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PLANE CURVES WITH NODES

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A smooth algebraic curve is birationally equivalent to a nodal plane curve. One of the main problems in the theory of plane curves is to describe the situation of nodes of an irreducible nodal plane curve (see [**16**, Art. 45], [**10**], [**7**, Book IV, Chapter I, §5], [**12**, p. 584], and [**3**]).

Let n denote the degree of a nodal curve and d the number of nodes. The case $(n, d) = (6, 9)$ has been analyzed by Halphen [**10**]. It follows from Lemma 3.5 and Proposition 3.6 that this is an exceptional case. The case $d \leqq n(n+3)/6$, $d \leqq (n-1)(n-2)/2$, and $(n, d) \neq (6, 9)$ was investigated by Arbarello and Cornalba [**3**]. We present a simpler proof (Corollary 3.8).

We consider the main case which is particularly important due to its applications to the moduli variety of curves, compare [**19**, Chapter VIII, Section 4]. Let $V_{n,d}$ be the variety of irreducible curves of degree n with d nodes and no other singularities such that each curve of $V_{n,d}$ can be degenerated into n lines in general position (see [**17**]). For $n(n+3)/6 \leqq d \leqq (n-1)(n-2)/2$ and $(n, d) \neq (6, 9)$, we prove that the map

$$V_{n,d} \longrightarrow \mathrm{Sym}^d(\mathbf{P}^2),$$

sending a curve to the set of its nodes, is a birational morphism onto its image (Theorem 3.9 (i)) and give a rough description of the image (Propositions 3.1 and 2.1 and Corollary 3.12) and of the generic curve of $V_{n,d}$

Recently, J. Harris proved that any plane nodal curve can be degenerated to a union of lines in general position. We do not use his result. Our results were announced at the A.M.S. Summer Institute on Algebraic Geometry (Bowdoin College, 1985) and the 75th Ontario Mathematics Meeting (Hamilton, February, 1986).

We now outline the main ideas of the proof. We assume

$$n(n+3)/6 \leqq d \leqq (n-1)(n-2)/2 \quad \text{and} \quad (n, d) \neq (6, 9).$$

A plane curve has d distinct singular points provided the coefficients of the corresponding polynomial satisfy $3d$ equations (see 1.1). We then obtain a curve which may however by reducible or nonreduced. It follows from Proposition 2.1 that if a reduced curve B with d assigned nodes is not a specialization of an irreducible curve with d assigned nodes, then the set of the assigned nodes of B does not belong to the dense stratum of Hilb^d. (We stratify the Hilbert scheme by the Hilbert function.) On the other hand, the set of nodes of a general

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curve of $V_{n,d}$ belongs to the dense stratum (Proposition 3.1). The nonreduced curves are treated in Proposition 3.6 which is the main technical result of the paper. In particular, we obtain a family of irreducible nodal curves which maps birationally onto its image in $\mathrm{Sym}^d(\mathbf{P}^2)$ under the map sending a curve to the set of its nodes. To complete the proof of Theorem 3.9, we compare $V_{n,d}$ with the family described above. Using the deformation theory of plane curves, we show that the two families coincide.

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1. Zero-dimensional subschemes of $\mathbf{P}^2$.

1.1. Let $f(x, y, z) = \sum a_{ijk} x^i y^j z^k$ be the homogeneous polynomial of degree n in three variables with general coefficients a_{ijk}. Let (a_{ijk}) denote the coordinates in $\mathbf{P}^N$ ($N = n(n+3)/2$) and $(x_1 : y_1 : z_1; \dots ; x_d : y_d : z_d)$ the coordinates in $(\mathbf{P}^2)^d$. We consider the following system S^d of $3d$ equations in a's:

$$f'_x(x_1, y_1, z_1) = 0, \quad f'_y(x_1, y_1, z_1) = 0, \quad \dots, f'_z(x_d, y_d, z_d) = 0.$$

Let $\mathbf{M}^d$ denote the corresponding $[3d \times (N+1)]$-matrix with entries in $\mathbf{C}[x_1, \dots, z_d]$.

Let $Q \in (\mathbf{P}^2)^d$. The system $S^d(Q)$ has a nontrivial solution in a's if and only if $\operatorname{rk} \mathbf{M}^d(Q) \leqq N$. Let $M^d_N \subset (\mathbf{P}^2)^d$ be the closed subscheme defined by the condition $\operatorname{rk} \mathbf{M}^d \leqq N$. We have (see, e.g., [**4**, Chapter II]):

1.2 Lemma. *Each irreducible component of M^d_N has dimension at least* $\min\{N-d, 2d\}$, *i.e.,*

$$\operatorname{codim}(M^d_N, (\mathbf{P}^2)^d) \leqq \max\{3d - N, 0\}.$$

1.3 *Notation.* We put

$$\nu(n, d) = 3n + (n-1)(n-2)/2 - d - 1 = N - d.$$

It is known that $\nu(n, d)$ is equal to the dimension of the variety $\Sigma_{n,d} \subset \mathbf{P}^N$ of all curves of degree n with d nodes and no other singularities [**17**].

Let τ be the smallest integer $\geqq n(n+3)/6$. If $d \geqq \tau$, then to τ triplets

$$(x_{i_1}, y_{i_1}, z_{i_1}), \dots, (x_{i_\tau}, y_{i_\tau}, z_{i_\tau}),$$

we can associate the $[3\tau \times (N+1)]$-submatrix of $\mathbf{M}^d$ consisting of the corresponding 3τ rows of $\mathbf{M}^d$.

1.4. Let Hilb^e denote the Hilbert scheme of zero-dimensional subschemes of degree e in $\mathbf{P}^2$. It is a smooth connected variety [8]. Let $\tilde{M}^e(k) \subset \mathrm{Hilb}^e$ be the subset consisting of zero-dimensional schemes lying on curves of degree k in $\mathbf{P}^2$. Let $M^e(k) \subset \tilde{M}^e(k)$ be the union of those irreducible components of $\tilde{M}^e(k)$ whose general members consist of e distinct points. If $V \in \tilde{M}^e(k)$ lies on a reduced curve of degree k, then $V \in M^e(k)$ [6].

We can stratify Hilb^e, $\tilde{M}^e(k)$, and $M^e(k)$ into a finite number of sets such that V and W belong to the same stratum if and only if

$$h^0\left(\mathbf{P}^2, \mathcal{J}_V(l)\right) = h^0\left(\mathbf{P}^2, \mathcal{J}_W(l)\right) \quad \text{for all } l.$$

Let $D^e \subset \mathrm{Hilb}^e$ $\left(D^e(k) \subset M^e(k)\right)$ denote the stratum corresponding to the Hilbert function $h^0\left(\mathbf{P}^2, \mathcal{J}_V(l)\right)$ shown in the table of Figure 1 (Figure 2).

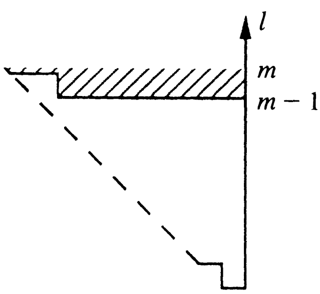

Figure 1

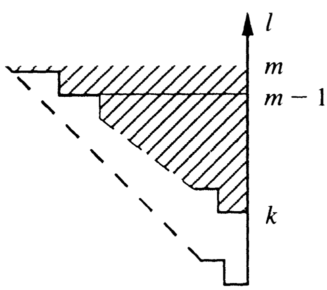

Figure 2

The tables describe Hilbert functions as follows. Let $\mathbf{P}^1 \subset \mathbf{P}^2$ be a general line. The exact standard sequence

$$0 \longrightarrow \mathcal{J}_V(l-1) \longrightarrow \mathcal{J}_V(l) \longrightarrow \mathcal{O}_{\mathbf{P}^1}(l) \longrightarrow 0$$

yields

$$(1) \qquad 0 \longrightarrow H^0\left(\mathbf{P}^2, \mathcal{J}_V(l-1)\right) \longrightarrow H^0\left(\mathbf{P}^2, \mathcal{J}_V(l)\right) \xrightarrow{\psi(l)} H^0\left(\mathbf{P}^1, \mathcal{O}(l)\right).$$

The number of shaded squares in the l-th row is equal to $\dim \operatorname{Im} \psi(l)$. Figure 1 (Figure 2) corresponds to the dense stratum of Hilb^e $\left(M(k)\right)$ and m denotes the row number of the first entirely shaded row; see Lemma 1.5 below. We have two canonical maps

$$\mathrm{Hilb}^e \xrightarrow{\phi} \mathrm{Sym}^e(\mathbf{P}^2) \xleftarrow{\sigma_e} (\mathbf{P}^2)^e.$$

We denote by U^e both the subscheme of Hilb^e whose elements correspond to reduced schemes and its image under ϕ.

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1.5 LEMMA. *Let $V \subset \mathbf{P}^2$ be a zero-dimensional subscheme. Then*

i) $\deg(V) = \sum_{l \geqq 0} \gamma(l)$, *where $\gamma(l)$ is the area of the unshaded part of the l-th row of the table for V;*

ii) *V is m-regular if and only if* $\dim \operatorname{Im} \psi(m) = h^0\left(\mathbf{P}^1, O(m)\right)$;

iii) $\overline{D^e} = \mathrm{Hilb}^e$ *and* $\overline{D^e(k)} = \overline{M^e(k)}$;

iv) *if*

$$V_e = y_1 + \cdots + y_e \in D^e(k) \cap U^e (D^e \cap U^e),$$

then there exists a subset $\{i_1, \ldots, i_{e-1}\} \subset \{1, \ldots, e\}$ with

$$y_{i_1} + \cdots + y_{i_{e-1}} \in D^{e-1}(k)(D^{e-1});$$

v) *$D^e \cap U^e$ is open in U^e and $D^e(k) \cap U^e$ is open in $M^e(k) \cap U^e$.*

Proof. i) Set

$$\beta(l) = h^0\left(\mathbf{P}^2, O(l)\right) - h^0\left(\mathbf{P}^2, \mathcal{J}_V(l)\right).$$

We have

$$\begin{aligned} \dim \operatorname{Im} \psi(l) &= h^0\left(\mathbf{P}^2, \mathcal{J}_V(l)\right) - h^0\left(\mathbf{P}^2, \mathcal{J}_V(l-1)\right) \\ &= h^0\left(\mathbf{P}^1, O(l)\right) - \beta(l) + \beta(l-1) \quad \text{for } l \geqq 0. \end{aligned}$$

Hence $\beta(l) - \beta(l-1) = \gamma(l)$. Since $\gamma(l) = 0$ for $l \gg 0$, we have

$$\sum_{i>0} \gamma(i) = \beta(l) = \deg(V) \quad \text{for } l \gg 0.$$

ii) The proof is obvious (cf. [**13**, §14]).

iii) We will prove that $\overline{D^e(k)} = \overline{M^e(k)}$. We assume $e > 4$ and proceed by induction on e. Let

$$V_e = z_1 + \cdots + z_e \in M^e(k) \cap U^e.$$

We set $V_{e-1} = z_1 + \cdots + z_{e-1}$. By hypothesis we can approximate V_{e-1} by a scheme

$$W_{e-1} = y_1 + \cdots + y_{e-1} \in D^{e-1}(k) \cap U^e$$

lying on a curve $C_k \subset \mathbf{P}^2$ of degree k. Let $\mathcal{L}$ denote a maximal linear system of curves of the smallest degree in $\mathbf{P}^2$ passing through W_{e-1} and not containing the curve C_k. We take any point $y_e \in C_k$ close to z_e and which is not a base point of $\mathcal{L}$. Then

$$y_1 + \cdots + y_{e-1} + y_e \in D^e(k).$$

iv) We will prove the assertion for

$$V_e = y_1 + \cdots + y_e \in D^e(k) \cap U^e.$$

Every $V_{e-1} \subset V_e$ with $\deg V_{e-1} = e-1$ is m-regular. Indeed, for arbitrary $e-2$ points of V_{e-1}, one can find a curve of degree $m-1$ passing only through these points of V_{e-1}. Since V_e is not $(m-1)$-regular, there exists a subset $\{i_1, \ldots, i_{e-1}\} \subset \{1, \ldots, e\}$ such that any curve of degree $m-2$ through $y_{i_1} + \cdots + y_{i_{e-1}}$ will pass through V_e. Thus $y_{i_1} + \cdots + y_{i_{e-1}}$ has the Hilbert function described in Figure 2.

v) The assertion follows from (iv) by induction on e. Since the proof is similar to the proof of (iii), we omit details.

1.6 *Definition.*

$$\dot{D}^e = \left\{ \sum_{i=1}^{e} P_i \in D^e \cap U^e \mid \sum_{j=1}^{d} P_{i_j} \in D^d \right.$$
$$\left. \text{for every } (i_1, \ldots, i_d) \subset (1, \ldots, e) \right\}.$$

It is easy to show by induction, as in Lemma 1.5 (iv), that $\dot{D}^e$ is an open subset of D^e.

Let $I_V \subset \mathbf{C}[x, y, z]$ denote the ideal of $V \in U^e$. The inclusion

$$\alpha : U^e \longrightarrow \mathrm{Hilb}^{3e}, \quad \alpha(I_V) = I_V^{(2)},$$

is an algebraic morphism and its image is a locally closed subset of Hilb^{3e} [5]; $I_V^{(2)}$ is the 2nd symbolic power of I_V.

1.7 LEMMA. i) $\dim M^\theta(k) = 2e$ *for* $e \leqq k(k+3)/2$ *and* $\dim M^e(k) = e + k(k+3)/2$ *for* $e \geqq k(k+3)/2 > 0$.

ii) *For* $d \geqq \tau$, *each nontrivial irreducible component of* $\alpha(U^d) \cap M^{3d}(n)$ *has dimension* $\geqq \nu(n, d)$.

Proof. i) We consider the system of e equations in b's:

$$\sum_{q+s+t=k} b_{qst} x_i^q y_i^s z_i^t = 0 \quad (1 \leqq i \leqq e),$$

where $(x_1 : y_1 : z_1; \ldots; x_e : y_e : z_e) \in (\mathbf{P}^2)^e$. We get

$$\dim M^e(k) = 2e \quad \text{for } e \leqq k(k+3)/2 \text{ and}$$
$$\dim M^e(k) \geqq e + k(k+3)/2 \quad \text{for } e > k(k+3)/2 > 0.$$

We now assume $e > k(k+3)/2 > 0$. Using Lemma 1.5, by induction on e we get

$$\dim M^e(k) \leqq e + k(k+3)/2.$$

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ii) Since Hilb^{3e} is smooth, for a component $K \subseteq \overline{\alpha(U^d)} \cap \overline{M^{3d}(n)}$,

$$\dim K \geqq \dim \overline{\alpha(U^d)} + \dim \overline{M^{3d}(n)} - \dim \mathrm{Hilb}^{3d} = n(n+3)/2 - d.$$

1.8. We can view S^d as a system of equations defining a closed subscheme of $\mathbf{P}^N \times (\mathbf{P}^2)^d$. We denoted by $\Lambda_{n,d}$ the projection of that subscheme to $\mathbf{P}^N \times \mathrm{Sym}^d(\mathbf{P}^2)$. According to Severi [**17**], $\Sigma_{n,d}$ is a locally closed smooth variety having several connected components. It follows the existence of the natural imbedding

$$\begin{array}{ccc} \Sigma_{n,d} & \hookrightarrow \Lambda_{n,d} \subset \mathbf{P}^N \times \mathrm{Sym}^d(\mathbf{P}^2) & \\ {\scriptstyle p_d}\searrow & \swarrow {\scriptstyle p_d} \quad \searrow & \\ & U^d \xrightarrow{\ \alpha\ } \alpha(U^d) & \end{array}$$

Here $p_d = p_{n,d}$ is the projection on the second factor. For a curve B of degree n with d singular points $P_1, \ldots, P_d$ and possibly other singular points, the corresponding pair $(B; \sum_{i=1}^d P_i) \in \Lambda_{n,d}$ is called a curve with d assigned singularities. We will identify $C \in \Sigma_{n,d}$ with its image in $\Lambda_{n,d}$.

2. Reducible curves. In order to describe the situation of nodes of irreducible curves, we have to consider reducible curves as well.

2.1 PROPOSITION. *Let $d \leqq (n-1)(n-2)/2$. Let $(B; \sum_{i=1}^d P_i) \in \Lambda_{n,d}$ be a reduced reducible curve of degree n with d assigned singular points. If $(B; \sum_{i=1}^d P_i)$ is not a specialization of an irreducible curve with d assigned nodes, then*

$$\phi^{-1}\left(\sum_{i=1}^{d} P_i\right) \notin D^d.$$

Proof. By [**1**] and [**14**, Theorem 1.4], we can find a nodal curve with d assigned nodes, perhaps a few unassigned nodes on intersections of its components, and no other singularities such that $(B; \sum_{i=1}^d P_i)$ is a specialization of it. By [**17**] and the assumption, we can assume that $(B; \sum_{i=1}^d P_i)$ is a specialization of a curve $C = C_1 + \cdots + C_r \quad (r \geqq 2)$ with d assigned nodes $Q_1, \ldots, Q_d$ and no other singularities.

We now suppose

$$\phi^{-1}\left(\sum_{i=1}^{d} P_i\right) \in D^d$$

and derive a contradiction. We get

$$\phi^{-1}\left(\sum_{i=1}^{d} Q_i\right) \in D^d.$$

If $r \geqq 3$, then, by Lemma 1.5 (iv), one can find $d-1$ nodes, say $Q_1, \ldots, Q_{d-1}$, such that

$$\phi^{-1}\left(\sum_{i=1}^{d-1} Q_i\right) \in D^{d-1}.$$

By [**17**], $(C; \sum_{i=1}^{d} Q_i)$ is a specialization of a curve with $d-1$ assigned nodes in D^{d-1} and no other singularities. After several steps, we obtain a curve $C' = C_1' + C_2'$ with d' assigned nodes in $D^{d'}$ and no other singularities such that C' has two irreducible components and B is a specialization of C'. By Lemma 1.5 (iv), we obtain a curve $C'' = C_1'' + C_2''$ with d'' assigned node in $D^{d''}$ and not other singularities such that C' is a specialization of C'' and C'' satisfies the following property: If we forget about a node of C'' which does not belong to $C_1'' \cap C_2''$, we obtain a set of points which does not belong to $D^{d''-1}$.

By Lemma 1.5 (iv), we can now forget about one node of $C_1'' \cap C_2''$ and obtain a set of nodes which belongs to $D^{d''-1}$. The Hilbert function of our set of d'' points is described in Figure 1 with

$$m(m-1)/2 < d'' \leqq m(m+1)/2.$$

Since $d'' \leqq (n-1)(n-2)/2$, we get $m-1 \leqq n-3$. By the Cayley-Bacharach theorem [**9**, p. 671], any curve of degree $m-1$ through $d''-1$ nodes of C'' will pass through the forgotten node. Therefore the proposition follows from Lemma 1.5 (i) and (ii).

2.2 PROPOSITION. *Let $d \geqq \tau$ and $n \geqq 6$. Let B be a reducible curve of degree n with d nodes and no other singularities. One can find τ nodes among the nodes of B such that a $[3\tau \times (N+1)]$-matrix associated to these τ nodes will have rank less than N.*

Proof. The idea is to choose as many nodes as possible on the intersections of components of B. Let k denote the degree of an irreducible component of B of minimal degree. Let $B = C_k + C_{n-k}$, where $\deg C_k = k$. We observe that if $k(n-k) \geqq \tau$, then the curve $2C_k$ has singularities at the $k(n-k)$ points of $C_k \cap C_{n-k}$, hence the rank of a $[3k(n-k) \times (N+1)]$-matrix associated to the points is less than N.

Case 1 ($k = n/2$). We can apply the previous observation.

Case 2 ($n \geqq 9, n/3 \leqq k < n/2$). Since $-6k^2 + 6kn - n^2 - 3n \geqq 0$, we get $k(n-k) \geqq \tau$.

Case 3 ($0 < k \leqq n/3, k(n-k) < \tau$). We will find a curve B_{n-2k} of degree $n-2k$ with $t = \tau - k(n-k)$ singular points at the nodes of $C_{n-k} \backslash C_k$. Then the curve $B_{n-2k} + 2C_k$ will have τ singular points of which $k(n-k)$ points lie on $C_k \cap C_{n-k}$ and $\tau - k(n-k)$ points lie on $B_{n-2k} \backslash C_k$, and we get a required submatrix of $\mathbf{M}^d$. We choose arbitrary t nodes of $B \backslash C_k$ and consider a matrix associated to the points. Since

$$n(n+3)/2 - [3\tau - 3k(n-k)] \geqq -1 + 3k(n-k), \tag{3}$$

there exist at least $-2 + 3k(n-k)$ curves of degree n with singularities at those points. Let $\mathcal{J}$ be the ideal of those points. We consider the sequence (see (1)):

$$0 \longrightarrow H^0\left(\mathbf{P}^2, \mathcal{J}^{(2)}(l-1)\right) \longrightarrow H^0\left(\mathbf{P}^2, \mathcal{J}^{(2)}(l)\right) \longrightarrow H^0\left(\mathbf{P}^1, O(l)\right) \longrightarrow$$

Since $h^o\left(\mathbf{P}^2, \mathcal{J}^{(2)}(n)\right) \geqq -2 + 3k(n-k)$, we get

$$\begin{aligned} h^0\left(\mathbf{P}^2, \mathcal{J}^{(2)}(n-2k)\right) &\geqq -2 + 3k(n-k) - (n+1) - n - \cdots \\ &- (n-2k+2) = -3 - 3k^2 + kn + (2k-2)(2k-1)/2 > 0 \end{aligned}$$

provided $(n, k) \neq (6, 1)$ or $(6, 2)$. However, if $(n, k) = (6, 1)$ or $(6, 2)$, then (3) is a strict inequality and

$$h^0\left(\mathbf{P}^2, \mathcal{J}^{(2)}(n-2k)\right) \neq 0.$$

Case 4 ($n = 7$ or $8; k = 3$). We get $k(n-k) = \tau$.

3. Irreducible curves. According to [**17**], $V_{n,d}$ is a smooth irreducible variety of dimension $\nu(n, d)$. Let

$$\overline{V}_{n,d} \subset \mathbf{P}^N \times \mathrm{Sym}^d(\mathbf{P}^2)$$

denote the closure of $V_{n,d}$.

3.1 PROPOSITION. *The set of d nodes of a general curve $C \in V_{n,d}$ belongs to $\dot{D}^d$. Therefore, the degree of the adjoint curves to C of minimal degree is equal to m provided $d = m(m+1)/2$, or $m-1$ provided*

$$m(m-1)/2 < d < m(m+1)/2.$$

In particular, it is independent of n.

Proof. We consider a curve $L = L_1 + \cdots + L_n$ which is a sum of n lines in general position. We take the point $P_1 = L_1 \cap L_2$. Then we pick two new points, P_2 and P_3, on the intersection $(L_1 + L_2) \cap L_3$, etc. We claim the first d points chosen above give an element of D^d for each $d \leqq (n-1)(n-2)/2$.

We first assume $d = m(m+2)/2$. Using Noether's $AF + BG$ theorem, it is easy to verify (compare [**7**, Book I, Chapter II, Theorem 24]) that no curve of degree $m-1$ will pass through the points of intersections of $L_1, \dots, L_{m+1}$. By Lemma 1.5, the points give an element of D^d.

It is now clear that for each $d' \leqq m(m+1)/2$, the set of the first d' points is an m-regular scheme. By Lemma 1.5, the first d points give an element of D^d for every $d \leqq (n-1)(n-2)/2$.

The curve L with d assigned nodes $P_1, \dots, P_d$ is virtually connected, hence it determines the variety $V_{n,d}$ [**17**]. So, by Lemma 1.5, the set of d nodes of C corresponds to an element of D^d. Let now $C_0 \in V_{n,\bar{d}} \subset \overline{V}_{n,d}(\bar{d} = (n-1)(n-2)/2)$ be a general rational curve. As C_0 varies, the nodes of C_0 can be interchanged [**17**, p. 348]. Hence the set of nodes of C corresponds to an element of $\dot{D}^d$. The assertion concerning adjoints follows at once from Figure 1.

3.2. We fix two integers $q \geqq e \geqq 0$. Let $D \subset \mathbf{P}^2$ be a reduced curve of degree n with e assigned singular points $P_1, \dots, P_e$, $q-e$ additional singular points $P_{e+1}, \dots, P_q$, and no other singularities. Let $\mathcal{N}_D$ be the normal sheaf of D in $\mathbf{P}^2$. Let

$$\mathcal{N}'_D = \operatorname{Ker}\left(\mathcal{N}_D \longrightarrow \mathbf{T}^1(D/k, O_D)\right)$$

and let $\mathcal{N}''_D(P_1, \dots, P_e)$ be the sheaf that coincides with $\mathcal{N}_D$ at $P_{e+1}, \dots, P_q$ and with $\mathcal{N}'_D$ elsewhere [**18**].

Let $\varphi : \tilde{D} \longrightarrow D$ be the normalization of D and $\tilde{\mathfrak{f}}$ the conductor. Let $D(P_1, \dots, P_e) \longrightarrow D$ denote the partial normalization at $P_1, \dots, P_e$ and $\mathfrak{f}$ the corresponding conductor. It follows from the definitions that

$$\text{(4)} \qquad \mathcal{N}''_D(P_1, \dots, P_e) \simeq J\mathcal{N}_D$$

where $J \subset O_D$ is the ideal which coincides with the Jacobian ideal at $P_1, \dots, P_e$ and $J_Q = O_{Q,D}$ elsewhere [**18**]. Let R be the ramification divisor of φ. One defines two sheaves on $\tilde{D}$, $\mathcal{N}_\varphi$ and $\mathcal{N}'_\varphi$, by the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Theta_{\tilde{D}} & \longrightarrow & \varphi^*(\Theta_{\mathbf{P}^2}) & \longrightarrow & \mathcal{N}_\varphi & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Theta_{\tilde{D}}(R) & \longrightarrow & \varphi^*(\Theta_{\mathbf{P}^2}) & \longrightarrow & \mathcal{N}'_\varphi & \longrightarrow & 0
\end{array}$$

where $\Theta_{\tilde{D}}$ and $\Theta_{\mathbf{P}^2}$ are the tangent sheaves [**3**]. By taking determinants of the lower row, we get

$$\text{(5)} \qquad \mathcal{N}'_\varphi \simeq \varphi^* O_{\mathbf{P}^2}(3) \otimes \omega_{\tilde{D}}(-R),$$

where

$$\omega_{\tilde{D}} \simeq \tilde{\mathfrak{f}} O_{\tilde{D}} \otimes \varphi^* \omega_D$$

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is the dualizing sheaf of $\tilde{D}$. Hence

$$(6)\qquad \begin{cases} \mathcal{N}'_{\varphi} \simeq \varphi^{*}\left(O_{\mathbf{P}^2}(3)\otimes\omega_D\right)\otimes\tilde{\mathfrak{f}}O_{\tilde{D}}(-R)\simeq\varphi^{*}\mathcal{N}_D\otimes\tilde{\mathfrak{f}}O_D(-R),\\ \varphi^{*}\mathcal{N}'_{\varphi}\simeq\mathcal{N}_D\otimes\varphi_{*}[\tilde{\mathfrak{f}}O_{\tilde{D}}(-R)],\ \varphi_{*}[\mathcal{N}'_{\varphi}\otimes O_{\tilde{D}}(R)]\simeq\tilde{\mathfrak{f}}\mathcal{N}_D.\end{cases}$$

It follows from (5), (6), and Serre duality that

$$(7)\qquad H^1(D,\tilde{\mathfrak{f}}\mathcal{N}_D)\simeq H^1\left(\tilde{D},\mathcal{N}'_{\varphi}\otimes O_{\tilde{D}}(R)\right)\simeq H^0\left(\tilde{D},\varphi^{*}O_{\mathbf{P}^2}(-3)\right)=0.$$

The exact sequence

$$\begin{aligned} H^0(\tilde{D},\mathcal{N}'_{\varphi})&\longrightarrow H^0\left(\tilde{D},\mathcal{N}'_{\varphi}\otimes O_{\tilde{D}}(R)\right)\\ &\longrightarrow H^0\left(\tilde{D},O_{\tilde{D}}(R)/O_{\tilde{D}}\right)\longrightarrow H^1(\tilde{D},\mathcal{N}'_{\varphi})\longrightarrow 0\end{aligned}$$

yields

$$h^1(\tilde{D},\mathcal{N}'_{\varphi})\leqq \text{ length }(O_{\tilde{D}}(R)/O_{\tilde{D}})=\deg R.$$

Let

$$\tilde{D}=\sum_{i=1}^{r}\tilde{D}_i,$$

where $\tilde{D}_i$'s are the irreducible components. We denote by g_i the geometric genus of $\tilde{D}_i$. By Riemann-Roch and (5),

$$(8)\qquad \begin{aligned} h^0(\tilde{D},\mathcal{N}'_{\varphi})&=3n+2\left(\sum_{i=1}^{r}g_i-r\right)\\ &-\deg R+r-\sum_{i=1}^{r}g_i+h^1(\tilde{D},\mathcal{N}'_{\varphi})\\ &\leqq 3n+\sum_{i=1}^{r}g_i-r.\end{aligned}$$

If $\deg R<3n$ then by the duality and (5),we get (cf. [**19**]):

$$(9)\qquad H^1(\tilde{D},\mathcal{N}'_{\varphi})=H^0\left(\tilde{D},\varphi^{*}O_{\mathbf{P}^2}(-3)\otimes O_{\tilde{D}}(R)\right)=0.$$

The next proposition is a generalization of [**2**, Theorems 3.1] and [**20**, Theorem 2]. It is already nontrivial if the curve C_n below is irreducible; see also (3.4).

3.3 Proposition. *Let $\mathcal{D}\subset\Lambda_{n,d}$ be an irreducible analytic family of curves of degree n with d assigned singular points whose general curve D is reduced* *with q singular points $P_1, \ldots, P_e, \ldots, P_d, \ldots, P_q$, of which the first d points are assigned. Moreover, $P_1, \ldots, P_e$ are nodes and $P_{e+1}, \ldots, P_d$ are not nodes. We assume:*

i) *there exists a curve C_n of degree n with d assigned singularities at the assigned singular points of D, and C_n and D have no common components;*

ii) $\dim \mathcal{D} \geqq \nu(n,d)-t$, *where* $t = \min\{d-e, n+1\}$. *Then* $\dim \mathcal{D} = \nu(n,d) - (d-e)$, *$q = d$, and $P_{e+1}, \ldots, P_d$ are cusps. Moreover, the number irreducible components of D is not greater than that of C_n, where each component of C_n is counted with its multiplicity.*

Proof. Let

$$\tilde{D} = \sum_{i=1}^{r} \tilde{D}_i,$$

where the $\tilde{D}_i$'s are irreducible curves of (geometric) genus g_i. We can assume $n \geqq 2$. Let

$$g(D) = \sum_{i-1}^{r} g_i - r + 1$$

denote the geometric genus of D. By [**11**],

$$g(D) \leqq (n-1)(n-2)/2 - d$$

with equality if and only if D has only d singular points which are nodes or cusps. We first assume D has more complicated singularities and prove (see Step 3) that it can only have nodes, cusps, and tacnodes. By (4), $\mathcal{N}'_D = \tilde{J}\mathcal{N}_D$ where $\tilde{J}$ is the Jacobian ideal. We have $\tilde{J} \subset \tilde{J}\mathcal{O}_{\tilde{D}} = \mathfrak{f}\mathcal{O}_{\tilde{D}}(-R)$ [**15**]. Hence, by (6),

$$\dim \mathcal{D} \leqq h^0(D, \mathcal{N}'_D) \leqq h^0(\tilde{D}, \mathcal{N}'_\varphi).$$

Step 1. Claim:

$$h^0(\tilde{D}_i, \mathcal{N}'_\varphi) \geqq g_i + 1 \quad \text{for } i = 1, \ldots, r.$$

This is trivial if D is irreducible, because

$$h^0(\tilde{D}, \mathcal{N}'_\varphi) \geqq \nu(n,d) - (n+1) > 2n + g(D) - 2.$$

We now suppose that $h^0(\tilde{D}_1, \mathcal{N}'_\varphi) \leqq g_1$ and derive a contraction. Let

$$C = \sum_{i=2}^{r} D_i, \quad \deg C = k, \quad \text{and}$$

$$g(C) = \sum_{i=2}^{r} g_i - r + 2.$$

By (8),

$$h^0(C, \mathcal{N}'_\varphi) \leqq 3k + \sum_{i=2}^{r} g_i - (r-1) = 3k + g(C) - 1,$$

hence

$$h^0(D_1, \mathcal{N}'_\varphi) \geqq \nu(n,d) - n - 1 - (3k + g(C) - 1) \geqq 2n - 3k + g_1 - 1.$$

If $2n - 3k \geqq 2$, we are done. We now consider the case $2n - 3k \leqq 1$, i.e., $k = (2n-1)/3 + x (x \geqq 0)$. Let u of the assigned singularities of C_n lie on D_1 and the remaining $v = d - u$ assigned singularities lie on $D \backslash D_1$. We will first prove that $u + n \geqq (D_1 \cdot C)$. If $u + n < (D_1 \cdot C)$, then $u + v + n < (D_1 \cdot C) + v$ hence

$$d + n < \sum_Q \delta(Q : D_1 + C),$$

by [**11**]. Since

$$(n-1)(n-2)/2 - d - n > (n-1)(n-2)/2 - \sum_Q \delta(Q : D) = g(D),$$

we get

$$\dim \mathcal{D} \geqq \nu(n,d) - n - 1 \geqq 3n + g(D) - 1,$$

in contradiction with [**20**] and [**2**]. We will conclude the proof of the claim for $n \geqq 20$ by showing that $u + n < (D_1 \cdot C)$ provided $\deg D_1 > 3$, the case $\deg D_1 \leqq 3$ being trivial. Since $(D_1 \cdot C_n) \geqq 2u$, it is enough to show that $2(D_1 \cdot C) - 2n > (D_1 \cdot C_n)$ for $k = (2n-1)/3 + x$ and $\deg D_1 > 3$. We have $4 \leqq \deg D_1 = n - k$ hence $x \leqq (n+1)/3 - 4$. Further,

$$(D_1 \cdot C_n) = n^2/3 + n/3 - xn.$$

It is enough to show that

$$\begin{aligned} f(x) &= 2[(n+1)/3 - x][(2n-1)/3 + x] - 2n - (n^2+n)/3 + xn \\ &= -2x^2 + x(n+4)/3 + (n^2 - 19n - 2)/9 > 0 \quad \text{for } n \geqq 20. \end{aligned}$$

The function $f(x)$ has maximum at $(n+4)/12, f(0) > 0$, and $f\left((n+1)/3 - 4\right) > 0$.

If $n \leqq 19$, then $\deg D_1 \leqq 6$. Since

$$\deg(R|D_1) \leqq (\deg D_1 - 1)(\deg D_1 - 2)/2 < 3 \deg D_1 - 3,$$

we get $H^1(D_1, \mathcal{N}'_\varphi) = 0$ by (9). Hence by (8),

$$h^0(\tilde{D}_1, \mathcal{N}'_\varphi) = 3 \deg D_1 + g_1 - 1 - \deg(R|D_1) > g_1 + 1.$$

Step 2. It follows that

$$H^1(\tilde{D}, \mathcal{N}_\varphi) = H^1(\tilde{D}, \mathcal{N}'_\varphi) = H^1(\tilde{D}_i, \mathcal{N}_\varphi) = H^1(\tilde{D}_i, \mathcal{N}'_\varphi) = 0.$$

We get

$$\nu(n, d) - t \leqq h^0(\tilde{D}, \mathcal{N}'_\varphi) = 3n + g(D) - 1 - \deg R.$$

Since $t \leqq n + 1$, $\deg R \leqq n + 1$.

Step 3. By (8) and (9),

$$3n + g(D) - 1 - \deg R \geqq h^0(D, \mathcal{N}'_D) \geqq \nu(n, d) - (d - e).$$

Hence

$$(10) \qquad (n-1)(n-2)/2 - \sum \delta_Q(Q : D) - \deg R \geqq (n-1)(n-2)/2 - d - (d - e).$$

By assumption, $P_{e+1}, \dots, P_d$ are not nodes. Hence

$$(11) \qquad \delta(P_1 : D) + \deg(R|P_i) \geqq 2, \quad i = e + 1, \dots, d.$$

Therefore, (10) and (11) are, in fact, equalities and

$$h^0(D, \mathcal{N}'_D) = h^0(\tilde{D}, \mathcal{N}'_\varphi) = \nu(n, d) - (d - e) = \dim \mathcal{D}.$$

The curve D has only d singular points which can be nodes, cusps, and tacnodes.

Step 4. Claim: D has no tacnodes. Suppose $P \in D$ is a tacnode. Then there are two points $Q_1, Q_2 \in \tilde{D}$, $\varphi(Q_1) = \varphi(Q_2) = P$, and the two branches of D at P have a contact of order two. By (5),

$$\deg \mathcal{N}'_\varphi = 3n + 2g(D) - 2 - \deg R > 2g$$

hence

$$\deg \mathcal{N}'_\varphi(-Q_1 - Q_2) > 2g(D) - 2.$$

Therefore

$$H^1\left(\tilde{D}, \mathcal{N}'_\varphi(-Q_1 - Q_2)\right) = 0,$$

and we derive a contradiction as in [**2**, p. 97] or [**20**, p. 221].

Step 5. The last assertion follows from the Enriques principle of degeneration ([**19**, p. 36], [**7**, p. 105]).

3.4 *Remark.* In Proposition 3.3, if D is a priori irreducible, we can drop condition (i) (see Step 1 of the proof) and put $t = \min\{d-e, 3(n-1\}$. However, in interesting applications D is a priori an arbitrary reduced curve.

In addition to reduced curves, we have to consider nonreduced curves as well. We need a simple lemma.

3.5 Lemma. *The inequality*

$$k(k+3) + \nu(l-2k, d-k(k+3)/2) \leqq \nu(l,d) - x \tag{12}$$

is equivalent to the inequality

$$7k^2 + 3k \leqq 4lk - 2x.$$

The inequalities are strict in the following cases: i) $x = 0$, $l \geqq 6$, $l \geqq 2k > 0$, *and* $(k,l) \neq (3,6)$*;* ii) $x \leqq l+1$, $l \geqq 2k+2$, *and* $k \geqq 3$*;* iii) $x \leqq l+1$, $l \geqq 2k+3$, *and* $k = 2$*;* iv) $x \leqq l+1$, $l \geqq 2k+5$, *and* $k = 1$, v) $x \leqq 3$, $l \geqq 7$, $l \geqq 2k+1$, *and* $k \geqq 2$*;* vi) $x \leqq 3$ *and* $l = 2k \geqq 10$.

3.6 Proposition. *We assume*

$$[n(n+3)/6] \leqq d \leqq (n-1)(n-2)/2 \quad \textit{and} \quad (n,d) \neq (6,9).$$

Let $V \subset M_N^d$ *be an irreducible component and* $Q = (Q_1, \ldots, Q_d) \in V$ *a general point. Let*

$$\left(C; \sum_{i=1}^{d} P_i\right) \in V_{n,d}$$

be a general curve.

i) *If* $\sigma_d(Q) \in \dot{D}^d$ *and* $d \geqq \tau$*, then a curve of degree* n *singular at* $Q_1, \ldots, Q_d$ *is an irreducible curve with* d *nodes and no other singularities and* $\dim V = \nu(n,d)$*. If* $d = [n(n+3)/6]$*, then* $\dim V = 2d$*,* $\sigma_d(Q) \in \dot{D}^d$*, and there exists an irreducible curve with* d *nodes in general position and no other singularities.*

ii) *If* l *is the degree of a nonreduced curve of minimal degree singular at* $P_1, \ldots, P_d$*, then* $l > n$ *unless* $(n,d) = (8,14)$ *or there is an irreducible pencil of curves of degree* $l-1$ *singular at* $P_1, \ldots, P_d$*. If* $(n,d) = (8,14)$*, then* $l = 8$ *and there is a unique nonreduced curve of degree* 8 *singular at* $P_1, \ldots, P_{14}$ *(a double quartic).*

Proof. We shall assume that $n \geqq 6$. For $n \leqq 5$, all the assertions are trivial. Let $B = F + E$ be a curve, where $\deg B = l$, F has only multiple components, $\deg F_{\text{red}} = k$, and E is reduced. We assume B has singularities at $Q_1, \ldots, Q_d$

$(P_1, \ldots, P_d)$, of which t points are on F_{red} and the remaining $d-t$ points are on $B \backslash F_{\text{red}}$. By assumptions and Proposition 3.1, we get $t \leqq k(k+3)/2$ (see Figure 1).

i) We put $n = l$. By assumptions $d \geqq k(k+3)/2$.

Case 1 ($n \geqq 6$ and $d \geqq \tau$). By Lemma 1.2, $\dim V \geqq \nu(n,d)$. By Lemma 1.7 (i), the left side of (12) is an upper bound on $\dim V$. Hence, by Lemma 3.5 (i), $k = 0$ or $n = 2k = 6$. In the latter case, $Q_1, \ldots, Q_{10}$ lie on a cubic, a contradiction. By **[20]** or **[2]**, B has only d singularities and they are nodes. By Proposition 2.1, B is irreducible.

Case 2 ($n \geqq 7$ and $d = [n(n+3)/6]$). By Lemma 1.2, $\dim V = 2d$ and $\sigma_d(Q) \in \dot{D}^d$. Since $\dim V \geqq \nu(n,d) - 3$, we get $k = 0$ or $n = 2k = 8$, by Lemma 3.5 (iv)–(vi). In the latter case, $Q_1, \ldots, Q_d$ lie on a quartic, and there is a reduced curve of degree 8 singular at Q, because $3d < n(n+3)/2$. By Proposition 2.1 and **[17]**, there is an irreducible curve with d nodes in general position and no other singularities.

ii) We suppose $(B; \sum_{i=1}^{d} P_i)$ is nonreduced, $l \leqq n$, and there is no pencil of curves of degree $l-1$ singular at $P_1, \ldots, P_d$. We will prove that $(n,d) = (8, 14)$.

It follows from sequence (1) and $\nu(m,d) - \nu(m-1,d) = m+1$ that we get at most

$$h = 2(l+2) + (l+3) + \cdots + (n+1) = l + 2 + \nu(n,d) - \nu(l,d)$$

curves of degree n singular at $P_1, \ldots, P_d$. By Lemma 1.7 (i), the left side of (12) is an upper bound on $\dim p_d(V_{n,d})$ for $d \geqq k(k+3)/2$. Because $k \leqq n/2$, we have $d \geqq k(k+3)/2$. Since

$$\dim p_d(V_{n,d}) \geqq \nu(n,d) - (h-1) = \nu(l,d) - l - 1,$$

we get a contradiction for $k \leqq 2$ and $l \geqq 7$, by Lemma 3.5 (iii) and (iv). We cannot have $n = 6$ and $k \leqq 2$. By Lemma 3.5 (ii), we also get a contradiction for $k \geqq 3$ and $l \geqq 2k+2$. If now $l \leqq 2k+1$, then all the singular points are on a curve of degree k and $l = 2k$. Hence, by Lemma 1.7 (i),

$$d \leqq k(k+3)/2 < [n(n+3)/6]$$

provided $(n,k) \neq (8,4)$, a contradiction. So, if there exists a nonreduced curve of degree $l \leqq n$ with d singular points $P_1, \ldots, P_d$, then $n = l = 2k = 8$, $d = 14$, and it is a unique double quartic.

3.7 *Remark.* The only property of the component $V_{n,d} \subset \Sigma_{n,d}$ used in the proof of Proposition 3.6 (ii) is that

$$\sum_{i=1}^{d} P_i \in \dot{D}^d.$$

3.8 COROLLARY. (cf. [**3**, Theorem 3.2]). *Let* $d \leqq n(n+3)/6$, $d \leqq (n-1)(n-2)/2$ *and* $(n,d) \neq (6,9)$. *Let* $P_1, \ldots, P_d$ *be general points in* $\mathbf{P}^2$. *Then there exists an irreducible curve of degree n having nodes at* $P_1, \ldots, P_d$ *and no other singularities.*

Proof. For $d = [n(n+3)/6]$ the assertion follows from Proposition 3.6 (i). The general case follows by Severi's regeneration [**17**], i.e., one can get rid of unassigned nodes.

3.9 THEOREM. i) *Let* $\tau \leqq d \leqq (n-1)(n-2)/2$ *and* $(n,d) \neq (6,9)$. *Then the morphism*

$$p_d : V_{n,d} \longrightarrow \mathrm{Sym}^d(\mathbf{P}^2)$$

maps $V_{n,d}$ *birationally onto its image, and for a general* $C \in V_{n,d}$,

$$p_d^{-1}\left(p_d(C)\right) \subset \Lambda_{n,d}$$

consists of a point.

ii) *Let* $d \leqq n(n+3)/6$, $d \leqq (n-1)(n-2)/2$, *and* $(n,d) \neq (6,9)$. *Let* $P_1, \ldots, P_d$ *be general points in* $\mathbf{P}^2$. *Then there exists* $C \in V_{n,d}$ *having nodes at* $P_1, \ldots, P_d$.

Proof. It is enough to prove (ii) for $d = [n(n+3)/6]$ and $n \geqq 7$, the case $n \leqq 6$ and $(n,d) \neq (6,9)$ being trivial. Furthermore, the second assertion of (i) follows at once from the first one.

(*). We fix d,

$$[n(n+3)/6] \leqq d \leqq (n-1)(n-2)/2$$

and $(n,d) \neq (6,9)$. We will prove the remaining assertions by induction on n for an arbitrary component $\Sigma \subset \Sigma_{n,d}$ whose general curve has d nodes in $\dot{D}^d$. (By (3.7), Σ satisfies (ii) of Proposition 3.6.)

Let $p_{n,d}(\Sigma) \subset \sigma_d(V)$, where $V \subset M_N^d$ is an irreducible component. By Proposition 3.6 (i), a curve singular at a general $(Q_1, \ldots, Q_d) \in V$ is an irreducible curve with d nodes and no other singularities. Moreover, $\dim V = \nu(n,d)$ for $d \geqq \tau$, and $\dim V = 2d$ for $d = [n(n+3)/6]$. Let Ω denote the component of $\Sigma_{n,d}$ whose general curve has nodes at $Q_1, \ldots, Q_d$. By Proposition 3.6 (i), p_d maps Ω birationally onto its image. We suppose $\Sigma \neq \Omega$ and derive a contradiction.

Let $W = \Sigma \cap \Omega$ and let D be a general curve of W. By assumptions and the standard sequence (see (1)), D is reduced. We consider D as a curve with d assigned singular points $P_1, \ldots, P_d$, where $P_1, \ldots, P_d$ are nodes of a general $C \in \Sigma$. Suppose

$$\dim p_{n,d}^{-1}\left(p_{n,d}(D)\right) \leqq n+2.$$

It follows from the standard sequences (see (1)) and assumptions the existence of a family of curves of degree $n-1$ with d assigned singularities at $P_1, \ldots, P_d$

whose dimension is at least $\nu(n-1,d)$ and such that a general curve of the family, B_{n-1}, is reduced. Hence B_{n-1} has only d singular points and they are nodes ([**20**] or [**2**]). By Proposition 2.1, B_{n-1} is irreducible. Furthermore,

$$\dim p_{n-1,d}^{-1}\left(p_{n-1,d}(B_{n-1})\right) \geqq 1$$

(see (1)), a contradiction. Thus

$$\dim p_{n,d}^{-1}\left(p_{n,d}(D)\right) \leqq n+1 \quad \text{and} \quad \dim W \geqq \nu(n,d)-n-1.$$

We assume $P_1,\ldots,P_d$ $(e \leqq d)$ are nodes of D and $P_{e+1},\ldots,P_d$ are not nodes. It follows from (4) and (7) that

$$H^1\left(D, \mathcal{N}_D''(P_1,\ldots,P_e)\right) = 0,$$

and D with e assigned nodes $P_1,\ldots,P_e$ determines a unique smooth analytic branch (local deformation space) $\mathcal{A}_D$ of dimension $\nu(n,e)$ ([**17**] and [**18**]). We consider two distinct analytic branches of dimension $\nu(n,d)$, one for Σ and the other one for Ω, passing through $(D;\sum_{i=1}^d P_i)$ and contained in $\mathcal{A}_D$. Since

$$\dim_D W \geqq \nu(n,e)-2(d-e) = \nu(n,d)-d+e,$$

D is an irreducible curve with e nodes, $d-e$ cusps, and no other singularities, by Proposition 3.3.

We now consider the variety

$$V(n,g) \subset \mathbf{P}^N, \quad g=(n-1)(n-2)/2-d,$$

of all irreducible plane curves of degree n and geometric genus g and the variety $V'_{n,d} \subset \mathbf{P}^N$ of all irreducible plane curves of degree n with d nodes and no other singularities. By [**2**] or [**20**],

$$V'_{n,d} \subset V(n,g) \subset \overline{V}'_{n,d} \subset \mathbf{P}^N.$$

We will conclude the proof of the theorem by showing that $V(n,g)$ is irreducible at D. Since D has at most $n+1$ cusps, we get

$$h^1(\tilde{D}, \mathcal{N}_\varphi) = 0.$$

Hence the local deformation space $\mathcal{B}$ of the couple $(\tilde{D},\varphi)$ is smooth and maps in a one-to-one fashion onto a neighborhood of D in $V(n,g)$ (cf. [**17**, Mumford's appendix to Chapter VIII], [**4**, vol. II]).

3.10 *Remark.* Theorem 3.9 (i) can be generalized to reducible curves. Let $\Sigma \subset \Lambda_{n,d}$ be an arbitrary complete irreducible algebraic system of curves of degree n with d nodes and no other singularities. Let $(B;\sum_{i=1}^d P_i)$ be a general curve. We assume the linear system $\mathcal{L}$ of curves of degree n with assigned singularities at $\sum_{i=1}^{d} P_i$ consists of reduced curves. Then the morphism

$$p_d : \Sigma \longrightarrow \mathrm{Sym}^d(\mathbf{P}^2)$$

maps Σ birationally onto its image unless $\dim \mathcal{L} \geqq 1$ and $\mathcal{L}$ has fixed components.

The proof is similar to the proof of Theorem 3.9. As above, we obtain the curve D with nodes and cusps which can however now be reducible. Then, instead of $V(n,g)$ and $V'_{n,d}$, we consider the corresponding quasi-direct sums (see [**20**, p. 222]) and conclude the proof as in Theorem 3.9.

3.11 COROLLARY. *Let*

$$\tau + 1 \leqq d \leqq (n-1)(n-2)/2 \quad \textit{and} \quad n \neq 6.$$

Let

$$\left(C; \sum_{i=1}^{d} P_i\right) \in V_{n,d}$$

be a general curve. A curve B of degree n with singularities at $d-1$ nodes of C will coincide with C.

Proof. We fix a section of σ_d over a small analytic neighborhood of $\sum_{i=1}^{d} P_i$. Suppose the linear system of curves of degree n with singularities at $\sum_{i-1}^{d-1} P_i$ has dimension $\geqq 1$. We obtain an analytic family $\mathcal{B}_{d-1} \subset \overline{V}_{n,d-1}$ of curves of degree n with $d-1$ singularities such that the fibers of $p_{d-1}|\mathcal{B}_{d-1}$ have dimension $\geqq 1$ and

$$\dim \mathcal{B}_{d-1} = \nu(n, d-1).$$

Hence $\mathcal{B}_{d-1}$ intersects any Zariski open subset of $V_{n,d-1}$, a contradiction.

Throughout the rest of the section, we assume

$$\tau \leqq d \leqq (n-1)(n-2)/2, \quad n \geqq 6,$$

and $(n,d) \neq (6,9)$.

3.12 COROLLARY. *$p_d(\overline{V}_{n,d})$ is an irreducible component of $\overline{\sigma_d(M_N^d) \cap \dot{D}^d}$.*

Proof. We apply Theorem 3.9 (i) and Propositions 3.1 and 3.6 (i).

3.13 *Remark.* It follows from Corollary 3.12 and Severi's statement ($V_{n,d} = V'_{n,d}$), recently proved by J. Harris, that

$$p_d(\overline{V}_{n,d}) = \overline{\sigma_d(M_N^d) \cap \dot{D}^d}.$$

In a future paper we intend to give another proof based on results on the structure of M_N^d.

3.14. We now consider the open subscheme

$$A = \cup_j A_j \subset M_N^d \cap \sigma_d^{-1}(\dot{D}^d),$$

where each A_j is defined by the nonvanishing of an $(N \times N)$-minor of $\mathbf{M}^d$. On each A_j, we can solve the system S^d in a's. The solutions will coincide on $A_i \cap A_j$, and we get a section s over $\sigma_d(A)$ of the trivial bundle

$$\mathbf{P}^N \times \mathrm{Sym}^d(\mathbf{P}^2) \longrightarrow \mathrm{Sym}^d(\mathbf{P}^2).$$

For the generic curve of $V_{n,d}$, the coefficients of the corresponding equation are symmetric functions of the nodes, and it is well known (and trivial) that every symmetric function of the nodes of the generic curve of $V_{n,d}$ can be expressed rationally by the coefficients.

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