Can. J. Math., Vol. XXXIX, No. 2, 1987, pp. 428-472

PROXIMAL ANALYSIS AND BOUNDARIES OF CLOSED SETS IN BANACH SPACE PART II: APPLICATIONS

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Introduction. This paper is a direct continuation of the article "Proximal analysis and boundaries of closed sets in Banach space, Part I: Theory", by the same authors. It is devoted to a detailed analysis of applications of the theory presented in the first part and of its limitations.

5. Applications in geometry of normed spaces. Theorem 2.1 has important consequences for geometry of Banach spaces. We start the presentation with a discussion of density and existence of R-proper points (Definition 1.3) for closed sets in Banach spaces. Our considerations will be based on the "lim inf" inclusions proven in the first part of our paper.

THEOREM 5.1. *If C is a closed subset of a Banach space E, then the K-proper points of C are dense in the boundary of C.*

Proof. If $\bar{x}$ is in the boundary of C, for each $r > 0$ we may find $\bar{y} \notin C$ with $\|\bar{y} - \bar{x}\| < r$. Theorem 2.1 now shows that $K_C(x_r) \neq E$ for some $x_r \in C$ with

$$\|\bar{x} - x_r\| \leqq 2r.$$

COROLLARY 5.1. ([**2**]) *If C is a closed convex subset of a Banach space E, then the support points of C are dense in the boundary of C.*

Proof. Since for any convex set C and $x \in C$ we have

$$T_C(x) = K_C(x) = P_C(x) = \overline{\mathbf{P}(C - x)} = WT_C(x) = WK_C(x) = WP_C(x),$$

the R-proper points of C at x, where $R_C(x)$ is any one of the above cones, are exactly the support points. Apply Theorem 5.1 to finish the proof.

COROLLARY 5.2. *Suppose that C is a closed subset of a Banach space and that for all $x \in C$*

$$d \liminf_{x' \xrightarrow{C} x} (R_C(x') + x') \subset K_C(x) + x,$$

as happens, if for all $x \in C$

$$\liminf_{x' \xrightarrow{C} x} R_C(x') \subset K_C(x).$$

Received October 29, 1985. The research of the first author was partially supported on NSERC Grant A5116 while that of the second author was partially supported on NSF Grant ECS 8300214.

Then the R-proper points of C are dense in the boundary of C.

Proof. Use Theorem 5.1 and the corresponding definitions.

Let us recall that a set C is called *tangentially regular at* $x \in C$ if $T_C(x) = K_C(x)$.

COROLLARY 5.3. *If C is a closed subset of a Banach space E such that for all* $x \in C$, $K_C(x)$ *is convex (in particular if C is tangentially regular at all its points), then the P-proper points of C are dense in the boundary of C.*

Proof. We have for all $x \in C$

$$K_C(x) = P_C(x) \supset \liminf_{x' \underset{C}{\to} x} P_C(x').$$

Use Corollary 5.2 to complete the proof.

COROLLARY 5.4. *If C is a closed, boundedly relatively weakly compact subset of a Banach space E then the P-proper points of C are dense in the boundary of C.*

If in addition E has an equivalent Frechet differentiable and Kadec norm or if C is weakly compact and E has an equivalent Frechet norm, then the WP-proper points of C are dense in the boundary of C.

Proof. Use Corollary 5.2 and Theorem 4.1, and then Corollary 5.2 and Theorem 3.4.

COROLLARY 5.5. *If C is a closed subset of a reflexive space then the WP-proper points of C are dense in the boundary of C.*

If in addition the norm of E is Kadec and Frechet differentiable, then the B-proper points are dense in the boundary of C.

Proof. The first statement follows from Corollary 5.4. Use Corollary 5.2 and Theorem 3.1 to justify the last statement.

We have formulated some assumptions on sets and spaces which guarantee the density (in particular existence) of R-proper points. We complete them with the following observations concerning the existence of R-proper points.

First let us recall that a Banach space E is said to have the *Radon-Nikodym property* if every closed and bounded convex subset of E is the closed convex hull of its strongly exposed points [**11**].

PROPOSITION 5.1. *Let C be a subset of a (locally convex vector) space E, then any support point of C is a WP-proper point of C.*

Proof. Note that if $x \in C$ and $x^* \in E^*$ are such that

$$(x^*, c - x) \leqq 0 \quad \text{for all } c \in C,$$

then $x^* \in WP_C^0(x)$.

COROLLARY 5.6. *If E is a Banach space with the Radon-Nikodym property, then every closed and bounded subset of E has a WP-proper point.*

Proof. Note that any strongly exposed point of $\overline{\text{co}}\, C$ must belong to C. Thus if E has the Radon-Nikodym property then any closed and bounded subset of E has a support point. Proposition 5.1 completes the proof.

Definition 5.1. A subset C of a (locally convex vector) space E is called *compactly epi-Lipschitzian* at x if there exist X, a neighbourhood of x, $\lambda > 0$, a compact set K and U, a neighbourhood of zero, such that

$$(5.1) \qquad X \cap C + tU \subset C + tK \quad \text{for all } t \in (0, \lambda).$$

If K may be chosen to be a one point set, then we say that C is *epi-Lipschitzian at x*.

The following important result is due to Rockafellar [**18**].

PROPOSITION 5.2. *If C is epi-Lipschitzian at x then*

$$\text{int } T_C(x) \neq 0$$

and for any $y \in$ int $T_C(x)$ there exist some $\lambda > 0$, X and U, neighbourhoods of x and 0, respectively, for which (5.1) is satisfied with $K = \{y\}$.

The properties of compactly epi-Lipschitzian sets are described in [**6**], where the following results are proven.

PROPOSITION 5.3. *If C is a closed subset of a Banach space E, $x \in C$, then*

(i) *C is compactly epi-Lipschitzian at x and*

$$\text{int } T_C(x) \neq 0$$

if and only if C is epi-Lipschitzian at x.

Moreover

$$\text{(ii)} \qquad \liminf_{\substack{x' \to x \\ C}} K_C(x) = T_C(x),$$

whenever C is compactly epi-Lipschitzian at x.

COROLLARY 5.7. *If C is a closed subset of a Banach space E which is compactly epi-Lipschitzian at $x \in C$, then $T_C(x) = E$ if and only if x lies interior to C.*

Proof. Use Proposition 5.3 (i) and Proposition 5.2.

PROPOSITION 5.4. *Suppose that C is a closed subset of a Banach space which is compactly epi-Lipschitzian at $x \in C$ and tangentially regular at x. If x lies in the boundary of C, then x is a P-proper point.*

https://doi.org/10.4153/CJM-1987-019-4 Published online by Cambridge University Press

Proof. As C is tangentially regular at x we have

$$T_C(x) = K_C(x) = P_C(x).$$

If x is in the boundary of C, Corollary 5.7 implies $T_C(x) \neq E$, hence $P_C(x) \neq E$ and the proof is finished.

COROLLARY 5.8. *Suppose that C is a closed subset of a Banach space E which is compactly epi-Lipschitzian at $x \in C$ and such that*

$$\liminf_{\substack{x' \to x \\ C}} K_C(x') = K_C(x).$$

If x lies in the boundary of C, then x is a P-proper point of C.

Proof. Under our assumptions C is tangentially regular at x on using Proposition 5.3 (ii), hence the conclusion follows from Proposition 5.4.

The existence of B-proper points of C is related to the existence of nearest points in C.

For any closed subset C of a normed space E, let

$$\text{Prox } C := \{c \in C | c \text{ is the nearest point of } C \text{ to some } z \notin C\}.$$

The following is an easy consequence of the definitions.

PROPOSITION 5.5. *Suppose C is a closed subset of a normed space E. Then x is a B-proper point of C if and only if $x \in$* Prox C.

If $\| \ \|$ is some norm on a space E let $B_C^{\| \|}(x)$ denote the Bony tangent cone to C at x with respect to the norm $\| \ \|$. Let $\mathcal{N}$ be the set of all equivalent norms of E.

PROPOSITION 5.6. *If C is a subset of a normed space E, $x \in C$ then*

$$\bigcap_{\| \| \in \mathcal{N}} B_C^{\| \|}(x) = K_C(x).$$

Proof. Inclusion

$$K_C(x) \subset \bigcap_{\| \| \in \mathcal{N}} B_C^{\| \|}(x)$$

follows from Proposition 1.1. So suppose that $y \notin K_C(x)$. Then

$$C \cap [x + [0, 2\lambda](y + 2\epsilon B)] = \{x\} \tag{5.2}$$

for some $\lambda > 0$, $\epsilon > 0$. Let

$$W := \text{co}\{\epsilon B, y, -y\}$$

and set $z := x + \lambda y$. Then x is the (unique) nearest point to z in $\| \ \|_W$. Indeed,

$$x = x + \lambda y - \lambda y \in z + \lambda W,$$

while

$$\begin{aligned} & C \cap (z + \lambda W) \\ & \subset C \cap (z - \lambda y + \lambda \operatorname{co}\{0, y + \epsilon B, 2y\}) \\ & \subset C \cap (x + \operatorname{co}\{0, \lambda y + \lambda\epsilon B, 2\lambda y\}) \\ & \subset C \cap (x + [0, 2\lambda](y + 2\epsilon B)) = \{x\} \end{aligned}$$

on using (5.2). Thus

$$y \notin B_{C}^{\|\,\|}{}_{W(x)}.$$

COROLLARY 5.9. *If C is a closed subset of a Banach space E, then there exists an equivalent norm on E in which C has a B-proper point.*

Proof. Apply Proposition 5.6 and Theorem 5.1.

Definition 5.2. A closed subset of a normed space E is *completely antiproximal* if Prox $C = \emptyset$ for all equivalent norms on E.

COROLLARY 5.10. *A Banach space contains no proper closed nonempty completely antiproximal subset.*

Proof. Use Corollary 5.9 and Proposition 5.5.

The question arises as to what can be said about the space if the norm in Corollary 5.9 may be chosen independently of the set C. The answer follows from the following theorem.

THEOREM 5.2. *Let E be a Banach space. Then the following are equivalent.*

(i) *There exists an equivalent norm on E for which the equality* (*inclusion*)

$$\liminf_{x' \xrightarrow{C} x} B_C(x') = T_C(x), \ (\liminf_{x' \xrightarrow{C} x} B_C(x') \subset T_C(x))$$

holds for any closed sets C, and $x \in C$;

(ii) *there exists an equivalent norm on E for which, for any closed set C, B-proper points exist densely in the boundary of C;*

(iii) *there exists an equivalent norm on E such that for any closed set C,* Prox *C is dense in the boundary of C;*

(iv) *there exists an equivalent norm on E such that for any closed set C,* Prox *C is nonempty;*

(v) *E is reflexive;*

(vi) *there exists an equivalent norm on E, such that for each closed subset C of E, the set of those points which have a nearest point in C is dense in E.*

Proof. (v) implies (i) by Theorem 3.1 and (i) implies (ii) by Corollary 5.2.

Implications (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) are obvious. We prove that (iv) $\Rightarrow$ (v). Assume that (iv) holds and consider E in the norm given by the assumption. Let $x^* \in E^*$, $\|x^*\| = 1$. Put

$$C := \{x \in E | (x^*, x) = 0\}.$$

Then by (iv) there exist $\bar{c} \in C$ and $\bar{x} \notin C$ such that for any $x \in E$, $(x^*, x) = 0$ implies

$$\|x - \bar{x}\| \geqq \|\bar{c} - \bar{x}\|.$$

Thus C and the ball $\bar{x} + \|\bar{x} - \bar{c}\|B$, where B is the closed unit ball in E, may be separated. Hence there exist $0 \neq y^* \in E^*$ and $\alpha \in \mathbf{R}$ such that for any $b \in B$, $c \in C$

$$(5.3) \quad (y^*, \bar{x} + \|\bar{x} - \bar{c}\|b) \leqq \alpha \leqq (y^*, c).$$

This is only possible if $\alpha = 0$ and $x^* = \lambda y^*$, for some $\lambda \neq 0$. Thus from (5.3) we get

$$(x^*, \bar{x} + \|\bar{x} - \bar{c}\|b) \geqq 0 \quad \text{for all } b \in B,$$

or

$$(x^*, -\bar{x} + \|\bar{x} - \bar{c}\|b) \geqq 0 \quad \text{for all } b \in B.$$

Hence as $(x^*, \bar{c}) = 0$ and $\|x^*\| = 1$ we get

$$\left(x^*, \frac{\bar{x} - \bar{c}}{\|\bar{x} - c\|}\right) = 1 \quad \text{or} \quad \left(x^*, \frac{\bar{c} - \bar{x}}{\|\bar{x} - c\|}\right) = 1.$$

We conclude that any functional from E^* attains its norm on a unit ball hence by an application of James' Theorem [**12**], E is reflexive. That is (v) holds. (v) implies (vi) by Proposition 3.3 and Proposition 3.4 and (vi) obviously implies (iv). The proof is complete.

We leave for a while considerations related to the existence and the density of R-proper points of closed subsets of Banach spaces and we will return to this subject in the next section in connection with differentiability and subdifferentiability properties of functions on Banach spaces.

We now turn to the theory of starshaped sets which, as will be shown, is another area of possible applications of our main results.

Let E be a normed space, $C \subset E$.

Definition 5.3. We say $x \in C$ *sees* c in C if the line segment $[x, c]$ is contained in C.

We say C has *α-visibility* (in $A \subset C$) if every subset of C of cardinality α is simultaneously seen by some point $c \in C$ ($a \in A$).

If C has k-visibility (in A) for every natural number k we say C has *finite visibility* (in A).

Comparing Definition 2.1 and Definition 5.3 we see that C of cardinality α is starshaped if and only if C has α-visibility.

Definition 5.4. A point $c \in C$ is said to be a *cone point of* C if there is a nonzero $x^* \in E^*$ such that

$$(x^*, c) = \sup\{(x^*, x) \mid x \text{ sees } c \text{ in } C\}.$$

As the set of those x's which see c in C is contained in $P_C(c) + c$, any P-proper point of C is a cone point of C.

Krasnoselski [**13**] showed that if C is a compact subset of $\mathbf{R}^n$ and every $n + 1$ cone points of C can be seen by some point in C, then C is starshaped.

We prove the following strengthening of Krasnoselski's result.

THEOREM 5.3. *Suppose that C is a norm-closed boundedly relatively weakly compact subset of a Banach space E. If there exists a bounded subset A of C such that for any finite number of P-proper points of C there is some point a in A which simultaneously sees them in C then*

$$\text{star } C \cap \text{wcl } A \neq \emptyset$$

and C is starshaped.

In particular, if C is a bounded closed subset of a reflexive space E and C has finite visibility, then C is starshaped.

Proof. By Corollary 4.1 we get

$$\begin{aligned} \text{star } C &= \bigcap_{x \in C} P_C(x) + x \\ &\supset \bigcap_{x \in C} (P_C(x) + x) \cap \text{wcl } A \\ &\supset \bigcap_{\substack{x \in C, \\ x \text{ is } P\text{-proper}}} ((P_C(x) + x) \cap \text{wcl } A). \end{aligned}$$

Define

$$A(x) := (P_C(x) + x) \cap \text{wcl } A, \quad x \in C.$$

Note that $A(x)$ is weakly compact for all $x \in C$. Let S be any finite subset of P-proper points of C. Then, as we have assumed, we may find $a \in A$ with $[a, s] \subset C$ for all $s \in S$. This in turn implies that $a - s \in K_C(s)$, therefore $a \in A(s)$. This means that

$$a \in \bigcap_{s \in S} A(s) \neq \emptyset$$

and by the finite intersection property we conclude that

$$\emptyset \neq \bigcap_{\substack{x \in C \\ x \text{ is } P\text{-proper}}} \subset \text{star } C \cap \text{wcl } A$$

therefore

$$\text{star } C \neq \emptyset.$$

Thus C is starshaped and the proof is finished.

COROLLARY 5.11. ([**5**]) *E is reflexive if and only if every closed bounded subset of E with finite visibility is starshaped.*

Proof. Assume that E is reflexive. Then every closed bounded subset of E with finite visibility is starshaped by Theorem 5.3. If the unit ball of E is not weakly compact, then the construction of a closed and bounded subset of E with finite visibility but not starshape was shown in [**5**], which finishes the proof.

Our next considered applications are related to the theory of vector fields and invariant flows, see [**10**] and [**17**], where additional details and references may be found.

Let C be a subset of a space E and let $A:C \to 2^E$ be a multifunction on C with values in E. We will say that A is *lowersemicontinuous* (LSC) *on C at $x \in C$* if

$$\liminf_{\substack{x' \to x \\ C}} A(x') = A(x).$$

We will say that A is *d-lowersemicontinuous* (dLSC) *on C at $x \in C$* if

$$d \liminf_{\substack{x' \to x \\ C}} A(x') = A(x).$$

THEOREM 5.4. *Let E be a normed space and let C be a closed subset of E. Suppose that V is a multifunction on C with values in E, which is* LSC *on C at all points of C. Consider the following statements.*

(i) $V(x) \subset T_C(x)$ *for all* $x \in C$.
(ii) $V(x) \subset K_C(x)$ *for all* $x \in C$.
(iii) $V(x) \subset P_C(x)$ *for all* $x \in C$.
(iv) $V(x) \subset WT_C(x)$ *for all* $x \in C$.
(v) $V(x) \subset WK_C(x)$ *for all* $x \in C$.
(vi) $V(x) \subset WP_C(x)$ *for all* $x \in C$.
(vii) $V(x) \subset B_C(x)$ *for all* $x \in C$.

Then

1) *if E is a Banach space,* (i) *and* (ii) *are equivalent;*

2) *if E is a reflexive Banach space,* (i)-(vi) *are equivalent;*

3) *if E is a reflexive Banach space and the norm of E is Frechet differentiable and Kadec,* (i)-(vii) *are equivalent;*

4) *if C is weakly compact, E is a Banach space which may be given an equivalent smooth norm then* (i)-(iii) *are equivalent;*

5) *if C is weakly compact, E is a Banach space (which may be given an equivalent Frechet differentiable norm), then* (i)-(vi) *are equivalent.*

Proof. Note that if

$$\liminf_{x' \underset{C}{\to} x} R_C(x') \subset T_C(x), \quad x \in C, \quad \text{and}$$

$$V(x') \subset R_C(x') \quad \text{for all } x' \in C$$

then with our assumptions on V we have

$$V(x) = \liminf_{x' \underset{C}{\to} x} V(x') \subset \liminf_{x' \underset{C}{\to} x} R_C(x') \subset T_C(x),$$

hence $V(x) \subset T_C(x)$.

This observation together with Theorem 1.1, Corollary 3.2, and Theorem 3.1 proves 1), 2), 3), respectively, and together with Theorem 3.4 it proves 4) and 5).

THEOREM 5.5. *Let E be a Banach space and let C be a norm-closed boundedly relatively weakly compact subset of E. Suppose that V is a multifunction on C with values in E, such that if*

$$\widetilde{V}(x) := V(x) + x, \quad x \in C,$$

then $\widetilde{V}$ is dLSC *on C at all points of C. With these assumptions* (i)-(iii) *of Theorem* 5.4 *are equivalent.*

If E may be given an equivalent Frechet differentiable and Kadec norm then (i)-(vi) *are equivalent.*

Proof. Note that if

$$d \liminf_{x' \underset{C}{\to} x} R_C(x') + x' \subset T_C(x) + x,$$

$x \in C$ and $V(x') \subset R_C(x')$ for all $x' \in C$, then with our assumptions on V we have

$$\begin{aligned} V(x) + x &= d \liminf_{x' \underset{C}{\to} x} V(x') + x' \\ &\subset d \liminf_{x' \underset{C}{\to} x} R_C(x') + x' \subset T_C(x) + x. \end{aligned}$$

Hence $V(x) \subset T_C(x)$.

This observation together with Theorem 4.1 implies the claimed equivalences.

As an interesting application of Theorem 5.4 and Theorem 5.5 we present their consequences in the theory of pseudoconvexity. First we prove the following basic relations of this theory in Banach spaces. Pseudoconvexity was defined in Definition 1.2.

THEOREM 5.6. *In a Banach space the closed convex sets coincide with the T-pseudoconvex and K-pseudoconvex sets.*

Proof. If C is K-pseudoconvex we have

$$C \subset \bigcap_{x \in C} K_C(x) + x.$$

By Corollary 2.1 we have

$$C \subset \bigcap_{x \in C} T_C(x) + x \subset \text{star } C,$$

which shows that $C = \text{star } C$. This means that C is convex and so T-pseudoconvex.

COROLLARY 5.12. *Let C be a closed subset of a normed space E. Then any one of the following assumptions* 1)-6) *imply that the equivalence*:

C is R-pseudoconvex if and only if C is convex,

holds;

1) *E is a Banach space and $T_C(x) \subset R_C(x) \subset K_C(x)$ for all $x \in C$,*

2) *E is a reflexive Banach space and*

$$T_C(x) \subset R_C(x) \subset WP_C(x) \quad \textit{for all } x \in C,$$

3) *E is a reflexive Banach space with a Frechet differentiable, Kadec norm and*

$$T_C(x) \subset R_C(x) \subset B_C(x) \quad \textit{for all } x \in C,$$

4) *C is weakly compact, E is a Banach space which has an equivalent Frechet norm and*

$$T_C(x) \subset R_C(x) \subset WP_C(x) \quad \textit{for all } x \in C,$$

5) *C is boundedly relatively weakly compact, E is a Banach space and*

$$T_C(x) \subset R_C(x) \subset P_C(x) \quad \textit{for all } x \in C,$$

6) *C is boundedly relatively weakly compact, E is a Banach space which has an equivalent Frechet, Kadec norm and*

$$T_C(x) \subset R_C(x) \subset WP_C(x) \quad \textit{for all } x \in C.$$

Proof. Note that if we define $V(x) := C - x$, $x \in C$, then V is LSC on C at all points of C. Use Theorem 5.4 and Theorem 5.6 to prove 1)-4). 5) and 6) follow from Theorem 5.6 and Theorem 5.5 as $\widetilde{V} = C$ is obviously dLSC on C at all points of C.

Theorem 5.6 and Corollary 5.12 complete results of [**5**], [**4**] and [**3**].

6. Differentiability and subdifferentiability. Let E be a locally convex topological vector space and let f be an extended real-valued function on E.

For

$$x \in \operatorname{dom} f := \{x \in E \mid |f(x)| < \infty\},$$

$y \in E$, $t \in (0, \infty)$ and $x^* \in E^*$ define

$$r(t, y) := \frac{f(x + ty) - f(x)}{t} - (x^*, y).$$

Let us recall that $x^* \in E$ is said to be the *Gateaux derivative of f at* $x \in \operatorname{dom} f$ if for all $y \in E$

$$(6.1) \qquad \lim_{t \downarrow 0} r(t, y) = 0.$$

If the convergence in (6.1) is uniform in y on all sequentially compact (bounded) sets, we say that x^* is the *Hadamard* (*Frechet*) *derivative of f at x*. If such an x^* exists we say f is Gateaux (Hadamard, Frechet) differentiable at x. (See **[21]** for details.)

We generalize these definitions as follows.

Definition 6.1. $x^* \in E^*$ is said to be a *Gateaux subderivative of f at* $x \in \operatorname{dom} f$ if for all $y \in E$

$$(6.2) \qquad \lim_{t \downarrow 0} \min\{r(t, y), 0\} = 0.$$

If the convergence in (6.2) is uniform in y on all sequentially compact (bounded) sets, we say that x^* is a *Hadamard* (*Frechet*) *subderivative of f at x*.

If E is considered in its weak topology then a corresponding Hadamard subderivative of f at $x \in \operatorname{dom} f$ will be called a *weak Hadamard* subderivative of f at x.

The set of all Hadamard (weak Hadamard, Frechet, Gateaux) subderivatives of f at $x \in \operatorname{dom} f$ will be called the *Hadamard* (*weak Hadamard*, *Frechet*, *Gateaux*) *subdifferential of f at x*. It will be denoted by

$$\partial^H f(x)(\partial^{WH} f(x), \partial^F f(x), \partial^G f(x)).$$

If $x \notin \operatorname{dom} f$, all the subdifferentials of f at x are empty, by convention.

We will say that f is *Hadamard* (*weakly Hadamard*, *Gateaux*, *Frechet*) *subdifferentiable* at x whenever the corresponding subdifferential is not empty.

As an easy consequence of the definitions we obtain the following relations.

PROPOSITION 6.1. *For any locally convex topological vector space E, any function f on E, and $x \in E$ the following inclusions always hold*

$$\partial^F f(x) \subset \partial^{WH} f(x) \subset \partial^H f(x) \subset \partial^G f(x).$$

https://doi.org/10.4153/CJM-1987-019-4 Published online by Cambridge University Press

(i) *If E is a reflexive Banach space then*

$$\partial^F f(x) = \partial^{WH} f(x);$$

(ii) *if E is finite dimensional then*

$$\partial^F f(x) = \partial^H f(x);$$

(iii) *if weakly convergent sequences converge in E (in particular if $E := l^1(S)$) then*

$$\partial^H f(x) = \partial^{WH} f(x);$$

(iv) *if f is Lipschitz on some neighbourhood of x then*

$$\partial^H f(x) = \partial^G f(x).$$

The following characterizations are easy to obtain.

PROPOSITION 6.2. (i) *x^* is the (weak) Hadamard subderivative of f at $x \in \operatorname{dom} f$ if and only if for all (weakly) convergent sequences y_n in E and all sequences $t_n \downarrow 0$ we have*

$$\liminf_{n\to\infty} r(t_n, y_n) \geqq 0;$$

(ii) *x^* is the Frechet subderivative of f at $x \in \operatorname{dom} f$ if and only if for all bounded sequences y_n in E and all sequences $t_n \downarrow 0$ we have*

$$\liminf_{n\to\infty} r(t_n, y_n) \geqq 0,$$

(iii) *x^* is the Gateaux subderivative of f at $x \in \operatorname{dom} f$ if and only if for all sequences $t_n \downarrow 0$ and $y \in E$ we have*

$$\liminf_{n\to\infty} r(t_n, y) \geqq 0.$$

Definition 6.2. Let $x \in \operatorname{dom} f$, $y \in E$. Then

$$\begin{aligned} f^G(x; y) &:= \inf_{t_n \downarrow 0} \liminf_{n\to\infty} \frac{f(x + t_n y) - f(x)}{t_n} \\ &= \liminf_{t \downarrow 0} \frac{f(x + ty) - f(x)}{t}, \end{aligned}$$

$$f^H(x; y) := \inf_{\substack{y_n \to y \\ t_n \downarrow 0}} \liminf_{n\to\infty} \frac{f(x + t_n y_n) - f(x)}{t_n},$$

$$f^{WH}(x; y) := \inf_{\substack{y_n \xrightarrow{w} y \\ t_n \downarrow 0}} \liminf_{n\to\infty} \frac{f(x + t_n y_n) - f(x)}{t_n},$$

(where $y_n \xrightarrow{w} y$ means y_n converges to y weakly), will be called respectively the *Gateaux*, the *Hadamard*, the *weak Hadamard directional subderivative of f at x with respect to y*.

As an obvious consequence of Proposition 6.2 and Definition 6.2 we obtain the following characterization.

PROPOSITION 6.3. (i) *x^* is a Gateaux subderivative of f at x if and only if*

$$f^G(x; y) \geqq (x^*, y) \quad \textit{for all } y \in E;$$

(ii) *x^* is a Hadamard subderivative of f at x if and only if*

$$f^H(x; y) \geqq (x^*, y) \quad \textit{for all } y \in E;$$

(iii) *x^* is a weak Hadamard subderivative of f at x if and only if*

$$f^{WH}(x; y) \geqq (x^*, y) \quad \textit{for all } y \in E.$$

Note that the following equivalent formulations of Definition 6.2 hold.

PROPOSITION 6.4. *Let $x \in \operatorname{dom} f$, $y \in E$. Then*
(i) $f^H(x; y) = \inf\{r \in \mathbf{R} \mid (y, r)$ *is the limit of some sequence*

$$t_n^{-1}(c_n - (x, f(x)))$$

with $t_n \downarrow 0$ and c_n in the epigraph of f $\}$;

(ii) $f^{WH}(x; y) = \inf\{r \in \mathbf{R} \mid (y, r)$ *is the weak limit of some sequence*

$$t_n^{-1}(c_n - (x, f(x)))$$

with $t_n \downarrow 0$ and c_n in the epigraph of f $\}$;

(iii) $f^G(x; y) = \inf\{r \in \mathbf{R} \mid$ *for some sequence $t_n \downarrow 0$, the sequence*

$$(x, f(x)) + t_n(y, r)$$

lies in the epigraph of f $\}$.

Proof. We give a proof of (i). The proofs of (ii) and (iii) are analogous.

Suppose that $x \in \operatorname{dom} f$. If $f^H(x; y) = +\infty$, then it is easy to see that the infimum in (i) is taken over the empty set and as such it is equal to $+\infty$. Therefore (i) holds in this case.

Assume now that $f^H(x; y) \neq +\infty$. Suppose that

$$(6.3) \qquad f^H(x; y) < r.$$

Then there exist sequences y_n converging to y and $t_n \downarrow 0$ such that

$$f(x + t_n y_n) < +\infty \quad \text{and}$$

$$f^{H}(x; y) \leqq \lim_{n\to\infty} t_n^{-1} (f(x + t_n y_n) - f(x)) < r,$$

where the limit above is finite or equals $-\infty$. Let

$$z_n := x + t_n y_n,$$
$$\gamma_n := \max\{f(z_n), f(x) + t_n r\},$$
$$c_n := (z_n, \gamma_n).$$

Then the sequence c_n lies in the epigraph of f and

$$(y, r) = \lim_{n\to\infty} t_n^{-1}(c_n - (x, f(x))).$$

Denoting the right-hand side of (i) by d we see that $d \leqq r$, which by (6.3) implies

(6.4) $\quad d \leqq f^{H}(x; y).$

This argument also shows that the set on the right-hand side of (i) is nonempty whenever $f^{H}(x; y) < +\infty$. So suppose that r' is in this set and let

$$(y, r') = \lim_{n\to\infty} t_n'^{-1}(c_n' - (x, f(x))),$$

where

$$c_n' = (z_n', \gamma_n'), \gamma_n' \geqq f(z_n'), t_n' \downarrow 0.$$

Put $y_n := t_n'^{-1}(z_n' - x)$, then

$$f^{H}(x; y) \leqq \liminf_{n\to\infty} t_n'^{-1}(f(x + t_n' y_n) - f(x))$$
$$\leqq \lim_{n\to\infty} t_n'^{-1}(\gamma_n' - f(x)) = r'.$$

Hence

$$f^{H}(x; y) \leqq d,$$

which together with (6.4) finishes the proof of (i).

We will be mostly interested in the case of normed spaces. The following observations will be helpful. If E is normed, then

$$f^{H}(x; y) = \liminf_{\substack{y'\to y\\ t\downarrow 0}} \frac{f(x + ty') - f(x)}{t},$$

and if f is locally Lipschitz around x then $f^{H}(x; \cdot)$ is continuous. Furthermore, this is a consequence of Eberlein-Smulian theorem [**12**] that if E is normed, then x^* is the weak Hadamard subderivative of f at x if and only if the convergence in (6.2) is uniform in y on all weakly compact sets.

In addition to the tangent cones listed in the Preliminaries in Part I we will also consider the *radial contingent cone to a set* C *at* $x \in C$, denoted $RK_C(x)$, which is the set of those y such that $x + t_n y \in C$ for some sequence $t_n \downarrow 0$.

For a function f on E, let epi f denote the epigraph of f.

Recalling the definitions of a contingent cone and of a weak contingent cone (Preliminaries, Part I) and using the fact that a contingent cone is closed we restate Proposition 6.4 as follows.

COROLLARY 6.1. *Let* E *be a normed space,* $x \in \operatorname{dom} f$, $y \in E$. *Then*

$$\operatorname{epi} f^H(x; \cdot) = K_{\operatorname{epi} f}(x, f(x));$$

$$f^{WH}(x; y) = \inf\{r \in \mathbf{R} \mid (y, r) \in WK_{\operatorname{epi} f}(x, f(x))\};$$

$$f^G(x; y) = \inf\{r \in \mathbf{R} \mid (y, r) \in RK_{\operatorname{epi} f}(x, f(x))\}.$$

Now we are ready to formulate the basic relations which tie the subdifferentiability theory to the theory of tangent cones.

THEOREM 6.1. *Let* E *be a normed space,* $x \in \operatorname{dom} f$. *Then*

(i) x^* *is a Hadamard subderivative of* f *at* x *if and only if*

$$(x^*, -1) \in K^0_{\operatorname{epi} f}(x, f(x));$$

(ii) x^* *is a weak Hadamard subderivative of* f *at* x *if and only if*

$$(x^*, -1) \in WK^0_{\operatorname{epi} f}(x, f(x));$$

(iii) x^* *is a Gateaux subderivative of* f *at* x *if and only if*

$$(x^*, -1) \in RK^0_{\operatorname{epi} f}(x, f(x)).$$

Proof. Use Proposition 6.3 and Corollary 6.1.

PROPOSITION 6.5. *If* E *is a normed space then* x^* *is a Frechet subderivative of* f *at* x *if and only if*

$$\liminf_{\|y\| \to 0,\, y \neq 0} \|y\|^{-1}(f(x + y) - f(x) - (x^*, y)) \geqq 0. \tag{6.5}$$

Thus, if f *is Frechet subdifferentiable at* x *then* f *is lowersemicontinuous at* x.

Proof. Assume that x^* is a Frechet subderivative of f at x. As for $y \neq 0$

$$\|y\|^{-1}(f(x + y) - f(x) - (x^*, y)) = r(\|y\|, \|y\|^{-1}y)$$

and the convergence in (6.2) is uniform on the unit ball, (6.5) follows.

Suppose that x^* is not a Frechet subderivative of f at x. Then there exist $\epsilon > 0$, a bounded sequence y_n in E and a sequence $t_n \downarrow 0$ such that inequality in Proposition 6.2 (i) is violated. Then

$$\liminf_{\|y\|\to 0, y\neq 0} r(\|y\|, \|y\|^{-1}y) \leqq \liminf_{n\to\infty} \|y_n\|^{-1}r(t_n, y_n) < 0$$

contradicts (6.5) and the proof of the equivalence is completed. The last statement follows from 6.5.

Remark. Note that with appropriate definitions of tangent cones, Theorem 6.1 holds in any locally convex vector space.

COROLLARY 6.2. *If E is a reflexive Banach space then the following are equivalent*:

(i) *x^* is a weak Hadamard subderivative of f at x*;

(ii) $(x^*, -1) \in WK^0_{\mathrm{epi}f}(x, f(x))$;

(iii) $f^{WH}(x; y) \geqq (x^*, y)$ *for all* $y \in E$;

(iv) *x^* is a Frechet subderivative of f at x*;

(v) $\liminf_{\|y\|\to 0, y\neq 0} \|y\|^{-1}(f(x + y) - f(x) - (x^*, y)) \geqq 0.$

Proof. Use Proposition 6.1 (i), Proposition 6.3 (iii), Theorem 6.1 (i), Proposition 6.5.

Let us recall that for any extended real-valued function f on a locally convex vector space E, $x^* \in E^*$ is said to be a *Clarke subgradient of f at* $x \in \mathrm{dom}\, f$ if

$$(x^*, -1) \in N_{\mathrm{epi}f}(x, f(x)).$$

The subgradient set $\partial f(x)$, consists of such functionals x^* (see **[9]** for details and references).

The proof of the following result may be found in **[16]**.

PROPOSITION 6.6. *If f is a lowersemicontinuous function on a Banach space E, then the set $\{x|\partial f(x) \neq \emptyset\}$ is dense in* $\mathrm{dom}\, f$.

Hence the following is also true.

COROLLARY 6.3. *Let f be a lowersemicontinuous function on a Banach space E. If the epigraph of f is tangentially regular at all its points then f is densely Hadamard subdifferentiable on* $\mathrm{dom}\, f$.

Proof. Use Proposition 6.6 and Theorem 6.1.

We prove a strengthening of this result for reflexive Banach spaces. Further on we use

$$\|(x, \alpha)\| := \sqrt{\|x\|^2 + \alpha^2} \quad \text{for } (x, \alpha) \in E \times \mathbf{R}.$$

THEOREM 6.2. *Every lowersemicontinuous function f on a reflexive Banach space E is densely Frechet subdifferentiable in* $\mathrm{dom}\, f$.

Proof. Let $x \in \operatorname{dom} f$. Then by Proposition 6.6 there exists $\bar{x}$ in $\operatorname{dom} f$ near x such that for some $x^* \in E^*$

(6.7) $\quad (x^*, -1) \in N_{\operatorname{epi} f}(\bar{x}, f(\bar{x}))$.

By Corollary 3.2

$$(6.8) \quad N_{\operatorname{epi} f}(\bar{x}, f(\bar{x})) = \overline{\mathrm{co}}\, w^* \limsup_{(x',\gamma') \underset{\operatorname{epi} f}{\to} (\bar{x}, f(\bar{x}))} WK^0_{\operatorname{epi} f}(x', \gamma')$$

$$= \overline{\mathrm{co}}\, w^* \limsup_{\substack{x' \to \bar{x} \\ f(x') \to f(\bar{x})}} WK^0_{\operatorname{epi} f}(x', f(x')),$$

where the last equality is justified by the lowersemicontinuity of f and the inclusion

$$WK_{\operatorname{epi} f}(x', \gamma') \supset WK_{\operatorname{epi} f}(x', f(x')),$$

whenever

$$\gamma' \geqq f(x') > -\infty.$$

(6.7) and (6.8) show that there exists x' in $\operatorname{dom} f$ near $\bar{x}$, $y^* \in E^*$ and $\gamma > 0$ such that

$$(y^*, -\gamma) \in WK^0_{\operatorname{epi} f}(x', f(x')).$$

By Corollary 6.2 we thus get $\gamma^{-1} y^* \in \partial^F f(x')$ and as x' lies in $\operatorname{dom} f$ near x we have that the set

$$\{x | \partial^F f(x) \neq \emptyset\}$$

is dense in $\operatorname{dom} f$ and the theorem is proven.

THEOREM 6.3. *Let f be a locally Lipschitz function on a weakly compactly generated Banach space E, then f is densely Hadamard subdifferentiable on E.*

Proof. Since E is a weakly compactly generated space, there exists a reflexive Banach space R and a one-to-one continuous linear operator $T:R \to E$ such that the range of T is dense in E [**11**]. Suppose f is locally Lipschitz on E. Let $x \in E$. Define

$$g(z) := f(T(z)), \quad z \in R.$$

Then g is locally Lipschitz on R and hence by Theorem 6.2 g is densely Frechet subdifferentiable on R. This together with the properties of T implies that there exist $\bar{x}$ near x in E, $\bar{z} \in R$; $z^* \in R^*$ such that $\bar{x} = T\bar{z}$ and

(6.9) $\quad (z^*, h) \leqq g^{WH}(\bar{z}; h) \quad$ for all $h \in R$.

As f is locally Lipschitz on E, there exists $K > 0$ such that

$$g^{WH}(\bar{z}; h) \leqq K\|Th\| \quad \text{for all } h \in R.$$

Thus

$$|(z^*, h)| \leqq K\|Th\| \quad \text{for all } h \in R,$$

which implies, by taking subgradients or directly, that $z^* = T^*y^*$ for some $y^* \in E^*$. Using this in (6.9) we get for any $h \in R$

$$\begin{aligned}(y^*, Th) &\leqq g^{WH}(\bar{z}; h) \leqq g^{G}(\bar{z}; h)\\ &= f^{G}(T\bar{z}; Th) = f^{H}(T\bar{z}; Th),\end{aligned}$$

where the last equality is due to the fact that f is locally Lipschitz on some neighbourhood of Tz. As observed earlier this property of f also implies that $f^{H}(\bar{x}, \cdot)$ is continuous on E. Hence as the range of T is dense in E we conclude that

$$(y^*, y) \leqq f^{H}(\bar{x}; y) \quad \text{for all } y \in E,$$

which shows that f is Hadamard subdifferentiable at $\bar{x}$ and the proof is finished.

We complete our considerations with some remarks about generic (i.e., on a dense G_δ subset of a domain) subdifferentiability and differentiability of functions.

First let us note that in general dense Frechet subdifferentiability in Theorem 6.2 can not be replaced with generic Frechet subdifferentiability. Such a theorem would be no longer true even for $E := \mathbf{R}$ (because for example there exists a continuous function nowhere differentiable on $[0, 1]$). Similarly, dense Gateaux (Hadamard) subdifferentiability in Theorem 6.3 can not be replaced with generic Gateaux subdifferentiability even for $E := \mathbf{R}$ (because there exists a locally Lipschitz function on $\mathbf{R}$ which is not generically Gateaux differentiable on $\mathbf{R}$, for example the one constructed in Proposition 1.9 of **[15]** as follows from Theorem 3.8 of the same paper). In view of the above remarks it is interesting to observe that as a consequence of Theorem 6.2 we obtain a simple proof for the following related result of Zhivkov.

COROLLARY 6.4 **[22]**. *If f is a locally Lipschitz function on an open subset D of a weakly compacted generated Banach space E then the subdifferential mapping*

$$\text{(6.10)} \quad \begin{aligned}\delta f(x) := \{x^* \in E^* | (x^*, y) \\ \leqq \limsup_{t\downarrow 0} t^{-1}(f(x + ty) - f(x)), \forall y \in E\}\end{aligned}$$

has nonempty images at the points of a dense G_δ subset of D.

Remark. Note that by Theorem 6.3 $\delta f(x)$ is nonempty densely in D.

Proof. Let R, T, and g be as in the proof of Theorem 6.3. Put $C := T^{-1}D$. Then C is open in R. Let $m, n \in \mathbf{N}$, $m, n \geqq 1$. Denote:

$$B_{mn} := \{z \in C| \text{ there is } z^* \in R^* \text{ such that}$$
$$g(z + h) - g(z) > (z^*, h) - m^{-1}\|h\| \text{ whenever } \|h\| \leqq n^{-1}\},$$
$$L := \{z \in C | \partial g^F(z) \neq \emptyset\},$$

and

$$G_m := \{x \in D| \text{ there are } n_m \geqq m \text{ and}$$
$$\bar{x} \in TB_{mn_m} \text{ with } \|\bar{x} - x\| < m^{-1}n_m^{-1}\}.$$

By Theorem 6.2 L is dense in C, thus TL is dense in D. Furthermore

$$TL \subset \bigcap_{m=1}^{\infty} G_m.$$

To see this assume $x \in TL$ and put $\bar{x} := x$ in definition of G_m. Obviously for all m G_m is an open set. Thus

$$G := \bigcap_{m=1}^{\infty} G_m$$

is a dense G_δ subset of D. Let $x \in G$. Then there exist sequences $z_m \in C$, $x_m = Tz_m$, $z_m^* \in E^*$ and $n_m \geqq m$ such that

$$\|x_m - x\| < m^{-1}n_m^{-1} \tag{6.11}$$

and

$$g(z_m + h) - g(z_m) > (z_m^*, h) - m^{-1}\|h\| \tag{6.12}$$

whenever $\|h\| \leqq n_m^{-1}$. Take any h in R with $\|h\| = 1$. Let L be a Lipschitz constant of f on some neighbourhood of x. Then for m sufficiently big we get by (6.11) and (6.12)

$$\begin{aligned} &f(x + n_m^{-1}Th) - f(x) \\ &\geqq f(x_m + n_m^{-1}Th) - f(x_m) - 2L\|x_m - x\| \\ &= g(z_m + n_m^{-1}h) - g(z_m) - 2L\|x_m - x\| \\ &\geqq (z_m^*, n_m^{-1}h) - m^{-1}n_m^{-1} - 2Lm^{-1}n_m^{-1}. \end{aligned}$$

Thus

$$\frac{f(x + n_m^{-1}Th) - f(x)}{n_m^{-1}} \geqq (z_m^*, h) - (2L + 1)m^{-1}. \tag{6.13}$$

Note that it follows from (6.13) that the sequence $\|z_m^*\|$ is bounded. Since R is reflexive the sequence z_m^* has a weak star convergent subsequence. Let z^* be its limit. Then from (6.13) we get

https://doi.org/10.4153/CJM-1987-019-4 Published online by Cambridge University Press

$$(6.14) \quad \limsup_{t \downarrow 0} \frac{f(x + tTH) - f(x)}{t} \geqq (z^*, h)$$

and we conclude easily that this inequality is valid for any h in R. In a similar way as in the proof of Theorem 6.3 we may argue that $z^* = T^*y^*$ for some $y^* \in E^*$. Then using the fact that f is Lipschitz on some neighbourhood of x and T is continuous with a dense range we get from (6.14)

$$\limsup_{t \downarrow 0} \frac{f(x + ty) - f(x)}{t} \geqq (y^*, y) \quad \text{for all } y \in E.$$

Thus $\delta f(x) \neq \emptyset$ whenever $x \in G$. This finishes the proof.

Definition 6.3. Let f be a function on a normed space E and let D be a subset of $\operatorname{dom} f$.

We say that f is *Frechet subconvex on* D if for each $x \in D$ there exists a proper convex function on E, $\Phi(x; \cdot)$ with $\Phi(x; 0) = 0$ and

$$(6.15) \quad \liminf_{\|y\| \to 0 \quad y \neq 0} \|y\|^{-1} (f(x + y) - f(x) - \Phi(x; y)) \geqq 0.$$

Obviously if f is Frechet subdifferentiable on D then by Proposition 6.5 it is Frechet subconvex on D.

Thus any convex function is Frechet subconvex on its domain of continuity.

THEOREM 6.4. *Let f be a function on a reflexive Banach space. If f is uppersemicontinuous and Frechet subconvex on (an open subset D of) $\operatorname{dom} f$ then f is densely Frechet differentiable on $\operatorname{dom} f$ (respectively on D).*

Proof. Let $\bar{x} \in \operatorname{dom} f$. Since $-f$ is a lowersemicontinuous function on a reflexive Banach space E, by Theorem 6.2 there exist $x \in \operatorname{dom} f$ near $\bar{x}$ and $x^* \in E^*$ such that

$$(6.16) \quad \limsup_{\|y\| \to 0, y \neq 0} \|y\|^{-1}(f(x + y) - f(x) - (x^*, y)) \leqq 0.$$

By subconvexity there exists a proper convex function $\Phi(x; \cdot)$ on E with $\Phi(x; 0) = 0$, such that f and Φ satisfy condition (6.15). Note that conditions (6.16) and (6.15) imply that $\Phi(x; \cdot)$ is finite on some neighbourhood of 0 in E. Thus the directional derivative of $\Phi(x; \cdot)$ at 0, denoted $\Phi'(x, 0; \cdot)$, is a proper convex function on E. It follows from (6.16) and (6.15) that

$$(6.17) \quad (x^*, y) \geqq \inf_{\lambda > 0} \lambda^{-1}\Phi(x; \lambda y) = \Phi'(x, 0; y),$$

therefore $\Phi'(x, 0; \cdot)$ being bounded by a continuous function on E is itself continuous. This together with (6.17) implies

(6.18) $x^* = \Phi'(x, 0; \cdot) \leqq \Phi(x; \cdot)$.

By (6.15), (6.16) and (6.18) we get

$$\lim_{\|y\| \to 0, y \neq 0} \|y\|^{-1}(f(x + y) - f(x) - (x^*, y)) = 0,$$

hence f is Frechet differentiable at x. The proof of the version in brackets is analogous.

Definition 6.4. Let f be a function on a normed space E and let D be a subset of dom f.

We say that f is *Gateaux subconvex on* D if for each $x \in D$ there exists a proper convex function on E, $\Phi(x; \cdot)$ with $\Phi(x; 0) = 0$ and

(6.19) $f^G(x; y) \geqq \Phi(x; y)$ for all y in E.

Obviously any Frechet subconvex function on D is Gateaux subconvex on D. Any function Gateaux subdifferentiable on D is Gateaux subconvex on D.

THEOREM 6.5. *Let f be a function on a weakly compactly generated Banach space. If f is locally Lipschitz on an open subset D of* dom f *and Gateaux subconvex on D then f is densely Hadamard differentiable on D.*

Proof. Let $\bar{x} \in E$. By Theorem 6.3 applied to the function $-f$ and by Proposition 6.3, there exist x near $\bar{x}$ in D and $x^* \in E^*$ such that

(6.20) $$\limsup_{t \downarrow 0} \frac{f(x + ty) - f(x)}{t} \leqq (x^*, y), \quad y \in E.$$

Let Φ be as in Definition 6.4. Then (6.19) and (6.20) imply that

(6.21) $\Phi(x; y) \leqq (x^*, y), \quad y \in E.$

Thus $\Phi(x; \cdot)$ is convex and continuous on E. As $\Phi(x; 0) = 0$, from (6.21) we get

$$\Phi(x; \cdot) = x^*.$$

This together with (6.19), (6.20), (6.21), and the fact that f is locally Lipschitz imply that x^* is the Hadamard derivative of f at x and the proof is finished.

Definition 6.5. Let T be a nonempty set and let E be a normed space. Suppose that F is an extended real-valued function on $E \times T$ and D is a subset of

$$\bigcap_{t \in T} \operatorname{dom} F(\cdot, t).$$

We say that F is *Frechet* (*Gateaux*) *equisubconvex on* D if for each $x \in D$ and each $t \in T$ there exists a proper convex function on E, $\Phi_t(x; \cdot)$ with $\Phi_t(x; 0) = 0$ and

$$(6.22) \quad \liminf_{\|y\| \to 0, y \neq 0} \inf_{t \in T} \|y\|^{-1}(F(x + y, t) - F(x, t) - \Phi_t(x; y)) \geqq 0.$$

(Respectively:

$$\liminf_{\lambda \downarrow 0} \inf_{t \in T} [(\lambda^{-1}F(x + \lambda y, t) - F(x, t)) - \Phi_t(x; y)] \geqq 0$$

for all $y \in E$.)

Proposition 6.7. *Let E be a normed space and let T be a nonempty set. Suppose that F is a function on $E \times T$ such that*

(i) *for each $x \in D$,*

$$T(x) := \{t \in T | F(x, t) = \sup_{t \in T} F(x, t)\}$$

is nonempty.

(ii) *F is Frechet (Gateaux) equisubconvex on D (or 6.22 holds with T replaced by $T(x)$).*
Then

$$f := \sup_{t \in T} F(\cdot, t)$$

is Frechet (Gateaux) subconvex on D.

Proof. We prove the case of Frechet subconvexity. The other one is analogous. Let $x \in D$. Put

$$\Phi(x;) := \sup_{t \in T(x)} \Phi_t(x; \cdot),$$

where $\Phi_t(x; \cdot)$, $t \in T$ are as in Definition 6.5. Then $\Phi(x; \cdot)$ is convex and proper on E. Also $\Phi(x; 0) = 0$ and furthermore

$$\begin{aligned}
&\liminf_{\|y\| \to 0, y \neq 0} \|y\|^{-1}(f(x + y) - f(x) - \Phi(x; y)) \\
&\geqq \liminf_{\|y\| \to 0, y \neq 0} \|y\|^{-1}(\sup_{t \in T(x)} F(x + y, t) - f(x) \\
&- \sup_{t \in T(x)} \Phi_t(x; y)) \\
&\geqq \liminf_{\|y\| \to 0, y \neq 0} \inf_{t \in T(x)} \|y\|^{-1}(F(x + y, t) \\
&- F(x, t) - \Phi_t(x, y)) \geqq 0,
\end{aligned}$$

where the last inequality follows from (6.22). Thus f is Frechet subconvex on D.

Note that (ii) is satisfied whenever T is compact and functions $F(x, \cdot)$ are uppersemicontinuous on T for all $x \in D$.

COROLLARY 6.5. *Let E be a reflexive Banach space and let T be a nonempty set. Suppose that F is a function on $E \times T$,*

$$f := \sup_{t \in T} F(\cdot, t)$$

and D denotes dom f *or an open subset of* dom f. *If F is Frechet equisubconvex on D, $T(x) \neq \emptyset$ for all $x \in D$ and f is uppersemicontinuous on D (which is the case if for example F is equiuppersemicontinuous on D, that is for any $x \in D$ and $\epsilon > 0$, there exists X, a neighbourhood of x in E, such that*

$$F(x', t) - F(x, t) < \epsilon$$

for all $t \in T$ whenever $x' \in X$), then f is densely Frechet differentiable on D.

Proof. By Proposition 6.7 f is Frechet subconvex on D. As it is also uppersemicontinuous on D, Theorem 6.4 finishes the proof.

COROLLARY 6.6. *Let E be a weakly compactly generated space and let T be a nonempty set. Suppose that F is a function on $E \times T$,*

$$f := \sup_{t \in T} F(\cdot, t)$$

and D denotes an open subset of dom f. *If F is Gateaux equisubconvex on D, $T(x) \neq \emptyset$ for all $x \in D$ and f is locally Lipschitz on D (as is the case if for example F is locally equi-Lipschitz on D, that is for any $x \in D$ there exists X, a neighbourhood of x in E and $K > 0$ such that*

$$|F(x', t) - F(x'', t)| < K\|x' - x''\|$$

for all $x', x'' \in X$ and all $t \in T$), then f is densely Hadamard differentiable on D.

Proof. By Proposition 6.7 f is Gateaux subconvex on D. As it is also locally Lipschitz on D, Theorem 6.5 finishes the proof.

COROLLARY 6.7. *Let E be a reflexive (weakly compactly generated) Banach space and let T be nonempty and finite. Suppose that F is a function on $E \times T$ and*

$$f := \max_{t \in T} F(\cdot, t).$$

If for each $t \in T$, $F(\cdot, t)$ is uppersemicontinuous (locally Lipschitz) and Frechet (Gateaux) subconvex on D, then f is densely Frechet (Hadamard) differentiable on D.

Proof. Since T is finite, conditions (i) and (ii) of Proposition 6.7 are satisfied. The proof follows from Corollary 6.5 (Corollary 6.6).

Let us recall that a Banach space E is called an Asplund space if every convex function on E is Frechet differentiable on a dense G_δ (dense) subset of its domain of continuity [**1**]. A Banach space is weak Asplund if every convex function on E is Gateaux differentiable on a dense G_δ subset of its domain of continuity. A Banach space is called a Gateaux (Minkowski) differentiability space if every convex (Minkowski) function on E is densely Gateaux differentiable on its domain of continuity [**14**]. Also E is a Minkowski differentiability space if and only if every equivalent norm on E has at least one point of Gateaux differentiability [**14**]. We will also say that a Banach space E is a weak Hadamard differentiability space if every convex function f is densely weak Hadamard differentiable on its domain of continuity. From Theorem 6.4 and Corollary 6.4 we obtain the following classical results.

COROLLARY 6.9. *Any reflexive Banach space is an Asplund space. Any weakly compactly generated space is a weak Asplund space.*

We will say that a Banach space E is an *R-proper space*, whenever for any closed subset C of E the R-proper points of C are dense in the boundary of C.

PROPOSITION 6.8. *If $E \times \mathbf{R}$ is a WP-proper (P-proper space), then E is a weak Hadamard (Gateaux) differentiability space.*

Proof. For example we prove the "weak" case. Let f be a convex function on E continuous at x. Put

$$C := \operatorname{epi}(-f).$$

Then there exists $\bar{x}$ close to x such that

$$WP_C(\bar{x}, -f(\bar{x})) \neq E$$

and f is continuous at x. As a consequence there exists

$$(-x^*, r) \in WK_C^0(\bar{x}, -f(\bar{x})), \quad (-x^*, r) \neq 0.$$

As $-f$ is locally Lipschitz around $\bar{x}$,

$$(-f)^{WH}(\bar{x}; y) < +\infty \quad \text{for all } y \in E$$

and using the characterization from Corollary 6.1 we get

$$(6.23) \quad (-x^*, y) \leqq r(-f)^{WH}(\bar{x}; y) \quad \text{for all } y \in E.$$

Thus $r > 0$, and without loss of generality we may assume $r = 1$.

We will prove that x^* is a weak Hadamard derivative of f at $\bar{x}$. To show this it is enough to argue that for all weakly converging sequences y_n and all sequences $t_n \downarrow 0$ we have

$$(6.24) \quad \lim_{n\to\infty} \left(\frac{f(\bar{x} + t_n y_n) - f(\bar{x})}{t_n} - (x^*, y_n) \right) = 0.$$

First note, that if y_n converges weakly to y and $t_n \downarrow 0$, by (6.23) where $r = 1$ and Definition 6.2, we get

$$(6.25) \quad (x^*, y) \geqq \limsup_{n\to\infty} \frac{f(\bar{x} + t_n y_n) - f(\bar{x})}{t_n}.$$

Let $\bar{x}^*$ be a subgradient of f at $\bar{x}_j$ then taking any $y \in E$ and $y_n = y$ for $n \in \mathbf{N}$ in (6.25) we get $(x^*, y) \geqq (\bar{x}^*, y)$. Hence $x^* = \bar{x}^*$. This together with (6.25) shows that

$$(x^*, y) = \lim_{n\to\infty} \frac{f(\bar{x} + t_n y_n) - f(\bar{x})}{t_n},$$

therefore (6.24) holds and the proof is finished.

7. The subgradient formulas in reflexive Banach spaces.

Definition 7.1. Let f be an extended real valued function on a Banach space E. Assume that f is lowersemicontinuous and let x be any point with $f(x)$ finite. We say that $x^* \in E^*$ is a *proximal subgradient to f at x* if $(x^*, -1)$ is a proximal normal functional (as defined in Definition 3.1) to the epigraph of f at $(x, f(x))$.

The proximal subgradient set of f at x denoted $\partial^P f(x)$ consists of all such functionals. If $x \notin \operatorname{dom} f$ then $\partial^P f(x)$ is empty by convention.

Thus we have: $x^* \in \partial^P f(x)$ if and only if

$$(x^*, -1) \in PN_{\operatorname{epi} f}(x, f(x))$$

and this definition extends the one given in [**19**] for $E := \mathbf{R}^m$.

An examination of the definitions shows that whenever the norm of E is Frechet differentiable then the following holds.

PROPOSITION 7.1. *Let the norm of E be Frechet differentiable and let f be a lowersemicontinuous function on E. If $x^* \in \partial^P f(\bar{x})$ then there is a neighbourhood $\bar{X}$ of $\bar{x}$ and a function g on E such that for any $x \in \bar{X}$ g has a Frechet derivative $g_x^* \in E^*$, the mapping $x \to g_x^*$ is norm to norm continuous on $\bar{X}$, $g_{\bar{x}}^* = x^*$, $g(\bar{x}) = f(\bar{x})$ and $g(x) \leqq f(x)$ for all $x \in \bar{X}$.*

Proof. As $(x^*, -1) \in PN_{\operatorname{epi} f}(\bar{x}, f(\bar{x}))$, there exists $(y, \alpha) \in E \times \mathbf{R}$ such that

$$d := \|(y - \bar{x}, \alpha - f(\bar{x}))\| = d_{\operatorname{epi} f}((y, \alpha))$$

and

$$(x^*, -1)(y - \bar{x}, \alpha - f(\bar{x})) = \|(x^*, -1)\| d.$$

Put

$$g(x) := \sqrt{d^2 - \|y - x\|^2} - \alpha,$$

use the fact that the norm of E is continuously differentiable.

THEOREM 7.1. *Let f be a lowersemicontinuous function on a reflexive Banach space E. Then the set $\{x|\partial^P f(x) \neq \emptyset\}$ is dense in* dom f, *whenever the norm of E is Kadec and Frechet differentiable.*

Proof. Let $x \in$ dom f. Then by Proposition 6.6 there exists $\bar{x}$ in dom f near x such that for some $x^* \in E^*$

(7.1) $\quad (x^*, -1) \in N_{\mathrm{epi}f}(\bar{x}, f(\bar{x}))$.

By Theorem 3.1

$$(7.2) \quad N_{\mathrm{epi}f}(\bar{x}, f(\bar{x})) = \overline{\mathrm{co}}\, w^* \limsup_{\substack{(x',\gamma')\to(\bar{x},f(\bar{x}))\\ \mathrm{epi}f}} PN_{\mathrm{epi}f}(x', \gamma')$$
$$= \overline{\mathrm{co}}\, w^* \limsup_{\substack{x'\to\bar{x}\\ f}} PN_{\mathrm{epi}f}(x', f(x')),$$

(we use notation: $x' \to_f x$ if and only if $x' \to x$ and $f(x') \to f(x)$), where the last equality is justified by the lowersemicontinuity of f and the inclusion

$$PN_{\mathrm{epi}f}(x', \gamma') \subset PN_{\mathrm{epi}f}(x', f(x')),$$

whenever

$$\gamma' \geqq f(x') > -\infty.$$

(7.1) and (7.2) show that there exist x' in dom f near $\bar{x}$, $y^* \in E^*$ and $\gamma > 0$ such that

$$(y^*, -\gamma) \in PN_{\mathrm{epi}f}(x', f(x')).$$

Hence

$$\gamma^{-1} y^* \in \partial^P f(x', f(x'))$$

and the proof is finished.

Proposition 7.1 and Theorem 7.1 together with renorming theorems (Proposition 3.3) obviously imply the following.

COROLLARY 7.1. *If f is a lowersemicontinuous function on a reflexive Banach space E then the set of points $\bar{x}$ for which there exist a neighbourhood $\bar{X}$ of $\bar{x}$ and a function g continuously differentiable on $\bar{X}$ with $g(\bar{x}) = f(\bar{x})$, $g(x) \leqq f(x)$ for $x \in \bar{X}$, is dense in* dom f.

We apply Corollary 7.1 to prove the following.

PROPOSITION 7.2. *Let E be a reflexive Banach space with Kadec norm and* *let K be a closed and bounded subset of E. Suppose that f is a lowersemicontinuous function on E bounded from below on K and somewhere finite on K. Then the set of those y for which the infimum*

$$(7.3) \qquad F(y) := \inf_{x \in K} (f(x) + \|x - y\|)$$

is attained on K is dense in E.

Proof. By the assumption F is lowersemicontinuous and finite on E. Thus Corollary 7.1 implies that there exists a dense subset G of E such that if $\bar{y} \in G$ then there exists a neighbourhood $\bar{Y}$ of $\bar{y}$ and a function g with norm to norm continuous Frechet derivative g_y^* on $\bar{Y}$, $g(\bar{y}) = F(\bar{y})$ and $g(y) \leqq F(y)$ for all $y \in \bar{Y}$. We will show that the infimum in (7.3) is attained for any $\bar{y}$ in G.

Let $x_n \in K$ and

$$f(x_n) + \|x_n - \bar{y}\| \to F(\bar{y}).$$

As K is bounded x_n has a weakly convergent subsequence. Without loss of generality assume that x_n converges weakly to some $\bar{x}$. It is enough to show that x_n converges to $\bar{x}$ in norm. Then $\bar{x} \in K$ and by lowersemicontinuity of f we get

$$f(\bar{x}) + \|\bar{x} - \bar{y}\| \leqq \liminf_{n \to \infty} (f(x_n) + \|x_n - \bar{y}\|) = F(\bar{y}).$$

So let us prove that x_n converges to $\bar{x}$ in norm. Put

$$\epsilon_n := (f(x_n) + \|x_n - \bar{y}\| - g(\bar{y}))^{1/2} \geqq 0.$$

If for some $n \in \mathbf{N}$ $\epsilon_n = 0$ we are done. So assume that $\epsilon_n > 0$ for all $n \in \mathbf{N}$. Without loss of generality assume that $\bar{Y}$ is closed. Applying Ekeland's variational principle as given in **[9]** to the function

$$y \to f(x_n) + \|x_n - y\| - g(y)$$

on $\bar{Y}$ yields y_n in $\bar{Y}$ such that $\|y_n - \bar{y}\| \leqq \epsilon_n$ and for all y in $\bar{Y}$ we have

$$(7.4) \qquad \|x_n - y\| - g(y) - (\|x_n - y_n\| - g(y_n)) \geqq -\epsilon_n\|y - y_n\|.$$

Without loss of generality assume that all y_n are in the interior of $\bar{Y}$. Then for all $t > 0$ sufficiently small

$$y_n + t(x_n - y_n) \in \bar{Y}.$$

By using (7.4) we get

$$t\|x_n - y_n\| \leqq -(g(y_n + t(x_n - y_n)) - g(y_n)) + \epsilon_n t\|x_n - y_n\|.$$

Hence dividing by t and letting t go to zero we obtain

$$(7.5) \qquad (1 - \epsilon_n)\|x_n - y_n\| \leqq g_{y_n}^*(y_n - x_n).$$

Let $y \in E$. For $t > 0$ sufficiently small $y_n + ty \in \bar{Y}$ and therefore it follows from (7.4) that

$$(1 + \epsilon_n) \|y\| \geqq g^*_{y_n}(y) \quad \text{for all } y \text{ in } E.$$

This and (7.5) implies

$$1 - \epsilon_n \leqq \|g^*_{y_n}\| \leqq 1 + \epsilon_n.$$

Hence as $g^*_{y_n}$ converges in norm to $g^*_{\bar{y}}$ we get

$$\|g^*_{\bar{y}}\| = 1.$$

Thus by (7.5) we have

$$(7.6) \quad \begin{aligned} \|\bar{y} - \bar{x}\| &\geqq g^*_{\bar{y}}(\bar{y} - \bar{x}) \\ &= g^*_{\bar{y}}(y_n - x_n) + (g^*_{\bar{y}}(\bar{y} - \bar{x}) - g^*_{\bar{y}}(y_n - x_n)) \\ &= g^*_{y_n}(y_n - x_n) + (g^*_{\bar{y}}(y_n - x_n) - g^*_{y_n}(y_n - x_n)) \\ &\quad + g^*_{\bar{y}}(\bar{y} - \bar{x}) - g^*_{\bar{y}}(y_n - x_n) \\ &\geqq \|y_n - x_n\| + \theta_n, \end{aligned}$$

where

$$\theta_n = -\epsilon_n \|y_n - x_n\| + g^*_{\bar{y}}(y_n - x_n) - g^*_{y_n}(y_n - x_n) + g^*_{\bar{y}}(\bar{y} - \bar{x}) - g^*_{\bar{y}}(y_n - x_n).$$

Note that θ_n converges to zero as n goes to infinity, therefore (7.6) implies

$$\|\bar{x} - \bar{y}\| \geqq \limsup_{n \to \infty} \|x_n - y_n\|.$$

Weak lowersemicontinuity of the norm implies now

$$\lim_{n \to \infty} \|x_n - y_n\| = \|\bar{x} - \bar{y}\|$$

and the Kadec property gives $x_n - y_n$ converges to $\bar{x} - \bar{y}$ in norm. Thus also x_n converges to $\bar{x}$ in norm and the proof is finished.

If $\| \ \|$ is some norm on E and $x \in C \subset E$ let $PN_C^{\| \|}(x)$ denote the set of proximal normal functionals to C at x with respect to the norm $\| \ \|$.

THEOREM 7.2. *Let f be a lowersemicontinuous function on a reflexive Banach space E. Assume that the norm of E is Kadec and Frechet differentiable. If f is finite at x and*

$$(y^*, 0) \in PN_{\mathrm{epi}f}(x, f(x))$$

then there exist sequences $x_n \to x$, $t_n \downarrow 0$, a sequence of equivalent norms on E $\| \ \|_n$, such that all of them are Kadec and Frechet differentiable and a sequence $y^*_n \in E^*$ with

$$(y_n^*, -1) \in PN_{\mathrm{epi}f}^{\|\ \|_n}(x_n, f(x_n)),$$

for which

$$(y^*, 0) = \lim_{n\to\infty} t_n(y_n^*, -1).$$

Proof. Without loss of generality let us assume that

$$(x, f(x)) = (0, 0) \quad \text{and} \quad \|y^*\| = 1.$$

Then there exists $(\bar{y}, 0) \notin \mathrm{epi}\, f$ such that

$$t := \|(\bar{y}, 0)\| = \mathrm{dist}_{\mathrm{epi}f}(\bar{y}, 0) \leqq 1$$

and $(y^*, 0)$ is the derivative of the norm of $E \times \mathbf{R}$ at $t^{-1}(\bar{y}, 0)$. Denote the closed unit ball of E by B.

Let us choose ϵ with $0 < \epsilon < 1/4$ and put $\lambda := 1$. Then

$$(7.7) \qquad x \in (0, \lambda](\bar{y} + 2\epsilon t B) \quad \text{implies } f(x) \geqq 0.$$

Let us also choose

$$\rho_n < \min\{(1/4)^{n+1}, (1 - \epsilon)t, \epsilon/4\}$$

and $\alpha_n > 0$ such that

$$\alpha_n < \min\{\epsilon\rho_n/3, ((1 - \epsilon)t - \rho_n)/2t\}.$$

Similarly as in the proof of Theorem 3.1 we define for $n \in \mathbf{N}$

$$W_n := \mathrm{co}(\epsilon t B, (1 + \alpha_n)^{-1}(\bar{y} + \alpha_n \epsilon t B),$$
$$(1 + \alpha_n)^{-1}(-\bar{y} + \alpha_n \epsilon t B)),$$

and we denote by $\|\ \|_n$ the norm of E associated with the unit ball W_n. Then

$$\epsilon t B \subset W_n \subset tB$$

and by Lemma 1 of Part I of the paper the norm $\|\ \|_n$ is an equivalent Kadec and Frechet differentiable norm of E.

Note that

$$(7.8) \qquad \gamma_n := \|\bar{y}\|_n = (1 + \alpha_n)(1 + \alpha_n\epsilon)^{-1}$$

and

$$(7.9) \qquad \gamma_n W_n \subset tB.$$

Assume without loss of generality that f is bounded below on the set $\{x \in E |\ \|x\| \leqq 1\}$ by some $-\beta > -\infty$. Choose $n \leqq k_n \in \mathbf{N}$ with

$$k_n^{-1} t^{-1} \gamma_n \beta < \alpha_n.$$

For $n \in \mathbf{N}$, $u \in E$ put

$$R_n := \{x \in E \mid \|x\| \leqq \rho_n\},$$

$$f_n(u) := \begin{cases} \sqrt{\gamma_n^2 - \|u\|_n^2} & \text{if } \|u\|_n \leqq \gamma_n(k_n^2 - 1)(k_n^2 + 1)^{-1}, \\ k_n(\gamma_n - \|u\|_n) & \text{if } \|u\|_n \geqq \gamma_n(k_n^2 - 1)(k_n^2 + 1)^{-1}. \end{cases}$$

Let us consider

(7.10) $\displaystyle F_n(y) := \inf_{x \in R_n} (t^{-1}\gamma_n f(x) - f_n(x - y)), \quad y \in E.$

For all n, $F_n(\bar{y}) \leqq 0$. If $F_n(\bar{y}) = 0$, then

$$0 = F_n(\bar{y}) \leqq f(0) - f_n(0 - \bar{y}) = 0,$$

and the infimum in (7.15) is attained at $x_n := 0$ for $y_n := \bar{y}$. Suppose that $F_n(\bar{y}) < 0$. Then there exists $\theta_n > 0$ such that for all y with $\|y - \bar{y}\|_n \leqq \theta_n$

(7.11) $\displaystyle F_n(y) = \inf_{x \in S_n} (t^{-1}\gamma_n f(x) - k_n(\gamma_n - \|x - y\|_n)),$

where

$$S_n := R_n \cap \{x \in E \mid \|x - \bar{y}\|_n \geqq \gamma_n\}.$$

Indeed, let $L_n \geqq k_n$ be such that

$$|f_n(u_1) - f_n(u_2)| \leqq L_n\|u_1 - u_2\|_n$$

whenever $\|u_i\|_n \leqq \gamma_n + 2$ for $i = 1, 2$. Choose $\theta_n > 0$ with

$$\theta_n < \min\{-(2L_n)^{-1}F_n(\bar{y}), 2\gamma_n(k_n^2 + 1)^{-1}, \alpha_n\}.$$

Then for all y for which $\|y - \bar{y}\|_n < \theta_n$ we have

(7.12) $\displaystyle F_n(y) = \inf_{x \in R_n} (t^{-1}\gamma_n f(x) - f_n(x - y)) < F_n(\bar{y}) + L_n\theta_n.$

Furthermore, if $\|x - \bar{y}\|_n \leqq \gamma_n$ we have

(7.13) $t^{-1}\gamma_n f(x) - f_n(x - \bar{y}) \geqq 0.$

Indeed, for $\|x - \bar{y}\|_n \leqq \gamma_n$

(7.14) $f_n(x - \bar{y}) \leqq \sqrt{\gamma_n^2 - \|x - \bar{y}\|_n^2}.$

Also it follows from (7.19) that

$$\|x - \bar{y}\| \leqq \|x - \bar{y}\|_n \cdot t\gamma_n^{-1}.$$

Thus

(7.15) $\|x - \bar{y}\| \leqq t \quad \text{and} \quad \sqrt{t^2 - \|x - \bar{y}\|^2} \leqq f(x).$

Also

$$\gamma_n^2 - \|x - \bar{y}\|_n^2 \leqq \gamma_n^2 - \gamma_n^2(t^2)^{-1}\|x - \bar{y}\|^2$$
$$= \gamma_n^2(t^2)^{-1}(t^2 - \|x - \bar{y}\|^2).$$

Using this, (7.14) and (7.15) we get

$$f_n(x - \bar{y}) \leqq \gamma_n t^{-1} \sqrt{t^2 - \|x - \bar{y}\|^2} \leqq \gamma_n t^{-1} f(x),$$

which implies (7.13).

Therefore if $\|x - \bar{y}\|_n \leqq \gamma_n$ then

$$(7.16) \quad t^{-1}\gamma_n f(x) - f_n(x - y)$$
$$= t^{-1}\gamma_n f(x) - f_n(x - \bar{y}) + [f_n(x - \bar{y}) - f_n(x - y)]$$
$$> 0 - L_n\theta_n \geqq F_n(\bar{y}) + L_n\theta_n.$$

(7.16) and (7.12) show that

$$(7.17) \quad F_n(y) := \inf_{x \in S_n} (t^{-1}\gamma_n f(x) - f_n(x - y)).$$

However, if $\|x - \bar{y}\|_n \geqq \gamma_n$ and $\|y - \bar{y}\|_n < \theta_n$ then

$$\|x - y\|_n \geqq \|x - \bar{y}\|_n - \theta_n \geqq \gamma_n - 2\gamma_n(k_n^2 + 1)^{-1}$$
$$= \gamma_n(k_n^2 - 1)(k_n^2 + 1)^{-1}$$

and therefore

$$f_n(x - y) = k_n(\gamma_n - \|x - y\|_n).$$

This together with (7.17) finishes the proof of (7.11).

Applying Proposition 7.2, we choose y_n in E with

$$\|y_n - \bar{y}\|_n < \theta_n$$

for which infimum in (7.11) is attained at some $x_n \in R_n$ such that

$$(7.18) \quad \|x_n - \bar{y}\|_n \geqq \gamma_n.$$

Then by (7.12) we get

$$F_n(y_n) < F_n(\bar{y}) + L_n\theta_n$$

and hence

$$t^{-1}\gamma_n f(x_n) - k_n(\gamma_n - \|x_n - \bar{y}\|_n) \leqq F_n(y_n) + L_n\theta_n$$
$$< F_n(\bar{y}) + 2L_n\theta_n < 0.$$

This together with (7.18) implies

$$-t^{-1}\gamma_n\beta \leqq t^{-1}\gamma_n f(x_n)$$
$$< t^{-1}\gamma_n f(x_n) - k_n(\gamma_n - \|x_n - \bar{y}\|_n) < 0.$$

Thus

(7.19) $f(x_n) < 0$

and

(7.20) $$\|x_n - \bar{y}\|_n < \gamma_n - k_n^{-1}t^{-1}\gamma_n f(x_n) < \gamma_n + k_n^{-1}t^{-1}\gamma_n\beta < \gamma_n + \alpha_n.$$

We will show that

(7.21) $\|x_n\| < \rho_n.$

As $\gamma_n \leqq 1 + \alpha_n$ we have by (7.20)

$$x_n = (x_n - \bar{y}) + \bar{y} \in (\gamma_n + \alpha_n)W_n + \bar{y} \subset \bar{y} + W_n + 2\alpha_n W_n.$$

Now following the lines of the proof of Theorem 3.1 we get

$$\bar{y} + W_n \subset (0, +\infty)(\bar{y} + \epsilon tB)$$

and

$$\begin{aligned} x_n &\in ((0, +\infty)(\bar{y} + \epsilon tB) + 2\alpha_n W_n) \cap \rho_n B \\ &\subset (0, 1)(\bar{y} + \epsilon tB) + 2\alpha_n W_n \\ &\subset \{ (0, 2\alpha_n\epsilon^{-1})(\bar{y} + \epsilon tB) + 2\alpha_n W_n \} \\ &\cup \{ [2\alpha_n\epsilon^{-1}, 1](\bar{y} + 2\epsilon tB) \}. \end{aligned}$$

This together with (7.7) and (7.19) implies

$$x_n \in (0, 2\alpha_n\epsilon^{-1})(\bar{y} + \epsilon tB) + 2\alpha_n tB.$$

Hence

$$\begin{aligned} \|x_n\| &\leqq 2\alpha_n\epsilon^{-1}t(1 + \epsilon) + 2\alpha_n t \\ &= t2\alpha_n\epsilon^{-1}(1 + 2\epsilon) \leqq 3\alpha_n\epsilon^{-1} < \rho_n \end{aligned}$$

and (7.21) is proven.

We conclude that the sequence x_n has the following property

(7.22) $x_n \to 0,$

thus by (7.19) and lowersemicontinuity of f

(7.23) $f(x_n) \to 0.$

Furthermore it follows from our construction via (7.21) that there exists a neighbourhood $X_n \subset R_n$ of x_n such that for all $x \in X_n$

(7.24) $$f(x) \geqq -t\gamma_n^{-1}k_n(\|x - y_n\|_n - \|x_n - y_n\|_n) + f(x_n) := h_n(x).$$

For $z \in E$ let φ_z^n and φ_z denote respectively the Frechet derivatives of the norm $\| \|_n$ and $\| \|$ at z.

Define

$$r_n := f(x_n) - t^{-1}\gamma_n k_n^{-1}\|x_n - y_n\|$$

and let h_n^* denote the Frechet derivative of the function h_n at x_n. Then

$$(7.25) \quad \|(h_n^*, -1)\|_n^{-1}(h_n^*, -1)(y_n - x_n, r_n - f(x_n))$$
$$= \sqrt{1 + (t^{-1}\gamma_n k_n^{-1})^2}\,\|x_n - y_n\|_n = \|(y_n - x_n, r_n - f(x_n))\|_n$$

and

$$(7.26) \quad \|x - y_n\|_n^2 + |h_n(x) - r_n|^2 = \|x - y_n\|_n^2$$
$$+ |-t\gamma_n^{-1}k_n(\|x - y_n\|_n - \|x_n - y_n\|_n)$$
$$+ t^{-1}\gamma_n k_n^{-1}\|x_n - y_n\|_n|^2$$
$$= (1 + (t^{-1}\gamma_n k_n^{-1})^2)\|x_n - y_n\|_n^2 + \mathcal{S}_n,$$

where

$$\mathcal{S}_n = ((t\gamma_n^{-1}k_n)^2 + 1)\|x - y_n\|_n^2 - 2((t\gamma_n^{-1}k_n)^2 + 1)$$
$$\times \|x - y_n\|_n\|x_n - y_n\|_n$$
$$+ ((t\gamma_n^{-1}k_n)^2 + 1)\|x_n - y_n\|_n^2$$
$$= ((t\gamma_n^{-1}k_n)^2 + 1)(\|x - y_n\|_n - \|x_n - y_n\|_n)^2 \geqq 0.$$

As h_n is a continuous function (7.26) and (7.25) show that

$$d_{\mathrm{epi}\,h_n}^{\| \|_n}(y_n, r_n) = \|(y_n - x_n, r_n - f(x_n))\|_n,$$

which together with (7.24) proves that

$$d_{\mathrm{epi}\,f}^{\| \|_n}(y_n, r_n) = \|(y_n - x_n, r_n - f(x_n))\|_n.$$

This in turn implies by (7.25) that

$$(7.27) \quad (h_n^*, -1) \in PN_{\mathrm{epi}\,f}^{\| \|_n}(x_n, f(x_n)).$$

Note that

$$(7.28) \quad h_n^* = t\gamma_n^{-1}k_n\varphi_{x_n - y_n}^n.$$

Now it may be easily argued (as in the proof of Lemma 1 in Part I) that

$$(7.29) \quad \varphi_{x_n - y_n}^n = \|x_n - y_n\|_n\|x_n - y_n\|^{-1}\varphi_{x_n - y_n}.$$

Put

$$s_n := \|x_n - y_n\|_n\|x_n - y_n\|^{-1} \quad \text{and}$$
$$t_n := s_n^{-1}(1 + (t\gamma_n^{-1}k_n)^2)^{-1/2},$$

then $\gamma_n \to 1$, $s_n \to t^{-1}$, $k_n \to \infty$. Thus

(7.30) $\quad t_n \downarrow 0.$

Furthermore Frechet differentiability of the norm $\| \ \|$ implies that

(7.31) $\quad \varphi_{x_n - y_n} \to \varphi_{-\bar{y}} = -y^*.$

Using (7.28), (7.29), (7.30), and (7.31) we get

(7.32) $\quad t_n(h_n^*, -1) \to (y^*, 0).$

Now (7.22), (7.23), (7.27), (7.30), and (7.32) show that the theorem is true.

Definition 7.2. Let f be an extended real valued function on a Banach space E with a Frechet differentiable norm. Assume that f is lowersemicontinuous and let x be any point with $f(x)$ finite. Denote by $\mathscr{F}$ the set of all equivalent and Frechet differentiable norms on E. We say that $x^* \in E^*$ is a *generalized proximal subgradient to f at x* if

$$(x^*, -1) \in \bigcup_{\| \ \| \in \mathscr{F}} PN_{\mathrm{epi} f}^{\| \ \|}(x, f(x)).$$

The set of all such functionals will be denoted by $\tilde{\partial}_f^P(x)$.

In the following theorem we state and prove the proximal subgradient formula which generalizes the finite dimensional version proven in [**19**]. Let us define

$$Y(x) := w^* \limsup_{x' \to_f x} \partial_f^P(x'),$$

$$V(x) := w^* \limsup_{x' \to_f x} \partial^F f(x'),$$

$$Y_0(x) := w^* \limsup_{\substack{x' \to_f x \\ \lambda \downarrow 0}} \lambda \tilde{\partial}^P f(x'),$$

$$V_0(x) := w^* \limsup_{\substack{x' \to_f x \\ \lambda \downarrow 0}} \lambda \partial^F f(x').$$

THEOREM 7.3. *Let f be a lowersemicontinuous function on a reflexive Banach space E. Assume that the norm of E is Kadec and Frechet differentiable. If f is finite at x then the formula*

(7.33) $\quad \partial f(x) = \overline{\mathrm{co}}(Y(x) + Y_0(x)) = \overline{\mathrm{co}}(V(x) + V_0(x))$

holds.

Proof. In any space with Frechet differentiable norm we have

$$\partial^{WH} f(x) \supset \partial^P f(x).$$

Hence as E is reflexive, equality

$$\partial^F f(x) = \partial^{WH} f(x)$$

implies

(7.34) $\overline{\text{co}}(Y(x) + Y_0(x)) \subset \overline{\text{co}}(V(x) + V_0(x))$.

Let $x^* \in V(x)$ and $\bar{x}^* \in V_0(x)$. Then sequences $x_n, \bar{x}_n \in E$, x_n^*, $\bar{x}_n^* \in E^*$ and $\lambda_n \downarrow 0$ exists such that

$$x_n \to_f x, \quad \bar{x}_n \to_f x, \quad x^* = w^* \lim_{n\to\infty} x_n^*,$$

$$\bar{x}^* = w^* \lim_{n\to\infty} \lambda_n \bar{x}_n^* \quad \text{and}$$

$$(x_n^*, -1) \in WK^0_{\text{epi}f}(x_n, f(x_n)),$$
$$(\bar{x}_n^*, -1) \in WK^0_{\text{epi}f}(\bar{x}_n, f(\bar{x}_n)).$$

This by Theorem 3.1 implies that

$$(x^*, -1) \in w^* \limsup_{x' \to_f x} WK^0(x', f(x')) \subset N_{\text{epi}f}(x, f(x)),$$

$$(\bar{x}^*, 0) \in w^* \limsup_{x' \to_f x} WK^0(x', f(x')) \subset N_{\text{epi}f}(x, f(x)).$$

Thus

$$(x^* + \bar{x}^*, -1) = (x^*, -1) + (\bar{x}^*, 0) \in N_{\text{epi}f}(x, f(x))$$

on using convexity of the Clarke normal cone. Therefore

$$x^* + \bar{x}^* \in \partial f(x).$$

This proves that

$$V(x) + V_0(x) \subset \partial f(x),$$

and consequently

(7.35) $\overline{\text{co}}(V(x) + V_0(x) \subset \partial f(x))$,

because the Clarke subgradient set is always convex and weak star closed.

By (7.34) and (7.35) it remains to show that

$$\partial f(x) \subset \overline{\text{co}}(Y(x) + Y_0(x)).$$

But this inclusion is a consequence of the equality

$$N_{\text{epi}f}(x, f(x)) = \overline{\text{co}}\{\lambda(y^*, -1) \mid y^* \in Y(x), \lambda > 0\}$$
$$\cup \{(y^*, 0) \mid y^* \in Y_0(x)\}$$

which follows from Theorem 3.1 and Theorem 7.2. This finishes the proof.

As any reflexive Banach space has an equivalent Kadec and Frechet differentiable norm we easily obtain from Theorem 7.3 the following result.

COROLLARY 7.2. *Let f be a lowersemicontinuous function on a reflexive Banach space. If f is finite at x then the following formula holds*:

$$\partial f(x) = \overline{\mathrm{co}}(V(x) + V_0(x)).$$

The subgradient formulas simplify for locally Lipschitz functions.

COROLLARY 7.3. *If f is a locally Lipschitz function on a reflexive Banach space with a Kadec and Frechet differentiable norm then*

$$\begin{aligned}(7.36) \quad \partial f(x) &= \overline{\mathrm{co}}\ w^* \limsup_{x' \to x} \partial^P f(x') \\ &= \overline{\mathrm{co}}\ w^* \limsup_{x' \to x} \partial^F f(x') \\ &= \overline{\mathrm{co}}\ w^* \limsup_{x' \to x} \partial^H f(x').\end{aligned}$$

Proof. We have $Y_0(x) \subset V_0(x)$ and as f is locally Lipschitz there exist X a neighbourhood of x and $M > 0$ such that for any $x' \in X$

$$f^H(x'; y) < M \quad \text{for all } y \in B,$$

(where B is a unit ball of E). Thus $V_0(x) = \{0\}$, and the first and the second equalities in (7.36) follow by (7.33), where $x' \to_f x$ may be replaced by $x' \to x$.

Furthermore for a locally Lipschitz function on a Banach space we have [**20**]

$$N_{\mathrm{epi}f}(x, f(x)) = (\liminf_{x' \to x} K_{\mathrm{epi}f}(x', f(x')))^0,$$

thus using (3.24) we get

$$N_{\mathrm{epi}f}(x, f(x)) \supset w^* \limsup_{x' \to x} K^0_{\mathrm{epi}f}(x', f(x')).$$

Hence using the appropriate definitions we obtain

$$\partial f(x) \supset \limsup_{x' \to x} \partial^H f(x'),$$

which together with the equalities already proven finishes the proof of (7.36).

Again by using renorming theorems we obtain easily the following.

COROLLARY 7.4. *If f is a locally Lipschitz function on a reflexive Banach space then*

$$\partial f(x) = \overline{\mathrm{co}}\ w^* \limsup_{x' \to x} \partial^F f(x')$$

$$= \overline{\text{co}}\ w^* \limsup_{x' \to x} \partial^H f(x').$$

The results of this section have many apparent consequences for non-smooth optimization and non-smooth analysis on reflexive Banach spaces which is the subject of our forthcoming paper.

8. Examples. In this section we present limiting examples for the theory developed in the paper. Our first examples will be based on the observation that the tangency properties of the set $\{x \in E| \, \|x\| \geqq 1\}$ are related to the differentiability properties of the norm $\| \ \|$ as is explained in the following considerations.

PROPOSITION 8.1. *Let E be a Banach space. Let $\| \ \|$ be a norm on E and*

$$C := \{x \in E| \, \|x\| \geqq 1\}.$$

If $\|\bar{x}\| = 1$ then the following are equivalent.

(i) *$\bar{x} \in C$ is a P-proper point of C;*

(ii) *$\bar{x} \in C$ is a cone point of C;*

(iii) *the norm $\| \ \|$ is Gateaux differentiable at $\bar{x}$.*

Proof. Obviously (i) implies (ii) as remarked after Definition 5.4. Assume that (ii) holds. Let x^* be such that

$$(8.1) \qquad (x^*, \bar{x}) = \sup\{ (x^*, x) \,|\, [\bar{x}, x] \subset C\}, \|x^*\| = 1.$$

Let $\bar{x}^*$ be a subgradient of the norm $\| \ \|$ at $\bar{x}$. Suppose that $y \in E$ and $(\bar{x}^*, y) \geqq 0$. Then for any t, $0 \leqq t \leqq 1$ we have

$$\| (1 - t)\bar{x} + t(y + \bar{x}) \| = \|\bar{x} + ty\| \geqq \|\bar{x}\| + t(\bar{x}^*, y) \geqq 1.$$

Hence by (8.1)

$$(x^*, \bar{x}) \geqq (x^*, \bar{x} + y).$$

Thus

$$(\bar{x}^*, y) \geqq 0 \text{ implies } (x^*, y) \leqq 0$$

and as $\|\bar{x}^*\| = \|x^*\| = 1$ we get $\bar{x}^* = -x^*$. This proves that the subgradient set of the norm $\| \ \|$ at $\bar{x}$ consists of exactly one functional and therefore the norm $\| \ \|$ is Gateaux differentiable at $\bar{x}$.

Assume now that (iii) holds. Let x^* be a Hadamard derivative of the norm $\| \ \|$ at $\bar{x}$. Suppose that $y \in K_C(\bar{x})$. Then

$$y = \lim_{n \to \infty} t_n^{-1}(c_n - \bar{x}) \quad \text{for some } c_n \in C, \, t_n \downarrow 0$$

and by Hadamard differentiability we have

$$(x^*, y) = \lim_{n \to \infty} \frac{\|c_n\| - \|\bar{x}\|}{t_n} \geqq 0.$$

This proves that $P_C(\bar{x}) \neq E$ and the proof is finished.

PROPOSITION 8.2. *Let* E, $\| \ \|$, C *be as in Proposition* 8.1. *If* $\|\bar{x}\| = 1$ *then the following are equivalent*:
(i) $\bar{x} \in C$ *is a WP-proper point of* C;
(ii) *the norm* $\| \ \|$ *is weak Hadamard differentiable at* $\bar{x}$.

Proof. Suppose that $WP_C(\bar{x}) \neq E$. Let $x^* \in WP_C^0(\bar{x})$ and $\|x^*\| = 1$. Then as in the proof of the implication (ii) $\Rightarrow$ (iii) in Proposition 8.1 we may prove that $-x^*$ is a Gateaux derivative of $\| \ \|$ at $\bar{x}$. Suppose that $-x^*$ is not a weak Hadamard derivative of $\| \ \|$ at $\bar{x}$. Then there exist $\epsilon > 0$ and sequences $t_n \downarrow 0$ and y_n converging weakly to some y such that

$$\frac{\|\bar{x} + t_n y_n\| - \|\bar{x}\|}{t_n} + (x^*, y_n) > \epsilon, \quad n \in \mathbf{N}. \tag{8.2}$$

a) Suppose that $(x^*, y) = 0$. As $\|\bar{x}\| = 1$, we conclude by 8.2 that

$$\frac{\|\bar{x} + t_n(y_n - (\epsilon/2)\bar{x})\| - \|\bar{x}\|}{t_n}$$
$$\geqq \frac{\|\bar{x} + t_n y_n\| - \|x\|}{t_n} - (\epsilon/2)\|\bar{x}\| > -(x^*, y_n) + \epsilon/2.$$

Therefore

$$\|\bar{x} + t_n(y_n - (\epsilon/2)\bar{x})\| \geqq 1 + t_n(-(x^*, y_n) + \epsilon/2) > 1,$$

for sufficiently big n. This implies that

$$y - (\epsilon/2)\bar{x} \in WP_C(\bar{x}),$$

hence

$$(x^*, (y - (\epsilon/2)\bar{x})) \leqq 0.$$

As $-(x^*, \bar{x}) = 1$, we get a contradiction from

$$-(x^*, (\epsilon/2)\bar{x}) = \epsilon/2 \leqq 0.$$

b) So suppose that $\alpha := (x^*, y) \neq 0$ and consider

$$z := y + \alpha\bar{x}, \ z_n := y_n + \alpha\bar{x}, \ x_n := \bar{x} + t_n y_n.$$

Using (8.2) we get

$$\frac{\|\bar{x} + t_n z_n\| - \|\bar{x}\|}{t_n} + (x^*, z_n) \tag{8.3}$$
$$> \frac{\|x_n + t_n\alpha\bar{x}\| - \|x_n\|}{t_n} + (x^*, \alpha\bar{x}) + \epsilon.$$

For $x \in E$ let f_x be the Gateaux derivative of the norm at x. Then the mapping $x \to f_x$ is norm to weak star continuous from the unit sphere of E to the unit sphere of E^* [**11**]. Thus we get

$$(8.4) \quad \frac{\|x_n + t_n\alpha\bar{x}\| - \|x_n\|}{t_n} \geqq \alpha f_{x_n\|x_n\|^{-1}}(\bar{x}).$$

Also $x_n/\|x_n\| \to \bar{x}$ implies

$$f_{x_n\|x_n\|^{-1}}(\bar{x}) \to f_{\bar{x}}(\bar{x}) = 1.$$

Thus (8.3) and (8.4) give

$$(8.5) \quad \frac{\|\bar{x} + t_nz_n\| - \|\bar{x}\|}{t_n} + (x^*, z_n) > \alpha(f_{x_n\|x_n\|^{-1}}(\bar{x}) - 1) + \epsilon > \epsilon/2,$$

whenever n is sufficiently big. As (8.5) is analogous to (8.2) and $(x^*, z) = 0$, we get a similar contradiction as in the first part of the proof. This proves that $\| \ \|$ is weak Hadamard differentiable at $\bar{x}$ whenever C is WP-proper at $\bar{x}$.

Replacing Hadamard by weak Hadamard, $P_C(\bar{x})$ by $WP_C(\bar{x})$ and norm convergence of $t_n^{-1}(c_n - x)$ by weak convergence in the proof of the implication (iii) $\Rightarrow$ (ii) in Proposition 8.1, we may prove that C is WP-proper at $\bar{x}$ whenever $\| \ \|$ is weak Hadamard differentiable at $\bar{x}$. This finishes the proof.

COROLLARY 8.1. *Let E be a Banach space. If any closed subset C of E has a cone point* (*in particular if E is a P-proper space*) *then E is a Minkowski differentiability space.*

Proof. Let $\| \ \|$ be any equivalent norm on E, then by our assumptions the set

$$C := \{x \in E | \ \|x\| \geqq 1\}$$

has a P-proper point $\bar{x} \in C$. By Proposition 8.1, $\| \ \|$ is Gateaux differentiable at $\bar{x}$. Thus any equivalent norm on E has a point of Gateaux differentiability, hence E is a Minkowski differentiability space.

Using Corollary 8.1 we may obtain a partial strengthening of Proposition 6.8.

COROLLARY 8.2. *Let E be a Banach space. If any closed subset of $E \times \mathbf{R}$ has a cone point then E is a Gateaux differentiability space.*

Proof. If our assumptions are satisfied then by Corollary 8.1, $E \times \mathbf{R}$ is a Minkowski differentiability space. This in turn is equivalent to E being a Gateaux differentiability space as proven in [**14**].

Example 1. Let E be any Banach space with a nowhere weak Hadamard differentiable norm. Let $\| \ \|$ be the norm of E. Put

$$C := \{x \in E| \ \|x\| \geqq 1\}.$$

Then C is a closed set which by Proposition 8.2:

(i) has no WP-proper points

(ii) is WP-pseudoconvex but not convex.

In particular the following spaces may serve as the example.

A. Consider $E := l_1(S)$, S uncountable. Then the norm of E is nowhere Gateaux [**12**] hence nowhere weak Hadamard differentiable.

B. Consider $E := C([0, 1])$. We will show that not only is the norm of E nowhere weak Hadamard differentiable and conditions (i) and (ii) hold but what is more, the set C:

(iii) has no WT-proper points,

(iv) is WT-pseudoconvex but not convex. ($WT_C(x)$ denotes the weak Clarke tangent cone to C at x as introduced in Part I.)

Proof. Let $f \in C$, $\|f\| = 1$. Take any $g \in E$. Suppose that sequences $t_n \downarrow 0$ and $f_n \in C$ converging (in norm) to f are given. Define

$$N_1 := \{n \in \mathbf{N}| \sup_{x \in [0,1]} f_n(x) \geqq 1\},$$

$$N_2 := \{n \in \mathbf{N}| \sup_{x \in [0,1]} -f_n(x) \geqq 1\}.$$

For $n \in N_1$ let

$$W_n := \{x| \ f_n(x) > 1 - t_n\}.$$

Then W_n is nonempty and open. Choose Q_n open, $Q_n \subset W_n$ with diam $Q_n < 1/n$, (where

$$\operatorname{diam} Q_n := \sup\{ |s' - s''| \, |s', s'' \in Q_n\}).$$

Suppose that N_1 is infinite. Then if

$$\bigcap_{n \in N_1} Q_n = \emptyset$$

put $U_n := Q_n$ for all $n \in N_1$. If

$$\bigcap_{n \in N_1} Q_n = y \quad \text{for some } y \in [0, 1],$$

put

$$U_n := Q_n \backslash y.$$

Then

$$(8.6) \qquad 1 - t_n < f_n(x) \quad \text{for } x \text{ in } U_n, n \in N_1$$

and

$$(8.7) \qquad \bigcap_{n \in N_1} U_n = \emptyset.$$

We will construct a sequence of functions h_n in C for which

(8.8) $\quad t_n^{-1}(h_n - f_n)$ converges weakly to g as n goes to ∞ in N_1.

For $n \in N_1$ consider functions

$$k_n := (1 - t_n)f_n + t_n(f + g).$$

If $\|k_n\| \geqq 1$ put $h_n := k_n$. If $\|k_n\| < 1$, pick $u_n \in U_n$ and put

$$h_n^1(x) := \begin{cases} k_n(x) & \text{if } x \in [0, 1], x \neq u_n, \\ 1 & \text{if } x = u_n, \end{cases}$$

$$h_n^2(x) := \begin{cases} k_n(x) & \text{if } x \in [0, 1]\setminus U_n, \\ 1 & \text{if } x \in U_n. \end{cases}$$

Then

$$k_n(x) \leqq h_n^1(x) \leqq h_n^2(x) \quad \text{for all } x \in [0, 1],$$

and

$$\|h_n^1(x)\| = \|h_n^2(x)\| = 1.$$

Also h_n^1 is uppersemicontinuous on [0, 1] and h_n^2 is lowersemicontinuous on [0, 1]. By Michael's selection theorem **[12]** there exists a continuous function h_n on [0, 1] such that

$$h_n^1 \leqq h_n \leqq h_n^2.$$

Note that all the constructed functions h_n are in C. We will show that (8.8) is satisfied. Indeed, let $K > 0$ be such that

$$\|f(x) + g(x) - f_n(x)\| \leqq K \quad \text{for all } x \in [0, 1].$$

If $h_n(x) = k_n(x)$, then

$$|t_n^{-1}(h_n(x) - f_n(x))| = |f(x) + g(x) - f_n(x)| \leqq K.$$

If $h_n(x) \neq k_n(x)$ then $x \in U_n$ and by (8.6) we get

$$\begin{aligned} -K &\leqq t_n^{-1}(k_n(x) - f_n(x)) \leqq t_n^{-1}(h_n(x) - f_n(x)) \\ &\leqq t_n^{-1}(h_n^2(x) - f_n(x)) \leqq t_n^{-1}(1 - f_n(x)) \leqq 1. \end{aligned}$$

Thus if n goes to infinity in N_1 the sequence

$$t_n^{-1}(h_n - f_n)$$

is norm bounded and as

$$\bigcap_{n \in N_1} U_n = \emptyset,$$

it converges pointwise to g. Therefore by Lebesgue's bounded convergence theorem it converges weakly to g.

Suppose now that N_2 is infinite, then replacing N_1 and f_n by N_2 and $-f_n$ in the construction of the sets U_n above, we get that there exist nonempty open sets U_n such that

$$1 - t_n < -f_n(x) \quad \text{for all } x \text{ in } U_n, \quad n \in N_2$$

and

$$\bigcap_{n \in N_2} U_n = \emptyset.$$

For $n \in N_2$ consider functions

$$k_n := (1 - t_n)(-f_n) + t_n(-f - g).$$

If $\|k_n\| \geqq 1$ put $h_n := k_n$. If $\|k_n\| < 1$, pick $u_n \in U_n$ and put

$$h_n^1(x) := \begin{cases} k_n(x) & \text{if } x \in [0, 1],\ x \neq u_n, \\ -1 & \text{if } x = u_n, \end{cases}$$

$$h_n^2(x) := \begin{cases} k_n(x) & \text{if } x \in [0, 1]\setminus U_n, \\ -1 & \text{if } x \in U_n. \end{cases}$$

Then for all $x \in [0, 1]$

$$h_n^2(x) \leqq h_n^1(x) \leqq k_n(x),$$

and

$$\|h_n^2(x)\| = \|h_n^1(x)\| = 1.$$

Also h_n^1 is lowersemicontinuous on [0, 1] and h_n^2 is uppersemincontinuous on [0, 1]. Again by Michael's selection theorem there exists a continuous function h_n on [0, 1] such that

$$h_n^2(x) \leqq h_n(x) \leqq h_n^1(x) \leqq k_n(x).$$

Note that all the functions h_n are in C and as in the first part of the proof one shows that if n goes to infinity in N_2 the sequence $t_n^{-1}(h_n - f_n)$ converges weakly to g. As the conclusion of the above considered cases we get $t_n^{-1}(h_n - f_n)$ converges weakly to g. As g was arbitrary in E, we get $WT_C(f) = E$, which proves also (iii) and (iv).

Note that the same argument works if $E := C(\Omega)$, where Ω is a compact, perfect, metric space.

Example 2. Let E be any non-Minkowski differentiable space. Let $\| \ \|$ be an equivalent norm of E which is nowhere Gateaux differentiable. Then by Proposition 7.1 the set

$$C := \{x \in E \mid \|x\| \geqq 1\}$$

(i) has no P-proper points;
(ii) has no cone points;
(iii) is closed and P-pseudoconvex but not convex.

In particular the following spaces and norms may serve as the example.

A. Let $E := L^{\infty}([0, 1], \mu)$, ($\mu$ is Lebesque measure). The norm of E is nowhere Gateaux differentiable [**21**].

B. Let $E := l^1(S)$, S uncountable. The norm of E is nowhere Gateaux differentiable.

C. Let $E := l^{\infty}(\mathbf{N})$. Define an equivalent norm on E by

$$\|x\| := \|x\|_{\infty} + \limsup_{n \to \infty} |x_n|, \quad x = (x_n) \in E.$$

Then $\| \ \|$ is nowhere Gateaux differentiable [**14**].

Example 1 shows that if in Theorem 1.1, Theorem 5.1, and Corollary 2.1 the contingent cones are replaced by weak Clarke tangent cones or weak pseudocontingent cones then their statements may fail even in separable (hence weakly compactly generated) space (as we have shown in $C([0, 1])$).

Example 2 shows that if in Theorem 1.1, Theorem 5.1, and Corollary 2.1 the contingent cones are replaced by pseudocontingent cones then their statements may fail even in a dual space with the Radon Nikodym property (as in $l^1(S)$, S uncountable).

In particular, the constructed sets have no WP-proper points or no P-proper points. Note however that those sets are unbounded and for example the nonexistence of WP-proper points of the considered set C in $l^1(S)$ is due to its unboundedness.

Example 3.

A. Let E and $\| \ \|$ be as in Example 1 or

B. Let E and $\| \ \|$ be as in Example 2. Consider

$$C := \{x \in E | 1 \leqq \|x\| \leqq 2\}.$$

Note that C is convex (hence WT-, P-, WP-pseudoconvex) on some neighbourhood of any x with $\|x\| = 2$. Thus in the case A, the set C is closed, bounded and

(i) has (WT-), WP-proper points but they are not dense in the boundary of C,

(ii) is (WT-), WP-pseudoconvex but not convex.

In the case B, the set C is closed, bounded and

(i) has P-proper points but they are not dense in the boundary of C,

(ii) is P-pseudoconvex but not convex.

However the following questions remain still open: does there exist a closed bounded subset of a Banach space, which has no WP-proper points or which has no P-proper points? Does there exist a closed bounded subset of a Banach space without cone points?

We have indicated several limiting examples for the theory in Banach spaces. The limiting examples for normed spaces were investigated in [**6**]. They are related to the existence of normed unsupportable spaces, the theory of which originated by Klee, was recently extended in [**8**]. The normed space is *unsupportable* if it contains a closed bounded convex set S with no support points. Most incomplete normed spaces are unsupportable. In [**8**] it is conjectured that all are.

Example 4. Let E be an unsupportable normed space. Let S be a closed convex bounded set with no support points and $0 \notin S$. Put

$$C_1 := S \cup -S \quad \text{and} \quad C_2 := \bigcup_{|\lambda| \leqq 1} \lambda S.$$

Then

A. One can check that $T_{C_1}(x) = E$ for all $x \in C_1$, hence for all $x \in C_1$,

$$\bigcap_{x \in C_1} T_{C_1}(x) + x = E \quad \text{while star } C_1 = \emptyset.$$

Thus C_1 is T-pseudoconvex but not convex and Corollary 2.1, Theorem 5.1 and Theorem 5.6 are violated outside Banach spaces.

B. Moreover, as in [**6**],

$$\bigcap_{x \in C_2} K_{C_2}(x) + x = \text{cone } S \cup \text{cone}(-S)$$

and

$$\bigcap_{x \in C_2} T_{C_2}(x) + x = \{0\}$$

while

$$\bigcap_{x \in C_2} P_{C_2}(x) + x = E.$$

This provides counter examples to Theorem 1.1 and Corollary 2.1, and exhibits a set with one K-proper and no P-proper points.

C. Any unsupportable convex set S is completely antiproximal and so provides a counter-example to Proposition 5.10.

References

1. E. Asplund, *Frechet-differentiability of convex functions*, Acta Math. *121* (1968), 31-47.

2. E. Bishop and R. R. Phelps, *The support functionals of a convex set*, Proc. Symp. Amer. Math. Soc. *139* (1963), 433-467.

3. J. M. Borwein, *Tangent cones, starshape and convexity*, Internat. J. Math. & Math. Sci. *1* (1978), 459-477.

4. J. M. Borwein and R. O'Brien, *Tangent cones and convexity*, Canadian Math. Bull. *19* (1976), 257-261.

5. J. M. Borwein, M. Edelstein and R. O'Brien, *Visibility and starshape*, J. London Math. Soc. (2), *4* (1976), 313-318.

6. J. M. Borwein and H. M. Strojwas, *Tangential approximations*, to appear in Nonlinear Anal. Th. Meth. Appl. *9* (1985), 1347-1366.

7. ——— *Proximal analysis and boundaries of closed sets in Banach space, Part I: Theory*, Can. J. Math. *38* (1986), 431-452.

8. J. M. Borwein and D. W. Tingley, *On supportless convex sets*, Proc. Amer. Math. Soc. *94* (1985), 471-476.

9. F. H. Clarke, *Optimization and nonsmooth analysis* (John Wiley, 1982).

10. ——— *Generalized gradients and applications*, Trans. Amer. Math. Soc. *205* (1975), 247-262.

11. J. Diestel, *Geometry of Banach spaces-selected topics* (Springer-Verlag, 1975).

12. R. B. Holmes, *Geometric functional analysis and its applications* (Springer-Verlag, 1975).

13. M. Krasnoselski, *Sur un critère pour qu'un domaine soit étaillée*, Rec. Math. [Math. Sbornik] N. S. *19* (1946), 309-310.

14. D. G. Larman and R. Phelps, *Gateaux differentiability of convex functions on Banach spaces*, J. London Math. Soc. (2), *20* (1979), 115-127.

15. G. Lebourg, *Generic differentiability of Lipschitzian functions*, Trans. Am. Math. Soc. *256* (1979), 125-144.

16. M. McLinden, *An application of Ekeland's theorem to mimimax problems*, Nonlinear. Anal. Th. Meth. Appl. *6* (1982), 189-196.

17. J. P. Penot, *A characterization of tangential regularity*, Nonlinear Anal., Theory, Meth. Appl. *5* (1981), 625-633.

18. R. T. Rockafellar, *Generalized directional derivatives and subgradients of nonconvex functions*, Can. J. Math. *32* (1980), 257-280.

19. ——— *Proximal subgradients, marginal values, and augmented Lagrangians in nonconvex optimization*, Mathematics of Operations Research *6* (1981).

20. J. S. Treiman, *Characterization of Clarke's tangent and normal cones in finite and infinite dimensions*, Nonlinear Anal. Th. Meth. Appl. *7* (1983), 771-783.

21. S. Yamamuro, *Differential calculus in topological linear spaces* (Springer-Verlag, 1970).

22. N. V. Zhivkov, *Generic Gateaux differentiability of directionally differentiable mappings*, (submitted to Rev. Roum. Math. Pures Appl., 1984).

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