\sum_{k=m}^{n} w_k^{-1/(p-1)}\right)^{p-1} \le K(n-m+1)^p \tag{1.3}$$

for all $m, m \in Z$, with $m \le n$, while for $p=1$, (1.2) is equivalent to A_1, where of course for $p=1$ the second factor on the left of (1.3) is understood to be $\sup\{w_k^{-1}: m \le k \le n\}$.

In this note we obtain a similar characterization of those sequences $w = \{w_k\}_{k\in Z^+}$ which satisfy the inequalities corresponding to (1.1) and (1.2) for T_0 and T_e. The conditions which we derive and denote by A_p^0 and A_p^e respectively are separately weaker than the A_p condition but a $w=\{w_k\}$ satisfying both A_p^0 and A_p^e must satisfy A_p (with m, n in (1.3) restricted to Z^+).

The inequalities for T quoted above were deduced in [4] from their non-discrete analogues (also contained in [4]) by the standard technique of (essentially) associating a suitable step function to the various sequences encountered and showing that the error thus introduced also satisfies the required inequalities. However, if one attempts to follow this approach directly in order to derive results for T_0 and T_e from their non-discrete analogues in [2] one encounters the same "error integral"

$$(Ef)(y) = \int_{|x-y|\ge\delta} \frac{|f(x)|}{(x-y)^2}\, dx \quad (\delta > 0)$$

that occurs in [4] for T, where it is only shown that the operation $f \to Ef$ satisfies inequalities of the form (1.1) and (1.2) for w satisfying A_p. The main objective then of §2 of this note is to prove a Lemma giving the required inequality under the (weaker) assumption that w satisfies A_p^0 and to indicate how the Lemma may be applied to derive the analogues of (1.1) and (1.2) for T_0 and T_e.

Several variants of the operator T appear in the literature (see for example [3] especially pp. 222–223). In the final section of the paper, we indicate some applications of our results to other discrete analogues of the Hilbert transform.

2. **Results for T_0 and T_e.**

In this section we shall derive the following theorems:

Theorem 1. *Let $w=\{w_k\}$ be a non-negative sequence on Z^+. For $1<p<\infty$,* (a), (b), *and* (c) *are equivalent, while for $p=1$,* (a) *and* (c) *are equivalent:*

(a) w satisfies the A_p^0 condition, i.e. There exists a constant K such that for all m, $n \in Z^+$ with $m \le n$ we have

$$\left(\sum_{k=m}^{n} w_k\right)\left(\sum_{k=m}^{n} (k^p w_k^{-1})^{1/(p-1)}\right)^{p-1} \le K\left(\frac{(m-n+1)(m+n)}{2}\right)^p.$$

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(b) *There exists a constant* B_p *such that*

$$\sum_1^\infty |(T_0a)_n|^p\, w_n \le B_p^p \sum_1^\infty |a_k|^p\, w_k.$$

(c) *There exists a constant* C_p *such that for all* $\alpha > 0$

$$\sum_{\{n\in Z^+ : |(T_0a)_n| > \alpha\}} w_n \le (C_p/\alpha)^p \sum_1^\infty |a_k|^p\, w_k.$$

THEOREM 2. *Let* $w = \{w_k\}$ *be a non-negative sequence on* Z^+. *For* $1 < p < \infty$, (a), (b), (c) *are equivalent, while for* $p = 1$, (c) *implies* (a):

(a) w *satisfies the* A_p^e *condition, i.e. There exists a constant* K *such that for all* $m, n \in Z^+$ *with* $m \le n$ *we have*

$$\left(\sum_{k=m}^{n} k^p w_k\right)\left(\sum_{k=m}^{n} w_k^{-1/(p-1)}\right)^{p-1} \le K\left(\frac{(m-n+1)(m+n)}{2}\right)^p.$$

(b) *There exists a constant* B_p *such that*

$$\sum_1^\infty |(T_ea)_n|^p\, w_n \le B_p^p \sum_1^\infty |a_k|^p\, w_k.$$

(c) *There exists a constant* C_p *such that for all* $\alpha > 0$

$$\sum_{\{n\in Z^+ : |(T_ea)_n| > \alpha\}} w_n \le (C_p/\alpha)^p \sum_1^\infty |a_k|\, w_k.$$

The second factor on the left of (a) in Theorem 1 is understood to be $\sup\{kw_k^{-1} : m \le k \le n\}$ when $p = 1$.

The first assertion of the next lemma may easily be verified by direct computation and the last statement is then a consequence of Lemma 1 of [2].

LEMMA 1. *Let* $w = \{w_k\}$ *be a non-negative sequence on* Z^+ *and define* $W(x)$ *on* $[\frac{3}{4}, \infty)$ *by* $W(x) = w_k$ *for* $|k - x| \le \frac{1}{4}$ *and linear in between. Then* w *satisfies the* A_p^0 *condition if and only if* $W(x)$ *satisfies*

$$\left(\int_a^b W(x)\right)\left(\int_a^b (x^p W(x)^{-1})^{1/(p-1)}\right)^{p-1} \le K\left(\frac{b^2 - a^2}{2}\right)^p \tag{2.1}$$

for some constant K *and all* $(a, b) \subset [\frac{3}{4}, \infty)$. *Moreover, if* w *satisfies the* $A_{p_0}^0$ *condition for some* p_0, $1 < p_0 < \infty$, *then* w *satisfies the* A_p^0 *condition for all* $p > p_0 - \varepsilon$, $\varepsilon > 0$, ε *sufficiently small.*

Now let $w = \{w_k\}$ and $W(x)$ be related as in Lemma 1 and for $a = \{a_k\}$ define

$f(x)=a_k$ when $|x-k|\leq\frac{1}{4}$ and zero otherwise. Then for $|n-y|\leq\frac{1}{4}$ we have

$$\frac{2}{\pi}\sum_{k=1}^{\infty}{}' \frac{ka_k}{k^2-n^2}=P.V.\frac{2}{\pi}\int_{3/4}^{\infty}\frac{2xf(x)}{x^2-y^2}\,dx-P.V.\frac{2}{\pi}\int_{n-1/4}^{n+1/4}\frac{2xf(x)}{x^2-y^2}\,dx$$
$$+\mathcal{O}\left(\int_{\substack{x\geq 3/4\\|x-y|\geq 1/2}}\frac{|f(x)|}{(x-y)^2}\,dx\right)$$

(where the notation $P.V.$ indicates that the integrals are understood to be Cauchy Principal values at $x=y$). For $|n-y|\leq\frac{1}{8}$ we have

$$\left|P.V.\frac{2}{\pi}\int_{n-1/4}^{n+1/4}\frac{2xf(x)}{x^2-y^2}\,dx\right|=|a_n|\,\mathcal{O}(1)=\mathcal{O}\left(\int_{1/8\leq|x-y|\leq 1/2}\frac{|f(x)|}{(x-y)^2}\,dx\right)$$

so that

$$|(T_0a)_n|\leq 2\,|(H_0f)(y)|+C((Ef)(y)),\qquad (|n-y|\leq\tfrac{1}{8})$$

where

$$(H_0f)(y)=P.V.\frac{2}{\pi}\int_0^{\infty}\frac{xf(x)}{x^2-y^2}\,dx$$

and

$$(Ef)(y)=\int_{\substack{|x-y|\geq 1/8\\x\geq 3/4}}\frac{|f(x)|}{(x-y)^2}\,dx.$$

Consequently,

$$\sum_{\{n\in Z^+:|(T_0a)_n|>\alpha\}}w_n\leq 4\int_{\{y\geq 7/8:|2(H_0f)(y)|+C|(Ef)(y)|>\alpha\}}W(y)\,dy$$

and therefore in order to show that (a) $\Rightarrow$ (c) in Theorem 1 it suffices to show that w satisfying A_p^0 implies

$$\int_{\{y:|(H_0f)(y)|>\alpha\}}W(y)\,dy\leq(C_p/\alpha)^p\int_0^{\infty}|f(x)|^p\,W(x)\,dx$$

and

$$\int_{\{y:|(Ef)(y)|>\alpha\}}W(y)\,dy\leq(C_p/\alpha)^p\int_0^{\infty}|f(x)|^p\,W(x)\,dx.$$

However, the first of these follows from Lemma 1 and Theorem 1 of [2] while the second, the crux of the proof, is contained in the next Lemma:

LEMMA 2. *Suppose $f(x)$ is non-negative, even and supported on $|x|\geq\delta$, $0<\delta<1$. For $y\in[2\delta,\infty)$ we have*

$$\int_{|x-y|\geq\delta}\frac{f(x)}{(x-y)^2}\,dx\leq C(Mf(\sqrt{|\cdot|}))(y^2)\tag{2.2}$$

where M denotes the Hardy-Littlewood Maximal function operator,

$$(Mg)(y)=\sup\left\{\frac{1}{b-a}\int_a^b |g| : y\in(a,b)\right\}$$

and C is a constant depending on δ but independent of f. Moreover, if W satisfies (2.1), *then there exists a constant C_p such that for all $\alpha>0$*

$$\int_{\{y:(Mf(\sqrt{|\cdot|}))(y^2)>\alpha\}} W(y)\,dy \le (C_p/\alpha)^p \int_0^\infty |f(x)|^p\, W(x)\,dx. \tag{2.3}$$

For the remainder of Theorem 1, the proof is entirely analogous to that of the non-discrete case as given in [2], moreover, Theorem 2 may be deduced from Theorem 1 just as Theorem 2 was deduced from Theorem 1 in [2]. It remains then to prove Lemma 2:

Proof of Lemma 2. Since f is even and supported on $|x|\ge\delta$ we have

$$\int_{|x-y|\ge\delta}\frac{f(x)}{(x-y)^2}\,dx=\int_{\substack{x\ge\delta\\|x-y|\ge\delta}}\frac{f(x)}{(x-y)^2}\,dx+\int_{x\ge\delta}\frac{f(x)}{(x+y)^2}\,dx$$

$$\le 2\int_{\substack{x\ge\delta\\|x-y|\ge\delta}}\frac{f(x)}{(x-y)^2}\,dx+\left(\frac{1}{2\delta}\right)^2\int_{y-\delta}^{y+\delta} f(x)\,dx$$

$$=\int_{\substack{|\sqrt{t}-y|\ge\delta\\t\ge\delta^2}}\frac{f(\sqrt{t})}{(\sqrt{t}-y)^2}\frac{dt}{\sqrt{t}}+\left(\frac{1}{2\delta}\right)^2\int_{(y-\delta)^2}^{(y+\delta)^2} f(\sqrt{t})\frac{dt}{2\sqrt{t}}$$

$$\le I_1+\left(\frac{y}{y-\delta}\right)\left(\frac{1}{2\delta}\right)\frac{1}{4\delta y}\int_{(y-\delta)^2}^{(y+\delta)^2} f(\sqrt{t})\,dt$$

$$\le I_1+\frac{1}{\delta}\,(Mf(\sqrt{|t|}))(y^2)$$

and it therefore suffices to prove the inequality for I_1. Now

$$I_1=\left(\int_{\delta^2}^{(y-\delta)^2}+\int_{(y+\delta)^2}^{\infty}\right)\frac{f(\sqrt{t})}{(\sqrt{t}-y)^2}\frac{dt}{\sqrt{t}}=I_2+I_3$$

and we consider I_2 and I_3 separately. Integrating by parts in I_2 we obtain

$$I_2=\frac{1}{(y-\delta)^2\delta}\int_{\delta^2}^{(y-\delta)^2} f(\sqrt{u})\,du+\int_{\delta^2}^{(y-\delta)^2}\frac{(3\sqrt{t}-y)\,dt}{2(y-\sqrt{t})^3t^{3/2}}\int_t^{(y-\delta)^2} f(\sqrt{u})\,du$$

$$\le\frac{y^2-\delta^2}{(y-\delta)^2\delta}\frac{1}{y^2-\delta^2}\int_{\delta^2}^{y^2} f(\sqrt{u})\,du+\int_{\delta^2}^{(y-\delta)^2}\frac{(3\sqrt{t}+y)\,dt}{2(y-\sqrt{t})^3t^{3/2}}\int_t^{y^2} f(\sqrt{u})\,du$$

$$\le 3(Mf(\sqrt{|u|}))(y^2)\left(\frac{1}{\delta}+\int_\delta^{y-\delta}\left[\frac{y+x}{x(y-x)}\right]^2 dx\right)$$

and this last integral is bounded by

$$4\int_{\delta}^{y/3}\frac{dx}{x^2}+16\int_{y/3}^{y-\delta}\frac{dx}{(y-x)^2}\leq 20\int_{\delta}^{\infty}\frac{dx}{x^2}=\frac{20}{\delta}.$$

Finally, the inequality for I_3 may be derived by assuming first that $f(x)=0$ for all large $|x|$ so that when I_3 is integrated by parts, the integrated term vanishes to yield

$$\begin{aligned}I_3&=\int_{(y+\delta)^2}^{\infty}\frac{(3\sqrt{t}-y)\,dt}{2(\sqrt{t}-y)^3t^{3/2}}\int_{(y+\delta)^2}^{t}f(\sqrt{u})\,du\\&\leq (Mf(\sqrt{|u|}))(y^2)\int_{(y+\delta)^2}^{\infty}\frac{(3\sqrt{t}-y)(\sqrt{t}+y)}{2(\sqrt{t}-y)^2t^{3/2}}\,dt\\&\leq (Mf(\sqrt{|u|}))(y^2)6\int_{(y+\delta)}^{\infty}\frac{dt}{(t-y)^2}\\&=\frac{6}{\delta}(Mf(\sqrt{|u|}))(y^2)\end{aligned}$$

from which the general case follows, and the proof of inequality (2.2) is complete. Now according to Lemma 1 of [2], if $W(y)$ satisfies (2.1) then $W(\sqrt{y})/(2\sqrt{y})$ satisfies the conditions of Theorem 1 of [5, p. 209] so that

$$\int_{\{y:(Mg)(y)>\alpha\}}W(\sqrt{y})\frac{dy}{2\sqrt{y}}\leq (C_p/\alpha)^p\int |g(y)|^p\,W(\sqrt{y})\frac{dy}{2\sqrt{y}}$$

which, via the change of variable $y=x^2$ implies (2.3).

3. **Applications to other discrete Hilbert transforms.** Among the variants of T noted by Hardy, Littlewood, and Polya [3, p. 222] is the operator

$$(T_\lambda a)_n=\frac{1}{\pi}\sum_{-\infty}^{\infty}{}'\frac{a_k}{k-n+\lambda}$$

where the $'$ is required only for λ integral. When λ is integral we have

$$(T_\lambda a)_n=(Ta)_{n-\lambda}$$

and if $w=\{w_k\}$ satisfies (1.3) it follows that

$$(w_k+w_{k+1})\left(\frac{1}{w_k^{1/(p-1)}}+\frac{1}{w_{k+1}^{1/(p-1)}}\right)^{p-1}\leq 2^pK,$$

and in particular

$$(1/2^pK)w_k\leq w_{k+1}\leq 2^pKw_k$$

for all k, and hence also

$$(1/2^pK)^{|\lambda|}w_k\leq w_{k+\lambda}\leq (2^pK)^{|\lambda|}w_k. \tag{3.1}$$

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But then it follows easily that T_λ (λ integral) satisfies (1.1) or (1.2) (with modified constants) whenever T does, and conversely. Now for λ non-integral, say $0<\lambda<1$, we have

$$|((T_\lambda - T)a)_n| \le C_\lambda \sum_{-\infty}^{\infty} \frac{|a_k|}{(k-n)^2} + \frac{|a_n|}{\pi\lambda}$$

so that in this case also (via Lemma 2) T and T_λ behave equivalently with regard to inequalities (1.1) and (1.2). Now it follows similarly that a sequence $\{w_k\}$ satisfying the A_p^0 or A_p^e conditions also satisfies an inequality of the form (3.1) and hence if $T_{\lambda,0}$ and $T_{\lambda,e}$ denote the restriction of T_λ to odd and even sequences respectively, it is easy to check that $T_{\lambda,0}$ and $T_{\lambda,e}$ may replace T_0 and T_e respectively in Theorems 1 and 2 of §2.

A more interesting discrete analogue of the Hilbert transform is the operator J given by

$$(Ja)_n = \frac{2}{\pi} \sum_{-\infty}^{\infty} \frac{\Delta(k-n)}{k-n} a_k$$

where $\Delta(k) = [1+(-1)^{k-1}]/2$. This transform shares more of the formal properties of the (non-discrete) Hilbert transform than does T, in particular, $J(Ja) = -a$ and if $a = \{a_k\}$ are the Fourier coefficients of $(Fa)(x)$, then $Ja = \{(Ja)_n\}$ are the Fourier coefficients of $-i \operatorname{sgn} x(Fa)(x)$. Moreover, J plays the same role in the boundary value theory of discrete analytic functions as that of the usual Hilbert transform in the classical theory of functions analytic in the upper half plane. From a different point of view this transform has also been studied in [1].

We claim that J may replace T in (1.1) and (1.2) and that J_0 and J_e:

$$\left.\begin{aligned}(J_0 a)_n &= \frac{2}{\pi} \sum_{1}^{\infty} \frac{\Delta(k-n)}{k^2-n^2} k a_k \\ (J_e a)_n &= \frac{2}{\pi} \sum_{1}^{\infty} \frac{\Delta(k-n)}{k^2-n^2} n a_k\end{aligned}\right\} \quad (n \in Z^+)$$

may replace T_0 and T_e respectively in Theorems 1 and 2. We give the details only for the implication: "$\{w_k\}$ satisfies $A_p \Rightarrow$ (1.2) holds with T replaced by J", the remaining parts being left to the reader.

We have

$$2(Ta)_n = (Ja)_n + (Ja)_{n+1} + (Sa)_n$$

where

$$(Sa)_n = \frac{2}{\pi}\left(a_n + \sum_{-\infty}^{\infty}{}' \frac{\Delta(n-k+1)a_k}{(k-n)(n-k+1)}\right).$$

Now put $a_k = a_k^0 + a_k^e$ where

$$a_k^0 = \begin{cases} a_k & \text{if } k \text{ is odd} \\ 0 & \text{otherwise.}\end{cases}$$

Then

$$\begin{aligned}\{n:|(Ja)_n|>\alpha\}&=\{n \quad \text{even}:|(Ja^0)_n|>\alpha\}\cup\{n\ \text{odd}:|(Ja^e)_n|>\alpha\}\\ &\subseteq\{n:|2(Ta^0)_n-(Sa^0)_n|>\alpha\}\cup\{n:|2(Ta^e)_n-(Sa^e)_n|>\alpha\}\end{aligned}$$

so that

$$\sum_{\{n:|(Ja)_n|>\alpha\}} w_n \leq \left(\frac{C}{\alpha}\right)^p\left\{\sum |a_k^0|^p\, w_k+\sum |a_k^e|^p\, w_k\right\}=\left(\frac{C}{\alpha}\right)^p\sum |a_k|^p\, w_k$$

since both T and S (via Lemma 2) satisfy the required weak type inequality whenever $\{w_k\}$ satisfies A_p.

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