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CHARACTERIZATIONS OF THE GENERALIZED HUGHES PLANES

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Let $\mathfrak{P}$ be a projective plane and $\mathfrak{Q}$ a subplane of $\mathfrak{P}$. If l is a line of $\mathfrak{P}$, we let $T(\mathfrak{Q}, l)$ denote the group of all elations in $\mathfrak{P}$ that have l as axis and leave $\mathfrak{Q}$ invariant. In [**12**, p. 921], Ostrom asked for a description of all finite planes $\mathfrak{P}$ that have a Baer subplane $\mathfrak{Q}$ with the property that $|T(\mathfrak{Q}, l)| = o(\mathfrak{Q})^2$ for all lines l of $\mathfrak{Q}$. Here $o(\mathfrak{Q})$ denotes the order of $\mathfrak{Q}$. Both the desarguesian planes of square order and the generalized Hughes planes have this property (Hughes [**10**], Ostrom [**14**], Dembowski [**6**]). One of the aims of this paper is to show that these are the only planes having such a Baer subplane.

Given such a plane, we let G be the group generated by all the groups $T(\mathfrak{Q}, l)$. Letting $q = o(\mathfrak{Q})$, we can show that, if $q \neq 7$, then the group G is isomorphic to $PSL(3, q)$. If $q = 7$, then either $G \cong PSL(3, 7)$, or $G \cong SL(3, 7)$. In all cases, the plane $\mathfrak{P}$ is uniquely determined up to isomorphism by the action of G on $\mathfrak{P}$. For this reason, we start by describing a class of planes, using only the groups $PSL(3, q)$, (or $SL(3, q)$), and certain of their subgroups. It is shown in the course of Theorem 2 and Corollary 2 that these planes are in fact precisely the desarguesian planes of square order and the generalized Hughes planes. However, the new description is more convenient for our purposes.

The proof of Theorem 2 yields Corollaries 1 to 10. Most of the results stated in Corollaries 1 and 3 to 9 are already known. For example, the full collineation group of a generalized Hughes plane has been determined by Rosati [**17**; **19**], who proved in addition, that all such planes are self-dual (Rosati [**18**, **19**]). Also, Corollary 7 was already known to Ostrom [**13**; **14**]. (In this context we must mention that the formulation of Theorem 5.4.3 of Dembowski [**4**, p. 248] is incorrect. It should read "generalized Hughes plane" instead of just "Hughes plane". This error in Dembowski's book probably stems from the fact that Ostrom [**13**] uses the words "Hughes planes" to describe what are usually called "generalized Hughes planes".) Corollary 8 is found nowhere in the literature, while Corollary 9 was previously known only for polarities.

Of exceptional interest is the plane of type **E**, for it admits a collineation group generated by elations and containing Baer collineations. By the same token, the planes of type **C**, **G**, and **H** are of interest, for in these cases the group of collineations generated by all the homologies contains Baer collineations.

Unfortunately, I was unable to make use of the results of Dembowski [**6**] and Unkelbach [**22**] in proving Theorem 2, since the proofs given by both these

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authors are incorrect, even though the results themselves are true. Yaqub [**24**] fills the gap in Dembowski's proof in the case $q \not\equiv 1 \bmod 3$ by taking $SL(3, q)$ instead of $GL(3, q)$. Unkelbach errs on page 156 in his exclusion of the special case $q = 19$. Also, his theorem on page 158, stating that $|\Delta_{P,1} \cap \Sigma| = 2(q+1)$ if $\Delta_{P,1}$ is not cyclic is true only if $q \equiv 3 \bmod 4$. This last error was pointed out to me by Mrs. Yaqub, whom I wish to thank here for the detailed conversation we exchanged recently at the Geometry Conference in Toronto. It was due to these talks that I once again turned my attention to a study of the Hughes planes. I also wish to thank her for her careful reading of the manuscript and the many valuable suggestions she made to improve the presentation of this paper. Finally, I would like to thank Dr. G. Gunther of Toronto for the translation of this paper.

I refer the reader to Dembowski [**4**] and Huppert [**11**] for those concepts and results that I do not specifically define or quote.

LEMMA 1. *Let V be a vector space over K. Suppose $V = P \oplus H$, where $P = wK$ for $w \neq 0$. Suppose further that we are given $\rho, \rho' \in \Gamma L(V)$ such that*

(1) $w^\rho = w^{\rho'} \in P$,

(2) $H^\rho = H = H^{\rho'}$, *and*

(3) *if* $\rho'\rho^{-1} \neq 1$, *then* $\rho'\rho^{-1}$ *is not a transvection.*

If σ and σ' are transvections with axis H, and τ and τ' are transvections with centre P, then $\sigma\rho\tau = \tau'\rho'\sigma'$ implies that $\sigma = \sigma' = 1$ or $\tau = \tau' = 1$.

Proof. Let φ be the linear form of V into K whose kernel is H and for which $\varphi(w) = 1$. We can then find $h, h' \in H$ so that $x^\sigma = x + h\varphi(x)$ and $x^{\sigma'} = x + h'\varphi(x)$ for all $x \in V$. We can also find linear forms ψ and ψ' of V into K with $\psi(w) = 0 = \psi'(w)$ such that $x^\tau = x + w\psi(x)$ and $x^{\tau'} = x + w\psi'(x)$. Since $\sigma\rho\tau = \tau'\rho'\sigma'$, we conclude that ρ and ρ' have the same companion automorphism α. Hence

$$x^{\sigma\rho\tau} = x^\rho + w[\psi(x^\rho) + \psi(h^\rho)\varphi(x)^\alpha] + h^\rho\varphi(x)^\alpha$$

and

$$x^{\tau'\rho'\sigma'} = x^{\rho'} + w^{\rho'}\psi'(x)^\alpha + h'[\varphi(x^{\rho'}) + \varphi(w^{\rho'})\psi'(x)^\alpha].$$

Since $\sigma\rho\tau = \tau'\rho'\sigma'$ and $w^\rho = w^{\rho'}$, we deduce

$$\text{(a)} \quad x^\rho + w[\psi(x^\rho) + \psi(h^\rho)\varphi(x)^\alpha] + h^\rho\varphi(x)^\alpha = x^{\rho'} + w^\rho\psi'(x)^\alpha + h'[\varphi(x^{\rho'}) + \varphi(w^\rho)\psi'(x)^\alpha] \quad \text{for all } x \in V.$$

Also, $w^\rho = w^{\rho'} \in P \subseteq \ker(\psi) \cap \ker(\psi')$ and $\varphi(w) = 1$, and hence we obtain from (a) the equation

$$w\psi(h^\rho) + h^\rho = h'\varphi(w^\rho).$$

Hence $w\psi(h^\rho) \in H \cap P = \{0\}$, and therefore $\psi(h^\rho) = 0$. Thus we have

(b) $\psi(h^\rho) = 0$ and $h^\rho = h'\varphi(w^\rho)$.

Combining (a) and (b), we obtain

(c) $x^\rho - x^{\rho'} + w\psi(x^\rho) - w^\rho\psi'(x)^\alpha + h^\rho[\varphi(x)^\alpha - \varphi(w^\rho)^{-1}\varphi(x^{\rho'}) - \psi'(x)^\alpha] = 0$ for all $x \in V$.

As $P^\rho = P$, we know that $w^\rho = wk$ for some $k \in K$. For $x \in H$, we know from (2) that $x^\rho - x^{\rho'} \in H$, and applying (c) we deduce

$$w[\psi(x^\rho) - k\psi'(x)^\alpha] \in H \cap P = \{0\}.$$

Hence $\psi(x^\rho) = k\psi'(x)^\alpha$ for all $x \in H$. But $P \subseteq \ker(\psi) \cap \ker(\psi')$ and $P^\rho = P$ allow us to write

(d) $\psi(x^\rho) = k\psi'(x)^\alpha$ for all $x \in V$.

Combining (c) and (d) yields

$$x^\rho - x^{\rho'} + h^\rho[\varphi(x)^\alpha - k^{-1}\varphi(x^{\rho'}) - \psi'(x)^\alpha] = 0 \quad \text{for all } x \in V.$$

This implies

$$x^{\rho'\rho^{-1}} = x + h[\varphi(x) - k^{-\alpha^{-1}}\varphi(x^{\rho'})^{\alpha^{-1}} - \psi'(x)].$$

From (b) and (d), we know $k\psi'(h)^\alpha = \psi(h^\rho) = 0$. Hence

$$\varphi(h) - k^{-\alpha^{-1}}\varphi(h^{\rho'})^{\alpha^{-1}} - \psi'(h) = 0,$$

implying that $\rho'\rho^{-1}$ is a transvection. By (3), we therefore conclude that $\rho'\rho^{-1} = 1$, and so

$$h[\varphi(x) - k^{-\alpha^{-1}}\varphi(x^{\rho'})^{\alpha^{-1}} - \psi'(x)] = 0 \quad \text{for all } x \in V.$$

If $h = 0$, then (b) implies that $h' = 0$ and hence $\sigma = \sigma' = 1$. If $h \neq 0$, then

$$\varphi(x) - k^{-\alpha^{-1}}\varphi(x^{\rho'})^{\alpha^{-1}} - \psi'(x) = 0 \quad \text{for all } x \in V.$$

But this implies that $\psi'(x) = 0$ for all $x \in H$. Since $P \subseteq \ker(\psi')$, we therefore conclude that $\psi' = 0$. But then $\psi = 0$ from (d), and hence $\tau = \tau' = 1$ in this case. This completes the proof.

LEMMA 2. *Let V be a vector space over K such that* $\dim V \geqq 3$. *Suppose $V = P \oplus H$ with* $\dim P = 1$. *Let S be a subgroup of $\Gamma L(V)_{P,H}$ that induces a group of fixed-point-free automorphisms on H. We denote by $T(H)$ the group of all transvections whose axis is H, and by $T(P)$ the group of all transvections whose centre is P. Let $A = T(H)S$ and $B = T(P)S$. Then $AB \cap BA = A \cup B$.*

Proof. Certainly $A \cup B \subseteq AB \cap BA$. Hence we need only show that $AB \cap BA \subseteq A \cup B$. Choose $\xi \in AB \cap BA$. Since S normalizes both $T(H)$ and $T(P)$, we know that $AB = T(H)ST(P)$, and $BA = T(P)ST(H)$. Hence we can find $\sigma, \sigma' \in T(H)$, $\rho, \rho' \in S$ and $\tau, \tau' \in T(P)$ such that $\sigma\rho\tau = \xi = \tau'\rho'\sigma'$. From Lemma 1, it is sufficient to show that $\rho = \rho'$. Suppose $P = wK$, and $w^\rho = wa$, $w^{\rho'} = wb$ with $a, b \in K$. Using the notation of Lemma 1, we then deduce

$$x^{\sigma\rho\tau} = x^\rho + w[\psi(x^\rho) + \psi(h^\rho)\varphi(x)^\alpha] + h^\rho\varphi(x)^\alpha$$

and

$$x^{\tau'\rho'\sigma'} = x^{\rho'} + wb\psi'(x)^\alpha + h'[\varphi(x^{\rho'}) + b\psi'(x)^\alpha].$$

Because $\sigma\rho\tau = \tau'\rho'\sigma'$, we therefore deduce

(a) $$x^\rho + w[\psi(x^\rho) + \psi(h^\rho)\varphi(x)^\alpha] + h^\rho\varphi(x)^\alpha = x^{\rho'} + wb\psi'(x)^\alpha + h'[\varphi(x^{\rho'}) + b\psi'(x)^\alpha] \quad \text{for all } x \in V.$$

This implies that $wa + w\psi(h^\rho) + h^\rho = wb + h'b$, and so we have

(b) $\psi(h^\rho) = b - a$ and $h^\rho = h'b$.

Applying (a) to any $x \in H$, we obtain

$$x^\rho + w\psi(x^\rho) = x^{\rho'} + wb\psi'(x)^\alpha + h'b\psi'(x)^\alpha.$$

Thus we have

(c) $x^\rho = x^{\rho'} + h'b\psi'(x)^\alpha$ and $\psi(x^\rho) = b\psi'(x)^\alpha$ for all $x \in H$.

Hence $x^{\rho'\rho^{-1}} = x - h\psi'(x)$ for all $x \in H$. Since $\dim V \geqq 3$, we can therefore find some $u \in \ker(\psi') \cap H$ such that $u \neq 0$. For this choice of u, we have $u^{\rho'\rho^{-1}} = u$. However, we know that $\rho'\rho^{-1} \in S$, and by assumption, S induces a group of fixed-point-free automorphisms on H. Hence we must conclude that $x^{\rho'\rho^{-1}} = x$ for all $x \in H$, implying that $h\psi'(x) = 0$ for all $x \in H$. If $h = 0$, then $h^\rho = 0$, and hence $0 = \psi(h^\rho) = b - a$ from (b). But then $w^{\rho'\rho^{-1}} = w$ and so $\rho'\rho^{-1} = 1$. If $h \neq 0$, then $\psi'(x) = 0$ for all $x \in H$. But then, from (c), we have $0 = b0 = b\psi'(h)^\alpha = \psi(h^\rho)$, and hence $a = b$ also in this case, again implying that $\rho = \rho'$.

Now let V be a 3-dimensional vector space over the field $GF(q)$. Suppose $V = P \oplus H$ with $\dim P = 1$, and let $G = SL(V)$. Let S be a subgroup of $G_{P,H}$ of order $q^2 - 1$ which induces a group of fixed-point-free automorphisms on H. Let $A = T(H)S$ and $B = T(P)S$. From Lemma 2, we have $AB \cap BA = A \cup B$. In addition,

$$|G : A| = |G : B| = (q^2 + q + 1)q(q - 1).$$

We also know that $A \cap B = S$, and hence

$$|G : A \cap B| = (q^2 + q + 1)q^3(q - 1).$$

We now consider the following incidence structure $\mathfrak{J} = \mathfrak{J}(G, S)$: The points of $\mathfrak{J}$ are the right cosets Ax for $x \in G$. The lines of $\mathfrak{J}$ are the right cosets By for $y \in G$. We say that Ax is incident with By exactly if $Ax \cap By \neq \emptyset$. From Higman and McLaughlin [**9**, Lemma 2 and Lemma 4] we know that $\mathfrak{J}$ is an incidence structure containing $(q^2 + q + 1)q(q - 1)$ points and the same number of lines. Every line of $\mathfrak{J}$ contains q^2 points, and every point of $\mathfrak{J}$ lies on q^2

lines. Since $AB \cap BA = A \cup B$, we also know that two distinct points are joined by at most one line. Furthermore, G operates flag-transitively on $\mathfrak{J}$ by means of the mappings $Ax \to Axg$ and $By \to Byg$.

LEMMA 3. *Let Ax be a point. Then the points of $\mathfrak{J}$ that are not connected with Ax, together with Ax, form an orbit of $x^{-1}G_{P,H}T(H)x$.*

Proof. The point Ax lies on q^2 lines, each of which contains $q^2 - 1$ points distinct from Ax. Hence the points of $\mathfrak{J}$ not connected with Ax, together with Ax, form a set containing $(q^2 + q + 1)q(q - 1) - q^2(q^2 - 1) = q(q - 1)$ points. Now the index of A in $G_{P,H}T(H)$ is $q(q - 1)$. Let $x_1, \ldots, x_{q(q-1)}$ be a set of coset representatives of S in $G_{P,H}$. Then it follows that $x_1, \ldots, x_{q(q-1)}$ is also a set of coset representatives of A in $G_{P,H}T(H)$, for $x_i x_j^{-1} \in A$ implies that $x_i x_j^{-1} \in ST(H) \cap G_{P,H} = S[T(H) \cap G_{P,H}] = S$. Hence $x^{-1}x_1x, \ldots, x^{-1}x_{q(q-1)}x$ is a set of coset representatives of $x^{-1}Ax$ in $x^{-1}G_{P,H}T(H)x$, and $x^{-1}Ax$ is the stabilizer of the point Ax in G. Hence the images of Ax under $x^{-1}G_{P,H}T(H)x$ are the points $Ax_1x, \ldots, Ax_{q(q-1)}x$. But the points Ax_ix and Ax_jx are connected if and only if $x_ix_j^{-1} \in ABA$ (cf. Dembowski [**4**, 1.2.8]). Since S normalizes both $T(H)$ and $T(P)$, we know that $ABA = ST(H)T(P)$-$T(H)$. Thus, if $x_ix_j^{-1} \in ABA$, then we can write $x_ix_j^{-1} = \sigma\tau_1\tau_2\tau_3$, with $\sigma \in S$, $\tau_1, \tau_3 \in T(H)$ and $\tau_2 \in T(P)$. Then we have

$$H = H^{x_ix_j^{-1}} = H^{\sigma\tau_1\tau_2\tau_3} = H^{\tau_2\tau_3},$$

and hence $H = H^{\tau_3^{-1}} = H^{\tau_2}$, implying that $\tau_2 = 1$. Also,

$$P = P^{x_ix_j^{-1}} = P^{\sigma\tau_1\tau_3} = P^{\tau_1\tau_3},$$

implying that $\tau_1\tau_3 = 1$. Hence $x_ix_j^{-1} = \sigma \in S$, and so $Ax_i = Ax_j$, or equivalently, $Ax_ix = Ax_jx$.

Similarly, we prove

LEMMA 4. *Let By be a line. Then the lines of $\mathfrak{J}$ that do not intersect By in a point, together with By, form an orbit of $y^{-1}G_{P,H}T(P)y$.*

Let $\mathfrak{Q}$ be the projective plane associated with V. We combine $\mathfrak{Q}$ and $\mathfrak{J}(G, S)$ to define the incidence structure $\mathfrak{P} = \mathfrak{P}(G, S)$, as follows: The points of $\mathfrak{P}$ are the points of $\mathfrak{Q}$ together with the points of $\mathfrak{J}$. The lines of $\mathfrak{P}$ are the lines of $\mathfrak{Q}$ together with the lines of $\mathfrak{J}$. Let X be a point and l be a line of $\mathfrak{P}$.

(a) If X and l are elements of $\mathfrak{Q}$, then X and l are incident in $\mathfrak{P}$ if and only if they are incident in $\mathfrak{Q}$.

(b) If X and l are elements of $\mathfrak{J}$, then X and l are incident in $\mathfrak{P}$ if and only if they are incident in $\mathfrak{J}$.

(c) If $X = P^x$ and $l = By$, then X and l are incident if and only if $P^x = P^y$.

(d) If $X = Ax$ and $l = H^y$, then X and l are incident if and only if $H^x = H^y$.

We shall denote the incidence by I, that is, $P \; I \; l$ if and only if P is incident with l and $P \not I \; l$ otherwise.

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Clearly, the definition of incidence given in (c) and (d) is independent of the choice of representatives x and y.

THEOREM 1. *$\mathfrak{P}(G, S)$ is a projective plane of order q^2 and $\mathfrak{Q}$ is a subplane of $\mathfrak{P}(G, S)$ of order q. The group G operates (not necessarily faithfully) on $\mathfrak{P}(G, S)$ as a group of collineations. For all $x \in G$, the groups $T(H^x)$ are groups of elations of $\mathfrak{P}(G, S)$ leaving $\mathfrak{Q}$ invariant.*

Proof. From the definition of $\mathfrak{P}(G, S)$, we obtain immediately that G operates as a collineation group on $\mathfrak{P}(G, S)$ and leaves $\mathfrak{Q}$ invariant. Now let X and Y be two distinct points of $\mathfrak{P}(G, S)$. We shall show that X and Y have a unique line in common.

Case 1: X and Y are points of $\mathfrak{Q}$. Since G acts transitively on $\mathfrak{Q}$, we may assume that $X = P$. Then $T = P^y$ with $P^y \neq P$. Trivially, $X + Y$ is the unique line of $\mathfrak{Q}$ joining X and Y. So suppose l is another such line. Then $l = Bz$. But then $P\ I\ Bz$ would imply that $P = P^z$, and also $P^y\ I\ Bz$ implies that $P^y = P^z$, yielding the contradiction $P^y = P^z = P \neq P^y$.

Case 2: Suppose $X = P$ and $Y = Ax$. From the definition of incidence in $\mathfrak{P}(G, S)$, we know that Y is on a unique line of $\mathfrak{Q}$, namely H^x. Suppose $P \subseteq H^x$, and suppose that l is a second line containing both P and Y. Then $l = By$. But $P\ I\ By$ implies that $P = P^y$, and $Ax\ I\ By$ implies that $Ax \cap By \neq \emptyset$. Hence there exist $a \in A$ and $b \in B$ such that $ax = by$. But $P = P^b$, and hence $P^{by} = P^y = P$, implying that $P = P^{ax}$. On the other hand, we have $H^a = H$, implying $H^{ax} = H^x$. But $P \not\subseteq H$, and so $P^{ax} \not\subseteq H^{ax}$, or equivalently, $P \not\subseteq H^x$. Thus we have shown that if $P \subseteq H^x$, then P and Y are joined by a unique line. Now suppose that $P \not\subseteq H^x$. Then $P^{x^{-1}} \not\subseteq H$, and so we can find some $a \in A$ such that $P^a = P^{x^{-1}}$, and so $P = P^{ax}$. Let $y = ax$. Then $P = P^y$ and so $P\ I\ By$. Since $Ax \cap By \neq \emptyset$, we conclude that $Ax\ I\ By$, and therefore P and Ax have at least one line in common. Now suppose Bz is a second line joining Ax and P. Then $P = P^z$ and $Ax \cap Bz \neq \emptyset$. Thus we can find $a_0 \in A$ and $b_0 \in B$ such that $a_0x = b_0z$. But then $P^{a_0x} = P^{b_0z} = P^z = P = P^{ax}$, and so $P^{a_0a^{-1}} = P$. But also $H^{a_0a^{-1}} = H$, and hence $a_0a^{-1} \in G_{P,H} \cap A = S$. Since $S \subseteq B$, we can write $a_0 = ca$ for some $c \in B$. But this implies $b_0z = a_0x = cax = cy$, and so $Bz = Bx$.

Case 3: $X = Ax$ and $Y = Ay$. Then H^x and H^y are the unique lines of Q containing X, respectively Y. If X and Y are not joined in $\mathfrak{J}$, then Lemma 3 states that we can find some $z \in G_{P,H}T(H)$ such that $Azx = Ay$. Hence $y = azx$ for some $a \in A$. But then $H^y = H^{azx} = H^x$, and so H^x is the unique line joining X and Y. Conversely, if $H^y = H^x$, then $yx^{-1} \in G_{P,H}T(H)$. Then we can write $y = zx$ with $z \in G_{P,H}T(H)$, implying by Lemma 3 that X and Y are not connected in $\mathfrak{J}$.

Thus we have shown that two distinct points of $\mathfrak{P}(G, S)$ are connected by a unique line. Similarly, one can show that two distinct lines of $\mathfrak{P}(G, S)$ intersect in a unique point. This is sufficient to conclude that $\mathfrak{P}(G, S)$ is a projective plane of order q^2.

The only thing left to show is that $T(H^x)$ has the required properties. It is sufficient to show this for $T(H)$. The points of $\mathfrak{Q}$ contained in H all stay invariant under $T(H)$. The remaining points of H are all of the form A^x where $H^* = H$. In other words, these are the points Ax with $x \in T(H)G_{P,H}$. If we choose $t \in T(H)$, then $xtx^{-1} \in T(H)$ for all $x \in T(H)G_{P,H}$, and hence $Axt = Axtx^{-1}x = Ax$, completing the proof.

We now ask for a construction of such subgroups S of $G_{P,H}$. Since $G = SL(V)$, we know that $\det \sigma = 1$ for all $\sigma \in S$. Now let $P = wK$, and let σ_0 be the restriction of σ to H. Then $w^\sigma = w(\det \sigma_0)^{-1}$. This implies that S operates faithfully on H. Since $|H| = q^2$ and $|S| = q^2 - 1$, we know therefore that S operates transitively on $H\backslash\{0\}$. Since S consists of K-linear transformations, where $K \cong GF(q)$, we see therefore that S and H define a near-field whose kernel contains a subfield isomorphic to $GF(q)$.

Conversely, suppose H is a nearfield of order q^2 whose kernel contains a subfield K isomorphic to $GF(q)$. Letting S_0 be the multiplicative group of H, we know that S_0 consists of K-linear transformations. Let $V = K \oplus H$. Given $\sigma_0 \in S_0$, define σ by $(k + h)^\sigma = k(\det \sigma_0)^{-1} + h\sigma_0$. Let S be the set of all these transformations σ; then S is a group of the required kind.

Types.—**A**) Let $H = GF(q^2)$. In this case, S is cyclic of order $q^2 - 1$. The collineation group induced by G in $\mathfrak{P}(G, S)$ is isomorphic to $PSL(3, q)$. This is trivial if $q \not\equiv 1 \bmod 3$, as $PSL(3, q) \cong SL(3, q)$ in this case. Now suppose that $q \equiv 1 \bmod 3$. In this case, S contains the transformation $x \to x\zeta$ with $\zeta \in GF(q)^*$ and $o(\zeta) = 3$. Hence $|\mathfrak{Z}(G)| = 3$. Moreover $\mathfrak{Z}(G) \subseteq S = A \cap B$. Since $G/\mathfrak{Z}(G)$ is simple, we know from Higman and McLaughlin [**9**, Prop. 1] that $G/\mathfrak{Z}(G)$ is isomorphic to the collineation group induced by G on $\mathfrak{P}(G, S)$. The collineation group induced by S in $\mathfrak{P}(G, S)$ is also cyclic, and has order $d^{-1}(q^2 - 1)$, where d is the greatest common divisor of 3 and $q - 1$. We shall see later that the plane $\mathfrak{P}(G, S)$ is desarguesian in this latter case.

B) Suppose q is odd. Let $H = GF(q^2)$. In H, we define a new multiplication $\circ$ by

$$x \circ y = \begin{cases} xy & \text{if } y \text{ is a square in } H, \\ x^q y & \text{if } y \text{ is not a square in } H. \end{cases}$$

It is well known and easily checked that $H(+, \circ)$ is a nearfield. Let S_0 be the multiplicative group of $H(+, \circ)$. In S_0, the set

$$\{x | x \in H\backslash\{0\}, x \text{ is a square in } GF(q^2)\}$$

is a cyclic subgroup of index 2 containing the multiplicative group of $GF(q)$. As for type **A**, we can show that the collineation group induced by G in $\mathfrak{P}(G, S)$ is isomorphic to $PSL(3, q)$. The collineation group induced by S has order $d^{-1}(q^2 - 1)$, where d is again the greatest common divisor of 3 and $q - 1$. In this case, S is not abelian; However, it contains a cyclic subgroup of index 2. It turns out that the planes of type **B** are exactly the planes constructed by Hughes [**10**].

For details on the exceptional nearfields used in the following examples, we refer the reader to Zassenhaus [**25**] (see also Lemma 7).

C) Let $q = 5$, and let H be the exceptional nearfield of order 5^2. Let S_0 be the multiplicative group of H. Then S_0 is isomorphic to $SL(2, 3)$. Since $5 \not\equiv 1 \bmod 3$, we know that $PSL(3, 5) \cong SL(3, 5)$, and hence the collineation group induced by G in $\mathfrak{P}(G, S)$ is isomorphic to $PSL(3, 5)$. In addition, $S \cong SL(2, 3)$.

D) Let $q = 11$, and let H be the exceptional nearfield of order 11^2, whose multiplicative group S_0 is soluble. Then we have $S_0 \cong Z_5 \times SL(2, 3)$, where Z_5 is the cyclic group of order 5. As $11 \not\equiv 1 \bmod 3$, we know that $G \cong PSL(3, 11)$, and that S operates faithfully on $\mathfrak{P}(G, S)$.

E) Let $q = 7$, and let H be the exceptional nearfield of order 7^2. In this case, S_0 contains a normal subgroup of index 2, which is isomorphic to $SL(2, 3)$. The Sylow 3-subgroup of S_0 is of order 3 and is not normal. Hence $S \cap \mathfrak{Z}(G) = \{1\}$. The collineation group induced by G is therefore isomorphic to $SL(3, 7)$, and S operates faithfully on $\mathfrak{P}(G, S)$.

F) Let $q = 23$, and let H be the exceptional nearfield of order 23^2. Since $SL(3, 23) \cong PSL(3, 23)$, we know that the group G operates faithfully on $\mathfrak{P}(G, S)$. Furthermore, we know that $S \cong Z_{11} \times S_0$, where S_0 is the group described under **E**).

G) Let $q = 11$, and let H be the exceptional nearfield of order 11^2 whose multiplicative group is isomorphic to $SL(2, 5)$. In this case, we have $G \cong PSL(3, 11)$; further, G and S both operate faithfully on $\mathfrak{P}(G, S)$.

H) Let $q = 29$, and let H be the exceptional nearfield of order 29^2. Then $G \cong PSL(3, 29)$, and $S \cong Z_7 \times SL(2, 5)$.

I) Let $q = 59$, and let H be the exceptional nearfield of order 59^2. Then $G \cong PSL(3, 59)$ and $S \cong Z_{29} \times SL(2, 5)$.

It will turn out that the planes of type **B** to **I** are precisely the generalized Hughes planes.

LEMMA 5. *Let q be a power of the prime p. Let Ω be a set containing $q(q - 1)$ elements. Let G be a group, isomorphic to $SL(2, q)$, operating on Ω. Suppose that the action of the Sylow p-subgroup of G on Ω is regular. Then the permutation group H induced in Ω by G is isomorphic to $PSL(2, q)$, except in the following cases: If $q = 2$, then $|H| = 2$ and if $q = 3$, then $|H| = 3$ or $H \cong PSL(2, 3)$.*

Proof. Suppose $p = 2$. If $q = 2$, then $|\Omega| = 2$, and hence $|H| = 2$. So suppose $q > 2$. Since q is even, we know $PSL(2, q) \cong SL(2, q)$, and since $q > 2$, we know that $PSL(2, q)$ is simple. Since the action of G on Ω is non-trivial, we therefore conclude that $H \cong PSL(2, q)$.

Now suppose $p > 2$. Since $G \cong SL(2, q)$, we know that G contains a unique involution σ. If $q > 3$, then $\langle \sigma \rangle$ is the only proper normal subgroup of G. If $q = 3$, then $\langle \sigma \rangle$ is contained in every proper normal subgroup of G. We now assume that σ does not induce the identity on Ω. Then $G = H$. Choosing $\alpha \in \Omega$, we then have $q(q^2 - 1) = |\alpha^H||H_\alpha|$. But $|\alpha^H| \leqq q(q - 1)$, and therefore $|H_\alpha| \geqq q + 1$. Furthermore $(q, |H_\alpha|) = 1$. Using this and assuming that $|H_\alpha|$ is odd, we deduce from Dickson [**7**, § 260] that $|H_\alpha| \leqq \frac{1}{2}(q + 1)$. Hence H_α has even order, implying $\sigma \in H_\alpha$. As this holds for all $\alpha \in \Omega$, we see that σ does induce the identity in Ω.

If $q > 3$, then $G/\langle\sigma\rangle$ is simple. Hence in this case we have $H \cong PSL(2, q)$.

We know that $PSL(2, 3) \cong A_4$. Hence we have in the case $q = 3$ that either $|H| = 3$ or $H \cong PSL(2, 3)$.

The next lemma can in essence already be found in Unkelbach [**22**, Lemma 4.4].

LEMMA 6. *Let q be a power of 2, and let Ω be a set containing $q(q - 1)$ elements. Suppose G is a group, operating on Ω, isomorphic to $PSL(2, q)$. Suppose further that the action of the Sylow 2-subgroups on Ω is regular. Then G operates transitively on Ω. For $\alpha \in \Omega$, the group G_α is cyclic of order $q + 1$, and G_α has exactly one other fixed point.*

Proof. Choose $\alpha \in \Omega$. Then $q(q^2 - 1) = |\alpha^G||G_\alpha|$. We have $|\alpha^G| \leqq q(q - 1)$ and hence $|G_\alpha| \geqq q + 1$. Since 2 is not a divisor of $|G_\alpha|$, we use Dickson [**7**, § 260] to deduce that G_α is cyclic of order $q + 1$. This implies that G operates transitively on Ω. Furthermore, again by Dickson (*loc. cit.*), we have $|\mathfrak{N}_G(G_\alpha) : G_\alpha| = 2$, so consequently, G_α has exactly one more fixed point.

We can prove an analogous lemma for odd q. However, in this case one finds over 30 different ways in which G can operate on Ω. As most of these cases have no bearing on the following, we prove a lemma for odd q that only yields those cases which are important in what follows. We should mention that this lemma, too, has a predecessor in the literature (Prohaska [**16**]).

LEMMA 7. *Let q be a power of the prime $p > 2$. Let Ω be a set containing $q(q - 1)$ elements. Suppose G is a group operating transitively on Ω. Suppose furthermore that G satisfies the following conditions:*

a) *The action of the Sylow p-subgroups of G on Ω is regular.*

b) *G contains a normal subgroup S which is isomorphic to $SL(2, q)$.*

c) *The group of automorphisms induced by G on S is isomorphic to $PGL(2, q)$. If H is the group induced by S on Ω, then either $q = 3 = |H|$ or H is isomorphic to $PSL(2, q)$. If $q = 3 = |H|$, then H has exactly two orbits of length 3 in Ω and the Sylow 2-subgroup of G operates trivially on Ω. If $H \cong PSL(2, q)$, then we have one of the following cases:*

1) *H operates transitively on Ω and for all $\alpha \in \Omega$, the group H_α is cyclic of order $\frac{1}{2}(q + 1)$. In addition, H_α has exactly one more fixed point.*

2) *$q \equiv 3 \bmod 4$ with $q > 7$, and H is transitive on Ω. The group H_α is the dihedral group of order $\frac{1}{2}(q + 1)$, and H_α has exactly one more fixed point.*

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3) *$q > 3$ and H has two orbits of length $\frac{1}{2}q(q-1)$. For $\alpha \in \Omega$, the stabilizer H_α of α is the dihedral group of order $q+1$, and H_α has exactly one other fixed point β. The points α and β lie in distinct orbits of H.*

4) *$q = 11$ and H has 10 orbits of length 11. Moreover, $H_\alpha \cong A_5$ and H_α has exactly five fixed points, no two of which lie in the same orbit of H.*

5) *$q = 19$ and H has 6 orbits of length $19 \cdot 3$. The stabilizer H_α is isomorphic to A_5 and has exactly 3 fixed points, no two of which lie in the same orbit of H.*

6) *$q = 29$ and H has 4 orbits of length $29 \cdot 7$. Furthermore, $H_\alpha \cong A_5$ and H_α has exactly two fixed points that lie in distinct orbits of H.*

7) *$q = 59$ and H has two orbits of length $59 \cdot 29$. We have $H_\alpha \cong A_5$ and H_α has exactly one fixed point.*

8) *$q = 7$ and H has 6 orbits of length 7. Moreover, $H_\alpha \cong S_4$ and H_α has exactly 3 fixed points, no two of which lie in the same orbit of H.*

9) *$q = 23$ and H has two orbits of length $23 \cdot 11$. If $\alpha \in \Omega$, then $H_\alpha \cong S_4$ and α is the only fixed point of H_α.*

10) *$q = 5$ and H has 4 orbits of length 5. For $\alpha \in \Omega$, we have $H_\alpha \cong A_4$, and H_α has exactly one fixed point in each orbit of H.*

11) *$q = 11$, and H has two orbits of length $11 \cdot 5$. For $\alpha \in \Omega$, we have $H_\alpha \cong A_4$, and H_α has exactly one fixed point in each orbit of H.*

Proof. Since the action of the Sylow p-subgroups of G is regular on Ω, and since G is transitive on Ω, we know that the Sylow p-subgroups of G all have order q. Since $|S| = q(q^2-1)$, we therefore conclude that all the Sylow p-subgroups of G lie in S. Hence we may use Lemma 5 to conclude that either $q = 3 = |H|$, or $H \cong PSL\ (2, q)$. If $q = 3 = |H|$, then $|\Omega| = 3 \cdot 2$, and so H has exactly two orbits of length 3. In this case, G contains exactly one Sylow 2-subgroup, which therefore operates trivially on Ω. For the following, we therefore assume that $H \cong PSL\ (2, q)$. Then we know that $|H| = \frac{1}{2}q(q^2-1)$. For $\alpha \in \Omega$, we then have $\frac{1}{2}q(q^2-1) = |\alpha^H||H_\alpha|$. Since q divides $|\alpha^H|$, we deduce that $|H_\alpha| \leqq \frac{1}{2}(q^2-1)$. Furthermore, $(q, |H_\alpha|) = 1$. On the other hand, we have $|\alpha^H| \leqq q(q-1)$, and so $|H_\alpha| \geqq \frac{1}{2}(q+1)$. Hence, by Dickson [**7**, § 260], H_α must be one of the following:

α) H_α is cyclic of order $\frac{1}{2}(q+1)$.

β) H_α is a dihedral group of order $q+1$.

γ) H_α is a dihedral group of order $q-1$.

δ) H_α is an elementary abelian group of order 4.

ϵ) $q \equiv 3 \bmod 4$ with $q > 3$, and H_α is a dihedral group of order $\frac{1}{2}(q+1)$.

ζ) $q^2 \equiv 1 \bmod 10$ and H_α is isomorphic to A_5.

η) $q^2 \equiv 1 \bmod 16$ and H_α is isomorphic to S_4.

ϑ) H_α is isomorphic to A_4.

In all cases, we have that H_α and H_β are conjugate in $\bar{G}$ for all $\alpha, \beta \in \Omega$, where $\bar{G}$ denotes the permutation group induced by G on Ω. Moreover, all the orbits of H have the same length. These two remarks follow from the fact that H is normal in $\bar{G}$ and that $\bar{G}$ acts transitively on Ω.

Ad α). In this case, H is transitive on Ω. We have $|\mathfrak{N}_G(H_\alpha) : H_\alpha| = 2$, and hence H_α has exactly two fixed points. This yields Case 1.

Ad β). Here H has exactly two orbits of length $\frac{1}{2}q(q-1)$. If $q > 3$, then H_α is a maximal subgroup of H and hence has exactly one fixed point in α^H. For $\beta \in \Omega\backslash\alpha^H$, the group H_α and H_β are conjugate in H, and hence H_α also has exactly one fixed point in $\Omega\backslash\alpha^H$. If $q = 3$, then H_α is the normal subgroup of H of order 4. Since it is a characteristic subgroup of H, we know that $\bar{G}$ normalizes H_α. Hence H_α induces the identity in Ω. But this means that the case $q = 3$ cannot occur, and thus we have Case 3.

Ad γ). Here all orbits would be of length $\frac{1}{2}q(q+1)$, which is obviously impossible, unless $q = 3$. But $q - 1 \geqq 4$, as H_α is a dihedral group.

Ad δ). In this case, $\frac{1}{2}(q+1) \leqq 4$, and hence $q = 3, 5$ or 7. If $q = 3$, we could show as above that $|H| = 3$, which is not the case. Hence we are left with $q = 5$ or $q = 7$. If $q = 5$, then H would have an orbit of length $5 \cdot 3$. But $5 \cdot 3$ is not a divisor of $|\Omega| = 5 \cdot 4$. If $q = 7$, then H would be transitive on Ω. But $7 \equiv -1 \bmod 8$. Using Dickson [**7**, § 260], we therefore know that $PSL\ (2, 7)$ contains two conjugate classes of Klein 4-groups which fuse under $PGL\ (2, 7)$ into one conjugate class. Combining this with c) yields a contradiction.

Ad ϵ). If $q = 7$, then H_α is a Klein 4-group, which, as we saw, is impossible. Hence $q > 7$. But this implies that $|\mathfrak{N}_H(H_\alpha) : H_\alpha| = 2$, and hence we have Case 2.

Ad ζ). Here we have $\frac{1}{2}(q+1) \leqq 60 \leqq \frac{1}{2}(q^2-1)$. Combining this with $q^2 \equiv 1 \bmod 10$ and $(q, 60) = 1$, we obtain

$$q \in \{11, 19, 29, 31, 41, 49, 59, 61, 71, 89, 101, 109\}.$$

But this, together with the fact that $q(q^2-1)/120$ is a divisor of $q(q-1)$, yields that $q \in \{11, 19, 29, 59\}$. In all these cases, we know from Dickson [**7**, § 259] that $PSL\ (2, q)$ contains exactly two conjugacy classes of groups that are isomorphic to A_5. These two conjugacy classes fuse under $PGL\ (2, q)$ into one conjugacy class. Moreover, all these groups are their own normalizers. Hence H_α will have a fixed point in exactly half the orbits, giving Cases 4), 5), 6), and 7).

Ad η). In this case, $\frac{1}{2}(q+1) \leqq 24 \leqq \frac{1}{2}(q^2-1)$. Combining this with $q^2 \equiv 1 \bmod 16$ yields $q \in \{7, 17, 23, 31, 41, 47\}$. But $q(q^2-1)/48$ is a divisor of $q(q-1)$, and hence $q = 7, 23$, or 47. Now, using Dickson [**7**, § 257], gives us Cases 8) and 9).

Ad ϑ). Here we have $\frac{1}{2}(q+1) \leqq 12 \leqq \frac{1}{2}(q^2-1)$, implying that $q \in \{5, 7, 11, 13, 17, 23\}$. But $q(q^2-1)/24$ is a divisor of $q(q-1)$. Therefore, $q+1$ is a divisor of 24. Hence $q \in \{5, 7, 11, 23\}$. If $q = 5$ or 11, then by Dickson [**7**, § 257], H contains exactly one conjugacy class of subgroups isomorphic to A_4, and in addition, all these groups are their own normalizers. This then yields Cases 10) and 11).

If $q = 7$, then H has precisely 3 orbits of length $7 \cdot 2$. Furthermore, by Dickson [**7**, § 257], H contains exactly two conjugacy classes of subgroups isomorphic to A_4. These two conjugacy classes fuse into one under $PGL\ (2, 7)$. Hence H_α has exactly two fixed points in each of exactly half of the orbits which is impossible, as the number of orbits is odd. By similar means, we can show that $q = 23$ cannot arise. Thus we have proved the lemma.

LEMMA 8. *Let $\mathfrak{P}$ be a projective plane. Let $\mathfrak{Q}$ be a subplane of $\mathfrak{P}$. Suppose that $\mathfrak{Q}$ is a Moufang plane and that every elation of $\mathfrak{Q}$ is induced by an elation of $\mathfrak{P}$. Let G be the group generated by all those elations of $\mathfrak{P}$ that induce elations in $\mathfrak{Q}$, and let G' be the commutator subgroup of G. Then $G = G'$.*

Proof. The following proof is due to P. Dembowski. He gave this proof several years ago at Oberwolfach, while showing that the little projective group of a Moufang plane is simple. To the best of my knowledge, this proof has been published only in my lecture notes [**12**]. Because of the limited circulation of these notes, I shall reproduce the proof here. Another proof which works only for finite planes has been published by Hering in [**8**].

First we observe that G operates transitively on the set of triples (P, Q, l) where P and Q are distinct points of $\mathfrak{Q}$ and where l is a line of $\mathfrak{Q}$ such that $P, Q \not I l$. Let σ and τ be two non-trivial elations of $\mathfrak{P}$ that induce elations in $\mathfrak{Q}$. Let l be the axis of σ and m be the axis of τ. Choose points P and Q in $\mathfrak{Q}$ with $P \not I l$ and $Q \not I m$. Then $P \neq P^\sigma \not I l$ and $Q \neq Q^\tau \not I m$. Hence there exists $\gamma \in G$ with $P^\gamma = Q$, $P^{\sigma\gamma} = Q^\tau$ and $l^\gamma = m$. Hence $\gamma^{-1}\sigma\gamma$ is an elation with axis m. Also, $Q^{\gamma^{-1}\sigma\gamma} = P^{\sigma\gamma} = Q^\tau$, and hence $\gamma^{-1}\sigma\gamma = \tau$. Therefore, all the non-trivial elations in G are conjugate in G.

Finally, let l and m be distinct lines of $\mathfrak{Q}$. Let $B = l \cap m$ and choose a second point $A \neq B$ in $\mathfrak{Q}$ lying on l. Let $\sigma \neq 1$ be an elation in G with centre A and axis l, and let $\tau \neq 1$ be an elation in G with centre B and axis m. Then $\tau^{-1}\sigma\tau$ is an elation with axis l, and hence also $\sigma^{-1}\tau^{-1}\sigma\tau$ is such an elation. If we assume $\sigma^{-1}\tau^{-1}\sigma\tau = 1$, then $A^\tau = A$ and hence $\tau = 1$. Therefore $\sigma^{-1}\tau^{-1}\sigma\tau \neq 1$. Hence one and therefore all elations of G are commutators, proving $G = G'$.

LEMMA 9. *Let q be a prime power such that $q \equiv 1 \bmod 3$. Suppose 3^r is the highest power of 3 that divides $q - 1$. Let Σ be a Sylow 3-subgroup of $PSL\ (3, q)$. Then we can find elements $\alpha, \beta \in \Sigma$ such that $\Sigma = \langle \alpha, \beta \rangle$ and*

a) $\alpha^3 = 1$.
b) $\beta^{3^r} = 1$.
c) $\alpha\beta^{3^{r-1}} = \beta^{3^{r-1}}\alpha$.
d) $\beta^{-1}\alpha^{-1}\beta\alpha = \alpha^{-1}\beta\alpha\beta^{-1}$.
e) $\alpha^{-1}\beta^{-1}\alpha\beta^{-1} = \alpha\beta\alpha^{-1}$.

If P, Q, R are three non-collinear points of the desarguesian projective plane of order q with $\{P, Q, R\}^\Sigma = \{P, Q, R\}$, then we can choose the generators α and β in such a way that $\beta^{-1}\alpha^{-1}\beta\alpha$ is a $(P, Q + R)$-homology of order 3^{r-1}. Conversely, if Σ_0 is a group with generators α and β and the presentation a) to e), then Σ_0 is isomorphic to Σ.

Proof. As $q \equiv 1 \bmod 3^r$, we have $q^2 + q + 1 \equiv 3 \bmod 3^r$. Hence $q^2 + q + 1$ is divisible by 3. If $r > 1$, then $q^2 + q + 1$ is not divisible by 9. If $r = 1$, then $q \equiv 4$ or $7 \bmod 9$, and hence $q^2 + q + 1 \equiv 3 \bmod 9$. Thus we see that also in this case 9 does not divide $q^2 + q + 1$. Since

$$|PSL\ (3, q)| = \tfrac{1}{3}(q^2 + q + 1)q^3(q + 1)(q - 1)^2$$

we deduce that $|\Sigma| = 3^{2r}$.

Let $\{b_1, b_2, b_3\}$ be a basis of the 3-dimensional vector space over $K = GF(q)$. Let $P = b_3K$, $Q = b_1K$ and $R = b_2K$. Let ζ be an element of order 3^r in the multiplicative group of K. Let α and β be the collineations defined with respect to the given basis by the matrices

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \quad \text{respectively} \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \zeta & 0 \\ 0 & 0 & \zeta^{-1} \end{bmatrix}.$$

Since $\det A = \det B = 1$, we know that $\alpha, \beta \in PSL\ (3, q)$. Obviously, $\alpha^3 = 1 = \beta^{3^r}$. In addition.

$$A^{-1}BA = \begin{bmatrix} \zeta^{-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \zeta \end{bmatrix}$$

and hence

$$B^{-1}A^{-1}BA = A^{-1}BAB^{-1} = \begin{bmatrix} \zeta^{-1} & 0 & 0 \\ 0 & \zeta^{-1} & 0 \\ 0 & 0 & \zeta^2 \end{bmatrix}.$$

Consequently, $\beta^{-1}\alpha^{-1}\beta\alpha = \alpha^{-1}\beta\alpha\beta^{-1}$ is a $(P, Q + R)$-homology. A simple check shows that its order is 3^{r-1}. Similarly, it is easy to see that $\alpha^{-1}\beta^{-1}\alpha\beta^{-1} = \alpha\beta\alpha^{-1}$. Hence $\alpha^{-1}\beta\alpha$ and β generate an abelian group T that is normalized by α. Since T leaves invariant the three points P, Q, and R, we deduce that $\alpha \notin T$. Hence the order of $T\langle\alpha\rangle$ is equal to $3|T|$.

From $\alpha^{-1}\beta^i\alpha = \beta^j$, we deduce

$$\begin{bmatrix} \zeta^{-i} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \zeta^i \end{bmatrix} = \begin{bmatrix} \gamma & 0 & 0 \\ 0 & \gamma & 0 \\ 0 & 0 & \gamma \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \zeta^j & 0 \\ 0 & 0 & \zeta^{-j} \end{bmatrix}.$$

But $\det B = 1 = \det\ (A^{-1}BA)$, and hence $\gamma^3 = 1$. Furthermore, we have $\zeta^{-i} = \gamma$, $1 = \gamma\zeta^j$ and $\zeta^i = \gamma\zeta^{-j}$. But this implies that $i = j$ and $3i \equiv 0 \bmod 3^r$, and therefore $i \equiv 0 \bmod 3^{r-1}$. Thus we have shown that $|\langle\alpha^{-1}\beta\alpha\rangle \cap \langle\beta\rangle| \leqq 3$. On the other hand, if $j = i = 3^{r-1}$ and $\gamma = \zeta^{-i}$, then clearly

$$A^{-1}B^iA = \begin{bmatrix} \gamma & 0 & 0 \\ 0 & \gamma & 0 \\ 0 & 0 & \gamma \end{bmatrix} B^i.$$

Hence c) holds; also $|\langle\alpha^{-1}\beta\alpha\rangle \cap \langle\beta\rangle| = 3$, implying that $|T| = 3^{2r-1}$. Consequently $T\langle\alpha\rangle$ is a Sylow 3-subgroup of $PSL(3, q)$. The second assertion of the lemma now follows, since $PSL(3, q)$ is transitive on the set of ordered triangles of the desarguesian plane of order q.

Now suppose G is a group generated by the elements α and β with the defining relations a) to e). Then Σ is a homomorphic image of G. From d), we know that $N = \langle\beta, \alpha^{-1}\beta\alpha\rangle$ is an abelian subgroup of G. From b) and c) we deduce that the order of N is 3^{2r-1}. From e) and a) it follows that N is a normal subgroup of G. Hence $G = N\langle\alpha\rangle$, and so from a), we conclude that $|G| = 3^{2r}$.

THEOREM 2. *Suppose* $\mathfrak{P}$ *is a finite projective plane. Then the following conditions are equivalent:*

a) $\mathfrak{P}$ *is isomorphic to one of the planes of type* **A** *to* **I**.

b) $\mathfrak{P}$ *contains a Baer-subplane* $\mathfrak{Q}$ *with the property that* $|T(\mathfrak{Q}, l)| = o(\mathfrak{Q})^2$ *for all lines* l *of* $\mathfrak{Q}$.

c) $\mathfrak{P}$ *contains a proper subplane* $\mathfrak{Q}$ *with the property that the subgroup of the full collineation group of* $\mathfrak{P}$ *which leaves invariant* $\mathfrak{Q}$ *operates flag-transitively on* $\mathfrak{P}\backslash\mathfrak{Q}$.

d) $\mathfrak{P}$ *contains a Baer-subplane* $\mathfrak{Q}$ *with the property that the subgroup of the full collineation group of* $\mathfrak{P}$ *which leaves* $\mathfrak{Q}$ *invariant is doubly transitive on the set of points of* $\mathfrak{Q}$.

Proof. Assume that $\mathfrak{P}$ is a plane of one of the types **A** to **I**. Then it follows from Theorem 1 that $\mathfrak{P}$ satisfies b). c) follows from b) by Dembowski [**5**, (4) p. 131]. (We note that Dembowski's proof of (4) only makes use of b).) We now show that d) follows from c). In this case, then, $\mathfrak{P}$ has a proper subplane $\mathfrak{Q}$ and a collineation group G that fixes $\mathfrak{Q}$ and operates flag-transitively on $\mathfrak{P}\backslash\mathfrak{Q}$. This means in particular that G is transitive on the set of lines of $\mathfrak{P}\backslash\mathfrak{Q}$. Let P be a point of $\mathfrak{Q}$ and let l be a line of $\mathfrak{P}\backslash\mathfrak{Q}$ passing through P. Such a line exists, since $\mathfrak{Q}$ is a proper subplane of $\mathfrak{P}$. Letting m be any line of $\mathfrak{P}\backslash\mathfrak{Q}$, we can find some $\gamma \in G$ such that $l^\gamma = m$. Hence m contains the point P^γ of $\mathfrak{Q}$. This implies that every line of $\mathfrak{P}$ carries a point of $\mathfrak{Q}$, and hence $\mathfrak{Q}$ is a Baer-subplane of $\mathfrak{P}$. Then d) follows from c) by Dembowski [**5**, proof of Theorem 5.1].

We now show that b) follows from d). For this, we let G be a collineation group of $\mathfrak{P}$ which leaves $\mathfrak{Q}$ invariant and which is doubly transitive on the set of points of $\mathfrak{Q}$. By Ostrom & Wagner [**15**, Theorem 5], we know that $\mathfrak{Q}$ is desarguesian, and the collineation group H induced on $\mathfrak{Q}$ by G contains the little projective group $PSL(3, q)$, where q is the order of $\mathfrak{Q}$. Since $PSL(3, q)$ operates doubly transitively on $\mathfrak{Q}$, we may assume that $H = PSL(3, q)$. Now let K be the kernel of the restriction of G to $\mathfrak{Q}$.

Case 1: Suppose q is a power of 2. Let l be a line of $\mathfrak{Q}$ and let Σ be a Sylow 2-subgroup of G that leaves l invariant. Then Σ contains a subgroup T that induces the group of all elations with axis l in $\mathfrak{Q}$. We have $|\mathrm{T}| \geqq q^2$. From Dembowski [**5**, Theorem 3.5] there exists $\tau \in \mathrm{T}$ which induces an involution in $\mathfrak{Q}$ and which in addition has a fixed point on l not belonging to $\mathfrak{Q}$. Furthermore, $\langle \tau \rangle$ operates faithfully on $\mathfrak{Q}$, implying $\tau^2 = 1$. Since τ has more than $q + 1$ fixed points on l, we deduce that τ is not a Baer involution. Hence $\tau \in T(Q, l)$, implying that $T(\mathfrak{Q}, l) \neq \{1\}$.

Case 2: Suppose q is odd. We first show that for every non-incident point-line pair (P, l) of $\mathfrak{Q}$ there exists in G an involutory (P, l)-homology of $\mathfrak{P}$. Let Σ be a Sylow 2-subgroup of $G_{P,l}$. Let 2^r be the highest power of 2 that divides $q - 1$, and 2^s the highest power of 2 that divides $|K|$. Then we can find a subgroup Z of Σ such that $|Z| = 2^{k+s}$ and Z induces a group of (P, l)-homologies of order 2^r in $\mathfrak{Q}$. There are $q(q - 1)$ points on l not belonging to $\mathfrak{Q}$, and hence there exists some A on l, A not in $\mathfrak{Q}$, such that $|A^Z| \leqq 2^r$. Hence $|Z_A| \geqq 2^s$. Let σ be an involution in Z_A. As σ has more than $q + 1$ fixed points on l, we know that σ is an involutory homology of $\mathfrak{P}$, and hence σ is an involutory (P, l)-homology of $\mathfrak{P}$. We may therefore assume that $Z_A = \{1\}$ for all A. Hence $|K|$ is odd and $|A^Z| = 2^r$ for all the possible choices of A. Now 2^{r+1} is not a divisor of $q(q - 1)$, and so we can find some A on l such that $|\Sigma| = 2^r|\Sigma_A|$. Since $|Z \cap \Sigma_A| = 1$, we deduce that $\Sigma = Z\Sigma_A$. We saw that $|K|$ is odd, and hence that Σ operates faithfully on $\mathfrak{Q}$. Let us now denote the groups induced in $\mathfrak{Q}$ by the same symbols Σ, Z and Σ_A. The group $H_{P,l}$ is isomorphic to a subgroup of $GL(2, q)$ which contains $SL(2,q)$. In particular, Z lies in the centre of $H_{P,l}$. Since Σ splits over Z, we can use Gaschütz' Theorem (see, for example, Huppert [**11**, Theorem 1.17.4]) to find a complement C of Z in $H_{P,l}$. This complement C will then contain $SL(2, q)$. But since $|SL(2, q) \cap Z| = 2$, we then deduce the contradiction $2 \leqq |C \cap Z| = 1$. This contradiction shows that there does exist an involutory (P, l)-homology of $\mathfrak{P}$. As this holds for all choices of non-incident point-line pairs (P, l), we therefore have $T(\mathfrak{Q}, l) \neq \{1\}$ for all lines l of $\mathfrak{Q}$ (André [**1**]).

As we saw, H is the little projective group of $\mathfrak{Q}$. Hence H is transitive on the set of triples (P, Q, l), where P and Q are distinct points of $\mathfrak{Q}$ and l is a line of $\mathfrak{Q}$ which contains neither P nor Q. This implies that $|T(\mathfrak{Q}, l)| = q^2$ for all lines l of $\mathfrak{Q}$, and hence we have b).

Finally, we show that b) implies a). Let q be the order of $\mathfrak{Q}$, so that q^2 is the order of $\mathfrak{P}$. In the following, we write $T(l)$ instead of $T(\mathfrak{Q}, l)$. We let $G = \langle T(l) | l \text{ is a line of } \mathfrak{Q} \rangle$. The plane $\mathfrak{Q}$ is desarguesian, and G induces the little projective group of $\mathfrak{Q}$ in $\mathfrak{Q}$. By Lemma 8, we know that $G = G'$. We let H be the restriction of G to $\mathfrak{Q}$, and K the kernel of this restriction.

(1) $K = \mathfrak{Z}(G)$.

Since $G/K \cong PSL(3, q)$, and $PSL(3, q)$ is simple, we know that $\mathfrak{Z}(G) \subseteq K$. On the other hand, choose $\tau \in T(l)$ and $\kappa \in K$. Then $\kappa^{-1}\tau\kappa \in T(l^\kappa) = T(l)$. If P is any point of $\mathfrak{Q}$ with $P\ I\ l$, then also $P^{\kappa^{-1}\tau\kappa} = P^{\tau\kappa} = P^\tau$, and hence $\kappa^{-1}\tau\kappa = \tau$, implying that $\kappa \in \mathfrak{Z}(G)$.

Let l and m be distinct lines of $\mathfrak{Q}$. Suppose $P = l \cap m$, and $T(P, m)$ is the group of all elations in $T(m)$ whose centre is P. Then $T(P, m)$ normalizes

$T(l)$. Hence $T(P, m)T(l)$ is a group of order q^3 which operates faithfully on $\mathfrak{Q}$, and hence its intersection with K is trivial. Now $q = p^r$ for some prime p. Let Π_0 be a Sylow p-subgroup of K. Then $\Pi = \Pi_0 T(P, m)T(l)$ is a group of order $|\Pi_0|q^3$, and so Π is a Sylow p-subgroup of G. By (1), Π_0 is an abelian normal subgroup of G. Also Π_0 has a complement in Π. Hence, by Gaschütz' Theorem (*loc. cit.*), we conclude that Π_0 has a complement C in G. This implies by (1) that C is normal in G. The factor group G/C is isomorphic to Π_0, and is therefore abelian. Hence $G = G' \subseteq C$, implying that $\Pi_0 = \{1\}$. Consequently, p does not divide $|K|$.

Now let l be a line of $\mathfrak{Q}$. Since $\mathfrak{Q}$ is a maximal subplane of $\mathfrak{P}$, we know that the action of K on the set of points of l not belonging to $\mathfrak{Q}$ is regular. Hence $|K|$ divides $q(q - 1)$. Since p does not divide $|K|$, we have therefore proved

(2) *$|K|$ divides $q - 1$.*

We next observe

(3) *If q is odd, then the Sylow 2-subgroup of H is isomorphic to a Sylow 2-subgroup of $GL(2, q)$.*

This follows from the observation that every Sylow 2-subgroup of H leaves invariant a non-incident point-line pair of $\mathfrak{Q}$, and from the fact that the index of $PSL(3, q)$ in $PGL(3, q)$ is either 1 or 3.

(4) *If $q \equiv 3 \bmod 4$, then $|K|$ is odd.*

Let Σ be a Sylow 2-subgroup of H. From (3) and from Carter & Fong [**2**, p. 142], we then deduce that Σ is a semi-dihedral group. From Schur [**21**, p. 108], we know that the Schur multiplier of this group must be $\{1\}$. Hence the Schur multiplier of $PSL(3, q)$ is of odd order (Schur [**20**, Theorem X, p. 49]). Since $G = G'$ and $K = \mathfrak{Z}(G)$, we therefore know from Schur [**20**, Theorem II, p. 31], that K is isomorphic to a subgroup of the Schur multiplier of $PSL(3, q)$. But this implies (4).

(5) *If $q = 2$ or 3, then $K = \{1\}$.*

This follows at once from (2), resp. (2) and (4).

(5) *Let (P, l) be a non-incident point-line pair of $\mathfrak{Q}$. Let L be the group generated by all the $T(Q, QP)$, where Q is a point of l belonging to $\mathfrak{Q}$. Then L is isomorphic to $SL(2, q)$.*

Let $\bar{L}$ be the restriction of L to $\mathfrak{Q}$. Since $\mathfrak{Q}$ is desarguesian, we know that $\bar{L} \cong SL(2, q)$. Also $\bar{L} \cong L/(L \cap K)$. If $q = 2$ or 3, then by (5) we know that $L \cap K = \{1\}$, and hence in these cases, we have $L \cong SL(2, q)$. So assume that $q > 3$, then $\bar{L}' = \bar{L}$, implying that $L = L'(L \cap K)$. Using (2), we then deduce that $T(Q, QP) \subseteq L'$ for all Q, and hence $L' = L$. But this implies that $L \cap K \subseteq L' \cap \mathfrak{Z}(L)$. By Schur (*loc. cit.*) we therefore conclude that $L \cap K$ is isomorphic to a subgroup of the Schur multiplier of $SL(2, q)$. Now suppose that

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$L \cap K \neq \{1\}$. Then from Schur [**21**, Theorem IX, p. 119] and the following remarks we could deduce that $q = 4$ and $|L \cap K| = 2$, or that $q = 9$ and $|L \cap K| = 3$, both in contradiction to (2). Thus $L \cap K = \{1\}$, implying that $L \cong \bar{L} \cong SL(2, q)$.

Let (P, l) be a non-incident point-line pair of $\mathfrak{Q}$. Then $G_l = T(l)\, G_{P,l}$ and so the groups G_l and $G_{P,l}$ induce the same permutation group on l. Let Ω be the set of points of l not belonging to $\mathfrak{Q}$. Then the proof of (4) in Dembowski [**5**, p. 131] shows that G_l, and hence $G_{P,l}$ operate transitively on Ω. Here one uses the fact that every point of Ω lies on exactly one line of $\mathfrak{Q}$, namely l. Let L be the group described in (6). Then L is normal in $G_{P,l}$ and the group of automorphisms induced in L be $G_{P,l}$ is isomorphic to $PGL(2, q)$. Using (2), we see that the Sylow p-subgroups of $G_{P,l}$ are the groups $T(Q, QP)$. As their action on Ω is regular, we may apply Lemmas 5, 6 and 7.

Our next aim is to prove

(7) *Suppose* $|K| > 2$. *Then* $q = 7$ *or* $q = 19$ *and* $|K| = 3$ *in either case. Let* $\bar{L}$ *be the permutation group induced on* Ω *by* L, *and choose* $\alpha \in \Omega$. *If* $q = 7$, *then the group* $\bar{L}_\alpha$ *is isomorphic to* S_4. *If* $q = 19$, *then* $\bar{L}_\alpha$ *is isomorphic to* A_5.

To show this, we choose some $\alpha \in \Omega$. Now $K = \mathfrak{Z}(G)$, and the action of K on Ω is regular. Hence $|K|$ divides the number of fixed points of L_α. Using Lemmas 6 and 7, we therefore know that $q = 11, 19, 7$ or 5. If $q = 7$ or 19, then $q \equiv 3 \bmod 4$, and hence, by (4), $|K|$ is odd. Thus Lemma 7 yields that $|K| = 3$, as well as the required properties of $\bar{L}_\alpha$.

Now suppose $q = 11$. By Lemma 7, we deduce that $|K| = 5$. Since $11 \not\equiv 1 \bmod 3$, we know that $G_{P,l}$ induces $GL(2, 11)$ on $\mathfrak{Q}$. Let Σ be the pre-image in G of the group of all (P, l)-homologies in $\mathfrak{Q}$. Let Π be a Sylow 5-subgroup of Σ. Then $|\Pi| = 25$. Since Π fixes all the points of l belonging to $\mathfrak{Q}$, we know that Π centralizes L. Let Π_1 be a Sylow 5-subgroup of $L\Pi$. Then $\Pi \subseteq \Pi_1$, and $\Pi_1 = \Pi\Pi_2$, where Π_2 is a Sylow 5-subgroup of L. But $|\Pi_2| = 5$, and hence Π_1 is abelian. The group $\Pi \cap K$ operates transitively on the five fixed points of L_α. Hence $\Pi = (\Pi \cap K)\Pi_\alpha$, implying that $\Pi_\alpha\Pi_2$ is a complement of $\Pi \cap K$ in Π_1. We have

$$|G| = 5|PSL(3, 11)| = 5(11^2 + 11 + 1)11^3(11 + 1)(11 - 1)^2,$$

and hence Π_1 is a Sylow 5-subgroup of G. By Gaschütz' Theorem (*loc. cit.*) we therefore conclude that $\Pi \cap K$ has a complement in G, in contradiction to $G = G'$.

We are left with the case $q = 5$. By Lemma 7, $|K| = 4$ in this case. Again, let Σ be the pre-image of the group of all (P, l)-homologies of $\mathfrak{Q}$. Then $|\Sigma| = 16$ and $\Sigma = K\Sigma_\alpha$. Furthermore, we know that Σ is abelian, since Σ/K is cyclic. Let m be a line of $\mathfrak{Q}$ passing through P, and let Q be a point of $\mathfrak{Q}$ on l other than $l \cap m$. Define L_1 with respect to Q and m the same way L was defined for P and l. Now $\frac{1}{2}5(5 + 1)$ is odd, and so there exists exactly one Sylow 2-subgroup T of L_1 satisfying $\{P, l \cap m\}^{\mathrm{T}} = \{P, l \cap m\}^{\mathrm{T}}$. Since Σ fixes all the points of l belonging to $\mathfrak{Q}$, we therefore conclude that Σ normalizes T. Also, T contains a unique involution. Since this involution has Q and $l \cap m$ as its only fixed points on l, we see that $\mathrm{T} \cap \Sigma = \{1\}$. We know $|SL(2, 5)| = 120$, and hence $|\mathrm{T}| = 8$, implying that $|\mathrm{T}\Sigma| = 16 \cdot 8$. We also have

$$|G| = 4|PSL(3, 5)| = 4(5^2 + 5 + 1)5^3(5 + 1)(5 - 1)^2,$$

and so we know that $\mathrm{T}\Sigma$ is a Sylow 2-subgroup of G. But note $\mathrm{T}\Sigma = K(\Sigma_\alpha \mathrm{T})$, and so, again using the Theorem of Gaschütz, we obtain a contradiction. This completes the proof of (7).

(8) *If $q \neq 7$ or 19, then $K = \{1\}$.*

Suppose $K \neq \{1\}$. Then $|K| = 2$ by (7), and by (2) and (4) $q \equiv 1 \bmod 4$. Let Σ be the pre-image of a Sylow 2-subgroup of the group of all (P, l)-homologies of $\mathfrak{Q}$ that belong to H. Then $|\Sigma| = 2^{r+1}$, where 2^r is the highest power of 2 that divides $q - 1$. Suppose there exists $\alpha \in \Omega$ with $|\alpha^\Sigma| = 2$. Then $\alpha^\Sigma = \alpha^K$ and thus $\Sigma = K\Sigma_\alpha$. Let m be some line of $\mathfrak{Q}$ passing through P. On l, choose a point Q of $\mathfrak{Q}$ distinct from $l \cap m$. Define L_1 with respect to Q and m the same way L was defined for P and l. Since $q \equiv 1 \bmod 4$, there exists a unique Sylow 2-subgroup T of L_1 satisfying $\{P, l \cap m\}^{\mathrm{T}} = \{P, l \cap m\}$. Hence T is normalized by Σ. The group T contains a unique involution. Its only fixed points on l are Q and $l \cap m$. Hence $\Sigma \cap \mathrm{T} = \{1\}$. This implies that the order of $\Sigma\mathrm{T}$ is 2^{2r+2}. We have

$$|G| = 2|PSL(3, q)| = d^{-1}2(q^2 + q + 1)q^3(q + 1)(q - 1)^2$$

where $d = (3, q - 1)$. Hence $\Sigma\mathrm{T}$ is a Sylow 2-subgroup of G. On the other hand, $\Sigma\mathrm{T} = K(\Sigma_\alpha\mathrm{T})$, and so the Theorem of Gaschütz again provides a contradiction. We conclude that $|\alpha^\Sigma|$ is divisible by 4 for all $\alpha \in \Omega$. Since Σ centralizes L, this implies that the number of fixed points of L_α is divisible by 4. Using Lemma 7, we therefore deduce that $q = 5$. Since the group Σ/K is cyclic of order 4, we can find $\sigma \in \Sigma$ such that $\Sigma\langle\sigma\rangle = K$. If $K \cap \langle\sigma\rangle = \{1\}$, then the argument used above again yields a contradiction. Hence $K \cap \langle\sigma\rangle \neq \{1\}$, and thus $K \subseteq \langle\sigma\rangle$, implying that Σ is a cyclic group of order 8. Since the unique involution of Σ is contained in K, we know that Σ has no fixed points in Ω. But this implies that 8 divides $|\Omega| = 5 \cdot 4$, which is a contradiction. Thus we have proved (8).

(9) *If $q = 19$, then $G \cong PSL(3, 19)$, and for $X \in \Omega$, the group L_X is not isomorphic to $SL(2, 5)$.*

If G is not isomorphic to $PSL(3, 19)$, then by (4) and (7), the order of K is equal to 3, and $L_X \cong SL(2, 5)$. If $G \cong PSL(3, 19)$, assume that $L_X \cong SL(2, 5)$.

Let Σ_0 be a Sylow 3-subgroup of L_X. Then there exist exactly two points Q and R of $\mathfrak{Q}$ lying on l that remain fixed by Σ_0. Let Σ be a Sylow 3-subgroup of $G_{\{P,Q,R\}}$ that contains Σ_0. Then Σ is also a Sylow 3-subgroup of G. Let

$\Sigma_1 = \Sigma_{P,Q,R}$. Then Σ_1 is a normal subgroup of index 3 in Σ, and Σ_1 normalizes the group $L_{Q,R}$. But this group is cyclic of order 18, and hence Σ_1 normalizes Σ_0. By Lemma 9, we know that Σ_1/K contains three cyclic groups of order 9, and thus Σ_1/K contains $3 \cdot 6 = 18$ elements of order 9. Furthermore, Σ_1/K contains an elementary abelian 3-group A of order 9. As $18 + 9 = 27 = |\Sigma_1/K|$, we see therefore, that $\Sigma_0K/K \subseteq A$. But Σ_0 does not induce a group of homologies in $\mathfrak{Q}$, since Σ_0 is a subgroup of L of order 3. Hence $\Sigma_0K/K = \mathfrak{Z}(\Sigma/K)$.

By Lemma 9, there exist $\alpha, \beta \in \Sigma$ and $i_1, \ldots, i_5 \in K$ so that $\Sigma/K = \langle \alpha K, \beta K \rangle$ and α, β satisfy the relations:

a) $\alpha^3 = i_1$.
b) $\beta^9 = i_2$.
c) $\alpha\beta^3 = i_3\beta^3\alpha$.
d) $\beta^{-1}\alpha^{-1}\beta\alpha = i_4\alpha^{-1}\beta\alpha\beta^{-1}$.
e) $\alpha^{-1}\beta^{-1}\alpha\beta^{-1} = i_5\alpha\beta\alpha^{-1}$.

Also we can choose α and β in such a way that $\beta^{-1}\alpha^{-1}\beta\alpha$ induces a (P, l)-homology of order 3 in $\mathfrak{Q}$.

$G_{P,l}$ induces a group of automorphisms in L which is isomorphic to the group $PGL(2, 19)$. Moreover, $G_{P,l} \subseteq GL(2, 19)$. We therefore conclude from Dickson [**7**, § 259], that $G_{X,XP} = Z \times L_X$, where Z is a cyclic group of order 1 or 3 which is contained in the centre of $G_{P,l}$. Since $G_{P,l}$ acts transitively on Ω, we therefore conclude that Z is a group of (P, l)-homologies of $\mathfrak{P}$.

Now let $K \neq \{1\}$. Then $|Z| = 3$. Hence there exists some $z \in Z$ and some $i \in K$ such that $\beta^{-1}\alpha^{-1}\beta\alpha = zi$, implying that $1 = (\beta^{-1}\alpha^{-1}\beta\alpha)^3$. This in turn implies that

$$\begin{aligned} 1 &= (\beta^{-1}\alpha^{-1}\beta\alpha)(\beta^{-1}\alpha^{-1}\beta\alpha)i_4\alpha^{-1}\beta\alpha\beta^{-1} = i_4\beta^{-1}(\alpha^{-1}\beta\alpha\beta^{-1})\alpha^{-1}\beta^2\alpha\beta^{-1} \\ &= i_4 i_4^{-1}\beta^{-1}(\beta^{-1}\alpha^{-1}\beta\alpha)\alpha^{-1}\beta^2\alpha\beta^{-1} = \beta^{-2}\alpha^{-1}\beta^3\alpha\beta^{-1} = i_3^{-1}. \end{aligned}$$

Hence β^3 lies in the centre of Σ. This conclusion is equally valid if we assume that $K = \{1\}$. But then $\langle \beta^3 \rangle K/K \subseteq \mathfrak{Z}(\Sigma/K) = \Sigma_0K/K$, and so $\Sigma_0K = \langle \beta^3 \rangle K$. Thus $\Sigma_0K = \mathfrak{Z}(\Sigma)$. But this implies in particular that $\Sigma_0^\alpha = \Sigma_0$. Hence Σ_0 fixes some subplane $\mathfrak{N}$ of order m pointwise, since Σ_0 has more than two fixed points on l. The points P, Q, and R are the only fixed points of Σ_0 in $\mathfrak{Q}$ and PQ, QR and RP are the only fixed lines of Σ_0 in $\mathfrak{Q}$. Now suppose W is some point of $\mathfrak{N}$ not belonging to $\mathfrak{Q}$. Then W lies on a unique line of $\mathfrak{Q}$. This line is fixed by Σ_0, since both W and $\mathfrak{Q}$ stay invariant under Σ_0. Hence W lies on one of the three lines PQ, QR or RP. Since these are lines of $\mathfrak{N}$, we know that each of them carries $m + 1$ points of $\mathfrak{N}$. Hence we have $3(m - 1) = m^2 + m + 1 - 3 = (m - 1)(m + 2)$. Thus $3 = m + 2$, or equivalently $m = 1$. This is a contradiction, and hence (9) is proved.

(10) *If $q \neq 7$, then $G \cong PSL(3, q)$. If $q = 7$, then either $G \cong PSL(3, 7)$ or $G \cong SL(3, 7)$.*

If G is not isomorphic to $PSL(3, q)$, then by (8) and (9), we know that

$q = 7$. Then (4) and (7) imply that $|K| = 3$. Since $PSL(3, 7) = PSL(3, 7)'$, we know from Schur [**20**, Theorem IV, p. 38] that $PSL(3, 7)$ has exactly one representation group D. By Schur [**21**, Theorem III, p. 99], there exists a subgroup Z of the centre of D such that $G \cong D/Z$. For the same reasons, there also exists a subgroup Z_0 of the centre of D with $SL\ (3, 7) \cong D/Z_0$. We must show $Z = Z_0$. Certainly, $Z \cap Z_0$ contains all elements of $\mathfrak{Z}(D)$ whose orders are not divisible by 3. Furthermore, $3|Z| = 3|Z_0| = |\mathfrak{Z}(D)|$. Hence we need only show that 9 is not a divisor of $|\mathfrak{Z}(D)|$. By Lemma 9, the Sylow 3-subgroup Σ of $PSL(3, 7)$ is an elementary abelian group of order 9. By Schur [**21**, Theorem VIII, p. 113], we know that the Schur multiplier of Σ is of order 3. Again by Schur [**20**, Theorem X, p. 49], we therefore conclude that 9 is not a divisor of $|\mathfrak{Z}(D)|$. This proves (10).

From Dembowski [**5**, (4) p. 131], we know that G is flag transitive on $\mathfrak{P}\backslash\mathfrak{Q}$. Let (X, y) be a flag of $\mathfrak{P}\backslash\mathfrak{Q}$. Let $A = G_X$, $B = G_y$ and $C = G_{X,y}$. Let l be the unique line of $\mathfrak{Q}$ through X, and let P be the unique point of $\mathfrak{Q}$ on y. Then $T(\mathfrak{Q}, l) \subseteq A$ and $T(\mathfrak{Q}, P) \subseteq B$, where $T(\mathfrak{Q}, P)$ denotes the group of all elations with centre P that leave $\mathfrak{Q}$ invariant. Thus $A = T(\mathfrak{Q}, l)C$, $B = T(\mathfrak{Q}, P)C$ and $A \cap B = C$. By Higman & McLaughlin [**9**] we see therefore that $\mathfrak{P}\backslash\mathfrak{Q}$ is uniquely determined by C. But by Dembowski [**4**, p. 317, footnote 1], we know that $\mathfrak{P}$ is uniquely determined by $\mathfrak{P}\backslash\mathfrak{Q}$ up to isomorphism. Hence, all we must show is that there exists a plane $\mathfrak{P}(G, S)$ of type **A** to **I**, so that C is conjugate in $G_{P,l}$ to the collineation group induced by S on $\mathfrak{P}(G, S)$.

If $G \cong PSL\ (3, q)$, then $|C| = (3, q - 1)^{-1}(q^2 - 1)$, if $G \cong SL(3, 7)$, then $|C| = 7^2 - 1$. Furthermore, $L_X = C \cap L$, and L_X is normal in C since L is normal in $G_{P,l}$.

If q is even, then by Lemma 6, $|L_X| = |\bar{L}_X| = q + 1$ and L_X is cyclic. The normalizer of L_X in $G_{P,l}$ has order $2(3, q - 1)^{-1}(q^2 - 1)$ (by Dickson [**7**, § 260]), and is a dihedral group. But C is contained in this normalizer and is of order $(3, q - 1)^{-1}(q^2 - 1)$, and hence is cyclic of this order. It follows from Dickson [**7**, § 260], that all the cyclic subgroups of order $(3, q - 1)^{-1}(q^2 - 1)$ are conjugate in $G_{P,l}$. Hence $\mathfrak{P}$ is of type **A** in this case.

Now suppose q is odd. The involutions of G are all contained in the same conjugacy class. By Lemma 5, the only involution in L is a (P, l)-homology. Hence all the involutions in G are homologies, that also induce homologies in $\mathfrak{Q}$. Now let σ be an involution in C. Then $X^\sigma = X$, and hence σ is a (P, l)-homology. Thus C contains exactly one involution. This implies that the Sylow 2-subgroups of C are either cyclic, or generalized quaternion groups.

Suppose $q \equiv 1 \bmod 4$. Let σ be an element of order 4 in C. Then $\sigma^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$. Since $q \equiv 1 \bmod 4$, we know that σ operates reducibly, and hence there exists an element ζ of order 4 in $GF(q)$, so that σ is conjugate to $\begin{bmatrix} \zeta & 0 \\ 0 & \zeta^i \end{bmatrix}$. This implies that $\zeta^2 = -1 = \zeta^{2i}$. Hence $i = 1$ or $i = 3$. If $i = 3$, then $\det \sigma = 1$, and so $\sigma \in L_X$.

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Now suppose $\bar{L}_X$ is of type 3) as in Lemma 7. Then the order of L_X is $2(q+1)$, and L_X contains elements of order 4. Choose some $\sigma \in C \backslash L_X$ such that $o(\sigma) = 4$. Then by the above argument, $\sigma = \begin{bmatrix} \zeta & 0 \\ 0 & \zeta \end{bmatrix}$, and hence σ lies in the centre of C. On the other hand, L_X contains cyclic groups of order 4, and this implies that the Sylow 2-subgroups of C are neither cyclic nor generalized quaternion groups. This is a contradiction. Consequently, all elements of order 4 belong to L_X. Since $|L_X| = 2(q+1)$, the Sylow 2-subgroups of L_X have order 4 and hence are cyclic. Since the elements of order 4 in C already lie in L_X, we deduce that the Sylow 2-subgroups of C are also cyclic. Suppose 2^r is the highest power of 2 that divides $q-1$. Then C contains an element $\rho = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ of order 2^{r+1}. This element must operate irreducibly, whereas ρ^2 acts reducibly. Hence we have

$$\begin{bmatrix} \lambda & 0 \\ 0 & \lambda^i \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a^2 + bc & (a+d)b \\ a(a+d) & cb + d^2 \end{bmatrix}.$$

Where λ is an element of $GF(q)^*$ of order 2^r. If $a + d \neq 0$, then $b = c = 0$, implying that ρ is reducible. Hence $a = -d$, and so $\lambda = \lambda^i$. But the element

$$\begin{bmatrix} \lambda^{2^{r-2}} & 0 \\ 0 & \lambda^{2^{r-2}} \end{bmatrix}$$

is of order 4, and hence lies in L_X. Therefore, the determinant of this matrix is 1. But this yields $1 = \lambda^{2^{r-1}} = -1$, a contradiction. Thus $\bar{L}_X$ is not of type 3).

If $\bar{L}_X$ is of type 1), then L_X is a cyclic group of order $q+1$, and hence contains no element of order 4. Therefore, the elements of order 4 in C lie in the centre of C, implying that the Sylow 2-subgroups of C are cyclic. If C itself is cyclic, then as above, we show that $\mathfrak{P}$ is of type **A**. So assume that C is not cyclic, and choose some element ρ in C such that the order of ρ is 2^{r+1}. Again as above, we see that ρ^2 lies in the centre of $G_{P,l}$. Moreover, ρ^2 generates the Sylow 2-subgroup of $\mathfrak{Z}(G_{P,l})$. Hence, as L_X has precisely two fixed points in Ω, we have $\mathfrak{Z}(G_{P,l}) \subseteq C$. The group $C/\mathfrak{Z}(G_{P,l})$ is not cyclic, as otherwise C would be abelian, and hence cyclic. Therefore, $C/\mathfrak{Z}(G_{P,l})$ is a dihedral group of order $q+1$. The group $PGL(2, q)$ contains exactly two conjugacy classes of dihedral groups of order $q+1$, the groups of one of these classes all lie in $PSL(2, q)$, whereas the groups of the other class do not. Now $\rho\mathfrak{Z}(G_{P,l}) \notin L\mathfrak{Z}(G_{P,l})/\mathfrak{Z}(G_{P,l})$, and hence all the groups C in question are conjugate in $G_{P,l}$. Thus $\mathfrak{P}$ is of type **B**.

Now suppose $q \equiv 3 \bmod 4$. In this case the Sylow 2-subgroups of $G_{P,l}$ are quasidihedral groups by Carter & Fong [**2**, p. 142]. These contain only one generalized quaternion group of index 2, which consequently lies in L. If $\bar{L}_X$ is a dihedral group, then the Sylow 2-subgroup of C is a generalized quaternion group, which therefore must lie in L_X, implying by comparing orders that $\bar{L}_X$ cannot be of type 2) in Lemma 7. Assume for the moment that $q = 3$. Then $|C| = 8$. If C is a quaternion group, then $C = L_X$, i.e. C is a Sylow 2-subgroup of L. Hence $\mathfrak{P}$ is of type **B**. If C is cyclic, then $\mathfrak{P}$ is of type **A**, since all the cyclic subgroups of order 8 are conjugate in $GL(2, 3)$. So suppose $q > 3$, and suppose $\bar{L}_X$ is of type 1), 2) or 3) of Lemma 7. Then by the above, $\bar{L}_X$ is of type 1) or 3). Suppose $\bar{L}_X$ is of type 3). Then $|L_X| = 2(q + 1)$ and $|L_X \cap \mathfrak{Z}(G_{P,l})| = 2$. This implies by Lemma 7, that $\mathfrak{Z}(G_{P,l}) \subseteq C$, and hence $C = \mathfrak{Z}(G_{P,l})L_X$. Since all subgroups of L that are isomorphic to L_X are conjugate to L_X in L, we deduce that all the groups C under discussion are conjugate in $G_{P,l}$. Hence $\mathfrak{P}$ is of type **B**.

Now suppose $\bar{L}_X$ is cyclic of order $\frac{1}{2}(q + 1)$. Then L_X is cyclic of order $q + 1$, and the Sylow 2-subgroups of C are cyclic. Let Σ be a Sylow 2-subgroup of C. Since Σ operates irreducibly, we know that the centralizer D of Σ in $G_{P,l}$ has order $(3, q - 1)^{-1}(q^2 - 1)$ (see, for example, Huppert [**11**, II.7.3]). Furthermore, D is cyclic. The number of fixed points of Σ in Ω is even, as $|\Omega|$ is even. It is actually 2. This can be seen as follows: Since all the cyclic subgroups of order $\frac{1}{2}(q + 1)$ of $PSL(2, q)$ are conjugate within $PSL(2, q)$, the group $\bar{L}$ acts on l as a subgroup of $PGL(2, q^2)$ in its usual action on the projective line over $GF(q^2)$. Hence the identity is the only permutation in $\bar{L}$ fixing more than two points on l. From this we infer that $\Sigma \cap L_X$ and hence Σ has exactly two fixed points in Ω. Hence both these fixed points are fixed points of D. Thus D and C are conjugate in $G_{P,l}$, implying that C is cyclic. Hence $\mathfrak{P}$ is of type **A**.

We are left with a discussion of types 4) to 11) of Lemma 7.

Ad 4: Here $L_X \cong SL(2, 5)$ and $|C| = 11^2 - 1 = |L_X|$. Hence $C = L_X$. From Dickson [**7**, § 259], we deduce that $\mathfrak{P}$ is of type **G**.

Ad 5: By (9), this case does not occur.

Ad 6: Here $L_X \cong SL(2, 5)$, and $|C| = 29^2 - 1 = 7|L_X|$. From Dickson [**7**, § 259] and $G_{P,l} \subseteq GL(2, 29)$, we deduce $C = ZL_X$, where Z is the subgroup of order 7 in the centre of $G_{P,l}$. Since all subgroups of $SL(2, 29)$ that are isomorphic to $SL(2, 5)$ are conjugate to L_X [**7**, § 259], we know that $\mathfrak{P}$ is of type **H**.

Ad 7: Here $L_X \cong SL(2, 5)$, and $|C| = 29|L_X|$. As under *Ad* 6, we deduce that $\mathfrak{P}$ is of type **I**.

Ad 8: Here $\bar{L}_X \cong S_4$, and $|C| = \frac{1}{3}d(7^2 - 1)$, where $d = 1$ if $G \cong PSL(3, 7)$ and $d = 3$ if $G \cong SL(3, 7)$. Since $|S_4| = 24$, we know that $|C|$ is divisible by 3. Hence $d = 3$, and so $G \cong SL(3, 7)$. From Dickson [**7**, § 257] and $G_{P,l} \subseteq GL(2, 7)$ we deduce that $\mathfrak{P}$ is of type **E**.

Ad 9: Again, $\bar{L}_X \cong S_4$. Further, we have $|C| = 23^2 - 1 = 11|L_X|$. From Dickson [**7**, § 257] and $G_{P,l} \subseteq GL(2, 11)$, we deduce that $C = ZL_X$, where Z is the subgroup of order 11 of $\mathfrak{Z}(G_{P,l})$. Hence we have type **F**.

Ad 10: We have $L_X \cong SL(2, 3)$, and $|C| = 24 = |L_X|$. Then Dickson [**7**, § 257] implies that we have type **C**.

Ad 11: We have $L_X \cong SL(2, 3)$ and $|C| = 11^2 - 1 = 5|L_X|$. Hence, $C = ZL_X$, where Z is the subgroup of order 5 of $\mathfrak{Z}(G_{P,l})$. This yields type **D**.

This completes the proof of the theorem.

The proof of Theorem 2 also yields:

COROLLARY 1. *Let* $\mathfrak{P}_i(i = 1, 2)$ *be two projective planes of order* q^2. *Let* $\mathfrak{Q}_i$ *be a Baer subplane of* $\mathfrak{P}_i$, *and let* $G_i = \langle T(\mathfrak{Q}_i, l) | l$ *is a line of* $\mathfrak{Q}_i \rangle$. *Let* S_i *be the stabilizer in* G_i *of a flag of* $\mathfrak{P}_i \backslash \mathfrak{Q}_i$. *If* G_i *acts flag-transitively on* $\mathfrak{P}_i \backslash \mathfrak{Q}_i$, *then* $\mathfrak{P}_1$ *and* $\mathfrak{P}_2$ *are isomorphic if and only if* S_1 *and* S_2 *are isomorphic.*

From Corollary 1 and Dembowski [**6**, Theorem 5.7], we obtain:

COROLLARY 2. *The planes of type* **A** *are the desarguesian planes of square order and the planes of type* **B** *to* **I** *are the generalized Hughes planes.*

The next corollary is a direct consequence of Corollary 1. It was proved by Rosati [**18**; **19**].

COROLLARY 3. *If* $\mathfrak{P}$ *is a generalized Hughes plane, then* $\mathfrak{P}$ *is self-dual.*

COROLLARY 4. *Let* C *and* L *be defined as above. Then* C *has exactly two fixed points on* Ω, *except in the following cases:*

a) $q = 11$, *and* $C = L_X \cong SL(2, 5)$. *In this case,* C *has exactly five fixed points.*
b) $q = 59$, *and* $L_X \cong SL(2, 5)$. *In this case,* C *has exactly one fixed point.*
c) $q = 7$, *and* $\bar{L}_X \cong S_4$. *Then* C *has exactly three fixed points.*
d) $q = 23$, *and* $\bar{L}_X \cong S_4$. *Then* C *has exactly one fixed point.*
e) $q = 5$, *and* $C = L_X \cong SL(2, 3)$. *Then* C *has exactly four fixed points.*
f) $q = 3$, *and* $C = L_X$ *is the quaternion group of order* 8. *Then* C *has exactly six fixed points.*

With the exception of Case f), all this was shown in the proof of Theorem 2. But f) follows from the fact that $SL(2, 3)$ contains precisely one quaternion group, and that $L_X \cong L_Y$ for all $X, Y \in \Omega$.

COROLLARY 5 (Rosati). *Let* K *be the group of all collineations of* $\mathfrak{P}$ *which fix* $\mathfrak{Q}$ *pointwise. Then* $|K| = 2$, *unless we have:*

a) $q = 11$, *and* $L_X \cong SL(2, 5)$. *Then* $|K| = 5$.
b) $q = 59$, *and* $L_X \cong SL(2, 5)$. *Then* $|K| = 1$.
c) $q = 7$, *and* $\bar{L}_X \cong S_4$. *Then* $|K| = 3$.
d) $q = 23$, *and* $\bar{L}_X \cong S_4$. *Then* $|K| = 1$.
e) $q = 5$, *and* $L_X \cong SL(2, 3)$. *Then* K *is cyclic of order* 4.
f) $q = 3$, *and* L_X *is the quaternion group of order* 8. *Then* $K \cong S_3$.

Proof. K centralizes G (see the proof of (1) in the proof of Theorem 2). Further, the action of K on Ω is regular. Hence $|K|$ divides the number of fixed points of C in Ω. Let X, Y be two fixed points of C in Ω. Then $A = G_X = T(\mathfrak{Q}, l)C = G_Y$, and $B = G_{PX} = T(\mathfrak{Q}, P)C = G_{PY}$. By Higman & McLaughlin [**9**], we can therefore find an isomorphism σ of $\mathfrak{P} \backslash \mathfrak{Q}$ onto $\mathfrak{J}(G, C)$ with $(X^\gamma)^\sigma = A\gamma$, and an isomorphism τ from $\mathfrak{P} \backslash \mathfrak{Q}$ onto $\mathfrak{J}(G, C)$ with $(Y^\gamma)^\tau = A\gamma$ for all $\gamma \in G$. Thus, $\sigma\tau^{-1}$ is a collineation of $\mathfrak{P} \backslash \mathfrak{Q}$ such that $X^{\gamma\sigma\tau^{-1}} = Y^\gamma$ for all $\gamma \in G$. But $l^\gamma = X^\gamma Y^\gamma = X^\gamma X^{\gamma\sigma\tau^{-1}}$. Since $\sigma\tau^{-1}$ has an extension to a collineation of $\mathfrak{P}$ (by Dembowski [**4**, p. 317, footnote 1]), we deduce that $\sigma\tau^{-1}$

is a collineation fixing all the lines of $\mathfrak{Q}$, as there passes exactly one line of $\mathfrak{Q}$ through each point of $\mathfrak{P} \backslash \mathfrak{Q}$. Since X and Y were arbitrarily chosen fixed points of C in Ω, we conclude that $|K|$ is the number of fixed points of C in Ω. This, combined with Corollary 3, yields the result except for showing that K is cyclic in Case e), and isomorphic to S_3 in Case f).

In Case e), $G \cong PGL(3, 5)$. By Dickson [**7**, § 257], we can find a 2-element ξ in the normalizer of C which induces an involutory automorphism in C such that $\xi\, \mathfrak{Z}(G_{P,l}) \notin L\mathfrak{Z}(G_{P,l})/\mathfrak{Z}(G_{P,l})$. Hence $o(\xi) = 8$ and $\xi^2 \in \mathfrak{Z}(G_{P,l})$. Thus the fixed points of C are the points X, X^{ξ}, X^{ξ^2} and X^{ξ^3}, if X is a fixed point of C. Now we can find $\kappa \in K$ such that $X^{\kappa} = X^{\xi}$. Since K centralizes the group G, we know that $X^{\xi^i \kappa} = X^{\kappa \xi^i} = X^{\xi^{i+1}}$. Hence $o(\kappa) = 4$. So we see that K is indeed cyclic in this case.

In Case f), $|K| = 6$. Also, $C = L_X$, and C fixes all the points of Ω. The group $G_{P,l}/C$ is of order 6, and is not abelian, and so $G_{P,l}/C$ is isomorphic to S_3. If we identify the action of $G_{P,l}/C$ on Ω with the right regular representation of $G_{P,l}/C$, then the transitivity of K on Ω together with the fact that K centralizes $G_{P,l}$ imply that the action of K on Ω is similar to the left regular representation of $G_{P,l}/C$. Hence K is isomorphic to S_3, and this completes the proof.

The full collineation group of the generalized Hughes planes were also determined by Rosati [**17**; **19**].

COROLLARY 6. *Let* $\mathfrak{P}$ *be a generalized Hughes plane of order* q^2. *Let* Π *be the full collineation group of* $\mathfrak{P}$, *and let* K *be the collineation group of* $\mathfrak{P}$ *which fix* $\mathfrak{Q}$ *pointwise. Then we have:*

a) *If* $G \cong PSL(3, q)$, *then* $\Pi = \Gamma \times K$, *where* Γ *is a group isomorphic to* $P\Gamma L(3, q)$.

b) *If* $G \cong SL(3, 7)$, *then* $|\Pi : G| = 3$, *and* Π *induces the group* $PGL(3, 7)$ *on* $\mathfrak{Q}$. *Also, we have* $K = \mathfrak{Z}(G) = \mathfrak{Z}(\Pi)$.

Proof. It follows from Wagner's celebrated theorem [**23**, Theorem 1] that Π fixes the subplane $\mathfrak{Q}$.

a) Let N be the normalizer of C in $P\Gamma L(3, q)$. Since P and l are the only fixed elements of C in the desarguesian plane $\mathfrak{Q}$, we have $N \subseteq P\Gamma L(3, q)_{P,l}$. Hence we see that N normalizes the groups $T(\mathfrak{Q}, l)$ and $T(\mathfrak{Q}, P)$, implying that N is a group of collineations of $\mathfrak{P}$. But all the subgroups of $G_{P,l}$ that are isomorphic to C, and that were used in the construction of the generalized Hughes planes, are already conjugate to C in $G_{P,l}$. Hence, using the Frattini argument, we deduce that $NG = P\Gamma L(3, q)$, and so we see that Π really contains a subgroup Γ which is isomorphic to $P\Gamma L(3, q)$. Since Γ induces in $\mathfrak{Q}$ the full collineation group of $\mathfrak{Q}$, we conclude that $\Pi = \Gamma K$. If $|K| \leqq 2$, then Γ is normal in Π. Also, $\Gamma \cap K = \{1\}$, and hence $\Pi = \Gamma \times K$. So suppose $|K| \geqq 3$. By Corollaries 5 and 4, we then know that $q = 11$, 5 or 3. In all three cases, we have $\Gamma = G$, and thus we always have $\Pi = \Gamma \times K$.

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b) The group $GL(3, 7)$ contains a subgroup Π_0 of index 2. This contains $SL(3, 7)$. As in a), we deduce that the normalizer N of C in Π_0 normalizes the groups $T(\mathfrak{Q}, l)$ and $T(\mathfrak{Q}, P)$, and hence is a subgroup of the collineation group of $\mathfrak{P}$. Again, we have $\Pi_0 = NG$, implying that Π_0 is a collineation group of $\mathfrak{P}$ which induces the group $PGL(3, 7)$ on $\mathfrak{Q}$. Thus $\Pi = \Pi_0 K = NGK = NG$.

COROLLARY 7 (Ostrom [**13**; **14**]). *Let $\mathfrak{P}$ be a generalized Hughes plane. Let $\mathfrak{Q}$ be defined as above. Then every homology of $\mathfrak{Q}$ can be extended to a homology of $\mathfrak{P}$.*

Proof. This is certainly correct if $o(\mathfrak{P}) = 9$, since $3 - 1 = 2$. So assume $q > 3$. Let G be the collineation group of $\mathfrak{P}$ which is generated by all the elations, and suppose first that $G \cong PSL(3, q)$. Then $G \subseteq \Gamma$. Also, Γ contains a group G^* such that $G \subseteq G^* \cong PSL(3, q)$. Let C^* be the stabilizer in G^* of a flag of $\mathfrak{P}\backslash\mathfrak{Q}$. Then $|C^*| = q^2 - 1$. But also, $|C^* : C| = (3, q - 1)$. If $\mathfrak{P}$ is of type **B**, then $C \supseteq \mathfrak{Z}(G_{P,l})$. Since $G_{P,l}$ operates transitively on Ω, we conclude that the elements of $\mathfrak{Z}(G_{P,l})$ are all (P, l)-homologies. We also know that in this case, C has exactly two fixed points in Ω. Since $|\mathfrak{Z}(G_{P,l}{}^*) : \mathfrak{Z}(G_{P,l})| = (3, q - 1)$, we conclude that the fixed points of C are also fixed points of $\mathfrak{Z}(G_{P,l}{}^*)$. Hence $\mathfrak{Z}(G_{P,l}{}^*)$ is a group of (P, l)-homologies. The order of this group is $q - 1$, which completes the proof in this case.

If $\mathfrak{P}$ is of type **D**, **F** or **I**, then $\mathfrak{Z}(G_{P,l}) \subseteq C$. In these cases $|C| = q^2 - 1$, and so the assertion follows.

If $\mathfrak{P}$ is of type **C**, we have $q = 5$, and $|\mathfrak{Z}(G_{P,l}) \cap C| = 2$. Choose $\delta \in \mathfrak{Z}(G_{P,l})$ with $o(\delta) = 4$, and let X be a fixed point of C in Ω. Then there exists $\lambda \in K$ such that $X^{\delta\lambda} = X$. Now $\delta\lambda$ centralizes the group $G_{P,l}$, and $G_{P,l}$ is transitive on Ω. Hence $\delta\lambda$ is a (P, l)-homology. Also, δ and $\delta\lambda$ induce the same collineation on $\mathfrak{Q}$. Hence $\delta\lambda$ is a homology of order 4. This proves Corollary 6 in this case. Moreover, since $|\mathfrak{Z}(G_{P,l}) \cap C| = 2$, we know that δ^2 lies in C, and hence δ^2 is the only involutory (P, l)-homology of $\mathfrak{P}$. On the other hand, we know that $(\delta\lambda)^2 = \delta^2\lambda^2$ is also an involutory (P, l)-homology. Hence $\lambda^2 = 1$. Since $G \cong PGL(3, 5)$, we therefore know that the index of $G\langle\lambda\rangle$ in GK is 2, and that $G\langle\lambda\rangle$ is the group generated by all homologies.

Similarly, for types **G** and **H**, we deduce that all (P, l)-homologies of $\mathfrak{Q}$ can be extended to (P, l)-homologies of $\mathfrak{P}$. Also we obtain in both cases that the full collineation group of $\mathfrak{P}$ is generated by homologies.

We are left with the case that $\mathfrak{P}$ is of type **E**. In this case $G \cong SL(3, 7)$. It is easily seen that $G_{P,l}{}^* = \mathfrak{Z}(G)L$, where $L \cong GL(2, 7)$ and where G^* is the full collineation group of $\mathfrak{P}$. This again yields that all homologies of $\mathfrak{Q}$ can be extended to homologies of $\mathfrak{P}$, and that the full collineation group of $\mathfrak{P}$ is generated by homologies. This proves Corollary 7.

Also, we have proved

COROLLARY 8. *Let $\mathfrak{P}$ be a generalized Hughes plane of order q^2. Then the collineation group Δ generated by all the homologies of $\mathfrak{P}$ is isomorphic to*

$PGL(3, q)$, unless $\mathfrak{P}$ is of type **C**, **E**, **G** *or* **H**. *If $\mathfrak{P}$ is of type* **C**, *then Δ has index 2 in the full collineation group. In the remaining cases, Δ is the full collineation group of $\mathfrak{P}$.*

The next corollary was proved by Ostrom [**14**] for orthogonal polarities of $\mathfrak{Q}$.

COROLLARY 9. *Let $\mathfrak{P}$ be a generalized Hughes plane. Let δ_0 be a duality of $\mathfrak{Q}$. Then δ_0 can be extended to a duality δ of $\mathfrak{P}$ such that $o(\delta) = o(\delta_0)$. In particular, every polarity of $\mathfrak{Q}$ extends to a polarity of $\mathfrak{P}$.*

Proof. We consider the group $P\Gamma^*L(3, q)$, consisting of all collineations and dualities of $\mathfrak{Q}$. If $G \cong PSL(3, q)$, then, again by using the Frattini argument, we deduce that $P\Gamma^*L(3, q) = N^*G$, where N^* is the normalizer of C in $P\Gamma^*L(3, q)$. Also, we see that $\{P, l\}^{N^*} = \{P, l\}$, and hence N^* is a group of collineations and dualities of $\mathfrak{P}$. This implies that $P\Gamma^*L(3, q)$ is a group of collineations and dualities of $\mathfrak{P}$. Thus the corollary holds in this case.

Now suppose that $G \cong SL(3, 7)$. By Corollary 2, we can find some duality ϵ of $\mathfrak{P}$. This must leave $\mathfrak{Q}$ invariant. Since the full collineation group Π of $\mathfrak{P}$ induces the group $PGL(3, 7)$ on $\mathfrak{Q}$, we can find a collineation $\pi \in \Pi$ such that δ_0 and $\epsilon\pi$ induce the same duality on $\mathfrak{Q}$. Hence δ_0 can be extended to a duality δ of $\mathfrak{P}$.

If 3 does not divide $o(\delta_0)$, then $\langle \delta_0 \rangle = \langle {\delta_0}^3 \rangle$, and of course, $o(\delta_0) = o({\delta_0}^3)$. Hence there exists some i, such that δ_0 and δ^{3i} induce the same duality on $\mathfrak{Q}$. Moreover, $o(\delta_0) = o(\delta^{3i})$.

Finally, we have to consider the case, where 3 divides $o(\delta_0)$. From Daues & Heineken [**3**] we then know that $o(\delta_0) = 3$ or $o(\delta_0) = 12$. Choose j in such a way that $o(\delta^j) = 3$. Again by [**3**], we know that ${\delta_0}^j \in PSL(3, q)_{P,l}$ for a suitable, non-incident point-line pair (P, l) of $\mathfrak{Q}$. Hence $\delta^j \in SL(3, q)$, and δ^j can be written as

$$\delta^j = \begin{bmatrix} a & 0 & 0 \\ 0 & b & c \\ 0 & d & e \end{bmatrix}.$$

Also,

$$\delta^{3j} = \begin{bmatrix} f & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & f \end{bmatrix},$$

where $f^3 = 1$. Hence we have $a^3 = f$, and so $a^9 = 1$. Therefore, $a^3 = 1$, implying that $f = 1$. Hence 9 does not divide $o(\delta)$. But this implies that $o(\delta_0) = o(\delta)$.

COROLLARY 10. *Suppose q is a prime power, and suppose $q \equiv 1 \bmod 3$. Let $\mathfrak{P}$ be a projective plane of order q^2, and suppose that G is a group of collineations of $\mathfrak{P}$ which is isomorphic to $SL(3, q)$. Then $q = 7$, and $\mathfrak{P}$ is the plane of type* **E**.

Proof. G fixes a Baer subplane $\mathfrak{Q}$ of $\mathfrak{P}$. The proof of this is precisely the same as the proof of Theorem 1 in Unkelbach [**22**]. This proof also shows that G acts doubly transitively on the set of points of $\mathfrak{Q}$. By Theorem 2, we therefore know that $\mathfrak{P}$ is either desarguesian, or a generalized Hughes plane. Since $q \equiv 1 \bmod 3$, we have $|\mathfrak{Z}(G)| = 3$, and thus $q = 7$ and $\mathfrak{P}$ is of type **E**.

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