J. Austral. Math. Soc. (Series A) 54 (1993), 321–333

BMO AND SINGULAR INTEGRALS OVER LOCAL FIELDS

JAMES E. DALY AND DOUGLAS S. KURTZ

(Received January 16, 1991)

Communicated by W. Moran

Abstract

This paper is concerned with the behavior of certain principal-value, singular integral operators on L^∞ and BMO defined over a local field. It is shown that unless the definition of these operators is changed appropriately, they may not be defined for some function in these spaces. Direct, constructive proofs of the existence and boundedness of the altered operators under certain smoothness conditions on the kernel are given.

1991 *Mathematics subject classification* (*Amer. Math. Soc.*): 43A70, 42B20, 42B30.

This paper is concerned with the behavior of certain principal-value, singular integral operators on L^∞ and BMO defined over a local field. We show that unless the definition of these operators is changed appropriately, they may not be defined for some functions in these spaces. We then give direct, constructive proofs of the existence and boundedness of the altered operators under certain smoothness conditions on the kernel. Previous work has concentrated upon the properties of these operators relative to Lebesgue and Hardy spaces, and then, providing results on L^∞ and BMO via duality arguments.

Let K denote a local field, λ the Haar measure on $(K, +)$, and $|\cdot|$ the associated modular function. There is a fixed prime power q so that for each $x \in K$, $|x| = q^n$ for some $n \in \mathbb{Z}$. For $n \in \mathbb{Z}$ we let

$$B^n = \{x \in K : |x| \le q^{-n}\} \quad \text{and} \quad D^n = \{x \in K : |x| = q^{-n}\}.$$

The measure λ is normalized so that $\lambda(B^n) = q^{-n}$. Since $|\cdot|$ is non-Archimedean, each B^n is a subgroup of K. Fix $\pi \in D^1$; that is, $|\pi| = 1/q$.

The standard results of local field analysis used here can be found in Taibleson [6].

We say a locally integrable function f has bounded mean oscillation, $f \in \mathrm{BMO}$, if

$$\sup_B \inf_c \left((1/\lambda(B) \int_B |f(x) - c| dx \right) < \infty, \tag{0.1}$$

where the supremum is taken over all balls $B \subset K$ and the infimum over all constants $c \in \mathbb{C}$. (See Coifman and Weiss [1].) Identifying functions which differ by an additive constant makes BMO into a Banach space with norm given by the left hand side of (0.1). An equivalent norm for BMO is obtained if the L^1-norm in (0.1) is replaced by the L^p-norm for $1 \le p < \infty$. Note that L^∞ is a proper subset of BMO.

Let k be the locally integrable function on $K^* = K\backslash\{0\}$ such that the integral operator T defined by

$$\begin{aligned} Tf(x) &= pv \int k(y) f(x-y)\, dy \\ &= \operatorname*{limit}_{n\to\infty} \operatorname*{limit}_{j\to\infty} \int_{q^{-n} \le |y| \le q^j} k(y) f(x-y)\, dy \\ &= \operatorname*{limit}_{n\to\infty} \operatorname*{limit}_{j\to\infty} T_{n,j}(x) \end{aligned}$$

is a bounded operator on L^2. We say k satisfies condition S_r, $1 \le r < \infty$, if k is locally in L^r on K^*, k has mean value zero, and there exists a non-negative function s on $(0,1)$ such that

$$\sum_1^\infty s(q^{-j}) < \infty \quad \text{and} \quad \int_{D^{-n}} |k(x-y) - k(x)|^r dx \le s(q^{-n}|y|) q^{-n/r'}$$

for $|y| < q^n$. S_∞ is defined by the obvious modification. Note that $k \in S_r$ implies $k \in S_t$ for all $t < r$. The pointwise behavior of these singular integral operators on local field have been studied previously for Lebesgue and Hardy spaces. See Daly and Phillips [2] and Taibleson [6]. In particular, for f in L^2 and k satisfying an S_r condition, then the sequence $T_{j,n}f(x)$ converges pointwise almost everywhere.

A large class of kernels k that satisfy the condition S_r are smooth homogeneous kernels $k(x) = \omega(x)/|x|$ where $\omega(\pi^j x) = \omega(x)$ for $x \ne 0$ and $j \in \mathbb{Z}$ and $\int_{D^0} \omega(x) dx = 0$. In particular, if ω is locally constant on K^* (that is $\omega(x) = \omega(x+y)$ for all $x \in D^0$ and $y \in B^n$ for some $n \in \mathbb{N}$) then $\omega/|\cdot|$ satisfies the S_r condition with $s(q^{-j}) = 0$ for $j > n$. These Calderón-Zygmund singular integral operators with locally constant kernels are known to map the Hardy space H^1 to itself boundedly; and, consequently, by duality, they are bounded on BMO. See Daly-Phillips [2] and Taibleson [6]

https://doi.org/10.1017/S1446788700031815 Published online by Cambridge University Press

for details. In this paper, we consider more general classes of kernels defined by the S_r and related conditions, and give direct proofs of the boundedness of the corresponding operators on BMO. Similar problems have been addressed by Garcia-Cuerva and Rubio de Francia [3] and Kurtz [4, 5] in the Euclidean case, where results analogous to Theorems 2 and 3 can be found for the case of L^∞ to BMO. Here we give extensions to the case of singular integral operators mapping BMO to BMO.

For a locally integrable function f on K and ball B, we let

$$f_B = 1/\lambda(B) \int_B f(x)dx$$

denote the average of f over B. We will use the following lemma.

LEMMA 1. *Let* $1 \leq p < \infty$. *There is a constant* $C = C(p, q)$ *such that if* $f \in \mathrm{BMO}$, $x_0 \in K$, *and* $n \geq 1$, *then*

$$\int_{x_0+B^{j-n}} |f(y) - f_B|p_{dy} \leq C^p n^p q^{n-j} (\|f\|_{\mathrm{BMO}})^p,$$

where $B = x_0 + B^j$.

PROOF. Let $B_0 = x_0 + B^j$ for $x_0 \in K$ and $j \in \mathbb{Z}$. Define $B_m = x_0 + B^{j-m}$ for $m \in \mathbb{Z}$. Consider the case $m = 1$. We have

(1.1)
$$\begin{aligned}\left(\int_{B_1} |f(x) - f_{B_0}|^p dx\right)^{1/p} &\leq \left(\int_{B_1} |f(x) - f_{B_1}|^p dx\right)^{1/p} + \lambda(B_1)^{1/p} \cdot |f_{B_1} - f_{B_0}| \\ &\leq C_1 \lambda(B_1)^{1/p} \cdot \|f\|_{\mathrm{BMO}} + \lambda(B_1)^{1/p} \cdot |f_{B_1} - f_{B_0}|.\end{aligned}$$

Estimate the factor $|f_{B_1} - f_{B_0}|$ as follows:

$$\begin{aligned}|f_{B_1} - f_{B_0}| &= |1/\lambda(B_0) \int_{B_0} f(x)dx - f_{B_1}| \\ &\leq q/\lambda(B_1) \int_{B_1} |f(x) - f_{B_1}|dx \qquad (1.2) \\ &\leq q \cdot \|f\|_{\mathrm{BMO}}.\end{aligned}$$

Combining (1.1) and (1.2), we have

$$\int_{B_1} |f(x) - f_{B_0}|^p dx \leq C^p \lambda(B_1) \|f\|^p_{\mathrm{BMO}}.$$

We now proceed by induction. Suppose

$$\int_{B_m} |f(x) - f_{B_0}|^p dx \leq C^p m^p \lambda(B_m) \|f\|^p_{\mathrm{BMO}}.$$

https://doi.org/10.1017/S1446788700031815 Published online by Cambridge University Press

For B_{m+1},

$$(1.3)\qquad \left(\int_{B_{m+1}} |f(x)-f_{B_0}|^p dx\right)^{1/p} \le \left(\int_{B_{m+1}} |f(x)-f_{B_m}|^p dx\right)^{1/p} + \lambda(B_{m+1})^{1/p}\cdot |f_{B_m}-f_{B_0}|.$$

The first term in the sum of (1.3) is bounded as before using (1.1) and (1.2), with $B_0' = B_m$ and $B_1' = B_{m+1}$. Thus

$$(1.4)\qquad \left(\int_{B_{m+1}} |f(x)-f_{B_m}|^p dx\right)^{1/p} \le C\lambda(B_{m+1})^{1/p}\cdot \|f\|_{\mathrm{BMO}}.$$

The factor $|f_{B_m}-f_{B_0}|$ can be bounded using the induction hypothesis:
(1.5)

$$\begin{aligned}|f_{B_m}-f_{B_0}| &= |(1/\lambda(B_m))\int_{B_m} f(x)dx - f_{B_0}| \le (1/\lambda(B_m))\int_{B_m} |f(x)-f_{B_0}|dx \\ &\le \left((1/\lambda(B_m))\int_{B_m} |f(x)-f_{B_0}|^p dx\right)^{1/p} \\ &\le C\cdot m\cdot \|f\|_{\mathrm{BMO}}.\end{aligned}$$

Combining (1.3)–(1.5), the proof is finished.

Given a set B, let χ_B denote its characteristic function and B^c its complement. Using the previous lemma, we prove the following theorem.

Theorem 2. *Let $f \in L^\infty$ and be supported on a set of finite measure. If k satisfies S_1, then Tf exists a.e., $Tf \in \mathrm{BMO}$, and $\|Tf\|_{\mathrm{BMO}} \le C\|f\|\infty$, where C is independent of f.*

Proof. Since $f \in L^2$, Tf exists a.e.. Let $E = \{x \in K : Tf(x) \text{ exists}\}$ and x_0 be a point of density of E. For $n_0 \in \mathbb{Z}$, consider the ball $B = x_0 + B^{n_0}$. Write f as

$$\begin{aligned} f(x) &= f_B + [f(x)-f_B]\chi_B(x) + [f(x)-f_B]\chi_{B^c}(x) \\ &= f_B + g_B(x) + h_B(x).\end{aligned}$$

Since f_B is a constant, $T(f_B)=0$ and hence exists a.e.. Using the fact T is bounded on L^2, we know Tg_B exists a.e. and

$$(2.1)\qquad \begin{aligned}\int_B |T(g_B)(y)|dy &\le \lambda(B)^{1/2}\|Tg_B\|_2 \le C\lambda(B)^{1/2}\|g_B\|_2 \\ &\le C\lambda(B)\cdot \|f\|_\infty.\end{aligned}$$

https://doi.org/10.1017/S1446788700031815 Published online by Cambridge University Press

Since x_0 is a point of density and Tg_B exists a.e., there is a point $y_0 \in B$ such that $Th_B(y_0)$ exists and $Th_B(y_0) = Tf(y_0) - Tg_B(y_0)$. Set $D_j = x_0 + D^{n-j} = \{|x_0 - z| = q^{j-n}\}$. Note that $\bigcup_{j=1}^{\infty} D_j = B^c$. For $x \in B$, we have

$$
\begin{aligned}
|Th_B(x) - Th_B(y_0)| &\leq \int |k(x-z) - k(y_0 - z)|\,|h_B(z)|dz \\
&\leq 2\|f\|_\infty \sum_1^\infty \int_{D_j} |k(x-z) - k(y_0-z)|dz \qquad (2.2)\\
&\leq C\|f\|_\infty \sum_1^\infty s(q^{-j}).
\end{aligned}
$$

By (2.1) and (2.2),

$$
\begin{aligned}
(1/\lambda(B)) \int_B |Tf(y) - Th_B(y_0)|dy &\leq (1/\lambda(B)) \int_B |Tg_B(y)|dy \\
&\quad + (1/\lambda(B)) \int_B |Th_B(y) - Th_B(y_0)|dy \\
&\leq C\|f\|_\infty
\end{aligned}
$$

since $k \in S_1$. Therefore, $Tf \in \mathrm{BMO}$ and $\|Tf\|_{\mathrm{BMO}} \leq C\|f\|_\infty$.

Theorem 2 can be extended to the case $f \in \mathrm{BMO}$ if we require additional smoothness of the kernel k. We say that the kernel k satisfies the S_r^+ condition if k satisfies S_r and $\sum_1^\infty js(q^{-j}) < \infty$. Using basically the same proof as before, we prove

THEOREM 3. *If k satisfies S_r^+ for some r, $1 < r \leq \infty$, and $f \in \mathrm{BMO}$, then either Tf exists only on a set of measure zero or $Tf \in \mathrm{BMO}$ with $\|Tf\|_{\mathrm{BMO}} \leq C\|f\|_{\mathrm{BMO}}$, where C is independent of f.*

PROOF. Let $E = \{x \in K : Tf(x) \text{ exists}\}$. Assume E has positive measure, since otherwise there is nothing to prove. Using the notation of Theorem 2, we fix x_0 and B and write $f = f_B + g_B + h_B$. The argument for g_B proceeds as before except one uses Lemma 1. For the estimate $|Th_B(x) - Th_B(y_0)|$ we use Lemma 1 and the fact $k \in S_r^+$:

https://doi.org/10.1017/S1446788700031815 Published online by Cambridge University Press

$$
\begin{aligned}
|Th_B(x) - Th_B(y_0)| &\le \int |k(x-z) - k(y_0 - z)|\,|h_B(z)|dz \\
&\le \sum_1^\infty \int_{D_j} |k(x-z) - k(y_0-z)|\,|f(z) - f_B|dz \\
&\le \sum_1^\infty \left(\int_{D_j} |k(x-z) - k(y_0-z)|^r dz \right)^{1/r} \\
&\quad \times \left(\int_{D_j} |f(z) - f_B|^{r'} dz \right)^{1/r'} \\
&\le \sum_1^\infty \left(s(q^{-j})(q^j q^{-n})^{-1/r'} \right) \\
&\quad \times \left((j+1)(q^{j+1} q^{-n})^{1/r'} \|f\|_{\mathrm{BMO}} \right) \\
&= C\|f\|_{\mathrm{BMO}} \sum_1^\infty js(q^{-j}) \\
&\le C\|f\|_{\mathrm{BMO}} .
\end{aligned}
$$

To complete the proof of the existence of Th_B, we use a variation of the previous argument for the boundedness of Th_B, but consider

$$|(T - T_{n,j})h_B(x) - (T - T_{n,j})h_B(y_0)| .$$

For $n > n_0$, we have

$$
\begin{aligned}
|(T - T_{n,j})h_B(x) &- (T - T_{n,j})h_B(y_0)| \\
&= |(T - T_{n_0,j})h_B(x) - (T - T_{n_0,j})h_B(y_0)| \\
&\le C\|f\|_{\mathrm{BMO}} \sum_{j+1}^\infty is(q^{-i}) .
\end{aligned}
$$

However, $(T - T_{n_0,j})h_B(y_0)$ goes to zero by assumption, thus so does $(T - T_{n,j})h_B(x)$. Therefore, Th_B exists a.e. on B and hence, Tf exists a.e. on B. As the radius of B is arbitrary, Tf exists a.e. on K.

The norm boundedness follows as in Theorem 2 before using the estimate (3.1) above in place of (2.2).

Theorem 3 shows that each operator T splits BMO into a subspace where T is a bounded operator and a subspace where it is infinite for each function. As an example of this we consider a special homogeneous kernel k. For the

3-adic field K, the cosets $1+B^1$ and $-1+B^1$ of D^0 are disjoint. Since each $x \in K^*$ can be written uniquely as $x = 3^n x'$ where $|x'| = 1$, we define the disjoint sets $P_1 = \{x : x' \in 1+B^1\}$ and $P_2 = \{x : x' \in -1+B^1\}$. Note that $P_1 \cup P_2 = K^*$. The kernel k given by $k(x) = (\chi_{P_1}(x) - \chi_{P_2}(x))/|x|$ is homogeneous of degree -1. The operator T is the 3-adic field analog of the Hilbert transform and is known to be well-defined and bounded on the Hardy spaces H^p for $0 < p < \infty$, and thus from BMO to BMO by duality. See, for example, Daly-Phillips [2]. However, considering $T(\chi_{P_1})(x)$, we have

$$\begin{aligned} T(\chi_{P_1})(x) &= \int_{P_1} \chi_{P_1}(x-y)dy/|y| - \int_{P_2} \chi_{P_1}(x-y)dy/|y| \\ &= (\mathrm{I}) + (\mathrm{II}), \end{aligned}$$

where the integrals are interpreted in the principal-value sense. Let $x \in P_1$. Then $x = 3^n(1+\omega)$ with $|\omega| < 1$. For integral (II), we have $y \in P_2$, so $y = 3^k(-1+\beta)$ with $|\beta| < 1$. Hence $x - y = 3^n(1+\omega) - 3^k(-1+\beta)$. If $n \neq k$ $(|y| \neq |x|)$, then with $N = \min(n, k)$, $x - y = \pi^N(1+\gamma)$ with $|\gamma| < 1$. So $x - y \in P_1$. If $n = k$ $(|y| = |x|)$, then $x - y = 3^n(-1 + (3 + \omega - \beta))$. So $x - y \in P_2$. Thus integral (II) is $\int_{P_2} -\chi_{\{|y|\neq|x|\}}(y)dy/|y|$. For integral (I), $x - y \in P_2$ if $|x| < |y|$. This can be seen from $x-y = 3^n(1+\omega) - 3^k(1+\beta) = 3^k(-1 + (-\beta + 3^{n-k} + 3^{n-k}\omega)) = 3^k(-1+\gamma)$ with $|\gamma| < 1$. So integral (I) becomes $\int_{P_1} \chi_{\{|y|\leq|x|\}}(y)dy/|y|$. Combining these,

$$T(\chi_{P_1})(x) = \int_{P_1} \chi_{\{|y|=|x|\}}(y)dy/|y| - \int_{P_2} \chi_{\{|y|>|x|\}}(y)dy/|y|,$$

which is infinite. The case of $y \in P_2$ is analogous. Hence $T(\chi_{P_1})$ is infinite everywhere on K^* as $P_1 \cup P_2 = K^*$. This example can be easily adapted to any local field K.

From the above example and Theorem 3, we know that an unaltered singular integral operator of the type under consideration may not be well-defined for a specific bounded function, much less one of bounded mean oscillation. To modify the operator and make the principle value integrals well-defined and bounded on BMO, we will exploit the facts that functions in BMO are equivalent up to additive constants and $T(1) = 0$. For a kernel k, we define the singular integral operator T' by

$$T'f(y) = \int (k(y-x) - k(-x)\chi(x))f(x)dx$$

where χ denotes the characteristic function of $(B^0)^c$. We will show that T' is bounded both from L^∞ and BMO to BMO under suitable smoothness

https://doi.org/10.1017/S1446788700031815 Published online by Cambridge University Press

conditions on the kernel k. Note that if

$$c_f = \int k(-x)\chi(x)f(x)dx = \int_{|x|>1} k(-x)f(x)dx$$

exists, then $T'f(y) = Tf(y) - c_f$. In this case, $T'f$ and Tf represent the same (equivalence class of) function(s) in BMO. In fact, the finiteness of c_f is exactly the condition on T that makes T itself well-defined and bounded on BMO.

First suppose that f is bounded. Then

$$\begin{aligned} T'f(y) &= \int (k(y-x) - k(-x)\chi(x))f(x)dx \\ &= \int_{|x|<|y|} (k(y-x) - k(-x)\chi(x))f(x)dx \\ &\quad + \int_{|x|=|y|} (k(y-x) - k(-x)\chi(x))f(x)dx \\ &\quad + \int_{|x|>|y|} (k(y-x) - k(-x)\chi(x))f(x)dx \\ &= (\mathrm{I}) + (\mathrm{II}) + (\mathrm{III}). \end{aligned}$$

For (I), we have

$$(\mathrm{I}) = \int_{|x|<\min(|y|,1)} k(y-x)f(x)dx + \int_{1\le|x|<|y|} (k(y-x) - k(-x))f(x)dx.$$

As f restricted to each of the sets $\{x : |x| < |y| \text{ and } |x| < 1\}$ and $\{x : 1 \le |x| < |y|\}$ is in L^2, each of the above two integrals is finite. Next, write (II) as

$$\int_{|x|=|y|} k(y-x)f(x)dx - \int_{1<|x|\le|y|} k(-x)f(x)dx.$$

If $|y| \le 1$, the second integral is 0. If $|y| > 1$, it is finite as before. The first integral can be identified as $T(f\chi_{D^n})(y)$ where $|y| = q^{-n}$. Since $f\chi_{D^n}$ is in L^2 and T is bounded on L^2, the first integral is also finite. Finally, (II) is equal to

$$\int_{|x|\le 1} k(y-x)\chi_{\{|\cdot|>|y|\}}(x)f(x)dx + \int_{|x|>\max(|y|,1)} (k(y-x) - k(-x))f(x)dx.$$

The first integral is either 0 (if $|y| \ge 1$) or finite by the local integrability of

k and the boundedness of f. The second integral is bounded by

$$\begin{aligned}
\|f\|_\infty \int_{|x|>\max(|y|,1)} &|k(y-x)-k(-x)|dx \\
&\le \|f\|_\infty \sum_{|y|<q^n} \int_{D^n} |k(y-x)-k(-x)|dx \\
&\le \|f\|_\infty \sum_{|y|<q^n} \int_{D^n} |k(x+y)-k(x)|dx \\
&\le \|f\|_\infty \sum_1^\infty s(q^{-n})
\end{aligned}$$

which is finite as long as k satisfies the condition S_1. Thus $T'f$ is finite a.e. if $f \in L^\infty$. In fact, arguing as in Theorem 2, T' is a bounded operator from L^∞ into BMO. We state this as Theorem 4.

THEOREM 4. *If $f \in L^\infty$ and k satisfies S_1, then $T'f$ exists a.e., $T'f \in$ BMO, and $\|T'f\|_{\mathrm{BMO}} \le C\|f\|_\infty$, where C is independent of f.*

PROOF. We again write f as $f(x) = f_B + g_B(x) + h_B(x)$ for a fixed ball B. To show that $T' : L^\infty \to$ BMO, we note that $T'(f_B) = 0$ since f_B is constant and k has the mean value zero property. Since $g_B = (f - f_B)\chi_B$ is bounded and has compact support,

$$c_1 = \int_{|x|\ge 1} k(-x) g_B(x)dx < \infty$$

by the local integrability of k. Therefore,

$$\begin{aligned}
\int_B |T'g_B(y) - c_1|dy = \int_B |Tg_B(y)|dy &\le \sqrt{\lambda(B)}\|Tg_B\|_2 \\
&\le C\sqrt{\lambda(B)}\|g_B\|_2 \le C\lambda(B)\|f\|_\infty .
\end{aligned}$$

Since $h_B \in L^\infty$, $T'h_B < \infty$ a.e.. Fix $y_0 \in B$ such that $T'h_B(y_0) < \infty$. Then,

$$\begin{aligned}
|T'h_B(y) - T'h_B(y_0)| &= \left| \int (k(y-x) - k(-x)\chi(x))h_B(x)dx \right. \\
&\qquad \left. - \int (k(y_0-x) - k(-x)\chi(x))h_B(x)dx \right| \\
&= \left| \int (k(y-x) - k(y_0-x))h_B(x)dx \right|
\end{aligned}$$

and now we can proceed as before.

https://doi.org/10.1017/S1446788700031815 Published online by Cambridge University Press

We now prove the analog of Theorem 3.

Theorem 5. *If k satisfies S_r^+ for some r, $1 < r \leq \infty$, and $f \in \mathrm{BMO}$, then $T'f \in \mathrm{BMO}$ and $\|T'f\|_{\mathrm{BMO}} \leq C\|f\|_{\mathrm{BMO}}$, where C is independent of f.*

Proof. We first show that $T'f$ exists a.e. for all $f \in \mathrm{BMO}$. We write, as before,

$$f = f_{B^0} + (f - f_{B^0})\chi_{B^0} + (f - f_{B^0})\chi_{(B^0)^c}.$$

Then

$$T'f = T'f_{B^0} + T'[(f - f_{B^0})\chi_{B^0}] + T'[(f - f_{B^0})\chi_{(B^0)^c}]. \tag{5.1}$$

Since $f_{B^0} \in L^\infty$, $T'f_{B^0}$ exists a.e. by Theorem 4. Next, we see that

$$\begin{aligned} T'[(f - f_{B^0})\chi_{B^0}](y) &= \int (k(y-x) - k(-x)\chi(x)(f(x) - f_{B^0}(x)\chi_{B^0}(x)dx \\ &= \int (k(y-x)(f(x) - f_{B^0})\chi_{B^0}(x))dx \\ &= T[(f - f_{B^0})\chi_{B^0}](y), \end{aligned}$$

because χ is supported on the complement of B^0. Since T is bounded on L^2, this term is finite a.e. For the last term of (5.1), we consider the integrals over over the sets $\{1 < |x| \leq |y|\}$ and $\{|x| > \max(|y|, 1)\}$ separately. First,

$$\begin{aligned} &\left| \int_{|x| > \max(|y|, 1)} (k(y-x) - k(-x)(f(x) - f_{B^0})dx \right| \\ &\quad \leq \sum_{\max(|y|, 1) < q^n} \int_{D^n} |k(y-x) - k(-x)|\,|f(x) - f_{B^0}|dx \\ &\quad \leq \sum_{\max(|y|, 1) < q^n} \left[\int_{D^n} |k(x+y) - k(x)|^r dx\right]^{1/r} \left[\int_{D^n} |f(x) - f_{B^0}|^{r'} dx\right]^{1/r'} \qquad (5.2) \\ &\quad \leq C\|f\|_{\mathrm{BMO}} \sum_1^\infty |q^{-n/r'} s(q^{-n})][nq^{n/r'}] \\ &\quad = C\|f\|_{\mathrm{BMO}} \sum_1^\infty n \cdot s(q^{-n}). \end{aligned}$$

The last term is finite since k satisfies S_r^+. Next choose N such that $|y| =$

q^N. Then, we have

$$\begin{aligned}
&\int_{1<|x|\le|y|}(k(y-x)-k(-x))(f(x)-f_{B^0})dx \\
(5.3)\qquad &= \int_{1<|x|\le|y|} k(y-x)(f(x)-f_{B^0})dx + f_{B^0}\cdot\int_{1<|x|\le q^N} k(-x)dx \\
&\quad - \int_{1<|x|\le q^N} k(-x)f(x)dx.
\end{aligned}$$

The first term is finite for a.e. y, $|y|=q^N$, since $(f-f_{B^0})\chi_{1<|x|\le q^N}$ is in L^2 and T is bounded on L^2. The second term is finite by the local integrability of k; and finally, the third term is finite by the facts that k is locally in L^r on K^* and f is locally in $L^{r'}$ on K. Consequently, $T'f$ is finite a.e.

The proof of the norm boundedness of T' is the same as that in Theorem 4 with the exception that the local integrability of k on K^* is replaced by the local integrability of $|k|^r$.

Theorem 5 can be extended to a more general class of kernels. We say k is locally in the Hardy space H^1 if $k\chi_{D^n}\in H^1(D^n)$ for all $n\in\mathbb{Z}$. The locally H^1 kernel k is said to satisfy the smoothness condition SH_1 if $\|k(\cdot-y)-k(\cdot)\|_{H^1(D^n)}\le s(q^{-n}|y|)$ for $|y|<q^{-n}$ with $\sum_1^\infty s(q^{-n})<\infty$. This is the natural generalization of the S_1 condition for local integrability to the Hardy space H^1.

Theorem 6. *If k satisfies SH_1 and $f\in\mathrm{BMO}$, then $T'f\in\mathrm{BMO}$ and $\|T'f\|_{\mathrm{BMO}}\le C\|f\|_{\mathrm{BMO}}$, where C is independent of f.*

Proof. The proof of Theorem 5 remains the same with the following exceptions. To estimate (5.2), one has

$$\begin{aligned}
&\left|\int_{|x|>\max(|y|,1)}(k(y-x)-k(-x))(f(x)-f_{B^0})dx\right| \\
&\quad\le \sum_{\max(|y|,1)<q^n}\left|\int_{D^n}(k(y-x)-k(-x))(f(x)-f_{B^0})dx\right| \\
&\quad\le \sum_{\max(|y|,1)<q^n}\|k(\cdot-y)-k(\cdot)\|_{H^1(D^n)}\|f\|_{\mathrm{BMO}} \\
&\quad\le C\|f\|_{\mathrm{BMO}}\sum_1^\infty s(q^{-n})
\end{aligned}$$

https://doi.org/10.1017/S1446788700031815 Published online by Cambridge University Press

which is finite as $k \in SH_1$. Also, for (5.3), one needs

$$\int_{1<|x|\leq q^n} k(-x)f(x)dx$$

to be finite. This follows as k is locally in H^1 on K^* and f is in BMO.

For homogeneous kernels slightly more is known. Suppose $k(x) = \omega(x)/|x|$, where ω is homogeneous of degree zero ($\omega(\pi^j x) = \omega(x)$ for $j \in \mathbb{Z}$), has integral zero on D^0, and satisfies the S_1 condition. Thus, if $|y| = 1$ and $n \geq 1$, k satisfies

$$\int_{D^{-n}} |k(x-y) - k(x)|dx \leq s(q^{-n});$$

or, equivalently,

$$\int_{D^{-n}} |\omega(x-y) - \omega(x)|dx/|x| \leq s(q^{-n}).$$

Writing x as $\pi^{-n}x'$ with $|x'| = 1$ and using the homogeneity of ω, the smoothness condition becomes

$$\int_{D^o} |\omega(x' - \pi^n y) - \omega(x')|dx \leq s(q^{-n}).$$

Thus

$$\sum_1^\infty \int_{D^0} \int_{D^0} |\omega(x' - \pi^n y) - \omega(x')|dxdy \leq \sum_1^\infty s(q^{-n}),$$

which is finite since k satisfies the S_1 condition. The finiteness of the left side of this inequality is precisely the smoothness condition on homogeneous kernels that was considered by Daly and Phillips [2]. There, it was proved that if this smoothness condition is satisfied then the Calderón-Zygmund singular integral operator T maps the Hardy space H^1 to itself boundedly. By the duality of H^1 and BMO and the fact that the adjoint T^*, T maps BMO to itself boundedly, where T is defined on BMO by $\langle g, Tf\rangle = \langle T^* g f\rangle$ for $g \in H^1$ and $f \in \text{BMO}$. It would be instructive to have a construction of T and T' and a direct proof of boundedness as in Theorems 5 and 6 that does not use duality and is valid for a wider class of kernels than the homogeneous ones. The methods used in the proofs presented here are not adequate.

References

[1] R. Coifman and G. Weiss, 'Extensions of Hardy spaces and their use in analysis', *Bull. Amer. Math. Soc.* **83** (1977), 569–645.

[2] J. Daly and K. Phillips, 'On singular integrals, multipliers, H^1, and Fourier series - a local field phenomenon', *Math. Ann.* **265** (1983), 181–219.

[3] J. Garcia-Cuerva and J. L. Rubio de Francia, *Weighted norm inequalities and related topics* (North-Holland, Amsterdam, 1985).

[4] D. S. Kurtz, 'Littlewood-Paley operators on BMO', *Proc. Amer. Math. Soc.* **99** (1987), 657–666.

[5] D. S. Kurtz, 'Singular integrals on BMO', *Proc. Centre Math. Anal. Austral. Nat. Univ.* **16** (1987), 169–172.

[6] M. Taibleson, *Fourier Analysis on Local Fields*, Math. Notes (Princeton Univ. Press, Princeton, 1975).

[7] S. Wang, 'Some properties of the Littlewood-Paley g-function', *Contemp. Math.* **42** (1985), 191–202.

Department of Mathematics
University of Colorado
Colorado Springs
Colorado, 80933
USA

Department of Mathematical Science
New Mexico State University
Las Cruces
New Mexico 88003
USA

https://doi.org/10.1017/S1446788700031815 Published online by Cambridge University Press