Canad. Math. Bull. Vol. **40** (1), 1997 pp. 47–53

A UNIVERSAL COEFFICIENT DECOMPOSITION FOR SUBGROUPS INDUCED BY SUBMODULES OF GROUP ALGEBRAS

MANFRED HARTL

ABSTRACT. Dimension subgroups and Lie dimension subgroups are known to satisfy a 'universal coefficient decomposition', *i.e.* their value with respect to an arbitrary coefficient ring can be described in terms of their values with respect to the 'universal' coefficient rings given by the cyclic groups of infinite and prime power order. Here this fact is generalized to much more general types of induced subgroups, notably covering Fox subgroups and relative dimension subgroups with respect to group algebra filtrations induced by arbitrary N-series, as well as certain common generalisations of these which occur in the study of the former. This result relies on an extension of the principal universal coefficient decomposition theorem on polynomial ideals (due to Passi, Parmenter and Seghal), to all additive subgroups of group rings. This is possible by using homological instead of ring theoretical methods.

It was first observed by Sandling [7] that dimension subgroups over an arbitrary commutative ring of coefficients can be decomposed in terms of the dimension subgroups over the 'universal' coefficient rings $\mathbb{Z}$ and $\mathbb{Z}/p^e\mathbb{Z}$, where p and e run through all primes and positive integers, respectively. Borrowing a notion from group cohomology this property may be conveniently termed by saying that dimension subgroups satisfy a *'universal coefficient decomposition'*. This property was conceptually proved and extended to Lie dimension subgroups by Parmenter, Passi and Sehgal [5], in developing a theory of polynomial ideals and their induced subgroups for this purpose. But still, important classes of induced subgroups are not covered by this theory, such as dimension subgroups with respect to arbitrary N-series, relative dimension subgroups or Fox subgroups. So in this paper we prove the universal coefficient decomposition for subgroups induced by a very general type of suitable submodules of group algebras (with respect to subgroups), which includes not only all types of induced subgroups mentioned before, but also certain common generalizations of them, such as *'relative dimension subgroups with respect to N-series'* or *'relative Fox dimension subgroups'* (some of which are explicitly computed in subsequent work). This result is based on a quite elementary homological lemma which extends the universal coefficient decomposition proved in [5] for polynomial ideals not only to *all* ideals, but even to all *additive subgroups* of group

---

Research supported by an individual fellowship of the Human Capital and Mobility Programme of the European Union, permitting a two-year stay at the Institut de Recherche Mathématique Avancée, Strasbourg.

Received by the editors September 27, 1995.

AMS subject classification: Primary: 20C07; secondary: 16A27.

Key words and phrases: induced subgroups, group algebras, Fox subgroups, relative dimension subgroups, polynomial ideals.

rings. So, surprisingly enough, it turns out that this nice property depends only on the ring structure of the coefficient rings, not the one of group algebras.

Throughout in this paper R denotes a commutative ring with identity 1_R. As usual, the *characteristic of* R is the least non-negative integer n such that $n1_R = 0$.

Let G be a group, $R(G)$ its group algebra with coefficients in R, $I_R(G)$ the augmentation ideal of $R(G)$, $I_R^n(G)$ its n-th associative and $I_R^{(n)}(G)$ its n-th Lie power, see [6, p. 2]. Write $i_R \colon \mathbb{Z}(G) \longrightarrow R(G)$ for the canonical ring homomorphism extending the identity map on G.

For an additive subgroup J of $\mathbb{Z}(G)$ let J_R denote the R-submodule of $R(G)$ spanned by $i_R(J)$.

THEOREM 1. *Let H be a subgroup of G and $J \subset \mathbb{Z}(G)I_{\mathbb{Z}}(H)$ be a right H-submodule, with the property that for all $h \in H$ there exists some $n = n(h) \geq 1$ such that $(h-1_{\mathbb{Z}})^n \in J$. Then for any commutative ring R with identity 1_R the following properties hold.*

(i) If characteristic of R is zero, then

$$G \cap (1_R + J_R) = \prod_{p \in \sigma(R)} \Big\{ H \cap t_p\Big(G \operatorname{mod}\big(G \cap (1_{\mathbb{Z}} + J)\big)\Big) \cap \Big(G \cap (1_{\mathbb{Z}/p^e\mathbb{Z}} + J_{\mathbb{Z}/p^e\mathbb{Z}})\Big)\Big\}$$

where $G \cap (1_R + J_R)$ and all factors on the right-hand side are subgroups which mutually commute. Here $\sigma(R) = \{p \mid p$ is a prime and $p^nR = p^{n+1}R$ for some $n \geq 0\}$, and for $p \in \sigma(R)$, p^e is the smallest power of p for which $p^eR = p^{e+1}R$. (If $\sigma(R)$ is empty then the right hand side is to be interpreted as $G \cap (1_{\mathbb{Z}} + J)$). By definition,

$$t_p\Big(G \operatorname{mod}\big(G \cap (1_{\mathbb{Z}} + J)\big)\Big) = \{g \in G \mid g^{p^k} \in G \cap (1_{\mathbb{Z}} + J) \text{ for some } k \geq 0\}.$$

(ii) If characteristic of R is $r > 0$, then

$$G \cap (1_R + J_R) = G \cap (1_{\mathbb{Z}/r\mathbb{Z}} + J_{\mathbb{Z}/r\mathbb{Z}}) = \bigcap_j G \cap (1_{\mathbb{Z}/p_j^{e_j}\mathbb{Z}} + J_{\mathbb{Z}/p_j^{e_j}\mathbb{Z}}),$$

where $r = \prod p_j^{e_j}$ is the prime factorization of r. ■

Before giving the proof we first discuss some

EXAMPLES 2. The following additive subgroups J of $\mathbb{Z}(G)$ satisfy the hypothesis of the theorem:

(i) associative powers $I_{\mathbb{Z}}^n(G)$ for all $n \geq 1$ and Lie powers $I_{\mathbb{Z}}^{(n)}(G)$ for $n \geq 2$, taking $H = G$ and $H = G'$, respectively; in this case the theorem is due to [7] and [5]. Indeed, for $n \geq 2$ one has $I_{\mathbb{Z}}^{(n)}(G) \subset I_{\mathbb{Z}}^{(2)}(G) = \mathbb{Z}(G)I_{\mathbb{Z}}(G')$ and $I_{\mathbb{Z}}^{n-1}(G') \subset I_{\mathbb{Z}}^{(n)}(G)$ by Sandling's formula for $I_R^{(n)}(G)$, *cf.* [6, I.1.8].

(ii) subgroups $J = MI_{\mathbb{Z}}(H)$, where $H \leq G$ and M is any right H-submodule of $I_{\mathbb{Z}}(G)$ with the property that for all $h \in H$ there exists some $n \geq 1$ such that $(h - 1_{\mathbb{Z}})^n \in M$. In this case we obtain in [3] a homological construction of a right H-submodule $J'_R \subset I_R^2(H)$ such that

$$G \cap \big(1_R + MI_R(H)\big) = H' \cap \Big(1_R + \big(M \cap I_R(H)\big)I_R(H) + J'_R\Big).$$

In the important case that H is free one even has $J'_R = 0$. This fact is further exploited in [1].

Thus we obtain

COROLLARY 3. *The following common generalizations of classical types of induced subgroups satisfy a universal coefficient decomposition:*

(1) relative dimension subgroups

$$D^{\mathcal{N}}_{n,R}(G,K) \overset{\text{def}}{=} G \cap \big(1 + I_R(K)I_R(G) + I^n_{R,\mathcal{N}}(G)\big)$$

with respect to a subgroup $K \leq G$ *and an N-series* $\mathcal{N}$ *of G; here* $\{I^i_{R,\mathcal{N}}(G)\}$ *denotes the ideal filtration of* $R(G)$ *induced by* $\mathcal{N}$*, cf. [6, III.1.5].*

(2) relative Fox dimension subgroups

$$G \cap \big(1 + I_R(K)I_R(H) + I^n_{R,\mathcal{N}}(G)I_R(H)\big)$$

with respect to any subgroups $K, H \leq G$ *and N-series* $\mathcal{N}$ *of G.*

We remark that such groups with an N-series different from the lower central series naturally arise in the study of classical Fox subgroups, namely when H is free (nilpotent) or is one of the two factors of a semidirect product, as is shown in [1] and in subsequent work. In [2] also the groups in (1), (2) above are calculated for $n = 3, 2$, respectively.

Now we turn to the proof of Theorem 1. As a key step we first obtain a generalization of the '*universal coefficient decomposition*' for polynomial ideals [5] to arbitrary additive subgroups of $\mathbb{Z}(G)$.

THEOREM 4. *Let* $J \leq \mathbb{Z}(G)$ *be any additive subgroup and R any commutative ring with identity* 1_R*. Then*

(1) if characteristic of R is zero,

$$i_R^{-1}J_R = \sum_{p\in\sigma(R)} \Big\{t_p\big(\mathbb{Z}(G) \bmod J\big) \cap \big(i^{-1}_{\mathbb{Z}/p^e\mathbb{Z}}J_{\mathbb{Z}/p^e\mathbb{Z}}\big)\Big\}.$$

If $\sigma(R)$ *is empty then the right hand side is to be interpreted as being J.*

(ii) If characteristic of R is $r > 0$*, then*

$$i_R^{-1}J_R = i^{-1}_{\mathbb{Z}/r\mathbb{Z}}J_{\mathbb{Z}/r\mathbb{Z}} = \bigcap_j i^{-1}_{\mathbb{Z}/p_j^{e_j}\mathbb{Z}}J_{\mathbb{Z}/p_j^{e_j}\mathbb{Z}}$$

where $r = \prod p_j^{e_j}$ *is the prime factorization of r.* ■

Theorem 4 rests on the following crucial homological

LEMMA 5. *Let A be an abelian group and R be any ring with identity* 1_R*. Consider the homomorphism* $j_R\colon A \longrightarrow R \otimes A$, $a \longmapsto 1_R \otimes a$*. Let r = characteristic of R. Then*

$$\text{Ker}(j_R) = rA + \sum_{p\in\sigma(R)} p^e t_p(A) \tag{1}$$

for $\sigma(R)$ *and* $e = e_p$ *as in Theorem 1. If* $\sigma(R)$ *is empty then* $r = 0$*, whence* $\text{Ker}(j_R) = 0$*. If* $r > 0$ *then* $\text{Ker}(j_R) = rA$*.*

PROOF. The right hand side of (1) is contained in $\mathrm{Ker}(j_R)$ since for $p \in \sigma(R)$, $a \in A$ and some $k \geq e$ such that $p^k a = 0$ we have $1_R \otimes p^e a = p^e 1_R \otimes a \in p^k R \otimes a = 0$. Conversely, let $u_R \colon \mathbb{Z}/r\mathbb{Z} \rightarrowtail R$, $u_R(1) = 1_R$. Then the map j_R factors as

$$j_R \colon A \twoheadrightarrow A/rA \cong \mathbb{Z}/r\mathbb{Z} \otimes A \xrightarrow{u_R \otimes A} R \otimes A. \tag{2}$$

Now consider the following part of a six-term exact sequence,

$$R/\langle 1_R \rangle * A \xrightarrow{\tau} \mathbb{Z}/r\mathbb{Z} \otimes A \xrightarrow{u_R \otimes A} R \otimes A \longrightarrow R/\langle 1_R \rangle \otimes A \longrightarrow 0,$$

where $*$ denotes the torsion product of abelian groups and τ is the appropriate connecting homomorphism. The inclusions of the torsion subgroups induce an isomorphism

$$t(R/\langle 1_R \rangle) * t(A) \xrightarrow{\cong} R/\langle 1_R \rangle * A,$$

as follows directly from the suitable six-term exact sequences. By the decomposition $t(X) = \bigoplus\{t_p(X) \mid p \text{ prime}\}$ for any abelian group X and by additivity of the torsion product we have

$$\mathrm{Im}(\tau) = \sum_{p \text{ prime}} \tau\big(t_p(R/\langle 1_R \rangle) * t_p(A)\big).$$

Let $\langle x, p^k, a \rangle$ be a canonical generator of $t_p(R/\langle 1_R \rangle) * t_p(A)$, *i.e.* $x \in R$, $a \in A$ such that $p^k x = n 1_R$ for some integer n and $p^k a = 0$, *cf.* [4, V.6]. Then $\tau\langle x, p^k, a \rangle = n1 \otimes a = 1 \otimes na$. Write $n = p^l m$ with $(p, m) = 1$. If $l \geq k$ then $\tau\langle x, p^k, a \rangle = 0$, so we need only to consider the case $l < k$. Let $m', p' \in \mathbb{Z}$ such that $mm' + pp' = 1$. Then

$$\begin{aligned} p^l 1_R &= p^l m m' 1_R + p^l p p' 1_R \\ &= n m' 1_R + p^{l+1} p' 1_R \\ &= p^k m' x + p^{l+1} p' 1_R \\ &= p^{l+1}(p^{k-l-1} m' x + p' 1_R), \end{aligned}$$

whence $p \in \sigma(R)$ and $l \geq e$. Thus $\tau\langle x, p^k, a \rangle = 1 \otimes (p^l m) a \in 1 \otimes p^e t_p(A)$, and $\mathrm{Ker}(u_R \otimes A) = \mathrm{Im}(\tau) \subset \sum_{p \in \sigma(R)} 1 \otimes p^e t_p(A)$. By (2) equality (1) is proved.

Now suppose $\sigma(R) = \emptyset$. Then $\mathrm{Ker}(j_R) = rA$. But if $r > 0$ then any prime not dividing r belongs to $\sigma(R)$, so $r = 0$.

Finally suppose $r > 0$. Let $p \in \sigma(R)$. If p does not divide r then $t_p(A) = r t_p(A)$. If p divides r write $r = p^s r'$ with $(p, r') = 1$. Assuming $e < s$ implies $r' p^{s-1} 1_R = r' p^{s-e-1} p^e 1_R \subset r' p^{s-e-1} p^{e+1} R = rR = 0$, which contradicts the fact that *characteristic of* $R = r$. Thus $e \geq s$, and $p^e t_p(A) \subset p^s t_p(A) = p^s r' t_p(A) = r t_p(A)$. Thus $\sum_{p \in \sigma(R)} p^e t_p(A) \subset rA$, whence $\mathrm{Ker}(j_R) = rA$ by (1) which completes the proof. ■

PROOF OF THEOREM 4. The map i_R induces a homomorphism $i'_R\colon \mathbb{Z}(G)/J \longrightarrow R(G)/J_R$, so we can write $i_R^{-1}J_R/J = \mathrm{Ker}(i'_R)$. But $R(G) \cong R\otimes\mathbb{Z}(G)$ and $J_R = \mathrm{Im}\big(R\otimes J \longrightarrow R\otimes\mathbb{Z}(G)\big)$, so by right exactness of the tensor product $R(G)/J_R \cong R\otimes\big(\mathbb{Z}(G)/J\big)$. Thus $\mathrm{Ker}(i'_R) = \mathrm{Ker}\Big(j_R\colon \mathbb{Z}(G)/J \longrightarrow R\otimes\big(\mathbb{Z}(G)/J\big)\Big)$, so we can apply Lemma 5 for $A = \mathbb{Z}(G)/J$. Just note that

$$\begin{aligned} p^e t_p(A) &= t_p(A)\cap p^e A \\ &= t_p(A)\cap \mathrm{Ker}(j_{\mathbb{Z}/p^e\mathbb{Z}}) \\ &= t_p\big(\mathbb{Z}(G)/J\big)\cap (i^{-1}_{\mathbb{Z}/p^e\mathbb{Z}}J_{\mathbb{Z}/p^e\mathbb{Z}}/J) \\ &= \big\{t_p\big(\mathbb{Z}(G) \bmod J\big)\cap (i^{-1}_{\mathbb{Z}/p^e\mathbb{Z}}J_{\mathbb{Z}/p^e\mathbb{Z}})\big\}/J, \end{aligned}$$

since $J \subset t_p\big(\mathbb{Z}(G) \bmod J\big)\cap (i^{-1}_{\mathbb{Z}/p^e\mathbb{Z}}J_{\mathbb{Z}/p^e\mathbb{Z}})$. ■

We still need the following useful little lemma which is well-known for $J \subset I_{\mathbb{Z}}(H)$, see [8].

LEMMA 6. *Let H be a subgroup of a group G and $J \subset R(G)I_R(H)$ be any subset. Then $G\cap(1_R + J) \subset H$.*

PROOF. Let T be a right transversal of H in G. Since $R(G)$ is a free right H-module with basis $\{[t], t\in T\}$ we have a composite isomorphism

$$\psi\colon R(G)/R(G)I_R(H) \cong R(G)\otimes_{R(H)} R \cong \bigoplus_{t\in T} R\cdot[t].$$

Now suppose $th \in (1_R + J) \subset \big(1_R + R(G)I_R(H)\big)$ for some $t\in T$, $h\in H$. Then $0 = \psi(th - 1_R) = \psi\big((t-1_R) + (h-1_R) + (t-1_R)(h-1_R)\big) = \psi(t-1_R) = [t]-[1]$, whence $t = 1$ as was to be shown. ■

PROOF OF THEOREM 1. Case (ii) follows immediately from Theorem 4(ii). In order to prove case (i) we shall proceed in several steps. Let us abbreviate

$$\begin{aligned} U_p &= H\cap t_p\Big(G \bmod \big(G\cap(1_{\mathbb{Z}} + J)\big)\Big)\cap\big(G\cap(1_{\mathbb{Z}/p^e\mathbb{Z}} + J_{\mathbb{Z}/p^e\mathbb{Z}})\big) \subset G, \\ W_p &= t_p\big(\mathbb{Z}(G) \bmod J\big)\cap(i^{-1}_{\mathbb{Z}/p^e\mathbb{Z}}J_{\mathbb{Z}/p^e\mathbb{Z}}) \subset \mathbb{Z}(G). \end{aligned}$$

STEP 1. For any commutative ring S with identity 1_S, $G\cap(1_S + J_S)$ is a subgroup of H. In fact, we have $G\cap(1_S+J_S)\subset H$ by Lemma 6. This implies that $G\cap(1_S+J_S)$ is a subgroup of H since for $g, h\in G\cap(1_S+J_S)$,

$$gh^{-1} - 1_S = (g-1_S) - (h-1_S)h^{-1} + (g-1_S)(h^{-1}-1_S) \in J_S$$

since J_S is a right H-submodule of $S(G)$.

STEP 2. Let $g \in G \cap (1_R + J_R)$. It will be shown that g is contained in the right-hand side of the decomposition in (i), viewed as an ordered product of subsets of G for the moment. Indeed, this is proved by a word-for-word copy of the proof of the corresponding statement for dimension subgroups given on page 17 of [6], replacing the reference to [6, Chapter I, Theorem 1.12] there by Theorem 4 above. The crucial point is that $g \in H$ by Lemma 6, so the number $n(g)$ is defined by hypothesis which implies

$$\begin{aligned} g^{r^s} - 1_{\mathbb{Z}} &= \sum_{i=1}^{r^s} \binom{r^s}{i} (g - 1_{\mathbb{Z}})^i \\ &\equiv \sum_{i=1}^{n(g)-1} \binom{r^s}{i} (g - 1_{\mathbb{Z}})^i \quad \bmod J \\ &\equiv 0 \quad \bmod J \end{aligned}$$

by construction of r and s. Moreover, it has to be noted in addition that the element $g_i = g^{q_i u_i}$ arising in the cited proof is contained in H and in $G \cap (1_{\mathbb{Z}/p^e\mathbb{Z}} + J_{\mathbb{Z}/p^e\mathbb{Z}})$ since g is and since both of these terms are subgroups, *cf.* step 1 above.

STEP 3. Now let $g \in U_p$, $p \in \sigma(R)$. Then for some $u \geq 0$, $g^{p^u} \in G \cap (1_{\mathbb{Z}} + J_{\mathbb{Z}})$. For $i \geq 1$ let K_i be the additive subgroup of $\mathbb{Z}(G)$ generated by the elements $(g - 1_{\mathbb{Z}})^j$, $j \geq i$. Now the equation

$$g^{p^u} - 1_{\mathbb{Z}} = \sum_{k=1}^{p^u} \binom{p^u}{k} (g - 1_{\mathbb{Z}})^k$$

shows that

$$p^u (g - 1_{\mathbb{Z}}) \in K_2 + J.$$

Therefore,

$$p^u K_i \subset (K_2 + J) K_{i-1} \subset K_2 K_{i-1} + J = K_{i+1} + J$$

as $K_{i-1} \subset I(H)$ and J is a right H-submodule. Thus for $n = n(g)$,

$$p^{(n-1)u} (g - 1_{\mathbb{Z}}) \in J + K_n \subset J + (g - 1_{\mathbb{Z}})^n \mathbb{Z}(H) \subset J$$

since $(g - 1_{\mathbb{Z}})^n \in J$. It follows that $g - 1_{\mathbb{Z}} \in W_p$. Hence by Theorem 4, $g - 1_{\mathbb{Z}} \in i_R^{-1} J_R$, *i.e.* $g \in G \cap (1_R + J_R)$. Since the latter term is a subgroup by step 1, we see that the product on the right-hand side of the decomposition in (i) is contained in $G \cap (1_R + J_R)$. So still regarding the right-hand side as an ordered product of subsets, the decomposition (i) is proved. It remains to show that the factors U_p are mutually commuting subgroups.

STEP 4. For proving that each factor U_p is a subgroup it is sufficient to apply the decomposition (i) just proved to a coefficient ring S which satisfies $\sigma(S) = \{p\}$ with the same number e as in R. Indeed, we then get $U_p = G \cap (1_S + J_S)$ which is a subgroup by step 1. Such a ring S can be obtained, for example, as a quotient of the polynomial ring $\mathbb{Z}[X]$, modulo the ideal generated by the element $p^e - p^{e+1} X$.

STEP 5. In order to show that the factors U_p mutually commute, let $p, q \in \sigma(R)$ and $a \in U_p$, $b \in U_q$. Then by step 3, $a - 1_{\mathbb{Z}} \in W_p$ and $b - 1_{\mathbb{Z}} \in W_q$, whence also $(a - 1_{\mathbb{Z}})b \in W_p$, noting that W_p is a right H-submodule since J is. Thus

$$(ab - 1_{\mathbb{Z}}) = (a - 1_{\mathbb{Z}})b + (b - 1_{\mathbb{Z}}) \in W_p + W_q.$$

Going through step 2 for $g = ab$ and $z_p = (a - 1_{\mathbb{Z}})b$, $z_q = (b - 1_{\mathbb{Z}})$ one finds elements $g_1 = g^{q_1 u_1} \in U_p$, $g_2 = g^{q_2 u_2} \in U_q$ such that $ab = g_1 g_2 = g_2 g_1$. Hence $U_p U_q \subset U_q U_p$ and, by symmetry, $U_q U_p = U_p U_q$. Thus the theorem is proved. ■

ACKNOWLEDGMENTS. I am indebted to the referee for pointing out an error in a former version of the paper and for suggesting a correction.

REFERENCES

1. M. Hartl, *Une approche homologique des sous-groupes et quotients de Fox*, C. R. Acad. Sci. Paris, to appear.

2. ______, *The third relative Fox dimension subgroup*, submitted.

3. ______, *The second Fox quotient of arbitrary group rings*, preprint.

4. S. Mac Lane, *Homology*, Springer Grundlehren **114**, Springer-Verlag, Berlin, Göttingen, Heidelberg, 1963.

5. M. M. Parmenter, I. B. S. Passi and S. K. Sehgal, *Polynomial ideals in group rings*, Canad. J. Math. **25**(1973), 1174–1182.

6. I. B. S. Passi, *Group Rings and Their Augmentation Ideals*, Lecture Notes in Math. **715**, Springer-Verlag, Berlin, Heidelberg, New York, 1979.

7. R. Sandling, *Dimension subgroups over arbitrary coefficient rings*, J. Algebra **21**(1972), 250–265.

8. ______, *Note on the integral group ring problem*, Math. Z. **124**(1972), 255–258.

U.R.A. au C.N.R.S. 751, et
Département de Mathématiques
Université de Valenciennes
le Mont Houy, BP 311
59304 Valenciennes Cedex
France