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CORRIGENDUM TO 'ON THE CONVERGENCE OF DIOPHANTINE DIRICHLET SERIES', PROC. EDINB. MATH. SOC. 55 (2012), 513–541

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I recently discovered that the proof of Proposition 3.1 in my 2012 paper 1[1] (stated on page 522) is not correct. The proof can be corrected but this introduces a factor on the left-hand side of Equation (3.2). This factor is not important for Proposition 3.1 itself but it is important for its applications. Unfortunately, it invalidates the end of the proof of Theorem 1.1. I could not find a complete correction for it, and I present a slightly weaker statement below. On the other hand, the statements of Theorems 1.2 and 1.3 are not modified because their proofs are not affected in a crucial way by this factor.

I use the notation and labels of the paper. These labels are of the form $(a.b)$ with $a \geq 1$. There will be no confusion with the labels of this *corrigendum* which are the form $(a.b)$ with $a = 0$. For the properties of continued fractions, I refer to §1.1 of the paper.

Correction to the proof of Proposition 3.1

In Equation (3.2) of Proposition 3.1, the implicit constant in the symbol $\ll$ does not depend on m but it depends on α, f, s and more importantly on k. The dependence on k was not noticed in the paper because of the error in the proof of Equation (3.2). Moreover, the independence of Equation (3.2) on k was explicitly used many times in the proofs of the main results of the paper, so that many arguments now have to be corrected.

We are given $f(x)$ defined on $\mathbb{R} \setminus (\pi\mathbb{Z})$ and such that $|f(x)| \leq c/|\sin(x)|^r$. The statement of Proposition 3.1 remains the same except that, for applications, Equation (3.2) must be made more precise. This is done as follows:

For every $\alpha \in (0, 1) \setminus \mathbb{Q}$, $s > r \geq 1$, $m \geq 0$, $k \geq 0$ and $N \geq 1$ such that $q_m \leq N < q_{m+1}$, we have

$$\sum_{n=1}^{N} \frac{|f(\pi n T^k(\alpha))|}{n^s} \leq C(f, s, r, \alpha) q_k(\alpha)^{s-r} \sum_{j=k}^{k+m} \frac{q_{j+1}(\alpha)^r}{q_j(\alpha)^s}, \tag{0.1}$$

where $C(f, s, r, \alpha) > 0$ need not be explicited. Since $s > r$, the factor $q_k(\alpha)^{s-r}$ is unfortunately unbounded as $k \to +\infty$.

I now give a proof of Equation (0.1), which at the same time corrects the proof of Proposition 3.1. The domination condition $|f(x)| \leq c/|\sin(x)|^r$ (ie, Equation (3.1) on page 522) shows it is enough to find a bound for

$$\sum_{n=1}^{N} \frac{1}{n^s |\sin\left(\pi n T^k(\alpha)\right)|^r}.$$

We have $|\sin(\pi n T^k(\alpha))| = \sin(\pi ||nT^k(\alpha)||) \asymp ||nT^k(\alpha)||$, where both implicit constants are absolute. Hence it is enough to find an upper bound for $\sum_{n=1}^{N}(1/n^s||nT^k(\alpha)||^r)$. By Lemma 3.2, for every integer $N \geq 1$,

$$\sum_{n=1}^{N} \frac{1}{n^s||nT^k(\alpha)||^r} \ll \sum_{j=0}^{m} \frac{\widehat{q}_{j+1}^{\,r}}{\widehat{q}_j^{\,s}}, \tag{0.2}$$

where $\widehat{q}_j := q_j(T^k(\alpha))$ is the denominator of the jth convergent to $T^k(\alpha)$. The implicit constant in Equation (0.2) depends at most on r, s, α and not on $m \geq 0$, $k \geq 0$. Now, the sequence $(\widehat{q}_j)_{j\geq -1}$ satisfies the recurrence relation $\widehat{q}_{j+1} = a_{k+j+1}\widehat{q}_j + \widehat{q}_{j-1}$. Since the sequences $(p_{j+k})_{j\geq 0}$ and $(q_{j+k})_{j\geq -1}$ are linearly independent solutions over $\mathbb{C}$ of the same recurrence, there exist two sequences of rational numbers $(u_k)_{k\geq 0}$ and $(v_k)_{k\geq 0}$ (independent of j) such that $\widehat{q}_j = u_k q_{j+k} + v_k p_{j+k}$ for any integers $j \geq -1$, $k \geq 0$. For $j = 0$, this yields $1 = u_k q_k + v_k p_k$ while for $j = -1$, this yields $0 = u_k q_{k-1} + v_k p_{k-1}$. Since $p_{k-1}q_k - p_k q_{k-1} = (-1)^k$, it follows that $u_k = (-1)^k p_{k-1}$ and $v_k = (-1)^{k+1} q_{k-1}$. Hence, for all $j \geq -1$, $k \geq 0$, we have

$$\widehat{q}_j = (-1)^k \left(p_{k-1}q_{j+k} - q_{k-1}p_{j+k}\right). \tag{0.3}$$

Equation (0.3) corrects the erroneous statement on pages 516 and 525 that $\widehat{q}_j = q_{j+k}$. If $k = 0$, then $\widehat{q}_j = q_j$ and Equation (0.2) proves Equation (0.1) in this case because $q_0(\alpha) = 1$. We now assume that $k \geq 1$, so that $q_{k-1} \neq 0$ and $q_{j+k} \neq 0$. We now rewrite Equation (0.3) as

$$\widehat{q}_j = (-1)^k q_{j+k} q_{k-1} \left(\frac{p_{k-1}}{q_{k-1}} - \alpha\right) + (-1)^{k+1} q_{j+k} q_{k-1} \left(\frac{p_{j+k}}{q_{j+k}} - \alpha\right).$$

Since $1/2q_n q_{n+1} < (-1)^n \left(p_n/q_n - \alpha\right) < 1/q_n q_{n+1}$ for all $n \geq 0$, there exist two absolute constants $c_1 > 0$ and $c_2 > 0$ such that for all $j \geq 0$, $k \geq 1$, we have

$$c_1 \frac{q_{j+k}}{q_k} \leq \widehat{q}_j \leq c_2 \frac{q_{j+k}}{q_k}.$$

It follows that for any integers $m \geq 0$, $k \geq 1$,

$$\sum_{j=0}^{m} \frac{\widehat{q}_{j+1}^{\,r}}{\widehat{q}_j^{\,s}} \leq c_2^r c_1^{-s} q_k^{s-r} \sum_{j=0}^{m} \frac{q_{j+k+1}^r}{q_{j+k}^s}.$$

This proves Equation (0.1) in this case as well, and the proof of Proposition 3.1 is complete.

About Theorem 1.1

The dependence on k in Equation (0.1) unfortunately invalidates the method used to prove that (for $s > 2$) $\Phi_s(\alpha) := \sum_{k=1}^{\infty} \cot(\pi n\alpha)/n^s$ converges if and only

$$\sum_{j=0}^{\infty} (-1)^j \frac{q_{j+1}(\alpha)}{q_j(\alpha)^s} \quad \text{converges,} \tag{0.4}$$

and it invalidates as well the proof of Identity (1.11). I prove here the following weaker version of Theorem 1.1 (the veracity of which remains unknown).

Theorem. *We fix an irrational number $\alpha \in (0, 1)$. If $s > 1$ and if*

$$\sum_{j=0}^{\infty} \frac{q_{j+1}(\alpha)}{q_j(\alpha)^s} < +\infty, \tag{0.5}$$

then the series $\Phi_s(\alpha)$ converges absolutely. Moreover, if $s > 2$ and if (0.5) holds, we have the identity

$$\Phi_s(\alpha) = \sum_{j=0}^{\infty} (-1)^j |q_{j-1}\alpha - p_{j-1}|^{s-1} G_s(T^j(\alpha)), \tag{0.6}$$

where the series on both sides converge absolutely.

Identity (0.6) is formally the same as Identity (1.11), except that it is proved to hold on a smaller set of irrational numbers.

I now present the proof of this theorem. Equation (0.5) is indeed a sufficient condition for the convergence of $\Phi_s(\alpha)$ when $s > 1$, as a direct application of Proposition 3.1 shows with $k = 0$, $f(x) = \cot(x)$ and $r = 1$. We first correct a misprint: in §4.2 on page 527, the function $F_s(z, \alpha)$ has (among others) poles at the points k/α where the positive integers k are such that $k \leq N\alpha + \alpha/2$, and not just $k \leq N\alpha$ as written. Consequently, on the right-hand side of Equations (4.1) and (4.5), $[N\alpha]$ must be replaced by $[N\alpha + \alpha/2]$. We thus have

$$\sum_{n=1}^{N} \frac{\cot(\pi n\alpha)}{n^s} = -\alpha^{s-1} \sum_{n=1}^{[N\alpha+\alpha/2]} \frac{\cot(\pi n T(\alpha))}{n^s} + G_s(\alpha) + \mathcal{O}\big(E_N(\alpha)\big). \tag{0.7}$$

The error term $E_N(\alpha)$ is still bounded as in (4.6) on page 528, the computations leading to (4.6) being unchanged. In particular, $E_N(\alpha) \to 0$ when $s > 2$. Observe now that by Equation (0.1), Assumption (0.5) implies the absolute convergence of all the series

$$\Phi_s\big(T^k(\alpha)\big) := \sum_{n=1}^{\infty} \frac{\cot(\pi n T^k(\alpha))}{n^s}, \quad k \geq 0,$$

when $s > 1$, hence a fortiori for $s > 2$. Hence, under (0.5) and for $s > 2$, we can in particular let $N \to +\infty$ in (0.7) and we get

$$\Phi_s(\alpha) = -\alpha^{s-1}\Phi_s(T(\alpha)) + G_s(\alpha).$$

We now iterate this equation: for all integer $\ell \geq 0$, we have

$$\Phi_s(\alpha) = (-1)^{\ell+1}\big(\alpha T(\alpha)\cdots T^{\ell}(\alpha)\big)^{s-1}\Phi_s\big(T^{\ell+1}(\alpha)\big) + \sum_{j=0}^{\ell}(-1)^j\big(\alpha T(\alpha)\cdots T^{j-1}(\alpha)\big)^{s-1}G_s\big(T^j(\alpha)\big). \tag{0.8}$$

Letting $N \to +\infty$ in Equation (0.1), we obtain

$$|\Phi_s\big(T^{\ell+1}(\alpha)\big)| \ll q_{\ell+1}^{s-1}\sum_{j=\ell+1}^{\infty}\frac{q_{j+1}}{q_j^s},$$

where the implicit constant is independent of ℓ. Since $\alpha T(\alpha)\cdots T^{\ell}(\alpha) = |q_\ell\alpha - p_\ell| < 1/q_{\ell+1}$, we deduce that

$$\Big|\big(\alpha T(\alpha)\cdots T^{\ell}(\alpha)\big)^{s-1}\Phi_s\big(T^{\ell+1}(\alpha)\big)\Big| \ll \sum_{j=\ell+1}^{\infty}\frac{q_{j+1}}{q_j^s} \to 0$$

when $\ell \to +\infty$ because of Assumption (0.5). From (0.8), we thus immediately deduce that the series

$$\sum_{j=0}^{\infty}(-1)^j\big(\alpha T(\alpha)\cdots T^{j-1}(\alpha)\big)^{s-1}G_s\big(T^j(\alpha)\big) \tag{0.9}$$

also converges under (0.5) and that it is equal to $\Phi_s(\alpha)$. In fact, since $|G_s(\alpha)| \ll 1/\alpha$, it is proved directly that the series (0.9) converges absolutely under (0.5). This completes the proof of the above theorem.

About Theorem 1.2

The statement of Theorem 1.2 remains valid but a few modifications have to be made to its proof. Given $s > 2$, we consider the hypothesis

$$\sum_{j=0}^{\infty}\frac{q_{j+1}(\alpha)^2}{q_j(\alpha)^s} < +\infty, \tag{0.10}$$

which corresponds to the convergence of the series (1.14) on page 518. Given $s > 2$, Equation (0.10) is a necessary and sufficient condition for the convergence of $\widehat{\Phi}_s(\alpha)$: this is still an immediate consequence of Lemma 3.2. From now on, s is assumed to be fixed and > 2; this fact will no longer be mentioned.

For latter use, we first prove a technical property: the series $\sum_{j=0}^{\infty} q_j q_{j+1}\sum_{n=j}^{\infty}(q_{n+1}/q_n^{s+1}$ converges under (0.10). For this, it is enough to prove that $\sum_{n=0}^{\infty}(q_{n+1}/q_n^{s+1})\sum_{j=0}^{n} q_j q_{j+1}$ converges under (0.10), and then to invoke Tonelli's theorem to justify that $\sum_{j=0}^{\infty} q_j q_{j+1}\sum_{n=j}^{\infty}(q_{n+1}/q_n^{s+1}) = \sum_{n=0}^{\infty}(q_{n+1}/q_n^{s+1})\sum_{j=0}^{n} q_j q_{j+1} < +\infty$. Now, it is proved that $\sum_{j=0}^{n} q_j \leq 3q_n$ on page 199 of B. Martin & M. Balazard, Comportement local moyen de la fonction de Brjuno, Fundamenta Mathematicae **218** (2012), 193–224. Hence $\sum_{j=0}^{n} q_j q_{j+1} \leq 3q_n q_{n+1}$ and $q_{n+1}/q_n^{s+1}\sum_{j=0}^{n} q_j q_{j+1} \leq 3(q_{n+1}^2/q_n^s)$. The result follows.

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We first correct a misprint: $[N\alpha]$ must be $[N\alpha+\alpha/2]$ in Equation (5.2), but this is harmless. Indeed, we can still let $N\to+\infty$ in the corrected Equation (5.2) because the convergence of the involved series holds under (0.10), by Proposition 3.1. Hence, the identity

$$\sum_{k=1}^{\infty}\frac{1}{k^s\sin^2(\pi k\alpha)}=\alpha^{s-2}\sum_{k=1}^{\infty}\frac{1}{k^s\sin^2(\pi kT(\alpha))}+\frac{s}{\pi}\alpha^{s-1}\sum_{k=1}^{\infty}\frac{\cot(\pi kT(\alpha))}{k^{s+1}}+\widehat{G}_s(\alpha)$$

remains valid. The convergence of $\sum_{k=1}^{\infty}(\cot(\pi kT(\alpha))/k^{s+1})$ holds (at least) when $\sum_{j=0}^{\infty}q_{j+1}/q_j^{s+1}<+\infty$ and $s>0$, hence at least when (0.10) holds. On page 535, it is said that Equation (5.5) is bounded independently of j by Proposition 3.1, but this is no longer true with the above correction. Hence, a certain number of modifications must then be made on pages 535 and 536. We start with some general considerations. It is true that $\Phi_{s+1}(T^j(\alpha))$ converges absolutely for any fixed $j\geq 0$ when (0.10) holds, and the corrected Proposition 3.1 now implies that

$$\left|\Phi_{s+1}(T^j(\alpha))\right|\ll q_j^s\sum_{k=j}^{\infty}\frac{q_{k+1}}{q_k^{s+1}} \tag{0.11}$$

where the implicit constant does not depend on j. We then have

$$\Phi_{s+1}(T^j(\alpha))=\sum_{n=0}^{\infty}(-1)^n\big(T^j(\alpha)T^{j+1}(\alpha)\cdots T^{j+n}(\alpha)\big)^sG_{s+1}(T^{j+n}(\alpha)),$$

where both series converge absolutely under (0.10). Since $|G_s(\alpha)|\ll 1/\alpha$, where the implicit constant depends on s only, we have as well

$$\sum_{n=0}^{\infty}\left|(T^j(\alpha)T^{j+1}(\alpha)\cdots T^{j+n}(\alpha))^sG_{s+1}(T^{j+n}(\alpha))\right|\ll q_j^s\sum_{k=j}^{\infty}\frac{q_{k+1}}{q_k^{s+1}} \tag{0.12}$$

where the implicit constant does not depend on j. We can now explain what must be changed to correct the proof of Theorem 1.2 starting from page 535. It is still true that we can let $N\to+\infty$ in (5.2) because the series $\Phi_{s+1}(T(\alpha))$ in (5.5) converges by (0.11) with $j=1$. For the same convergence reasons, Equation (5.6) holds as well and the next task is to justify that we can let $J\to+\infty$ in (5.6). On page 536, the convergence of $\sum_{j=0}^{\infty}(\alpha T(\alpha)\cdots T^j(\alpha))^{s-2}\widehat{G}_s(\alpha)$ holds under the convergence of the series (1.14) because $|\widehat{G}_s(\alpha)|\ll 1/\alpha^2$. We still have

$$\lim_{J\to+\infty}(\alpha T(\alpha)\cdots T^J(\alpha))^{s-2}\widehat{\Phi}_s(T^{J+1}(\alpha))=0$$

because, by (0.1), we have

$$\left|\widehat{\Phi}_s(T^{J+1}(\alpha))\right|\ll q_{J+1}(\alpha)^{s-2}\sum_{j=J+1}^{\infty}\frac{q_{j+1}^2}{q_j^s}$$

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where the implicit constant is independent of J. Since $\alpha T(\alpha)\cdots T^J(\alpha) = |q_J\alpha - p_J| < 1/q_{J+1}$, we deduce that

$$\Big|\big(\alpha T(\alpha)\cdots T^J(\alpha)\big)^{s-2}\widehat{\Phi}_s\big(T^{J+1}(\alpha)\big)\Big| \ll \sum_{j=J+1}^{\infty}\frac{q_{j+1}^2}{q_j^s} \to 0$$

when $J\to+\infty$ because of the convergence of the series (1.14). Finally, the series

$$\sum_{j=0}^{\infty}\big(\alpha T(\alpha)\cdots T^{j-1}(\alpha)\big)^{s-2}(T^j(\alpha))^{s-1}\Phi_{s+1}\big(T^{j+1}(\alpha)\big)$$

also converges because by (0.11) (with j changed to $j+1$), we have

$$\Big|\big(\alpha T(\alpha)\cdots T^{j-1}(\alpha)\big)^{s-2}(T^j(\alpha))^{s-1}\Phi_{s+1}\big(T^{j+1}(\alpha)\big)\Big| \ll q_j q_{j+1}\sum_{n=j+1}^{\infty}\frac{q_{n+1}}{q_n^{s+1}}$$

and, as shown above, the series $\sum_{j=0}^{\infty} q_j q_{j+1}\sum_{n=j+1}^{\infty} q_{n+1}/q_n^{s+1}$ converges under (0.10). This corrects the argument given on the first seven lines of page 536, and consequently Equation (5.7) remains true. The rest of the argument on pages 536–537 consists in justifying that we can exchange the summations in a double series. Again the given argument does not hold because on lines -7 and -6, it is in fact not true that the series is bounded independently of j. But as mentioned above, Equation (0.1) leads to the upper bound (0.12) which is enough to prove that the series

$$\sum_{j=0}^{\infty}\big|(\alpha T(\alpha)\cdots T^{j-1}(\alpha))^{s-2}(T^j(\alpha))^{s-1}\big|\sum_{m=0}^{\infty}\big|(T^{j+1}(\alpha)\cdots T^{m+j}(\alpha))^{s}G_{s+1}(T^{m+j+1}(\alpha))\big|$$

converges, and the exchange of summations is still justified by Fubini's theorem. After that point, the rest of the argument is not changed. This corrects the proof of Theorem 1.2.

About Theorem 1.3

The statement of Theorem 1.3 remains valid but a few modifications have to be made to its proof. In Equations (6.1)–(6.3), $[N\alpha]$ must be changed to $[N\alpha+\alpha/2]$, but again this is harmless because we can still let $N\to+\infty$ to obtain (6.4)–(6.6). The proof of Theorem 1.3 then goes unchanged till the final sentence '*We conclude this proof with the remark that it is important that not only are the three series on the left-hand sides of (6.4)–(6.6) convergent but their sums are bounded independently of k (this is ensured by Proposition 3.1)*'. As shown above, the bounds deduced from (0.1) do depend on k and the conclusion can not be obtained as quickly. However, these bounds are of the same form as those used to correct Theorem 1.1 and the same method proves Theorem 1.3.

Reference

1. T Rivoal, On the convergence of Diophantine Dirichlet series, *Proc. Edinburgh Math. Soc.* **55**(2) (2012), 513–541.