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# A CONSTRUCTIVE DEFINITION OF THE APPROXIMATELY CONTINUOUS DENJOY INTEGRAL

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1. **Introduction.** The author has defined [2] the approximately continuous Denjoy integral (AD-integral) which includes exactly the general Denjoy integral and the AP-integral defined by Burkill [2].

The aim of this paper is to give a constructive definition of the AD-integral.

2. **The AD-integral.** A real valued function $F(x)$ is said to be (ACG) on the interval $[a, b]$ if $[a, b]$ is the sum of a countable number of closed sets on each of which $F(x)$ is absolutely continuous. An extended real valued function $f(x)$ is said to be AD-integrable on $[a, b]$ if there exists a function $F(x)$ which is approximately continuous, (ACG) on $[a, b]$ and

$$\text{AD } F(x) = f(x) \quad \text{a.e.},$$

where by AD we mean the approximate derivative. We call the function $F(x)$ an indefinite integral of $f(x)$, and the definite integral of $f(x)$ on $[a, b]$, denoted by (AD) $\int_a^b f(t)\,dt$, is defined as $F(b)-F(a)$ (see [2]).

We have established the following properties of the AD-integral in [3]. If $I=[\alpha, \beta]$, $I^0$ is the open interval $(\alpha, \beta)$.

THEOREM 1. *If $f(x)$ is AD-integrable on every interval $[a, \beta]$, where $a<\beta<b$, and*

$$\operatorname{app}\lim_{\beta\to b} (\text{AD}) \int_a^\beta f(t)\,dt = l,$$

*then $f(x)$ is AD-integrable on $[a, b]$ and*

$$(\text{AD}) \int_a^b f(t)\,dt = l.$$

THEOREM 2. *Let $E$ be a closed set in $I_0=[a, b]$ and $\{I_k=[a_k, b_k]\}$ the sequence of contiguous closed intervals of $E$ with respect to $I_0$ and let $f(x)$ be a function which is Lebesgue integrable on $E$ and AD-integrable on each $I_k$. Suppose that the following conditions are satisfied:*

(i) $\sum_{k=1}^\infty |(\text{AD}) \int_{I_k} f(t)\,dt| < \infty$;
(ii) *if $x \in E$ is a limit point of $\{I_k\}$, then there exists a set $E_x$ which has unit density*

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*at x and contains all the end points of* $\{I_k\}$ *in a sufficiently small neighbourhood of x, such that*

$$\lim_{k\to\infty} O(\text{AD}, f, E_x \cap I_k) = 0,$$

*where* $O(\text{AD}, f, E_x \cap I_k)$ *means the oscillation of the indefinite* AD-*integral of f on* $E_x \cap I_k$.

*Then* $f(x)$ *is* AD-*integrable on* $I_0$ *and we have*

$$(\text{AD})\int_{I_0} f(t)\,dt = (L)\int_E f(t)\,dt + \sum_{k=1}^{\infty} (\text{AD})\int_{I_k} f(t)\,dt.$$

THEOREM 3. *If* $f(x)$ *is* AD-*integrable on* $I_0 = [a, b]$, *then for any closed set* $E \subset I_0$, *there exists a portion* $J^0 \cap E$ *which satisfies the following three conditions:*

(i) $f(x)$ *is L-integrable on* $J \cap E$;

(ii) *Let* $\{I_k\}$ *be the sequence of contiguous closed intervals of* $J \cap E$ *with respect to J. Then*

$$\sum_{k=1}^{\infty} \left| (\text{AD})\int_{I_k} f(t)\,dt \right| < \infty;$$

(iii) *If x is a limit point of* $\{I_k\}$, *then there exists a set* $E_x$ *which has unit density at x and contains all the end points of* $I_k$ *in a sufficiently small neighbourhood of x, such that*

$$\lim_{k\to\infty} O(\text{AD}, f, E_x \cap I_k) = 0.$$

3. **A constructive definition of the AD-integral.** Let $T$ be a functional operation by which there corresponds to each closed subinterval $I = [\alpha, \beta]$ of the fixed interval $I_0 = [a, b]$ a class of functions defined on $I$, $K(T, I)$ and to each functions $f \in K(T, I)$ a finite real number. This class of functions will be called domain of the operation $T$ on $I$, and the number associated with $f$ will be denoted by $T(f, I)$ or $T_\alpha^\beta(f)$.

Throughout this section we mean $I$ and $J$ to be closed intervals.

An operator $T$ is termed an approximately continuous integral if the following two conditions are fulfilled;

(i) If $f \in K(T, I)$ then $f \in K(T, J)$ for all $J \subset I$. The function $F(x) = T_\alpha^x(f)$ $(\alpha \le x \le \beta)$ is approximately continuous on $I = [\alpha, \beta]$.

(ii) If $I_1$ and $I_2$ are abutting intervals and if $f \in K(T, I_1) \cap K(T, I_2)$ then $f \in K(T, I_1 \cup I_2)$ and

$$T(f, I_1 \cup I_2) = T(f, I_1) + T(f, I_2).$$

If $T$ is an approximately continuous integral, any function belonging to $K(T, I)$ is termed $T$-integrable on $I$ and the number $T(f, I)$ is called definite $T$-integral of $f$ on $I$.

Given two integrals $T_1$ and $T_2$, we shall say that the integral $T_1$ includes the integral $T_2$, written $T_2 \subset T_1$, if $f \in K(T_2, I_0)$ implies $f \in K(T_1, I_0)$ and $T_1(f, I_0) = T_2(f, I_0)$.

Let $T$ be an integral and $f$ a function defined on $I_0$. Then we shall say that a point $x_0 \in I_0$ is a $T$-singular point of $f$ if there exist arbitrarily small intervals containing $x_0$ in its interior on each of which $f$ is not $T$-integrable. Denoting by $S$ the set of these points, we see that the set $S$ is closed and that $f$ is $T$-integrable on every interval $I$ which contains no points of $S$.

With each integral $T$ we now associate two generalized integrals $T^C$ and $T^H$. Given any interval $I$, the domain of $T^C$, $K(T^C, I)$ is the class of all the functions $f$ which fulfil the following conditions:

($c_1$) the set $S$ of $T$-singular points of $f$ is at most finite;

($c_2$) there exists an approximately continuous function $F(x)$ on $I$ such that if $[\alpha, \beta]$ is any interval containing no points of $S$ then

$$T_\alpha^\beta(f) = F(\beta)-F(\alpha).$$

Let $G(x)$ be any other function satisfying the condition ($c_2$) and let $a_1 < a_2 < \cdots < a_n$ be the $T$-singular points of $f$. Then, for $a_i < \alpha_i < \beta_i < a_{i+1}$, we have

$$T_{\alpha_i}^{\beta_i}(f) = F(\beta_i)-F(\alpha_i) = G(\beta_i)-G(\alpha_i).$$

Hence, by the approximate continuity of $F$ and $G$,

$$F(a_{i+1})-F(a_i) = G(a_{i+1})-G(a_i)$$

which implies

$$G(b)-G(a) = F(b)-F(a).$$

Therefore we may define

$$T^C(f, I) = F(b)-F(a) \quad \text{for } I = [a, b].$$

We see easily that the operation $T^C$ is an approximately continuous integral.

THEOREM 4. *If the* AD-*integral includes the approximately continuous integral $T$ then the* AD-*integral also includes the $T^C$-integral.*

**Proof.** Let $f \in K(T^C, I)$ and let $a_1 < a_2 < \cdots < a_n$ be a finite sequence of $T$-singular points of $f$. Then we have

$$T_\alpha^\beta(f) = F(\beta)-F(\alpha)$$

for $a_i < \alpha < \beta < a_{i+1}$. Since $T \subset \text{AD}$, we obtain

$$(\text{AD}) \int_\alpha^\beta f(t)\, dt = F(\beta)-F(\alpha).$$

Hence

$$\operatorname*{app\,lim}_{\substack{\alpha \to a_1 + \\ \beta \to a_{1+1} -}} (\text{AD}) \int_\alpha^\beta f(t)\, dt = F(a_{i+1})-F(a_i).$$

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By Theorem 1, $f(x)$ is AD-integrable on $[a_i, a_{i+1}]$. Therefore $f \in K(\text{AD}, I)$ and

$$(\text{AD}) \int_I f(t)\, dt = T^C(f, I).$$

Next we define the operation $T^H$. The domain of the operation $T^H$ on $I$ is defined as the class of functions $f$ which fulfil the following conditions:

($h_1$) if $S$ denotes the set of all $T$-singular points of $f$ on $I$, $f$ is $L$-integrable on $S$ and is $T$-integrable on each of contiguous closed intervals $I_k$ of $S$ with respect to $I$;

($h_2$) $\sum_{k=1}^{\infty} |T(f, I_k)| < \infty$;

($h_3$) if $x$ is a limit point of $\{I_k\}$, there exists a set $E_x$ which has unit density at $x$ and contains all the end points of $I_k$ in a sufficiently small neighbourhood of $x$, such that

$$\lim_{k\to\infty} O(T, f, E_x \cap I_k) = 0.$$

For any such functions, we write by definition

$$T^H(f, I) = (L) \int_S f(t)\, dt + \sum_{k=1}^{\infty} T(f, I_k).$$

We see that the operation $T^H(f, I)$ is an approximately continuous integral and that $T^H(f, I)$ includes $T$-integral.

THEOREM 5. *If the* AD-*integral includes an approximately continuous integral $T$ then the* AD-*integral also includes the $T^H$-integral.*

**Proof.** Let $f \in K(T^H, I)$ and $S$ be the set of all $T$-singular points of $f$ on $I$. Also let $\{I_k\}$ be the contiguous closed intervals of $S$ with respect to $I$. Then we have

$$T(f, I_k) = (\text{AD}) \int_{I_k} f(t)\, dt$$

since $T \subset \text{AD}$. By ($h_2$), we obtain

$$\sum_{k=1}^{\infty} \left| (\text{AD}) \int_{I_k} f(t)\, dt \right| < \infty,$$

and

$$\lim_{k\to\infty} O(T, f, E_x \cap I_k) = \lim_{k\to\infty} O(\text{AD}, f, E_x \cap I_k).$$

It follows from Theorem 2 that $f(x)$ is AD-integrable on $I$ and

$$\begin{aligned} T^H(f, I) &= (L) \int_S f(t)\, dt + \sum_{k=1}^{\infty} T(f, I_k) \\ &= (L) \int_S f(t)\, dt + \sum_{k=1}^{\infty} (\text{AD}) \int_{I_k} f(t)\, dt \\ &= (\text{AD}) \int_I f(t)\, dt. \end{aligned}$$

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Let $\{T_\xi\}$ be a sequence of approximately continuous integral defined on $I_0$, in general transfinite, such that $T_\xi \subset T_\eta$ whenever $\xi < \eta$. We denote by $\sum_{\xi<\alpha} T_\xi$ the operation $T$ whose domain on $I_0$ is the sum of the domain of the operation $T_\xi$ for $\xi < \alpha$, and which is defined for every function $f$ of its domain by the relation $T(f, I) = T_{\xi_0}(f, I)$, where $\xi_0$ is the least of indices $\xi < \alpha$ such that $f$ is $T_\xi$-integrable on $I_0$. The operation $T = \sum_{\xi<\alpha} T_\xi$ is an approximately continuous integral.

THEOREM 6. *Let $\{L_\xi\}$ be a transfinite sequence defined by an induction as follows;*

$$L_0 = L, \qquad L_\alpha = \Big(\sum_{\xi<\alpha} L_\xi\Big)^{CH}.$$

*Then*

$$\mathrm{AD} = \sum_{\xi<\Omega} L_\xi,$$

*where $\Omega$ is the smallest ordinal number of the third class.*

**Proof.** Since $L \subset \mathrm{AD}$, we have $L_\xi \subset \mathrm{AD}$ for every $\xi < \Omega$ by Theorems 4 and 5. Hence $\sum_{\xi<\Omega} L_\xi \subset \mathrm{AD}$. In order to prove the theorem, it is sufficient to show that every function $f$ which is AD-integrable on $I_0$, is $L_\xi$-integrable on $I_0$ for some $\xi < \Omega$.

Let $S_\xi$ be the set of $L_\xi$-singular points of $f$ on $I_0$. Since $\{S_\xi\}$ is the nonincreasing transfinite sequence, there exists an index $\nu < \Omega$ such that $S_\nu = S_{\nu+1} = \cdots$. We must show that $S_\nu = \varnothing$.

Suppose that $S_\nu \neq \varnothing$. The function $f$ being AD-integrable on $I_0$, by Theorem 3, there exists an interval $J$ with $J^0 \cap S_\nu \neq \varnothing$ such that

(i) $f$ is $L$-integrable on $S_\nu \cap J$;

(ii) if $\{I_k\}$ be the contiguous closed intervals of $S_\nu \cap J$ with respect to $J$, then

$$\sum_{k=1}^{\infty} \left| (\mathrm{AD}) \int_{I_k} f(t)\, dt \right| < \infty;$$

(iii) if $x$ is a limit point of $\{I_k\}$, there exists a set $E_x$ which has unit density at $x$ and contains all the end points of $I_k$ in a sufficiently small neighbourhood of $x$, such that

$$\lim_{k\to\infty} O(\mathrm{AD}, f, E_x \cap I_k) = 0.$$

Since $f$ is $L_\nu$-integrable on $I \subset I_k^0$, it follows from the relation $L_\nu \subset \mathrm{AD}$ that, for the interval $I = [\alpha, \beta] \subset I_k^0 = (a_k, b_k)$,

$$(L_\nu) \int_\alpha^\beta f(t)\, dt = (\mathrm{AD}) \int_\alpha^\beta f(t)\, dt.$$

By Theorem 1, we get

$$\operatorname*{app\,lim}_{\substack{\alpha\to a_k+\\ \beta\to b_k-}} (L_\nu) \int_\alpha^\beta f(t)\, dt = (\mathrm{AD}) \int_{a_k}^{b_k} f(t)\, dt.$$

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Hence $f$ is $(L_\nu)^C$-integrable on each $I_k$, and also $L_{\nu+1}$-integrable. The above three properties are also true if $S_\nu$ is replaced by $S_{\nu+1}$ and AD by $L_{\nu+1}$. Hence $f$ is, by the definition, $(L_{\nu+1})^H$-integrable and therefore $L_{\nu+2}$-integrable on $J$. But

$$S_{\nu+2} \cap J^0 = S_{\nu+1} \cap J^0 = S_\nu \cap J^0 \neq \varnothing,$$

which is a contradiction. We have thus $S_\nu = \varnothing$ and the theorem is proved.

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