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UPPER AND LOWER SOLUTIONS FOR THE SINGULAR p-LAPLACIAN WITH SIGN CHANGING NONLINEARITIES VIA INEQUALITY THEORY[1]

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Abstract. In this paper, general existence theorems are presented for the singular equation

$$\begin{cases} -(\varphi_p(u'))' = f(t, u, u'), \ 0 < t < 1 \\ u(0) = u(1) = 0. \end{cases}$$

Throughout, our nonlinearity is allowed to change sign. The singularity may occur at $u = 0$, $t = 0$ and $t = 1$.

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1. Introduction. In this paper, we study the singular boundary value problem

$$\begin{cases} -(\varphi_p(u'))' = f(t, u, u'), \ 0 < t < 1 \\ u(0) = u(1) = 0 \end{cases} \tag{1.1}$$

where $\varphi_p(s) = |s|^{p-2}s$, $p > 1$. The singularity may occur at $u = 0$, $t = 0$ and $t = 1$, and the function f is allowed to change sign. Note f may not be a Carathéodory function because of the singular behavior of the u variable. In the literature [6, 7, 10], (1.1) has been discussed extensively when $f(t, u, v) \equiv f(t, u)$ and f is positive i.e. $f : (0, 1) \times (0, \infty) \to (0, \infty)$. Recently [1, 11], (1.1) was discussed when $f(t, u, v) \equiv f(t, u)$ and $f : (0, 1) \times (0, \infty) \to R$. The case when f depends on the u' variable has received very little attention in the literature, see [2, 5] and references therein. This paper presents a new and very general existence result for (1.1) when $f : (0, 1) \times (0, \infty) \times R \to R$. Equation of the above form occur in the study of the $p-$Laplace equation, non-Newtonian fluid theory, and the turbulent flow of a gas in a porous medium [9]. The

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case $p = 2$ and $p \neq 2$ are quite different. For example, (i) there exists a Green's function when $p = 2$ but not if $p \neq 2$; (ii) $\varphi_p^{-1}(x)$ is continuously differentiable for $1 < p \leq 2$ but $\varphi_p^{-1}(x)$ is not continuously differentiable for $p > 2$. As a result the argument in the case $p \neq 2$ is more difficult. Other differences between $p = 2$ and $p \neq 2$, can be found in [**12**].

2. General Existence Theorem. First we consider the boundary value problem

$$\begin{cases} -(\varphi_p(u'))' = g(t, u, u'), \ 0 < t < 1 \\ u(0) = a, \ u(1) = b \end{cases}$$

where $g : (0, 1) \times R^2 \to R$ is continuous and suppose that there exist positive continuous functions $q \in C(0, 1)$ and $\Psi : [0, +\infty) \to (0, \infty)$ with

$$\int_0^1 q(t)\, dt < +\infty$$

and

$$|g(t, u, v)| \leq q(t)\Psi(|v|) \text{ for all } (t, u, v) \in (0, 1) \times R^2.$$

For all $\rho \in (0, 1]$, define the operator

$$N_\rho : C[0, 1] \to C[0, 1]$$

by

$$\left(N_\rho u\right)(t) := \varphi_p^{-1}\left(A_u + \rho \int_0^t g\left(\tau, (Ju)(\tau), u(\tau)\right) d\tau\right),$$

where

$$J(u)(\tau) = b - \int_\tau^1 u(s)\, ds$$

for all $0 \leq \tau \leq 1$, and $A_u \in (-\infty, \infty)$ is such that

$$\int_0^1 \varphi_p^{-1}\left(A_u + \rho \int_0^t g\left(\tau, (Ju)(\tau), u(\tau)\right) d\tau\right) dt = b - a.$$

LEMMA 2.1. [**5**] (1) $N_\rho : C[0, 1] \to C[0, 1]$ *is completely continuous.*

(2) *If* $\Omega \subset \{z \in C[0, 1] \mid (N_\rho z)(t) = z(t)\}$ *and* $\sup\{\sup_{[0,1]} |z(t)| \mid z \in \Omega\} < \infty$, *then* Ω *is a relatively compact set in* $C[0, 1]$.

LEMMA 2.2. [**11**] *Let* $e_n = [\frac{1}{2^{n+1}}, 1]\ (n \geq 1)$, $e_0 = \emptyset$. *If there exist a sequence* $\{\varepsilon_n\} \downarrow 0$ *and* $\varepsilon_n > 0$ *for* $n \geq 1$, *then there exist a function* $\lambda \in C^1[0, 1]$ *such that*

(1) $\varphi_p(\lambda') \in C^1[0, 1]$ *and* $\max_{0 \leq t \leq 1} |(\varphi_p(\lambda'(t)))'| > 0$, *and*

(2) $\lambda(0) = \lambda(1) = 0$ *and* $0 < \lambda(t) \leq \varepsilon_n$, $t \in e_n \backslash e_{n-1}$, $n \geq 1$.

We next present a general existence theorem for BVP (1.1).

THEOREM 2.1. *Let $n_0 \in \{1, 2, \dots\}$ be fixed and suppose the following conditions are satisfied:*

$$f : (0, 1) \times (0, \infty) \times R \to R \text{ is continuous} \tag{2.1}$$

$$\begin{cases} \text{let } n \in \{n_0, n_0 + 1, \dots\} \equiv N_0 \text{ and associated with each } n \in N_0 \\ \text{we have a constant } \rho_n \text{ such that } \{\rho_n\} \text{ is a nonincreasing} \\ \text{sequence with } \lim_{n\to\infty} \rho_n = 0 \text{ and such that for} \\ \frac{1}{2^{n+1}} \le t \le 1 \text{ and } v \in R \text{ we have } f(t, \rho_n, v) \ge 0 \end{cases} \tag{2.2}$$

$$\begin{cases} \exists\, \alpha \in C[0, 1] \cap C^1(0, 1),\ \varphi_p(\alpha') \in C^1(0, 1),\ \alpha(0) = 0 = \alpha(1), \\ \alpha > 0 \text{ on } (0, 1) \text{ such that} \\ -(\varphi_p(\alpha'))' \le f(t, \alpha(t), v) \text{ for } (t, v) \in (0, 1) \times R \end{cases} \tag{2.3}$$

$$\begin{cases} \exists\, \beta \in C^1[0, 1],\ (\varphi_p(\beta'))' \in C(0, 1), \\ \text{with } \beta(t) \ge \alpha(t),\ \beta(t) \ge \rho_{n_0} \text{ for } t \in [0, 1] \text{ and} \\ -(\varphi_p(\beta'))' \ge f(t, \beta(t), \beta'(t)) \text{ for } t \in (0, 1) \text{ and} \\ -(\varphi_p(\beta'))' \ge f(\frac{1}{2^{n_0+1}}, \beta(t), \beta'(t)) \text{ for } t \in (0, \frac{1}{2^{n_0+1}}) \end{cases} \tag{2.4}$$

and

$$\begin{cases} \text{there exist } q \in C(0, 1) \text{ and} \\ \text{for any } 0 < \varepsilon < a_0 = \sup_{t\in[0,1]} \beta(t), \text{ there exists continuous function} \\ \Psi_\varepsilon : [0, \infty) \to (0, \infty) \text{ such that} \\ |f(t, u, v)| \le q(t)\Psi_\varepsilon(|v|) \text{ for } (t, u, v) \in (0, 1) \times [\varepsilon, a_0] \times R, \\ \int_0^1 q(s)\, ds < \infty \text{ and } \int_0^1 q(s)\, ds < \int_0^\infty \frac{du}{\Psi_\varepsilon(\varphi_p^{-1}(u))} \end{cases} \tag{2.5}$$

where φ_p^{-1} is the inverse function of φ_p. Then (1.1) has a solution $u \in C[0, 1] \cap C^1(0, 1)$, $(\varphi_p(u'))' \in C(0, 1)$ with $\alpha(t) \le u(t) \le \beta(t)$ for $t \in [0, 1]$.

Proof. For $n = n_0, n_0 + 1, \dots$ let

$$e_n = \left[\frac{1}{2^{n+1}}, 1\right] \text{ and } \theta_n(t) = \max\left\{\frac{1}{2^{n+1}}, t\right\},\ 0 \le t \le 1$$

and

$$f_n(t, x, y) = \max\{f(\theta_n(t), x, y), f(t, x, y)\}.$$

Next we define inductively

$$g_{n_0}(t, x, y) = f_{n_0}(t, x, y)$$

and

$$g_n(t, x, y) = \min\left\{f_{n_0}(t, x, y), \dots, f_n(t, x, y)\right\},\ n = n_0 + 1,\ n_0 + 2, \dots.$$

Notice

$$f(t, x, y) \le \dots \le g_{n+1}(t, x, y) \le g_n(t, x, y) \le \dots \le g_{n_0}(t, x, y)$$

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for $(t, x, y) \in (0, 1] \times (0, \infty) \times R$ and

$$g_n(t, x, y) = f(t, x, y) \text{ for } (t, x, y) \in e_n \times (0, \infty) \times R.$$

Without loss of generality assume $\rho_{n_0} \leq \min_{t\in[\frac{1}{3},\frac{2}{3}]} \alpha(t)$. Fix $n \in \{n_0, n_0+1, \ldots\}$. Let $t_n \in [0, \frac{1}{3}]$ and $s_n \in [\frac{2}{3}, 1]$ be such that

$$\alpha(t_n) = \alpha(s_n) = \rho_n \text{ and } \alpha(t) \leq \rho_n \text{ for } t \in [0, t_n] \cup [s_n, 1].$$

Define

$$\alpha_n(t) = \begin{cases} \rho_n \text{ if } t \in [0, t_n] \cup [s_n, 1] \\ \alpha(t) \text{ if } t \in (t_n, s_n). \end{cases}$$

We begin with the boundary value problem

$$\begin{cases} -(\varphi_p(u'))' = g^*_{n_0}(t, u, u'), \ 0 < t < 1 \\ u(0) = u(1) = \rho_{n_0} \end{cases} \tag{2.6}$$

where

$$g^*_{n_0}(t, u, v) = \begin{cases} g_{n_0}(t, \alpha_{n_0}, v^*) + r(\alpha_{n_0} - u), \ u(t) \leq \alpha_{n_0}(t) \\ g_{n_0}(t, u, v^*), \qquad \alpha_{n_0}(t) \leq u(t) \leq \beta(t) \\ g_{n_0}(t, \beta, v^*) + r(\beta - u), \ u(t) \geq \beta(t) \end{cases}$$

with

$$v^* = \begin{cases} M_{n_0}, \ v > M_{n_0} \\ v, \ -M_{n_0} \leq v \leq M_{n_0} \\ -M_{n_0}, \ v < -M_{n_0} \end{cases}$$

and $r : R \to [-1, 1]$ is the radial retraction defined by

$$r(u) = \begin{cases} u, \quad |u| \leq 1 \\ \dfrac{u}{|u|}, \quad |u| > 1, \end{cases}$$

and $M_{n_0} \geq \sup_{[0,1]} |\beta'(t)|$ is such that (with $\varepsilon = \min_{[0,1]} \alpha_{n_0}(t)$)

$$\int_0^{\varphi_p(M_{n_0})} \frac{du}{\Psi_\varepsilon\left(\varphi_p^{-1}(u)\right)} > \int_0^1 q(s)\, ds. \tag{2.7}$$

From [5], we know problem (2.6) has a solution $u_{n_0} \in C^1[0, 1]$ with $(\varphi_p(u'_{n_0}))' \in C(0, 1)$. We first show

$$u_{n_0}(t) \geq \alpha_{n_0}(t) \text{ for } t \in [0, 1]. \tag{2.8}$$

Suppose (2.8) is not true. Then $u_{n_0} - \alpha_{n_0}$ has a negative absolute minimum at $\tau \in (0, 1)$. Now since $u_{n_0}(0) - \alpha_{n_0}(0) = 0 = u_{n_0}(1) - \alpha_{n_0}(1)$ there exists $\tau_0, \tau_1 \in [0, 1]$ with

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$\tau \in (\tau_0, \tau_1)$ and

$$u_{n_0}(\tau_0) - \alpha_{n_0}(\tau_0) = u_{n_0}(\tau_1) - \alpha_{n_0}(\tau_1) = 0$$

and

$$u_{n_0}(t) - \alpha_{n_0}(t) < 0,\ \ t \in (\tau_0, \tau_1).$$

We now claim

$$(\varphi_p(u'_{n_0}))' - (\varphi_p(\alpha'_{n_0}))' < 0 \text{ for a.e. } t \in (\tau_0, \tau_1). \tag{2.9}$$

If (2.9) is true, then (2.8) holds. Let

$$w_{n_0}(t) = u_{n_0}(t) - \alpha_{n_0}(t) < 0 \text{ for } t \in (\tau_0, \tau_1).$$

Then

$$\int_{\tau_0}^{\tau_1} ((\varphi_p(u'_{n_0}))' - (\varphi_p(\alpha'_{n_0}))')w_{n_0}(t)\,dt \geq 0.$$

On the other hand, using the inequality

$$(\varphi_p(b) - \varphi_p(a))(b-a) \geq 0 \text{ for } a, b \in R$$

and the fact that there exists $\tau^* \in (\tau_0, \tau_1)$ with $u'_{n_0}(\tau^*) \neq \alpha'_{n_0}(\tau^*)$, we have

$$\begin{aligned} &\int_{\tau_0}^{\tau_1} ((\varphi_p(u'_{n_0}))'(t) - (\varphi_p(\alpha'_{n_0}))'(t))w_{n_0}(t)\,dt \\ &= -\int_{\tau_0}^{\tau_1} (\varphi_p(u'_{n_0}(t)) - \varphi_p(\alpha'_{n_0}(t)))(u'_{n_0}(t) - \alpha'_{n_0}(t))\,dt \\ &< 0, \end{aligned}$$

which is a contradiction. As a result if we show that (2.9) is true then (2.8) will follow.

To see that (2.9) is true we will in fact prove more, i.e., we will prove that

$$(\varphi_p(u'_{n_0}))'(t) - (\varphi_p(\alpha'_{n_0}))'(t) < 0 \text{ for } t \in (\tau_0, \tau_1) \text{ provided } t \neq t_{n_0} \text{ or } t \neq s_{n_0}.$$

Fix $t \in (\tau_0, \tau_1)$ and assume $t \neq t_{n_0}$ or $t \neq s_{n_0}$. Then

$$\begin{aligned} &(\varphi_p(u'_{n_0}))'(t) - (\varphi_p(\alpha'_{n_0}))'(t) \\ &\quad = -[g_{n_0}(t, \alpha_{n_0}(t), (u'_{n_0}(t))^*) + r(\alpha_{n_0}(t) - u_{n_0}(t)) + (\varphi_p(\alpha'_{n_0}))'(t)] \\ &\quad = \begin{cases} -[g_{n_0}(t, \alpha(t), (u'_{n_0}(t))^*) + r(\alpha(t) - u_{n_0}(t)) + (\varphi_p(\alpha'))'(t)] \text{ if } t \in \big(t_{n_0}, s_{n_0}\big) \\ -[g_{n_0}(t, \rho_{n_0}, (u'_{n_0}(t))^*) + r\big(\rho_{n_0} - u_{n_0}(t)\big)] \text{ if } t \in (0, t_{n_0}) \cup (s_{n_0}, 1). \end{cases} \end{aligned}$$

Case (1). $t \in [\frac{1}{2^{n_0+1}}, 1)$.

Then since $g_{n_0}(t, u, v) = f(t, u, v)$ for $(u, v) \in (0, \infty) \times R$ (note $t \in e_{n_0}$) we have

$$\begin{aligned} &(\varphi_p(u'_{n_0}))'(t) - (\varphi_p(\alpha'_{n_0}))'(t) \\ &\quad = \begin{cases} -[f(t, \alpha(t), (u'_{n_0}(t))^*) + r(\alpha(t) - u_{n_0}(t)) + (\varphi_p(\alpha'))'(t)] \text{ if } t \in \big(t_{n_0}, s_{n_0}\big) \\ -[f(t, \rho_{n_0}, (u'_{n_0}(t))^*) + r\big(\rho_{n_0} - u_{n_0}(t)\big)] \text{ if } t \in \big(0, t_{n_0}\big) \cup (s_{n_0}, 1) \end{cases} \\ &\quad < 0, \end{aligned}$$

from (2.2) and (2.3).

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Case (2). $t \in (0, \frac{1}{2^{n_0+1}})$.

Then since

$$g_{n_0}(t, u, v) = \max\left\{f\left(\frac{1}{2^{n_0+1}}, u, v\right), f(t, u, v)\right\}$$

we have $g_{n_0}(t, u, v) \geq f(t, u, v)$ and $g_{n_0}(t, u, v) \geq f(\frac{1}{2^{n_0+1}}, u, v)$ for $(u, v) \in (0, \infty) \times R$. Thus we have

$$\begin{aligned} &(\varphi_p(u'_{n_0}))'(t) - (\varphi_p(\alpha'_{n_0}))'(t) \\ &\leq \begin{cases} -[f(t, \alpha(t), (u'_{n_0}(t))^*) + r(\alpha(t) - u_{n_0}(t)) + (\varphi_p(\alpha'))'(t)] \text{ if } t \in (t_{n_0}, s_{n_0}) \\ -[f(\frac{1}{2^{n_0+1}}, \rho_{n_0}, (u'_{n_0}(t))^*) + r(\rho_{n_0} - u_{n_0}(t))] \text{ if } t \in (0, t_{n_0}) \cup (s_{n_0}, 1) \end{cases} \\ &< 0, \end{aligned}$$

from (2.2) and (2.3).

Now case (1) and (2) guarantee that (2.9) holds, so (2.8) is satisfied. Thus

$$\alpha(t) \leq \alpha_{n_0}(t) \leq u_{n_0}(t) \text{ for } t \in [0, 1]. \tag{2.10}$$

Next we show

$$u_{n_0}(t) \leq \beta(t) \text{ for } t \in [0, 1]. \tag{2.11}$$

If (2.11) is not true then $u_{n_0} - \beta$ would have a positive absolute maximum at say $\tau_0 \in (0, 1)$, in which case $(u_{n_0} - \beta)'(\tau_0) = 0$ and

$$(\varphi_p(u'_{n_0}))'(\tau_0) - (\varphi_p(\beta'))'(\tau_0) \leq 0. \tag{2.12}$$

See the proof in [5].

There are two cases to consider, namely $\tau_0 \in [\frac{1}{2^{n_0+1}}, 1)$ and $\tau_0 \in (0, \frac{1}{2^{n_0+1}})$.

Case (1). $\tau_0 \in [\frac{1}{2^{n_0+1}}, 1)$.

Then $u_{n_0}(\tau_0) > \beta(\tau_0)$, $u'_{n_0}(\tau_0) = \beta'(\tau_0)$ together with $g_{n_0}(\tau_0, u, v) = f(\tau_0, u, v)$ for $(u, v) \in (0, \infty) \times R$ and $M_{n_0} \geq \sup_{[0,1]} |\beta'(t)|$ gives

$$\begin{aligned} &(\varphi_p(u'_{n_0}))'(\tau_0) - (\varphi_p(\beta'))'(\tau_0) \\ &\quad = -[g_{n_0}(\tau_0, \beta(\tau_0), (u'_{n_0}(\tau_0))^*) + r(\beta(\tau_0) - u_{n_0}(\tau_0))] - (\varphi_p(\beta'))'(\tau_0) \\ &\quad = -[(\varphi_p(\beta'))'(\tau_0) + f(\tau_0, \beta(\tau_0), \beta'(\tau_0))] - r(\beta(\tau_0) - u_{n_0}(\tau_0)) \\ &\quad > 0 \end{aligned}$$

from (2.4), which is a contradiction.

Case (2). $\tau_0 \in (0, \frac{1}{2^{n_0+1}})$.

Then $u_{n_0}(\tau_0) > \beta(\tau_0)$ together with

$$g_{n_0}(\tau_0, u, v) = \max\left\{f\left(\frac{1}{2^{n_0+1}}, u, v\right), f(\tau_0, u, v)\right\}$$

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for $(u, v) \in (0, \infty) \times R$ gives

$$\begin{aligned}&\left(\varphi_p\left(u'_{n_0}\right)\right)'(\tau_0) - (\varphi_p(\beta'))'(\tau_0)\\&\quad = -\left[\max\left\{f\left(\frac{1}{2^{n_0+1}}, \beta(\tau_0), \beta'(\tau_0)\right), f\left(\tau_0, \beta(\tau_0), \beta'(\tau_0)\right)\right\} + r\left(\beta(\tau_0) - u_{n_0}(\tau_0)\right)\right]\\&\qquad - (\varphi_p(\beta'))'(\tau_0) > 0\end{aligned}$$

from (2.4), which is a contradiction.

Thus (2.11) holds. Next we show that

$$\left|u'_{n_0}\right|_\infty = \sup_{t\in[0,1]}\left|u'_{n_0}(t)\right| \le M_{n_0}. \tag{2.13}$$

Suppose that (2.13) is false. Let $\varepsilon = \min_{[0,1]}\alpha_{n_0}(t)$. Without loss of generality assume $u'_{n_0}(t) \not\le M_{n_0}$ for some $t \in [0, 1]$. Then since $u_{n_0}(0) = u_{n_0}(1) = \rho_{n_0}$ there exists $\tau_1 \in (0, 1)$ with $u'_{n_0}(\tau_1) = 0$ and so there exists $\tau_2, \tau_3 \in (0, 1)$ with $u'_{n_0}(\tau_3) = 0$, $u'_{n_0}(\tau_2) = M_{n_0}$ and $0 \le u'_{n_0}(s) \le M_{n_0}$ for s between τ_3 and τ_2. Without loss of generality assume $\tau_3 < \tau_2$. Now since $\alpha_{n_0}(t) \le u_{n_0}(t) \le \beta(t)$ for $t \in [0, 1]$ and

$$g_{n_0}(t, u, v) = \max\left\{f\left(\frac{1}{2^{n_0+1}}, u, v\right), f(t, u, v)\right\}$$

for $(t, u, v) \in (0, 1) \times (0, \infty) \times R$, we have for $s \in (\tau_3, \tau_2)$ that

$$(\varphi_p(u'_{n_0}))'(\tau_0) \le q(s)\Psi_\varepsilon(u'_{n_0}(s)),$$

and so

$$\int_0^{\varphi_p(M_{n_0})} \frac{du}{\Psi_\varepsilon\left(\varphi_p^{-1}(u)\right)} = \int_{\tau_3}^{\tau_2} \frac{\left(\varphi_p\left(u'_{n_0}\right)\right)'}{\Psi_\varepsilon\left(u'_{n_0}(s)\right)} ds \le \int_0^1 q(s)\, ds.$$

This contradicts (2.7). The other cases are treated similarly. As a result $\alpha(t) \le u_{n_0}(t) \le \beta(t)$ for $t \in [0, 1]$ and $\left|u'_{n_0}\right|_\infty \le M_{n_0}$. Thus u_{n_0} satisfies

$$-\left(\varphi_p\left(u'_{n_0}\right)\right)' = g_{n_0+1}\left(t, u_{n_0}, u'_{n_0}\right),\ \ 0 < t < 1$$

Next we consider the boundary value problem

$$\begin{cases} -(\varphi_p(u'))' = g^*_{n_0+1}(t, u, u'),\ \ 0 < t < 1\\ u(0) = u(1) = \rho_{n_0+1}\end{cases} \tag{2.14}$$

where

$$g^*_{n_0+1}(t, u, v) = \begin{cases} g_{n_0+1}\left(t, \alpha_{n_0+1}, v^*\right) + r\left(\alpha_{n_0+1} - u\right),\ \ u(t) \le \alpha_{n_0+1}(t)\\ g_{n_0+1}\left(t, u, v^*\right),\ \ \rho_{n_0+1} \le u(t) \le u_{n_0}(t)\\ g_{n_0+1}\left(t, u_{n_0}, v^*\right) + r\left(u_{n_0} - u\right),\ \ u(t) \ge u_{n_0}(t)\end{cases}$$

with

$$v^* = \begin{cases} M_{n_0+1}, & v > M_{n_0+1}\\ v, & -M_{n_0+1} \le v \le M_{n_0+1}\\ -M_{n_0+1}, & v < -M_{n_0+1};\end{cases}$$

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here $M_{n_0+1} \geq M_{n_0}$ is such that (with $\varepsilon = \min_{[0,1]} \alpha_{n_0+1}(t)$) and Ψ_ε and q are as described in (2.5))

$$|f(t,u,v)| \leq q(t)\Psi_\varepsilon(|v|) \text{ for } (t,u,v) \in (0,1) \times [\varepsilon,\infty) \times R$$

and

$$\int_0^1 q(s)\,ds < \int_0^{\varphi_p(M_{n_0+1})} \frac{du}{\Psi_\varepsilon\left(\varphi_p^{-1}(u)\right)}. \tag{2.15}$$

From [5] we know there exists a solution $u_{n_0+1} \in C^1[0,1]$ with $(\varphi_p(u'_{n_0+1}))' \in C(0,1)$ to (2.14). We first show that

$$u_{n_0+1}(t) \geq \alpha_{n_0+1}(t), \ \ t \in [0,1]. \tag{2.16}$$

Suppose that (2.16) is not true. Then there exists $\tau_0, \tau_1 \in [0,1]$ with

$$u_{n_0+1}(\tau_0) - \alpha_{n_0+1}(\tau_0) = u_{n_0+1}(\tau_1) - \alpha_{n_0+1}(\tau_1) = 0$$

and

$$u_{n_0+1}(t) - \alpha_{n_0+1}(t) < 0, \ \ t \in (\tau_0, \tau_1).$$

If we show

$$\left(\varphi_p\left(u'_{n_0}\right)\right)' - \left(\varphi_p\left(\alpha'_{n_0}\right)\right)' < 0 \text{ for a.e. } t \in (\tau_0,\tau_1), \tag{2.17}$$

then as before (2.16) is true. Fix $t \in (\tau_0,\tau_1)$ and assume $t \neq t_{n_0}$ or $t \neq s_{n_0}$. Then

$$\begin{aligned}&\left(\varphi_p\left(u'_{n_0}\right)\right)'(t) - \left(\varphi_p\left(\alpha'_{n_0}\right)\right)'(t)\\ &= \begin{cases} -\left[g_{n_0+1}\left(t,\alpha(t),\left(u'_{n_0+1}(t)\right)^*\right) + r(\alpha(t) - u_{n_0+1}(t)) + (\varphi_p(\alpha'))'(t)\right] & \text{if } t \in (t_{n_0+1}, s_{n_0+1})\\ -\left[g_{n_0+1}\left(t,\rho_{n_0+1},\left(u'_{n_0+1}(t)\right)^*\right) + r(\rho_{n_0+1} - u_{n_0+1}(t))\right] & \text{if } t \in (0,t_{n_0+1}) \cup (s_{n_0+1},1).\end{cases}\end{aligned}$$

Case (1). $t \in [\frac{1}{2^{n_0+2}}, 1)$.

Then since $g_{n_0+1}(t,u,v) = f(t,u,v)$ for $(u,v) \in (0,\infty) \times R$ (note $t \in e_{n_0+1}$) we have

$$\begin{aligned}&\left(\varphi_p\left(u'_{n_0+1}\right)\right)'(t) - \left(\varphi_p\left(\alpha'_{n_0+1}\right)\right)'(t)\\ &= \begin{cases} -\left[f\left(t,\alpha(t),\left(u'_{n_0+1}(t)\right)^*\right) + r(\alpha(t) - u_{n_0+1}(t)) + \left(\varphi_p(\alpha')\right)'(t)\right] & \text{if } t \in (t_{n_0+1}, s_{n_0+1})\\ -\left[f\left(t,\rho_{n_0+1},\left(u'_{n_0+1}(t)\right)^*\right) + r\left(\rho_{n_0+1} - u_{n_0+1}(t)\right)\right] & \text{if } t \in (0,t_{n_0+1}) \cup \left(s_{n_0+1},1\right)\end{cases}\\ &< 0,\end{aligned}$$

from (2.2) and (2.3).

Case (2). $t \in (0, \frac{1}{2^{n_0+2}})$.

Then since $g_{n_0+1}(t_1,u,v)$ equals

$$\min\left\{\max\left\{f\left(\frac{1}{2^{n_0+1}},u,v\right), f(t,u,v)\right\}, \max\left\{f\left(\frac{1}{2^{n_0+2}},u,v\right), f(t,u,v)\right\}\right\}$$

we have

$$g_{n_0+1}(t,u,v) \geq f(t,u,v)$$

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and

$$g_{n_0+1}(t,u,v) \geq \min\left\{f\left(\frac{1}{2^{n_0+1}},u,v\right),\ f\left(\frac{1}{2^{n_0+2}},u,v\right)\right\}$$

for $(u,v) \in (0,\infty) \times R$. Thus we have

$$\begin{aligned}&\left(\varphi_p\left(u'_{n_0+1}\right)\right)'(t) - \left(\varphi_p\left(\alpha'_{n_0+1}\right)\right)'(t)\\ &\leq \begin{cases} -\left[f\left(t,\alpha(t),\left(u'_{n_0+1}(t)\right)^*\right) + r\left(\alpha(t) - u_{n_0+1}(t)\right) + \left(\varphi_p\left(\alpha'\right)\right)'(t)\right] & \\ \hfill \text{if } t \in \left(t_{n_0+1}, s_{n_0+1}\right) & \\ -\left[\min\left\{f\left(\frac{1}{2^{n_0+1}},\rho_{n_0+1},\left(u'_{n_0+1}(t)\right)^*\right), f\left(\frac{1}{2^{n_0+2}},\rho_{n_0+1},\left(u'_{n_0+1}(t)\right)^*\right)\right\}\right. & \\ \left. \quad + r\left(\rho_{n_0+1} - u_{n_0+1}(t)\right)\right] \hfill \text{if } t \in \left(0,t_{n_0+1}\right) \cup \left(s_{n_0+1},1\right) & \end{cases}\\ &< 0,\end{aligned}$$

from (2.2) and (2.3) since

$$f\left(\frac{1}{2^{n_0+1}},\rho_{n_0+1},\left(u'_{n_0+1}(t)\right)^*\right) \geq 0 \text{ and } f\left(\frac{1}{2^{n_0+2}},\rho_{n_0+1},\left(u'_{n_0+1}(t)\right)^*\right) \geq 0$$

because

$$f\left(t,\rho_{n_0+1},\left(u'_{n_0+1}(t)\right)^*\right) \geq 0 \text{ for } t \in \left[\frac{1}{2^{n_0+2}},1\right]$$

and $\frac{1}{2^{n_0+1}} \in [\frac{1}{2^{n_0+2}},1]$.

Consequently (2.16) is true. Thus

$$\alpha(t) \leq \alpha_{n_0+1}(t) \leq u_{n_0+1}(t) \text{ for } t \in [0,1]. \tag{2.18}$$

Next we show that

$$u_{n_0+1}(t) \leq u_{n_0}(t) \text{ for } t \in [0,1]. \tag{2.19}$$

If (2.19) is not true then $u_{n_0+1} - u_{n_0}$ would have a positive absolute maximum at say $\tau_0 \in (0,1)$, in which case $(u_{n_0+1} - u_{n_0})'(\tau_0) = 0$ and

$$\left(\varphi_p\left(u'_{n_0+1}\right)\right)'(\tau_0) - \left(\varphi_p\left(u'_{n_0}\right)\right)'(\tau_0) \leq 0. \tag{2.20}$$

The proof is as above. Then $u_{n_0+1}(\tau_0) > u_{n_0}(\tau_0)$ together with $g_{n_0}(\tau_0,u,v) \geq g_{n_0+1}(\tau_0,u,v)$ for $(u,v) \in (0,\infty) \times R$ gives (note $(u'_{n_0+1}(\tau_0))^* = (u'_{n_0}(\tau_0))^* = u'_{n_0}(\tau_0)$ since $M_{n_0+1} \geq M_{n_0}$ and $|u'_{n_0}|_\infty \leq M_{n_0}$)

$$\begin{aligned}&\left(\varphi_p\left(u'_{n_0+1}\right)\right)'(\tau_0) - \left(\varphi_p\left(u'_{n_0}\right)\right)'(\tau_0)\\ &\quad = -\left[g_{n_0+1}\left(\tau_0,u_{n_0}(\tau_0),\left(u'_{n_0+1}(\tau_0)\right)^*\right) + r\left(u_{n_0}(\tau_0) - u_{n_0+1}(\tau_0)\right)\right] - \left(\varphi_p\left(u'_{n_0}\right)\right)'(\tau_0)\\ &\quad \geq -\left[\left(\varphi_p\left(u'_{n_0}\right)\right)'(\tau_0) + g_{n_0}\left(\tau_0,u_{n_0}(\tau_0),u'_{n_0}(\tau_0)\right)\right] - r\left(u_{n_0}(\tau_0) - u_{n_0+1}(\tau_0)\right)\\ &\quad = -r\left(u_{n_0}(\tau_0) - u_{n_0+1}(\tau_0)\right)\\ &\quad > 0,\end{aligned}$$

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which is a contradiction. Thus (2.19) holds. Next we show that

$$\left|u'_{n_0+1}\right|_\infty \leq M_{n_0+1}. \tag{2.21}$$

Essentially the same argument as before guarantees that (2.21) holds. As a result

$$-\left(\varphi_p\left(u'_{n_0+1}\right)\right)' = g_{n_0+1}\left(t, u_{n_0+1}, u'_{n_0+1}\right) \text{ on } (0,1).$$

Now proceed inductively to construct $u_{n_0+2}, u_{n_0+3}, \cdots$ as follows. Suppose we have u_k for some $k \in \{n_0+1, n_0+2\}$ with $\alpha(t) \leq \alpha_k(t) \leq u_k(t) \leq u_{k-1}(t) (\leq \beta(t))$ for $t \in [0,1]$.

Then consider the boundary value problem

$$\begin{cases} -\left(\varphi_p\left(u'\right)\right)' = g^*_{k+1}\left(t, u, u'\right) \ (0 < t < 1), \\ u(0) = u(1) = \rho_{k+1}, \end{cases} \tag{2.22}$$

where

$$g^*_{k+1}(t,u,v) = \begin{cases} g_{k+1}(t, \rho_{k+1}, v^*) + r(\rho_{k+1} - u), \ u(t) \leq \rho_{k+1} \\ g_{k+1}(t, u, v^*), \ \rho_{k+1} \leq u(t) \leq u_k(t) \\ g_{k+1}(t, u_k, v^*) + r(u_k - u), \ u(t) \geq u_k(t) \end{cases}$$

with

$$v^* = \begin{cases} M_{k+1}, \quad v > M_{k+1} \\ v, \quad -M_{k+1} \leq v \leq M_{k+1} \\ -M_{k+1}, \quad v < -M_{k+1}; \end{cases}$$

here $M_{k+1} \geq M_k$ is such that (with $\varepsilon = \min_{[0,1]} \alpha_{k+1}(t)$ and Ψ_ε and q are as described in (2.5))

$$|f(t,u,v)| \leq q(t)\Psi_\varepsilon(|v|) \text{ for } (t,u,v) \in (0,1) \times [\varepsilon, \infty) \times R$$

and

$$\int_0^1 q(s)\,ds < \int_0^{\varphi_p(M_{k+1})} \frac{du}{\Psi_\varepsilon\left(\varphi_p^{-1}(u)\right)}.$$

There exists a solution $u_{k+1} \in C^1[0,1]$ with $(\varphi_p(u'_k))' \in C(0,1)$ to (2.22) and essentially the same reasoning as above yields

$$\alpha(t) \leq \alpha_{k+1}(t) \leq u_{k+1}(t) \leq u_k(t), \ \left|u'_{k+1}(t)\right| \leq M_{k+1} \text{ for } t \in [0,1] \tag{2.23}$$

with

$$-(\varphi_p(u'_{k+1}))' = g_{k+1}(t, u_{k+1}, u'_{k+1}) \text{ for } 0 < t < 1.$$

Now consider the interval $\left[\frac{1}{2^{n_0+1}}, 1 - \frac{1}{2^{n_0+1}}\right]$. We claim that

$$\begin{cases} \{u_n^{(j)}\}_{n=n_0+1}^\infty, \ j = 0, 1, \text{ is a bounded, equicontinuous} \\ \text{family on } \left[\frac{1}{2^{n_0+1}}, 1 - \frac{1}{2^{n_0+1}}\right]. \end{cases} \tag{2.24}$$

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First note that

$$|u_n|_\infty \le \left|u_{n_0}\right|_\infty \le \sup_{[0,1]} \beta(t) = a_0 \text{ for } t \in [0,1] \text{ and } n \ge n_0 + 1. \tag{2.25}$$

Let

$$\varepsilon = \min_{t \in \left[\frac{1}{2^{n_0+1}}, 1 - \frac{1}{2^{n_0+1}}\right]} \alpha(t).$$

Then (2.5) guarantees the existence of Ψ_ε and q (as described in (2.5)) with

$$|f(t,u,v)| \le q(t)\Psi_\varepsilon(|v|) \text{ for } (t,u,v) \in (0,1) \times [\varepsilon, \infty) \times R.$$

This implies that

$$|g_n(t, u_n(t), u_n'(t))| \le q(t)\Psi_\varepsilon(|u_n'(t)|) \text{ for } t \in [a,b] = \left[\frac{1}{2^{n_0+1}}, 1 - \frac{1}{2^{n_0+1}}\right] \subseteq e_{n_0}$$

and $n \ge n_0 + 1$. As a result

$$|(\varphi_p(u_n'))'| \le q(t)\Psi_\varepsilon(|u_n'(t)|) \text{ for } t \in [a,b] \text{ and } n \ge n_0 + 1. \tag{2.26}$$

The mean value theorem implies that there exists $\tau_{1,n} \in (a,b)$ with

$$|u'(\tau_{1,n})| = \frac{|u(b) - u(a)|}{b-a} \le \frac{2a_0}{b-a} = d_{n_0} \text{ for } n \ge n_0.$$

Fix $n \ge n_0 + 1$ and let $t \in [a,b]$. Without loss of generality assume that $u_n'(t) > d_{n_0}$. Then there exists $\tau_1 \in (a,b)$ with $u_n'(\tau_1) = d_{n_0}$ and $u_n'(s) > d_{n_0}$ for s between τ_1 and t. Without loss of generality assume that $\tau_1 < t$. From (2.26) we have

$$\frac{(\varphi_p(u_n'(s)))'}{\Psi_\varepsilon(|u_n'(s)|)} \le q(s) \text{ for } s \in (\tau_1, t),$$

so integration from τ_1 to t yields

$$\int_{\varphi_p(d_{n_0})}^{\varphi_p(u_n'(t))} \frac{du}{\Psi_\varepsilon\left(\varphi_p^{-1}(u)\right)} \le \int_0^1 q(s)\,ds.$$

Let $I_{n_0}(z) = \int_{\varphi_p(d_{n_0})}^{\varphi_p(z)} \frac{du}{\Psi_\varepsilon(\varphi_p^{-1}(u))}$, so

$$\left|u_n'(t)\right| \le I_{n_0}^{-1}\left(\int_0^1 q(s)\,ds\right) \equiv R_{n_0}. \tag{2.27}$$

A similar bound is obtained for the other cases, so

$$|u_n'(s)| \le R_{n_0} \text{ for } s \in [a,b] = \left[\frac{1}{2^{n_0+1}}, 1 - \frac{1}{2^{n_0+1}}\right]$$

and $n \ge n_0 + 1$. Now (2.25), (2.26) and (2.27) guarantee that (2.24) holds. The Arzela-Ascoli theorem guarantees the existence of a subsequence N_{n_0} of integers and a function

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$z_{n_0} \in C^1[\frac{1}{2^{n_0+1}}, 1 - \frac{1}{2^{n_0+1}}]$ with $u_n^{(j)}$, $j = 0, 1$, converging uniformly to $z_{n_0}^{(j)}$ on $[\frac{1}{2^{n_0+1}}, 1 - \frac{1}{2^{n_0+1}}]$ as $n \to \infty$ through N_{n_0}. Similarly

$$\begin{cases} \{u_n^{(j)}\}_{n=n_0+2}^{\infty},\ j = 0, 1, \text{ is a bounded, equicontinuous} \\ \text{family on } [\frac{1}{2^{n_0+2}}, 1 - \frac{1}{2^{n_0+2}}], \end{cases}$$

so there is a subsequence N_{n_0+1} of N_{n_0} and a function

$$z_{n_0+1} \in C^1\left[\frac{1}{2^{n_0+2}}, 1 - \frac{1}{2^{n_0+2}}\right]$$

with $u_n^{(j)}$, $j = 0, 1$, converging uniformly to $z_{n_0+1}^{(j)}$ on $[\frac{1}{2^{n_0+2}}, 1 - \frac{1}{2^{n_0+2}}]$ as $n \to \infty$ through N_{n_0+1}. Note $z_{n_0+1} = z_{n_0}$ on $[\frac{1}{2^{n_0+1}}, 1 - \frac{1}{2^{n_0+1}}]$ since $N_{n_0+1} \subseteq N_{n_0}$. Proceed inductively to obtain subsequences of integers

$$N_{n_0} \supseteq N_{n_0+1} \supseteq \ldots\ldots \supseteq N_k \supseteq \ldots\ldots$$

and functions

$$z_k \in C^1\left[\frac{1}{2^{n_0+1}}, 1 - \frac{1}{2^{n_0+1}}\right]$$

with

$$u_n^{(j)},\ j = 0, 1, \text{ converging uniformly to } z_k^{(j)} \text{ on } \left[\frac{1}{2^{n_0+1}}, 1 - \frac{1}{2^{n_0+1}}\right]$$

as $n \to \infty$ through N_k, and

$$z_k = z_{k-1} \text{ on } \left[\frac{1}{2^k}, 1 - \frac{1}{2^k}\right].$$

Define a function $u : [0, 1] \to [0, \infty)$ by $u(t) = z_k(t)$ on $\left[\frac{1}{2^{k+1}}, 1 - \frac{1}{2^{k+1}}\right]$ and $u(0) = u(1) = 0$. Notice u is well defined and

$$\alpha(t) \le u(t) \le u_{n_0}(t) \le \beta(t) \text{ for } t \in (0, 1).$$

Now let $[a, b] \subset (0, 1)$, be a compact interval. There is an index n^* such that $[a, b] \subset \left[\frac{1}{2^{n+1}}, 1 - \frac{1}{2^{n+1}}\right]$ for all $n > n^*$ and therefore, for all $n > n^*$

$$-(\varphi_p(u_n'))' = f(t, u_n, u_n') \text{ for } a \le t \le b.$$

On the other hand, $\alpha \in C[0, 1]$, $\alpha(t) > 0$ for all $0 < t < 1$ so let $r = \min_{a \le t \le b} \alpha(t) > 0$. Moreover, (2.5) guarantees that there exists q and $\Psi_\varepsilon(|v|)$ (with $\varepsilon = r$) such that

$$|f(t, u, v)| \le q(t)\Psi_\varepsilon(|v|),\ (t, u, v) \in (0, 1) \times [\varepsilon, \infty) \times R.$$

It is easy to see that there exists a continuous function $\bar{f} : (0, 1) \times R^2 \to R$ such that

$$|\bar{f}(t, u, v)| \le q(t)\Psi_\varepsilon(|v|),\ (t, u, v) \in (0, 1) \times R^2$$

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and

$$\overline{f}(t, u, v) = f(t, u, v) \text{ for all } (t, u, v) \in (0, 1) \times [\varepsilon, \infty) \times R.$$

It is clear that $u_n(t) \geq \varepsilon$, $a \leq t \leq b$ for all $n \geq n_0$. Moreover

$$-\left(\varphi_p\left(u_n'\right)\right)' = \overline{f}\left(t, u_n, u_n'\right) \text{ for } a \leq t \leq b.$$

There exists a subsequence S of $\{n^* + 1, n^* + 2, \cdots\}$ with

$$\max_{a \leq t \leq b} |u_n(t) - u(t)| \to 0 \text{ and } \max_{a \leq t \leq b} \left|u_n'(t) - u'(t)\right| \to 0 \text{ as } n \to \infty.$$

Now $(\varphi_p(u'))' \in C[a, b]$ and

$$-(\varphi_p(u'))' = f(t, u, u') \text{ for } a \leq t \leq b.$$

Since $[a, b] \subset (0, 1)$ is arbitrary, we find that

$$(\varphi_p(u'))' \in C(0, 1) \text{ and } -(\varphi_p(u'))' = f(t, u, u') \text{ for } 0 < t < 1.$$

It remains to show u is continuous at 0 and 1. Let $\varepsilon > 0$ be given. Now since $\lim_{n\to\infty} u_n(0) = 0$ there exists $n_1 \in \{n_0, n_0 + 1, \ldots\}$ with $u_{n_1}(0) < \frac{\varepsilon}{2}$. Next since $u_{n_1} \in C[0, 1]$ there exists $\delta_{n_1} > 0$ with

$$u_{n_1}(t) < \frac{\varepsilon}{2} \text{ for } t \in \left[0, \delta_{n_1}\right].$$

Now for $n \geq n_1$ we have, since $\{u_n(t)\}_{n \in N_0}$ is nonincreasing for each $t \in [0, 1]$,

$$\alpha(t) \leq u_n(t) \leq u_{n_1}(t) < \frac{\varepsilon}{2} \text{ for } t \in \left[0, \delta_{n_1}\right].$$

Consequently

$$\alpha(t) \leq u(t) \leq \frac{\varepsilon}{2} < \varepsilon \text{ for } t \in (0, \delta_{n_1}]$$

and so u is continuous at 0. Similarly u is continuous at 1. As a result $u \in C[0, 1]$.

Suppose that (2.1)–(2.3), (2.5) hold and in addition assume the following conditions are satisfied:

$$-(\varphi_p(\alpha'))' < f(t, u, \alpha'(t)) \text{ for } (t, u) \in (0, 1) \times \{u \in (0, \infty) : u < \alpha(t)\} \tag{2.28}$$

and

$$\begin{cases} \exists\, \beta \in C^1[0, 1],\ (\varphi_p(\beta'))' \in C(0, 1), \\ \text{with } \beta(t) \geq \rho_{n_0} \text{ for } t \in [0, 1] \text{ and} \\ -(\varphi_p(\beta'))' \geq f(t, \beta(t), \beta'(t)) \text{ for } t \in (0, 1) \text{ and} \\ -(\varphi_p(\beta'))' \geq f\left(\frac{1}{2^{n_0+1}}, \beta(t), \beta'(t)\right) \text{ for } t \in \left(0, \frac{1}{2^{n_0+1}}\right). \end{cases} \tag{2.29}$$

Then the result in Theorem 2.1 is again true. This follows immediately from Theorem 2.1 once we show that (2.5) holds i.e. once we show that $\beta(t) \geq \alpha(t)$ for $t \in [0, 1]$. Suppose it is false. Then $\alpha - \beta$ would have a positive absolute maximum

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at say $\tau_0 \in (0,1)$, so $(\alpha-\beta)'(\tau_0)=0$ and $\left(\varphi_p(\alpha')\right)'(\tau_0)-\left(\varphi_p(\beta')\right)'(\tau_0)\le 0$. Now $\alpha(\tau_0)>\beta(\tau_0)$ and (2.28) implies that

$$f(\tau_0,\beta(\tau_0),\beta'(\tau_0))+(\varphi_p(\alpha'))'(\tau_0)=f(\tau_0,\beta(\tau_0),\alpha'(\tau_0))+(\varphi_p(\alpha'))'(\tau_0)>0,$$

and this together with (2.29) yields the inequality

$$(\varphi_p(\alpha'))'(\tau_0)-(\varphi_p(\beta'))'(\tau_0)\ge(\varphi_p(\alpha'))'(\tau_0)+f(\tau_0,\beta(\tau_0),\beta'(\tau_0))>0,$$

which is a contradiction. Thus we have the following result.

COROLLARY 2.2. *Let* $n_0\in\{1,2,\dots\}$ *be fixed and suppose* (2.1)–(2.3), (2.5), (2.28) *and* (2.29) *hold. Then* (1.1) *has a solution* $u\in C[0,1]\cap C^1(0,1)$ *with* $(\varphi_p(u'))'\in C(0,1)$ *and with* $\alpha(t)\le u(t)\le\beta(t)$ *for* $t\in[0,1]$.

REMARK 2.1. (*i*) If in (2.2) we replace $\frac{1}{2^{n+1}}\le t\le 1$ with $0\le t\le 1-\frac{1}{2^{n+1}}$ then one would replace (2.4) with

$$\begin{cases}\exists\,\beta\in C^1[0,1],\ (\varphi_p(\beta'))'\in C(0,1),\\ \text{with } \beta(t)\ge\alpha(t),\ \beta(t)\ge\rho_{n_0} \text{ for } t\in[0,1] \text{ and}\\ -\left(\varphi_p(\beta')\right)'\ge f(t,\beta(t),\beta'(t)) \text{ for } t\in(0,1) \text{ and}\\ -\left(\varphi_p(\beta')\right)'\ge f\left(1-\frac{1}{2^{n_0+1}},\beta(t),\beta'(t)\right) \text{ for } t\in\left(1-\frac{1}{2^{n_0+1}},1\right).\end{cases}\tag{2.30}$$

(*ii*) If in (2.2) we replace $\frac{1}{2^{n+1}}\le t\le 1$ with $\frac{1}{2^{n+1}}\le t\le 1-\frac{1}{2^{n+1}}$ then one would replace (2.4) with

$$\begin{cases}\exists\,\beta\in C^1[0,1],\ (\varphi_p(\beta'))'\in C(0,1),\\ \text{with } \beta(t)\ge\alpha(t),\ \beta(t)\ge\rho_{n_0} \text{ for } t\in[0,1] \text{ and}\\ -\left(\varphi_p(\beta')\right)'\ge f(t,\beta(t),\beta'(t)) \text{ for } t\in(0,1) \text{ and}\\ -\left(\varphi_p(\beta')\right)'\ge f\left(\frac{1}{2^{n_0+1}},\beta(t),\beta'(t)\right) \text{ for } t\in\left(0,\frac{1}{2^{n_0+1}}\right)\\ -\left(\varphi_p(\beta')\right)'\ge f\left(1-\frac{1}{2^{n_0+1}},\beta(t),\beta'(t)\right) \text{ for } t\in\left(1-\frac{1}{2^{n_0+1}},1\right).\end{cases}\tag{2.31}$$

This is clear once one change the definition of e_n and θ_n. For example in case (*ii*), take

$$e_n=\left[\frac{1}{2^{n+1}},1-\frac{1}{2^{n+1}}\right] \text{ and } \theta_n(t)=\max\left\{\frac{1}{2^{n+1}},\min\left\{t,1-\frac{1}{2^{n+1}}\right\}\right\}.$$

3. Construction of α and β. Suppose the following condition is satisfied:

$$\begin{cases}\text{let } n\in\{n_0,n_0+1,\dots\} \text{ and associated with each } n \text{ we}\\ \text{have a constant } \rho_n \text{ such that } \{\rho_n\} \text{ is a decreasing}\\ \text{sequence with } \lim_{n\to\infty}\rho_n=0 \text{ and there exists a constant}\\ k_0>0 \text{ such that for } \frac{1}{2^{n+1}}\le t\le 1,\ 0<u\le\rho_n \text{ and } v\in R \text{ we have}\\ f(t,u,v)>k_0.\end{cases}\tag{3.1}$$

We will show if (3.1) holds then (2.3) (and of course (2.2)) and (2.28) are satisfied.

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Using Lemma 2.2, we know there exists a function $\lambda \in C^1[0,1]$ such that $\varphi_p(\lambda') \in C^1[0,1]$, $\lambda(0) = \lambda(1) = 0$, $M = \max_{0\le t\le 1} |(\varphi_p(\lambda'(t)))'| > 0$ and

$$0 < \lambda(t) \le \rho_n, \ \ t \in e_n \backslash e_{n-1} \text{ for } n \ge 1.$$

Let $r = \sup_{[0,1]} |\lambda'(t)|$. From (3.1) there exists $k_0 > 0$ with

$$f(t,u,v) > k_0 \text{ for } t \in (0,1), \ \ 0 < u < \lambda(t) \text{ and } v \in R.$$

Let

$$m = \min\left\{1, \left(\frac{k_0}{M}\right)^{\frac{1}{p-1}}\right\}.$$

Let $\alpha(t) \equiv m\lambda(t)$ for $t \in [0,1]$. Then

$$\begin{aligned}|(\varphi_p(\alpha'))'| &= \varphi_p(m)|(\varphi_p(\lambda'))'|\\ &\le \varphi_p(m)M\\ &\le \frac{k_0}{M}M = k_0,\end{aligned}$$

so

$$(\varphi_p(\alpha'))' + f(t,\alpha(t),v) \ge k_0 - k_0 = 0 \text{ for } (t,v) \in (0,1)\times R \tag{3.2}$$

i.e. (2.3) is satisfied. On the other hand

$$\begin{aligned}(\varphi_p(\alpha'))' + f(t,u,\alpha'(t)) &\ge f(t,u,\alpha'(t)) - k_0\\ &> k_0 - k_0\\ &= 0 \text{ for } (t,u) \in (0,1)\times\{u \in (0,\infty) : u < \alpha(t)\},\end{aligned}$$

so (2.28) is satisfied.

Now we discuss the existence of an upper solution β.

Consider the following conditions:

$$\begin{cases}\text{there exist continuous functions } q:(0,1)\to[0,\infty),\ \Psi:[0,\infty)\to(0,\infty) \text{ and}\\ \text{there exist } h>0 \text{ continuous and nondecreasing on } [0,\infty) \text{ such that}\\ |f(t,u,v)| \le q(t)h(u)\Psi(|v|) \text{ for } (t,u,v)\in(0,1]\times[\rho_{n_0},\infty)\times R\end{cases} \tag{3.3}$$

$$\begin{cases}\text{there exist } M > \rho_{n_0} \text{ and } N > 0 \text{ such that}\\ h(M)\int_0^1 q(s)\,ds < \int_0^{\varphi_p(N)} \dfrac{du}{\Psi\left(\varphi_p^{-1}(u)\right)}\end{cases} \tag{3.4}$$

$$M - \rho_{n_0} > \varphi_p^{-1}(Ch(M))b_0 \tag{3.5}$$

where

$$\begin{cases}b_0 = \max\left\{\int_0^{\frac12}\varphi_p^{-1}\left(\int_s^{\frac12} q(r)\,dr\right)ds,\ \int_{\frac12}^1 \varphi_p^{-1}\left(\int_{\frac12}^s q(r)\,dr\right)ds\right\} \text{ and}\\ C = \max_{-N\le v\le N}\Psi(|v|)\end{cases} \tag{3.6}$$

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$$\begin{cases} \text{for any } \varepsilon > 0, \text{ there exists a continuous function} \\ \Psi_\varepsilon : [0, \infty) \to (0, \infty) \text{ such that} \\ |f(t, u, v)| \leq q(t)\Psi_\varepsilon(|z|) \text{ for } (t, u, v) \in (0, 1) \times [\varepsilon, M] \times R, \\ \int_0^1 q(s)ds < \infty \text{ and } \int_0^1 q(s)\, ds < \int_0^\infty \frac{du}{\Psi_\varepsilon(\varphi_p^{-1}(u))} \end{cases} \tag{3.7}$$

and

$$f(t, u, v) \text{ is nonincreasing on } \left(0, \frac{1}{2^{n_0+1}}\right) \text{ for each fixed } (u, v) \in [\rho_{n_0}, M] \times [-N, N]. \tag{3.8}$$

We show if conditions (3.3)–(3.5), (3.7), (3.8) (here b_0 and C are as in (3.6)) hold then (2.4) and (2.5) hold.

Consider the problem

$$\begin{cases} -(\varphi_p(u'))' = f^*(t, u, u'), \ 0 < t < 1 \\ u(0) = u(1) = \rho_{n_0} \end{cases} \tag{3.9}$$

where

$$f^*(t, u, v) = \begin{cases} f(t, \rho_{n_0}, v^*) + r(\rho_{n_0} - u), & u \leq \rho_{n_0} \\ f(t, u, v^*), & \rho_{n_0} \leq u \leq M \\ f(t, M, v^*) + r(M - u), & u \geq M \end{cases}$$

with

$$v^* = \begin{cases} N, & v > N \\ v, & -N \leq v \leq N \\ -N, & v < -N. \end{cases}$$

From [**5**] we know that (3.9) has a solution $u \in C^1[0, 1]$ with $(\varphi_p(u'))' \in C(0, 1)$. We first show that

$$u(t) \geq \rho_{n_0}, \ t \in [0, 1]. \tag{3.10}$$

Suppose that (3.10) is not true. Then there exists a $t_0 \in (0, 1)$ with $u(t_0) < \rho_{n_0}$, $u'(t_0) = 0$ and

$$(\varphi_p(u'))'(t_0) \geq 0.$$

However note

$$\begin{aligned} (\varphi_p(u'))'(t_0) &= -[f(t_0, \rho_{n_0}, (u'(t_0))^*) + r(\rho_{n_0} - u(t_0))] \\ &= -\left[f(t_0, \rho_{n_0}, 0) + r(\rho_{n_0} - u(t_0))\right] \\ &< 0, \end{aligned}$$

a contradiction.

Consequently (3.10) is true. Next we show

$$u(t) \leq M \text{ for } t \in [0, 1]. \tag{3.11}$$

Suppose (3.11) is false. Now since $u(0) = u(1) = \rho_{n_0}$ there exists either (i) $t_1, t_2 \in (0, 1)$ with $\rho_{n_0} \leq u(t) \leq M$ for $t \in [0, t_2)$, $u(t_2) = M$ and $u(t) > M$ on (t_2, t_1) with

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$u'(t_1) = 0$; or (ii) $t_3, t_4 \in (0, 1)$, $t_4 < t_3$ with $\rho_{n_0} \le u \le M$ for $t \in (t_3, 1]$, $u(t_3) = M$ and $u(t) > M$ on (t_4, t_3) with $u'(t_4) = 0$.

We can assume without loss of generality that either $t_1 \le \frac{1}{2}$ or $t_4 \ge \frac{1}{2}$. Suppose that $t_1 \le \frac{1}{2}$. Notice that for $t \in (t_2, t_1)$ we have

$$(\varphi_p(u'))' = f^*(t, u, u') \le Cq(t)h(M) \ (C \text{ is defined in } (3.6)). \tag{3.12}$$

Integrate (3.12) from t_2 to t_1 to obtain

$$\varphi_p(u'(t_2)) \le Ch(M) \int_{t_2}^{t_1} q(s)\, ds$$

and this together with the fact that $u(t_2) = M$ yields

$$\varphi_p(u'(t_2)) \le Ch(M) \int_{t_2}^{t_1} q(s)\, ds. \tag{3.13}$$

Also for $t \in (0, t_2)$ we have

$$\begin{aligned} -(\varphi_p(u'))' &= f^*(t, u, u') \\ &\le Cq(t)h(u(t)) \\ &\le Cq(t)h(M). \end{aligned}$$

Integrate from t $(t \in (0, t_2))$ to t_2 to obtain

$$-\varphi_p(u'(t_2)) + \varphi_p(u'(t)) \le Ch(M) \int_t^{t_2} q(s)\, ds,$$

so

$$\varphi_p(u'(t)) \le Ch(M) \int_t^{t_2} q(s)\, ds + \varphi_p(u'(t_2)).$$

This together with (3.13) yields

$$\varphi_p(u'(t)) \le Ch(M) \int_t^{t_1} q(s)\, ds \text{ for } t \in (0, t_2).$$

Thus

$$u'(t) \le \varphi_p^{-1}(Ch(M))\varphi_p^{-1}\left(\int_t^{t_1} q(s)\, ds\right) \text{ for } t \in (0, t_2).$$

Integrate from 0 to t_2 to obtain

$$M - \rho_{n_0} \le \varphi_p^{-1}(Ch(M)) \int_0^{t_2} \varphi_p^{-1}\left(\int_t^{t_1} q(s)\, ds\right).$$

That is

$$\begin{aligned} M - \rho_{n_0} &\le \varphi_p^{-1}(Ch(M)) \int_0^{\frac{1}{2}} \varphi_p^{-1}\left(\int_t^{\frac{1}{2}} q_{\rho_{n_0}}(s)\, ds\right) dt \\ &\le \varphi_p^{-1}(Ch(M))\, b_0. \end{aligned}$$

This contradicts (3.5) so (3.11) holds (a similar argument yields a contradiction if $t_4 \ge \frac{1}{2}$).

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Thus we have

$$\rho_{n_0} \leq u(t) \leq M \text{ for } t \in [0, 1].$$

Next we show that

$$|u'|_\infty = \sup_{t\in[0,1]} |u'(t)| \leq N. \tag{3.14}$$

Suppose (3.14) is false. Without loss of generality assume $u'(t) \not\leq N$ for some $t \in [0, 1]$. Then since $u(0) = u(1) = \rho_{n_0}$ there exists $\tau_1 \in (0, 1)$ with $u'(\tau_1) = 0$, and so there exists $\tau_2, \tau_3 \in (0, 1)$ with $u'(\tau_3) = 0$, $u'(\tau_2) = N$ and $0 \leq u'(s) \leq N$ for s between τ_3 and τ_2. Without loss of generality assume that $\tau_3 < \tau_2$. Now since $\rho_{n_0} \leq u(t) \leq M$ for $t \in [0, 1]$ and (with $\varepsilon = \rho_{n_0}$)

$$(\varphi_p(u'))' \leq q(t)h(M)\Psi_\varepsilon(\varphi_p(u'(t))),$$

and so

$$\int_0^{\varphi_p(N)} \frac{du}{\Psi_\varepsilon(\varphi_p^{-1}(u))} = \int_{\tau_3}^{\tau_2} \frac{(\varphi_p(u'))'}{\Psi_\varepsilon(u'(s))} ds \leq h(M) \int_0^1 q(s)\, ds.$$

This contradicts (3.4). The other cases are treated similarly. As a result $\rho_{n_0} \leq u(t) \leq M$ for $t \in [0, 1]$ and $|u'|_\infty \leq N$.

Let $\beta(t) = u(t)$ for $t \in [0, 1]$. Then

$$\begin{cases} \beta \in C^1[0, 1],\ (\varphi_p\beta'))' \in C(0, 1), \\ \text{with } \beta(t) \geq \rho_{n_0} \text{ for } t \in [0, 1] \text{ and} \\ -(\varphi_p(\beta'))' = f(t, \beta(t), \beta'(t)) \text{ for } t \in (0, 1) \end{cases}$$

and

$$-\left(\varphi_p\left(\beta'\right)\right)' = f\left(t, \beta(t), \beta'\left(t\right)\right) \geq f\left(\frac{1}{2^{n_0+1}}, \beta\left(t\right), \beta'(t)\right) \text{ for } t \in \left(0, \frac{1}{2^{n_0+1}}\right).$$

As a result (2.4) and (2.5) are satisfied.

THEOREM 3.1. *Suppose* (2.1), (3.1) *and* (3.3) – (3.5), (3.7), (3.8) (*here* b_0 *and* C *are as in* (3.6)) *hold. Then problem* (1.1) *has a solution* $u \in C[0, 1] \cap C^1(0, 1)$ *with* $(\varphi_p(u'))' \in C(0, 1)$.

4. Examples.

EXAMPLE 1. Consider the boundary value problem

$$\begin{cases} -u'' = \frac{1}{\sqrt{t}}\left(\frac{1}{u^2} - 1\right) h(u)(|u'| + 1),\ \ 0 < t < 1 \\ u(0) = u(1) = 0 \end{cases} \tag{4.1}$$

with

$$h(u) = \begin{cases} \frac{\sqrt{2}u}{40} + 0.05 \ \text{ for } 0 \leq u \leq \sqrt{2} \\ u^2 - 1.9 \ \text{ for } \sqrt{2} < u. \end{cases}$$

Then (4.1) has a solution $u \in C[0, 1] \cap C^1(0, 1)$ with $(\varphi_p(u'))' \in C(0, 1)$.

To see that (4.1) has a solution we will apply Theorem 3.1. Let $n \in \{1, 2, \cdots\}$, $p = 2$ and $\rho_n = \frac{1}{\sqrt{n+1}}$. Let $k_0 = 0.05$. Then, for $\frac{1}{2^{n+1}} \le t \le 1$, $0 < u \le \rho_n$ and $v \in R$ we have

$$f(t, u, v) = \frac{1}{\sqrt{t}}\left(\frac{1}{u^2} - 1\right) h(u)(|v| + 1)$$
$$\ge h(u)\,((n+1) - 1) \ge 0.05 = k_0,$$

so (3.1) is satisfied.

Let $n_0 = 1$ so $\rho_{n_0} = \frac{\sqrt{2}}{2}$, and let $M = \sqrt{2}$ and $N = 10$. Let $q(t) = \frac{1}{\sqrt{t}}$ and $\Psi(v) = |v| + 1$. Then

$$C = \max_{v \in [-N,N]} \Psi(v) = 11, \ \int_0^1 \frac{dt}{\sqrt{t}} = 2, \ b_0 = \int_0^{\frac{1}{2}} \int_s^{\frac{1}{2}} \frac{dt}{\sqrt{t}}\, ds = \frac{\sqrt{2}}{6},$$

so

$$|f(t, u, v)| \le q(t) h(u)\, \Psi(|v|) \text{ for } (t, u, v) \in (0, 1] \times [\rho_1, \infty) \times R.$$

Also notice that

$$h(M) \int_0^1 q(t)\, dt = 0.2,$$

$$\int_0^N \frac{du}{\Psi(v)} = \ln 10 \cong 2.3026,$$

$$M - \rho_1 = \sqrt{2} - \frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{2}$$

and

$$Ch(M)b_0 = 11 \times 0.1 \times \frac{\sqrt{2}}{6} = \frac{11\sqrt{2}}{60}.$$

As a result (3.3)–(3.5) are satisfied. We next establish (3.7).

Let $\Psi_\varepsilon(v) = \left(\frac{1}{\varepsilon^2} + 1\right)(|v| + 1)$. Then

$$|f(t, u, v)| \le q(t)\Psi_\varepsilon(|v|) \text{ for } (t, u, v) \in (0, 1) \times [\varepsilon, M] \times R.$$

Also

$$\int_0^K \frac{dv}{\Psi_\varepsilon(v)} = \frac{\varepsilon^2}{1+\varepsilon^2} \int_0^K \frac{dv}{v+1}$$
$$\ge \frac{\varepsilon^2}{1+\varepsilon^2} \ln (K+1) \to \infty \ \ (\text{as } K \to \infty)$$

i.e.

$$\int_0^\infty \frac{dv}{\Psi_\varepsilon\left(\varphi_p^{-1}(v)\right)} = \infty$$

and

$$\int_0^1 \frac{dt}{\sqrt{t}} = 2.$$

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Then

$$\int_0^1 q(s)\,ds < \infty \text{ and } \int_0^1 q(s)\,ds < \int_0^\infty \frac{du}{\Psi_\varepsilon\left(\varphi_p^{-1}(u)\right)},$$

so (3.7) holds. Finally $f(t,u,v)$ is nonincreasing on $\left(0,\frac{1}{4}\right)$ for each fixed $(u,v)\in\left[\rho_{n_0},M\right]\times[-N,N]$, so (3.8) is satisfied. Theorem 3.1 guarantees that (4.1) has a solution $u\in C[0,1]\cap C^1(0,1)$ with $(\varphi_p(u'))'\in C(0,1)$.

EXAMPLE 2. Consider the boundary value problem

$$\begin{cases} -(|u'|^{p-2}u')' = \frac{1}{\sqrt{t}u^\alpha} + |u'|^\beta - r(t),\ 0<t<1 \\ u(0)=u(1)=0 \end{cases} \tag{4.2}$$

with $p>1$, $\alpha>0$, $r\in C[0,1]$ and $\beta>0$ is such that

$$\int_0^\infty \frac{dv}{\left(v^{\frac{1}{p-1}}+1\right)^\beta} = \infty.$$

Then (4.2) has a solution $u\in C[0,1]\cap C^1(0,1)$.

Let $n\in\{1,2,\cdots\}$ and $\rho_n = \frac{1}{n(1+C_1)^{1/\alpha}}$ where $C_1 = \max_{t\in[0,1]}|r(t)|$. Also let $k_0=1$, so for $\frac{1}{2^{n+1}}\le t\le 1$, $0<u\le\rho_n$ and $v\in R$ we have

$$\begin{aligned} f(t,u,v) &= \frac{1}{\sqrt{t}u^\alpha} + |v|^\beta - r(t) \\ &\ge \frac{1}{\sqrt{t}u^\alpha} - C_1 \\ &\ge \frac{1}{u^\alpha} - C_1 \ge 1 = k_0 \end{aligned}$$

and so (3.1) holds. Next let

$$h(u) = 1+\frac{1}{\rho_1^\alpha}+C_1,\ \ q(t)=\frac{1}{\sqrt{t}}$$
$$\text{and } \Psi(v)=(v+1)^\beta \ \text{ for } v\in[0,\infty).$$

For $(t,u,v)\in(0,1]\times[\rho_1,\infty)\times R$, we have

$$\begin{aligned} |f(t,u,v)| &\le \frac{1}{\sqrt{t}\rho_1^\alpha} + C_1 + \Psi(|v|) \\ &\le \frac{1}{\sqrt{t}}\left[\frac{1}{\rho_1^\alpha}+C_1+\Psi(|v|)\right] \\ &\le \frac{1}{\sqrt{t}}\left(1+\frac{1}{\rho_1^\alpha}+C_1\right)\Psi(|v|). \end{aligned}$$

Let $N>0$ be such that

$$\int_0^{\varphi_p(N)} \frac{dv}{\left(v^{\frac{1}{p-1}}+1\right)^\beta} > 2\left(1+\frac{1}{\rho_1^\alpha}+C_1\right)$$

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and $M > 0$ be such that

$$M > \rho_1 + b_0 (N+1)^{\frac{\beta}{p-1}} \left(1 + \frac{1}{\rho_1^\alpha} + C_1\right)^{\frac{1}{p-1}}$$

where

$$b_0 = \max\left\{\int_0^{\frac{1}{2}} \left(\sqrt{2} - 2\sqrt{s}\right)^{\frac{1}{p-1}} ds,\ \int_{\frac{1}{2}}^1 \left(2\sqrt{s} - \sqrt{2}\right)^{\frac{1}{p-1}} ds\right\}$$

Then (3.3)–(3.5) are satisfied. We next establish (3.7).

For any $\varepsilon > 0$, let

$$\Psi_\varepsilon(v) = \left(1 + \frac{1}{\varepsilon^\alpha} + C_1\right)(v+1)^\beta \text{ for } v \in [0, \infty).$$

Now for $(t, u, v) \in (0, 1] \times [\varepsilon, M] \times R$, we have,

$$\begin{aligned}|f(t,u,v)| &\le \tfrac{1}{\sqrt{t\varepsilon^\alpha}} + C_1 + (|v|+1)^\beta \\ &\le \tfrac{1}{\sqrt{t}}\left(\tfrac{1}{\varepsilon^\alpha} + C_1 + (|v|+1)^\beta\right) \\ &\le q(t)\left(1 + \tfrac{1}{\varepsilon^\alpha} + C_1\right)(|v|+1)^\beta \\ &= q(t)\Psi_\varepsilon(|v|).\end{aligned}$$

Also

$$\int_0^K \frac{dv}{\Psi_\varepsilon\left(\varphi_p^{-1}(v)\right)} = \frac{\varepsilon^\alpha}{1 + (1+C_1)\varepsilon^\alpha} \int_0^K \frac{dv}{\left(v^{\frac{1}{p-1}} + 1\right)^\beta} \to \infty \text{ (as } K \to \infty)$$

so

$$\int_0^\infty \frac{dv}{\Psi_\varepsilon\left(\varphi_p^{-1}(v)\right)} = \infty.$$

As a result

$$\int_0^1 q(s)\,ds < \infty \text{ and } \int_0^1 q(s)\,ds < \int_0^\infty \frac{du}{\Psi_\varepsilon(\varphi_p^{-1}(u))},$$

so (3.7) holds. Finally $f(t, u, v)$ is nonincreasing on $\left(0, \frac{1}{4}\right)$ for each fixed $(u, v) \in [\rho_1, M] \times [-N, N]$, so (3.8) is satisfied. Theorem 3.1 guarantees that (4.2) has a solution $u \in C[0, 1] \cap C^1(0, 1)$ with $(\varphi_p(u'))' \in C(0, 1)$.

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