5

Spontaneous breaking of symmetries

> If my view is correct, the universe may have a kind of domain structure. In one part of the universe you may have one preferred direction of the axis; in another part the direction of the axis may be different.
>
> *(Y. Nambu)*

In the gauge theory of the previous chapter, all gauge bosons and fermions are massless. In the real world the only massless vector particle is the photon. Evidently we must devise a procedure for giving masses to gauge bosons and other particles. During the past few decades, substantial progress has been made in understanding the connection between particle masses and symmetries. In theories with global symmetries, it is possible for the states to have the same symmetry as the operators of the theory, as is, for instance, the case with strong isospin. This, however, is not the only mode in which a symmetry manifests itself. In field theories the symmetry can be broken by giving a non-vanishing vacuum expectation value to some field, i.e.

$$\langle \Omega | \phi | \Omega \rangle \neq 0.$$

We say now that the symmetry is spontaneously broken.[1] In this case the operators of the theory exhibit the symmetry, but the physical states do not. In other words, for a symmetry which is spontaneously broken, remnants of the symmetry occur explicitly in the commutation relations of the operators, but are realized in the particle spectrum in a subtle way. In this chapter we study two such cases: the Goldstone mode and the Higgs phenomenon.

In many cases a symmetry does not allow the introduction of a mass term. The breaking of the symmetry generates a mass term. This is demonstrated in Section 5.1, where the Lagrangian is invariant under a discrete symmetry. The

[1] The word spontaneous is used to communicate the idea that the phenomenon happens without any evident external cause, for example spontaneous combustion, spontaneous emission, . . .

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selection of the vacuum state breaks the symmetry and at the same time generates a mass.

For theories with continuous global symmetries the situation is different. The selection of a non-trivial vacuum generates masses, but at least one of the scalar particles must remain massless. This is the Goldstone phenomenon described in Section 5.2. Finally, in gauge theories, the particles that would become Goldstone mesons are eliminated by a gauge transformation and produce masses for gauge bosons (the Higgs mechanism).

5.1 Spontaneous breaking of global symmetries: discrete symmetry

Before we describe the general case, it is instructive to discuss a few simple examples in which the main ideas are transparent. Consider a real scalar field $\phi(x)$ and the classical Lagrange function

$$\mathcal{L} = \frac{1}{2}\frac{\partial\phi}{\partial x^\mu}\frac{\partial\phi}{\partial x_\mu} - U(\phi(x)), \tag{5.1}$$

with $U(\phi)$ a potential depending on ϕ. We are interested in finding the ground state. To this end we construct the Hamiltonian

$$\mathcal{H} = \frac{1}{2}(\partial_0\phi)^2 + \frac{1}{2}(\vec{\nabla}\phi)^2 + U(\phi). \tag{5.2}$$

The field with lowest energy is a constant field, whose value minimizes the potential $U(\phi)$. All this is classical. In the quantum theory $\phi(x)$ is an operator with a conjugate momentum. The field and its conjugate momentum satisfy commutation relations. The fields operate on the eigenstates of the Hamiltonian. For simple field theories it is possible to construct the eigenstates explicitly. The lowest energy is the ground state, also called the "vacuum." The word vacuum is somewhat misleading, because the vacuum state is not empty but, rather, is a complicated superposition of many particles. The term vacuum is appropriate in free-field theory, where it corresponds to the state with no particles, but for interacting fields the vacuum is a complicated state with many particles present. The vacuum and other states for simple Hamiltonians are constructed explicitly in Problems 1–3 at the end of this chapter. In this book, by vacuum we mean the lowest energy state, which will be denoted by $|\Omega\rangle$ or simply $|\ \rangle$.

In the class of theories of Eq. (5.1) we discuss two cases:

$$U(\phi) = \frac{\lambda}{4!}\phi^4 \pm \frac{\mu^2}{2}\phi^2 \quad \text{with} \quad \lambda > 0. \tag{5.3}$$

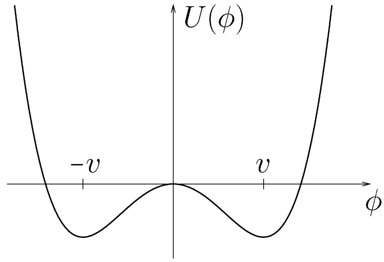

Figure 5.1. The Higgs potential $U(\phi)$.

(I) Case 1. We select

$$U(\phi) = \frac{\lambda}{4!}\phi^4 + \frac{\mu^2}{2}\phi^2. \tag{5.4}$$

This is the familiar theory for a field ϕ with mass μ^2 and an interaction term $(\lambda/4!)\phi^4$. The Feynman rules and other properties of this theory occur in many textbooks. The symmetry is explicit, with the Lagrangian being invariant under the transformation

$$\phi \to -\phi.$$

All solutions are also invariant under this transformation.

(II) Case 2. Now select

$$U(\phi) = \frac{\lambda}{4!}\phi^4 - \frac{\mu^2}{2}\phi^2. \tag{5.5}$$

In this case there is no mass term and $U(\phi)$ must be considered as a potential. The shape and the minima of the potential at

$$\phi = \pm\sqrt{\frac{3!\mu^2}{\lambda}} = \pm v$$

are shown in Fig. 5.1. We can select one of the minima as the ground state and study small oscillations around the minimum. This choice of the ground state breaks the symmetry, since the vacuum is no longer symmetric under the transformation $\phi \to -\phi$. We look for a solution in the neighborhood of v, and make the substitution

$$\phi = v + \phi' \quad \text{with} \quad v = \langle\Omega|\phi|\Omega\rangle. \tag{5.6}$$

This describes small oscillations. In terms of the new field,

$$U(\phi') = \frac{\lambda}{4!}\phi'^4 + \frac{\lambda v}{3!}\phi'^3 + \frac{1}{2}\frac{\lambda v^2}{3}\phi'^2 - \frac{\lambda v^4}{4!}.$$

We note that the term linear in ϕ' does not appear, but instead the new field acquired the mass $\sqrt{\lambda v^2/3}$. Another result of the shift is the appearance of a cubic self-coupling, which spoils the symmetry of the original Lagrangian. In this case the original symmetry is not present in the solution that we have chosen.

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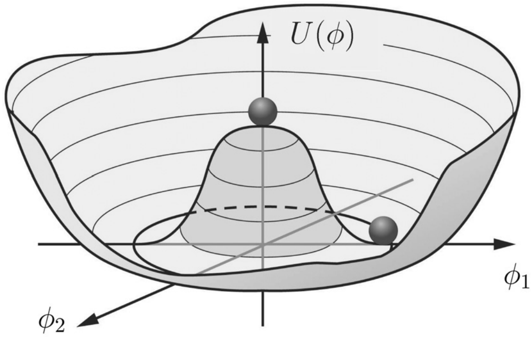

Figure 5.2. The Higgs potential $U(\phi_1, \phi_2)$.

This simple example demonstrates explicitly the breaking of the symmetry. It is based on a discreet symmetry, i.e. the reflection of the potential. New phenomena occur when the Lagrangian possesses either a continuous global symmetry or a local symmetry. In the following we discuss both cases.

5.2 Continuous global symmetries

The SO(2) model Next we consider a theory based on a continuous symmetry and then we break it spontaneously. Let us consider a theory with two real scalar fields, $\phi_1(x)$ and $\phi_2(x)$, and with the potential

$$U(\phi) = \frac{\lambda}{4!}\left(\phi_1^2 + \phi_2^2 - v^2\right)^2; \tag{5.7}$$

ϕ_1 and ϕ_2 are massless. This theory is invariant under rotation of ϕ_1 and ϕ_2, i.e. invariant under the group SO(2). The rotations are described by the angle θ,

$$\begin{pmatrix} \phi_1' \\ \phi_2' \end{pmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}. \tag{5.8}$$

The potential is shown in Fig. 5.2 and has the shape of a Mexican hat. The minima of the potential lie on the circle

$$\phi_1^2 + \phi_2^2 = v^2. \tag{5.9}$$

We show in Fig. 5.3 the locus of minima at the bottom of the hat. The lowest energy state is any vector $\vec{\phi}$ in the ϕ_1–ϕ_2 plane which ends at the circumference C.

We consider next the quantum-mechanical case and select a minimum in a specific direction. Without loss of generality, we select a coordinate system with the ϕ_1-axis parallel to the vacuum state, then

$$\langle \phi_1 \rangle = v \quad \text{and} \quad \langle \phi_2 \rangle = 0.$$

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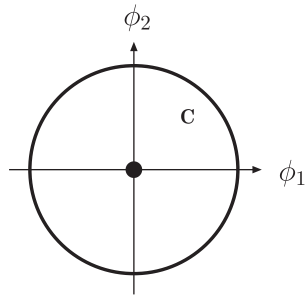

Figure 5.3. The locus of the minima of the Higgs potential.

Next we shift the fields,

$$\phi_1 = \phi_1' + v, \quad \phi_2 = \phi_2', \tag{5.10}$$

and find

$$U(\phi') = \frac{\lambda}{4!}\left(\phi_1'^2 + \phi_2'^2 + 2v\phi_1'\right)^2. \tag{5.11}$$

On expanding this, we see that the ϕ_1 field has mass, but the ϕ_2 field is massless. There again appear cubic terms in the fields, which break the original SO(2) symmetry.

This is a transparent example of a more general phenomenon and it is worthwhile to elaborate on the general case. One starts with a Lagrangian invariant under global transformations of a group G. The minima of the potential have the same symmetry. Then we break the symmetry by selecting one of the minima to be the vacuum, which is invariant under the subgroup H of G. In this analysis there are generators $\{g_i\}$ of G that do not belong to H. They are broken by the selection of the vacuum. To each *broken generator* $\{g_i\}$ there corresponds a *massless field*. These fields are called the Goldstone bosons. The Goldstone bosons transform under G like the coset or factor space of K = (G/H). We demonstrate this general phenomenon with several examples.

Example 1 In the SO(2) model, that we discussed, there is one generator

$$I(\theta) = \mathrm{e}^{\mathrm{i}\theta\sigma_y} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}. \tag{5.12}$$

After breaking of the symmetry, the potential term in (5.11) is not invariant under the transformation

$$\begin{pmatrix} \tilde{\phi}_1 \\ \tilde{\phi}_2 \end{pmatrix} = I(\theta)\begin{pmatrix} \phi_1' \\ \phi_2' \end{pmatrix}. \tag{5.13}$$

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The generator is broken and there exists one massless particle ϕ_2. When we change the orientation of the vacuum, there is still a massless particle, which is a linear superposition of ϕ_1 and ϕ_2.

Example 2 We consider the model with three real fields (ϕ_1, ϕ_2, ϕ_3) invariant under the group SO(3), that is, invariant under rotations in a three-dimensional space. We represent the state by a column matrix

$$\varphi = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix} \tag{5.14}$$

and the Lagrangian by

$$\mathcal{L} = \frac{1}{2}\partial^\mu \varphi^+ \partial_\mu \varphi - U(\varphi^+ \varphi), \tag{5.15}$$

with

$$U(\varphi^+\varphi) = \frac{\lambda}{4!}\left(\varphi^+\varphi - v^2\right)^2. \tag{5.16}$$

We next break the symmetry by giving a vacuum expectation value to $\langle \phi_3 \rangle \neq 0$. After shifting of the fields we obtain the potential

$$U(\phi') = \frac{\lambda}{4!}\left(\phi_1'^2 + \phi_2'^2 + \phi_3'^2 + 2v\phi_3'\right)^2. \tag{5.17}$$

This expression is still invariant under rotations around the 3-axis, but $U(\phi)$ and the Lagrangian change when we rotate around the first and second axes. We can represent a general rotation through the Euler angles (α, β, γ), which consists of the following three successive rotations:

(i) a rotation through an angle α about the 3-axis with the transformation matrix

$$R(0,0,\alpha) = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{5.18}$$

(ii) a rotation through β about the 2-axis with $R(0, \beta, 0)$, and
(iii) a rotation through γ about the 1-axis with $R(\gamma, 0, 0)$.

The product of the three matrices gives the complete rotation

$$R(\alpha, \beta, \gamma) = R(\gamma, 0, 0)R(0, \beta, 0)R(0, 0, \alpha). \tag{5.19}$$

The model is invariant under the rotations $R(0, 0, \alpha)$, but the $R(\gamma, 0, 0)$ and $R(0, \beta, 0)$ generators are broken. To the last two generators there correspond two massless particles, as follows from the form of the potential in (5.17).

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Example 3 The last example is a scalar theory invariant under global SU(2). For the field we consider a complex scalar doublet

$$\phi(x) = \begin{bmatrix} \phi_+ \\ \phi_0 \end{bmatrix} = \begin{bmatrix} \phi_1 + \mathrm{i}\phi_2 \\ \phi_3 + \mathrm{i}\phi_4 \end{bmatrix}. \tag{5.20}$$

Each of the fields has a real and an imaginary part. The Lagrangian is given by

$$\mathcal{L} = \partial_\mu \phi^\dagger \partial^\mu \phi - \lambda\left(\phi^\dagger \phi - v^2\right)^2. \tag{5.21}$$

Again we break the symmetry by giving a vacuum expectation value to the real part of ϕ_0,

$$\langle \mathrm{Re}\,\phi_0 \rangle = \langle \phi_3 \rangle = v. \tag{5.22}$$

When we shift the field ϕ_0 as before, the potential in terms of the components ϕ_+ and ϕ_0' becomes

$$U(\phi') = \lambda\left[|\phi_+|^2 + |\phi_0'|^2 + 2(\mathrm{Re}\,\phi_0')v\right]^2. \tag{5.23}$$

The new potential is not invariant under transformations of SU(2) with the two fields in $\phi_+(x)$ as well as Im ϕ_0 remaining massless, as is verified by expanding Eq. (5.23).

To sum up, we found that in field theories with continuous global symmetries the breaking of the symmetry requires the existence of scalar particles of zero mass. In fact, to every broken generator, there corresponds a massless particle. These are representative examples of Goldstone's theorem, which follows from general properties of field theory.

Goldstone's theorem If there is a continuous global symmetry transformation under which the Lagrangian is invariant, then either the vacuum state is invariant under the transformation, or there must exist spinless particles of zero mass.

We demonstrate the content of the theorem by studying the symmetry properties of a general potential.

(i) We assume that the potential $V(\phi_i)$ contains a set of real fields that transform according to a representation T^a of the group G

$$\phi_i'(x) = \phi_i(x) + \mathrm{i}\varepsilon^a T^a_{ij}\phi_j(x). \tag{5.24}$$

When the fields belong to the adjoint representation, the number of fields equals the number of generators.

(ii) We assume that the potential is invariant under the group G. Then

$$\delta V(\phi_i) = \frac{\partial V}{\partial \phi_i}\,\delta\phi_i = \mathrm{i}\frac{\partial V}{\partial \phi_i}\varepsilon^a T^a_{ij}\phi_j(x) = 0. \tag{5.25}$$

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Since the ε^a are arbitrary and continuous variables, it follows that

$$\frac{\partial V}{\partial \phi_i} T^a_{ij} \phi_j(x) = 0. \tag{5.26}$$

(iii) At the minimum of the potential

$$\left. \frac{\partial V}{\partial \phi_i} \right|_{\phi_i = v_i} = 0 \tag{5.27}$$

for each ϕ_i. Differentiating Eq. (5.26) again gives

$$\frac{\partial^2 V}{\partial \phi_i \partial \phi_k} T^a_{ij} \phi_j + \frac{\partial V}{\partial \phi_i} T^a_{ik} = 0. \tag{5.28}$$

(iv) At the minimum of the potential, the second term vanishes and

$$\left. \frac{\partial^2 V}{\partial \phi_i \partial \phi_k} \right|_{\phi_i = v_i} T^a_{ij} v_j = 0. \tag{5.29}$$

The mass matrix is

$$M^2_{ik} = \left. \frac{\partial^2 V}{\partial \phi_i \partial \phi_k} \right|_{\phi_i = v_i .} \tag{5.30}$$

Equation (5.29) is an eigenvalue equation with $T^a_{ij} v_j$ being the eigen-vectors. There are two important possibilities now. The first is

$$(\alpha) \quad T^a_{ij} v_j = 0, \tag{5.31}$$

which means that the generator T^a annihilates the vacuum. The states corresponding to the generators T^a have the symmetry of the group. In the second possibility there are generators T^b for which

$$(\beta) \quad T^b_{ij} v_j \neq 0, \tag{5.32}$$

in which case the symmetry generated by the T^b is not a symmetry of the vacuum. In this case $T^b_{ij} v_j$ is an eigenvector with eigenvalue zero, i.e. the states $T^b_{ij} \phi_j$ have zero mass. In Lagrangian theories this is a proof of the theorem, which at the same time demonstrates which particles remain massless.

Physical examples of the phenomenon occur in non-relativistic many-body systems. The Heisenberg ferromagnet is an example that consists of an infinite array of spin-$\frac{1}{2}$ magnetic dipoles. The Hamiltonian is rotationally invariant but the magnets in the ground state are aligned with all spins parallel, thus breaking the rotational symmetry. In this case the frequency of the spin waves goes to zero with the wave-number.

In particle physics the phenomenon is relevant to understanding the connection of symmetries of the hadronic currents. In Section 2.2 we showed that, in the limit

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of zero pion mass ($m_\pi = 0$), the axial current is conserved. The conserved vector and axial currents generate an $SU(2)_L \times SU(2)_R$ algebra. Therefore, the symmetry is present in the currents and leads to many important predictions. It is absent from the particle spectrum, since there are no parity-degenerate multiplets. Think of the (ρ^+, ρ^0, ρ^-) and (A^+, A^0, A^-) isospin multiplets. The ρs transform into each other with $SU(2)_{\text{vector}}$ and the As with $SU(2)_{\text{axial}}$, but there is no connection between the two multiplets through $SU(2)_L \times SU(2)_R$ transformations. In other words, the operators are $SU(2)_L \times SU(2)_R$-symmetric but the particle states are not ($m_\rho \neq m_A$). This physical situation can be understood in $SU(2)_L \times SU(2)_R$ theory with spontaneously broken symmetry. The $SU(2)_A$ generators are broken by non-zero vacuum expectation values and must be accompanied by zero-mass Goldstone bosons. This is in agreement with the fact that the pions have masses much smaller (nearly zero) than those of all other hadrons. Strictly speaking, the pions should be massless, but corrections to the potential or radiative corrections can produce small masses.

5.3 Spontaneous breaking of local symmetries

A new phenomenon occurs in local gauge theories, which is crucial for constructing theories with massive gauge bosons. The simplest example is scalar electrodynamics, which was introduced in Section 4.2. We consider the Lagrangian of Eq. (4.36) with the potential

$$V(\phi^*\phi) = -\mu^2\phi^*\phi + \lambda(\phi^*\phi)^2. \tag{5.33}$$

The theory is invariant under the gauge transformation defined in Eq. (4.39). In addition, the Lagrangian is invariant under a global rotation of the scalar field

$$\phi \to e^{i\alpha}\phi, \tag{5.34}$$

with α independent of space and time. We first demonstrate properties of the theory under global transformations and then indicate the changes introduced in a gauge theory.

For the vacuum state we select one of the minima of the potential. By a global rotation we can transform $\langle\phi\rangle$ to a real value. We represent the field and its vacuum state by

$$\phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2), \quad \langle\phi_1\rangle = v = \left(\frac{\mu^2}{2\lambda}\right)^{\frac{1}{2}} \quad \text{and} \quad \langle\phi_2\rangle = 0. \tag{5.35}$$

As in the previous cases, we translate ϕ,

$$\phi_1(x) = \phi_1'(x) + v, \tag{5.36}$$

and leave ϕ_2 unchanged. In terms of the new field, the potential becomes

$$V(\phi) = -\frac{\mu^4}{4\lambda} + \lambda\Big[\big(\phi_1'^2 + \phi_2^2\big)^2 + 4v^2\phi_1'^2 + 4v\phi_1'\big(\phi_1'^2 + \phi_2^2\big)\Big]. \tag{5.37}$$

The ϕ_1' field acquired a mass and there are also trilinear interaction terms. In addition there are changes in the kinetic terms, which we describe in Problem 5.4; important among them is the property that A_μ acquires a mass.

Alternatively, we can study this theory as a gauge theory. We define two real fields $\theta(x)$ and $\rho(x)$ by the relation

$$\phi(x) = \mathrm{e}^{\mathrm{i}\theta(x)/v}\,\frac{\rho(x) + v}{\sqrt{2}} \tag{5.38}$$

and give $\rho(x)$ a vacuum expectation value.

Then we observe that the local gauge transformation

$$\begin{aligned} \phi' &= \mathrm{e}^{-\mathrm{i}\theta(x)/v}\phi(x) = \frac{\rho(x) + v}{\sqrt{2}}, \\ A_\mu'(x) &= A_\mu(x) + \frac{1}{ev}\partial_\mu\theta(x) \end{aligned} \tag{5.39}$$

eliminates $\theta(x)$ completely. This transformation is unusual, because the field itself occurs in the gauge transformation; but it is legitimate in all respects. After the gauge transformation, the first two terms of Eq. (4.36) retain their form, with primed fields replacing the old ones. The net effect is

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}'F'^{\mu\nu} + (D'^{\mu}\phi')^*(D_\mu'\phi') + \frac{\mu^2}{2}[\rho(x) + v]^2 - \frac{\lambda}{4}[\rho(x) + v]^4, \tag{5.40}$$

with $D_\mu' = \partial_\mu + \mathrm{i}eA_\mu'$. The second term,

$$(D'^{\mu}\phi')^*(D_\mu'\phi') = \frac{1}{2}\partial_\mu\rho\partial^\mu\rho + \frac{1}{2}e^2A_\mu'A'^{\mu}\big(\rho^2 + 2\rho v + v^2\big), \tag{5.41}$$

generates a mass for the A_μ' field. The Goldstone field disappeared and the vector field became massive.

We have described a second important case of spontaneous symmetry breaking. We started with a locally invariant theory describing a charged scalar field (two degrees of freedom) and a massless gauge field with two polarizations. After spontaneous symmetry breaking there is one real scalar field and a massive gauge field with three polarizations. The spontaneous breaking of the symmetry redistributed the degrees of freedom: one of the two real fields forming the complex scalar field was transformed into the longitudinal polarization of the vector field. This example illustrates that the spontaneous breaking of local symmetry does not produce

Goldstone mesons, but gives masses to the gauge bosons. It will be used later on in order to create masses for the intermediate gauge bosons.

The spontaneous breaking of scalar electrodynamics is physically unrealistic, because electric charge is not conserved. This is evident from the presence of the $A^2\rho$ term in (5.33). It is a consequence of the fact that we introduced a non-zero vacuum expectation value for a charged field,

$$\langle\Omega|\phi(x)|\Omega\rangle \neq 0, \tag{5.42}$$

which in itself violates charge conservation. In realistic theories we can preserve the conservation laws by giving non-zero expectation values to fields that carry the vacuum quantum numbers.

This phenomenon was introduced in the sixties in order to evade the Goldstone theorem and maintain gauge invariance, despite the fact that the vector meson acquires a mass. At that time it was thought to be relevant for the strong interactions. Later it was extended to non-Abelian gauge theories. It is now used in order to break the gauge symmetry of the electroweak theory and produce masses for the intermediate gauge bosons. It is referred to as the Higgs mechanism and we describe it in Chapter 7.

Problems for Chapter 5

1. Consider a one-dimensional harmonic oscillator. Its Hamiltonian

$$H = \frac{1}{2}\left(p^2 + \omega_0^2 x^2\right)$$

can easily be rewritten in terms of the classical variables

$$a = \sqrt{\frac{1}{2\omega_0}}(\omega_0 x + \mathrm{i}p) \quad \text{and} \quad a^+ = \sqrt{\frac{1}{2\omega_0}}(\omega_0 x - \mathrm{i}p).$$

In quantum mechanics a and a^+ are operators satisfying the commutation relations

$$\left[a,\, a^+\right] = 1 \quad \text{and} \quad [a,\, a] = \left[a^+,\, a\right] = 0.$$

Compute

(i) the eigenstates of this Hamiltonian,

(ii) the time development of the operators a and a^+, and

(iii) the matrix elements of a and a^+ between arbitrary states.

When you have done all this, then you have solved this quantum field theory completely.

2. The Hamiltonian for an asymmetric oscillator is

$$H = \frac{1}{2}\left(p^2 + \omega_0^2 x^2\right) + kx.$$

Replace again the position and momentum variables with the operators a and a^+. The Hamiltonian reads

$$H = \frac{1}{2}\omega_0(a^+a + aa^+) + k(a + a^+).$$

The problem now is to find a unitary transformation such that

$$UaU^+ = a - k/\omega_0,$$
$$Ua^+U^+ = a^+ - k/\omega_0.$$

With the help of U it is possible to eliminate the linear term kx and reduce this problem to the previous one.

3. The vacuum state, $|\Omega\rangle$, is not always the empty state, $|0\rangle$. This is demonstrated with the Hamiltonian

$$H = \frac{5}{3}a^+a + \frac{2}{3}(a^+)^2 + \frac{2}{3}a^2.$$

The number operator $N = a^+a$ does not commute with the Hamiltonian. Consequently the eigenstates of N are not eigenstates of H.

Find the lowest-energy state of H. To this end, construct two operators with the properties

$$\left[H,\ Q^\pm\right] = \pm Q^\pm.$$

These operators raise and lower the energy by one unit. The lowest-energy state is defined as usually by the condition

$$Q^-|\Omega\rangle = 0\,.$$

You can represent the vacuum as $\sum_n c_n(a^+)^n|0\rangle$, and then use the above condition to give an explicit formula for $|\Omega\rangle$. It is possible to write the normalized vacuum state in closed form. Finally, construct all higher-energy states.

4. Work out the kinetic term for scalar electrodynamics using the new fields of Eq. (5.36). Show that the field A_μ acquired a mass and demonstrate the appearance of trilinear interaction terms.

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