EMBEDDING THEOREMS FOR ABELIAN GROUPS

C. G. CHEHATA

1. Introduction. Given an abelian group G and a mapping θ that maps a subgroup A of G homomorphically onto another subgroup B of G, then it is known **(3)** that there always exists an embedding group $G^* \supseteq G$ which is abelian and possesses an endomorphism θ^* which coincides with θ on A, i.e. $a\theta = a\theta^*$ whenever $a\theta$ is defined. θ is called *a partial endomorphism* of G and $\theta\eta$ a *total endomorphism* or simply an *endomorphism* that *extends or continues* θ. If, moreover, we require that $\theta\eta$ acts as an isomorphism on the group $G^*(\theta^*)^n$ for some positive integer n, then a necessary and sufficient condition is that whenever $a\theta^{n+1}$ is defined and $a\theta^{n+1} = e$ then $a\theta^n = e$, where e denotes the unit element of G **(2)**.

In this paper we consider an abelian group G and a set of partial endomorphisms $\theta(\alpha)$, where α ranges over some well-ordered set Σ, and derive necessary and sufficient conditions for the following two extensions to be possible.

(I) We require that every $\theta(\alpha)$ be extendable to a total endomorphism $\theta^*(\alpha)$ of one and the same abelian group $G^* \supseteq G$ such that for every $\alpha \in \Sigma$, $\theta^*(\alpha)$ is an isomorphism on $G^*[\theta^*(\alpha)]^{n(\alpha)}$, $n(\alpha)$ being a given positive integer.

(II) We require that all the $\theta(\alpha)$ be extendable to a single total endomorphism τ of an abelian group $G^* \supseteq G$ such that τ is an isomorphism on $G^*\tau^m$, again m being a given positive integer.

In a previous paper **(1)**, sufficient conditions are obtained for extending partial endomorphisms $\theta(\alpha)$ of a group G, not necessarily abelian, to endomorphisms $\theta^*(\alpha)$ with similar restrictions imposed on them in case every subgroup H of G is an E-subgroup (H is an E-subgroup of G if for every normal subgroup N in H, $N^G \cap H = N$ where N^G is the normal closure of N in G). If G is abelian, then every subgroup of G is an E-subgroup and we were able to deduce (Corollary, Theorem 2) that G can be embedded in G^* (not necessarily abelian) with the required conditions fulfilled if whenever $x[\theta(\alpha)]^{n(\alpha)+1}$ is defined and is equal to e then $x[\theta(\alpha)]^{n(\alpha)} = e$.

The result we obtain in case (I) shows more. It proves that this condition is also necessary, and moreover that the embedding group G^* can be chosen abelian.

The main tool throughout the work is the direct product of two groups with one amalgamated subgroup.

2. Extension in case (I). We shall assume in this section and the next section that we are given an abelian group G and a well-ordered set Σ of

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ordinal σ; also we assume that for each $\alpha \in \Sigma$, G possesses a partial endomorphism $\theta(\alpha)$ which maps the subgroup $A_\alpha \subseteq G$ onto the subgroup $B_\alpha \subseteq G$. For every $\alpha \in \Sigma$, $n(\alpha)$ is a given positive integer.

THEOREM 1. *Each $\theta(\alpha)$ is extendable to a total endomorphism $\theta^*(\alpha)$ of one and the same abelian supergroup $G^* \supseteq G$ such that $\theta^*(\alpha)$ is an isomorphism on $G^*[\theta^*(\alpha)]^{n(\alpha)}$ if and only if*

(2.1) *whenever $x[\theta(\alpha)]^{n(\alpha)+1}$ is defined and is equal to e then*

$$x[\theta(\alpha)]^{n(\alpha)} = e.$$

Proof. The necessity of the conditions is immediate, for if G^*, $\theta^*(\alpha)$ exist and fulfil the required conditions, and if $x \in A$ such that $x[\theta(\alpha)]^{n(\alpha)+1}$ is defined and is equal to e, then

$$x[\theta(\alpha)]^{n(\alpha)+1} = x[\theta^*(\alpha)]^{n(\alpha)+1} = e.$$

Since $\theta^*(\alpha)$ is an isomorphism on $x[\theta^*(\alpha)]^{n(\alpha)}$, then

$$x[\theta^*(\alpha)]^{n(\alpha)} = e$$

and hence

$$x[\theta(\alpha)]^{n(\alpha)} = e.$$

To prove that conditions (2.1) are sufficient, let α be an arbitrary element in Σ. Assume that $K_{\theta(\alpha)}$ is the kernel of the homomorphism $\theta(\alpha)$ and put

$$H = G/K_{\theta(\alpha)}.$$

Denote by $\phi(\alpha)$ the natural mapping of G onto $G/K_{\theta(\alpha)}$. The homomorphism $\theta(\alpha)$ of A_α onto B_α induces an isomorphism $\chi(\alpha)$ of $A_\alpha/K_{\theta(\alpha)}$ onto B_α.

We then form the direct product of the groups G and H amalgamating the subgroup $B_\alpha \subseteq G$ with the subgroup $A_\alpha/K_{\theta(\alpha)} \subseteq H$ according to the isomorphism $\chi(\alpha)$, and denote this product by

$$G_\alpha = \{G \times G/K_{\theta(\alpha)},\ B_\alpha = A_\alpha/K_{\theta(\alpha)}\}.$$

Evidently G_α is abelian and the partial endomorphism $\phi(\alpha)$ of G_α extends $\theta(\alpha)$. In **(2)** we proved the following lemma.

LEMMA 1. *In the group G_α, if $x[\phi(\alpha)]^{n(\alpha)+1}$ is defined and $x[\phi(\alpha)]^{n(\alpha)+1} = e$, then $x[\phi(\alpha)]^{n(\alpha)} = e$.*

This means that $\phi(\alpha)$, which extends $\theta(\alpha)$, preserves the property (2.1).

We shall describe the process of embedding G in the abelian group G_α and hence extending the partial endomorphism $\theta(\alpha)$ to another partial endomorphism $\phi(\alpha)$ by saying that G_α is an α-extension of G.

For any $\lambda \in \Sigma$, we define an abelian group G_λ in the following manner. If O is the first element of Σ, then we construct G_1, which is the O-extension of G.

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Inductively, if for $\lambda \in \Sigma$, G_λ is defined and thus possesses for every $\alpha \in \Sigma$ a partial endomorphism, $\theta(\lambda, \alpha)$ say, mapping a subgroup $A_{\lambda,\alpha} \subseteq G$ onto another subgroup $B_{\lambda,\alpha} \subseteq G$ and satisfying the condition corresponding to (2.1), then we form the λ-extension of G_λ and call it $G_{\lambda+1}$. Thus $G_{\lambda+1}$ is the direct product

$$G_{\lambda+1} = \{G_\lambda \times G_\lambda/K_{\theta(\lambda)};\quad B_{\lambda,\lambda} = A_{\lambda,\lambda}/K_{\theta(\lambda)}\}.$$

For every $\alpha \in \Sigma$, $G_{\lambda+1}$ possesses the partial endomorphism $\theta(\lambda+1, \alpha)$ which maps a subgroup $A_{\lambda+1,\alpha} \subseteq G_{\lambda+1}$ onto another subgroup $B_{\lambda+1,\alpha} \subseteq G_{\lambda+1}$, where

$$A_{\lambda+1,\lambda} = G_\lambda,$$

$$B_{\lambda+1,\lambda} = G_\lambda/K_{\theta(\lambda)},$$

and $\theta(\lambda+1, \lambda)$ is the natural mapping of G_λ onto $G_\lambda/K_{\theta(\lambda)}$; and for $\alpha \neq \lambda$,

$$A_{\lambda+1,\alpha} = A_{\lambda,\alpha},$$

$$B_{\lambda+1,\alpha} = B_{\lambda,\alpha},$$

$$\theta(\lambda+1, \alpha) \text{ is } \theta(\lambda, \alpha).$$

Also $\theta(\lambda+1, \alpha)$, for every $\alpha \in \Sigma$, satisfies the condition (2.1).

If ρ is a limit ordinal and G_λ, with the partial endomorphisms $\theta(\lambda, \alpha)$ mapping $A_{\lambda,\alpha} \subseteq G_\lambda$ onto $B_{\lambda,\alpha} \subseteq G_\lambda$, are defined for every $\alpha \in \Sigma$ and $\lambda < \rho$, then we put

$$G_\rho = \bigcup_{\lambda<\rho} G_\lambda, \qquad A_{\rho,\alpha} = \bigcup_{\lambda<\rho} A_{\lambda,\alpha}, \qquad B_{\rho,\alpha} = \bigcup_{\lambda<\rho} B_{\lambda,\alpha}$$

and define a homomorphic mapping $\theta(\rho, \alpha)$ of $A_{\rho,\alpha}$ onto $B_{\rho,\alpha}$ as follows. If $x \in A_{\rho,\alpha}$, that is to say if $x \in A_{\lambda,\alpha}$ for some suitable $\lambda < \rho$, we put

$$x\,\theta(\rho, \alpha) = x\,\theta(\lambda, \alpha).$$

LEMMA 2. *If $x[\theta(\rho, \alpha)]^{n(\alpha)+1}$ is defined and $x[\theta(\rho, \alpha)]^{n(\alpha)+1} = e$, then*

$$x[\theta(\rho, \alpha)]^{n(\alpha)} = e.$$

Proof. Since $x[\theta(\rho, \alpha)]^{n(\alpha)+1}$ is defined, then there exists a suitable $\pi < \rho$ such that

$$x[\theta(\rho, \alpha)]^{n(\alpha)+1} = x[\theta(\pi, \alpha)]^{n(\alpha)+1} = e,$$

and since $\theta(\pi, \alpha)$ satisfies (2.1) then

$$x[\theta(\pi, \alpha)]^{n(\alpha)} = e,$$

and this in turn gives

$$x[\theta(\rho, \alpha)]^{n(\alpha)} = e.$$

Now we continue the process of forming an abelian group G_λ corresponding to every $\lambda \in \Sigma$ until we finally form G_σ, where σ is the ordinal type of Σ. Put

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$$G^{(0)} = G, \qquad \theta(\alpha) = \theta_0(\alpha),$$

$$G^{(1)} = G_\sigma^{(0)}, \qquad \theta_0(\sigma, \alpha) = \theta_1(\alpha),$$

and form inductively

$$G^{(i)} = G_\sigma^{(i-1)}, \qquad \theta_{i-1}(\sigma, \alpha) = \theta_i(\alpha)$$

for any positive integer i. Let

$$G^* = \bigcup_i G^{(i)}.$$

G^* is an abelian group containing G. For every $\alpha \in \Sigma$ we define a mapping $\theta^*(\alpha)$ as follows. If $g \in G^*$, i.e. if $g \in G^{(j)}$ for some suitable positive integer j, we put

$$g\,\theta^*(\alpha) = g\,\theta_j(\alpha).$$

Obviously $\theta^*(\alpha)$ are total endomorphisms of G^* which extend the $\theta(\alpha)$. For any $g \in G^*$,

$$g[\theta^*(\alpha)]^{n(\alpha)+1} = g[\theta_j(\alpha)]^{n(\alpha)+1}$$

for some suitable j; and if this element is equal to e, then by Lemma 2 we get

$$g[\theta_j(\alpha)]^{n(\alpha)} = g[\theta^*(\alpha)]^{n(\alpha)} = e.$$

This completes the proof of Theorem 1.

3. Extension in case (II).

THEOREM 2. *All $\theta(\alpha)$, α runs over Σ, are extendable to one and the same total endomorphism τ of an abelian supergroup $G^* \supseteq G$ such that τ is an isomorphism on $G^*\tau^m$, where m is a given positive integer, if and only if when we define κ to map any word $w(a_\mu) \in \{A_\alpha\}$ onto $w(a_\mu\theta(\mu)) \in \{B_\alpha\}$, where $a_\mu \in A_\mu$, μ ranges over some finite set $I \subset \Sigma$, and α ranges over Σ, then*

(3.1) *κ is a homomorphic mapping of $\{A_\alpha\}$ onto $\{B_\alpha\}$,*

(3.2) *whenever $a\kappa^{m+1}$ is defined for an element $a \in \{A_\alpha\}$ and $a\kappa^{m+1} = e$, then $a\kappa^m = e$.*

Proof. Assume that G^* and τ exist with the required property fulfilled. If $g \in G^*$, then $g\tau$ is uniquely defined. In particular the map $w(a_\mu)\tau$ of any word $w(a_\mu) \in \{A_\alpha\} \subseteq G^*$ is uniquely defined, and since τ extends $\theta(\alpha)$ for every $\alpha \in \Sigma$, then

$$w(a_\mu)\tau = w(a_\mu\theta(\mu)) = w(a_\mu)\kappa.$$

Thus κ is uniquely defined. That κ is a homomorphism follows immediately from the fact that τ extends κ. This proves the necessity of (3.1). The necessity of (3.2) follows from **(2)**.

Now assume (3.1) and (3.2). If we put

$$\{A_\alpha\} = A', \qquad \{B_\alpha\} = B',$$

then κ becomes a partial endomorphism of the abelian group G which satisfies (3.2) and this ensures from **(2)** the existence of an abelian group $G^* \supseteq G$ and a total endomorphism τ of G^* extending κ and acting as an isomorphism on $G^*\tau^m$.

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Faculty of Science, The University,
Alexandria, Egypt