

EXISTENCE THEOREMS FOR OPTIMAL CONTROL OF QUASILINEAR PARABOLIC PARTIAL DIFFERENTIAL EQUATIONS

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Abstract

The question on existence of optimal controls for a system governed by quasilinear parabolic partial differential equations which is linear in the control variables is considered. It is shown that whenever the controls converge in the weak * topology of L^∞ , the corresponding solutions converge uniformly. Using this result and results on lower semi-continuity of integral functionals, existence theorems for optimal controls are proved.

1. Introduction

In this paper, we shall consider the existence of optimal controls for an optimal control problem in which a given cost functional will be minimized subject to a system governed by the following quasilinear parabolic partial differential equations with first boundary conditions

$$\begin{aligned}\mathcal{L}_u \phi(u)(x, t) &= 0, & (x, t) \in Q &= \Omega \times (0, T), \\ \phi(u)(x, t) &= \psi(x, t) & \text{on } \Gamma,\end{aligned}\tag{1.1}$$

where Ω is an open bounded connected subset of R^n ;

$$\Gamma = \{(x, 0): x \in \Omega\} \cup \partial\Omega \times [0, T];$$

$\partial\Omega$ denotes the boundary of Ω ; $u \in D$, D is the set of admissible controls to be defined later; ψ is given function defined on \bar{Q} and for each $u \in D$, the operator

\mathcal{L}_u is defined by

$$\mathcal{L}_u \phi = \phi_t - \sum_{i,j=1}^n \frac{\partial}{\partial x_j} (a_{ij}(x, t, \phi) \phi_{x_i}) + a(x, t, \phi) u(x, t), \quad (x, t) \in \bar{Q} \quad (1.2)$$

in which $a = (a_1, \dots, a_m)$, $u = (u_1, \dots, u_m)$ and $au = \sum_{i=1}^m a_i u_i$.

Related control problems to the problem considered in this paper can be found in [1, 4, 7, 8, 9, 10]. In [4, 7, 8, 10], the authors considered systems of ordinary differential equations. In [1], Berkovitz considered an abstract control problem with L^p -bounded controls ($p \geq 1$). However, the class of problems considered in [1] does not include the problem considered here. Indeed, in this paper the existence of solutions for the system is considered and the control restraint set depends also on the solution of the system, while in [1] the existence of solutions is not considered and the control restraint does not depend on the solution of the system. In [9], Noussair *et al.* considered a similar control problem to ours in which the existence of optimal controls was established using Filippov's implicit function lemma [5]. Our results are complementary to that given in [9] and overlap with the results in [1]. Further, our results can be considered as an extension of results in [4, 7, 8, 10] for ordinary differential systems to a quasilinear parabolic partial differential system.

2. Notations

Let R^n denote the n -dimensional Euclidean space. For any $x \in R^n$, let $|x| = (\sum_{i=1}^n |x_i|^2)^{1/2}$. The abbreviation a.e. means almost everywhere with respect to Lebesgue measure; $|E|$ denotes the Lebesgue measure of the measure set $E \subset R^s$ ($s \geq 1$), ∂B denotes the boundary of the set B and \bar{B} its closure.

A function $f: X \times Y \rightarrow R^n$ is said to be a Carathéodory function if $f(\cdot, t)$ is measurable on X for every $t \in Y$ and $f(s, \cdot)$ is continuous on Y for almost all $s \in X$.

Let G denote a bounded connected subset of R^n ; and denote by $C^l(G)$ ($1 \leq l < \infty$), the class of all l -times continuously differentiable real-valued functions on G . Also $C_0^l(G)$ denotes the class of all functions from $C^l(G)$ with compact support in G .

Let $L^q(G; R^m)$ ($m, q \geq 1$) be the Banach space of all measurable functions $z: G \rightarrow R^m$ that are q th-power integrable on G . Its norm is defined by

$$\|z\|_{q,G} = \left\{ \int_G |z(x)|^q dx \right\}^{1/q} \quad \text{for } 1 \leq q < \infty,$$

$$\|z\|_{\infty,G} = \text{ess sup}_{x \in G} |z(x)| \quad \text{for } q = \infty.$$

Let $L^{q,r}(G \times I; R^m)$ ($m, q, r \geq 1$) be the Banach space of all measurable functions $z: G \times I \rightarrow R^m$ such that $\|z\|_{q,r,G} < \infty$, where

$$\|z\|_{q,r,G \times I} = \left\{ \int_I \left(\int_G |z(x,t)|^q dx \right)^{r/q} dt \right\}^{1/r}, \quad 1 \leq q, r < \infty,$$

$$\|z\|_{q,\infty,G \times I} = \text{ess sup}_{t \in I} \|z(\cdot, t)\|_{q,G}, \quad 1 \leq q < \infty, \quad r = \infty,$$

$$\|z\|_{\infty,r,G \times I} = \left\{ \int_I (\|z(\cdot, t)\|_{\infty,G})^r dt \right\}^{1/r}, \quad q = \infty, \quad 1 \leq r < \infty,$$

and

$$\|z\|_{\infty,\infty,G \times I} = \text{ess sup}_{(x,t) \in G \times I} |z(x,t)| \quad \text{for } q = r = \infty.$$

The space $L^{q,q}(G \times I; R^m)$ will be denoted by $L^q(G \times I; R^m)$ and the norm $\|\cdot\|_{q,q,G \times I}$ by $\|\cdot\|_{q,G \times I}$. Also $L^q(G; R^1)$ ($L^{q,r}(G \times I; R^1)$, respectively) will be denoted by $L^q(G)$ ($L^{q,r}(G \times I)$).

We define $W_q^{2,1}(G \times I)$, $1 \leq q < \infty$, to be the Banach space of all functions z from $L^q(G \times I)$ having generalized derivatives of the form $(\partial^r/\partial t^r)(\partial^s/\partial x^s)$ with any r and s satisfying the inequality $2r + s \leq 2$. The norm in it is defined by

$$\|z\|_{q,G \times I}^{(2)} = \|z\|_{q,G \times I} + \left\| \frac{\partial z}{\partial t} \right\|_{q,G \times I} + \|z_x\|_{q,G \times I} + \|z_{xx}\|_{q,G \times I}.$$

Let $H^\lambda(\bar{G})$, $0 < \lambda < 1$, be the Banach space of all functions z that are continuous in \bar{G} and have a finite norm

$$|z|_{\bar{G}}^{(\lambda)} = \sup_{x, x' \in \bar{G}} \frac{|z(x) - z(x')|}{|x - x'|^\lambda} + \max_{x \in \bar{G}} |z(x)|;$$

and let $H^{\lambda,\lambda/2}(\overline{G \times I})$, $0 < \lambda < 1$, be the Banach space of all functions z that are continuous in $\overline{G \times I}$ and have a finite norm

$$|z|_{\overline{G \times I}}^{(\lambda)} = \|z\|_{\overline{G \times I}}^{(\lambda)} + \|z\|_{\overline{G \times I}}^{(\lambda/2)} + \max_{\overline{G \times I}} |z|,$$

where

$$\|z\|_{\overline{G \times I}}^{(\lambda)} = \sup_{(x,t), (x',t') \in \overline{G \times I}} \frac{|z(x,t) - z(x',t')|}{|x - x'|^\lambda},$$

$$\|z\|_{\overline{G \times I}}^{(\lambda/2)} = \sup_{(x,t), (x,t') \in \overline{G \times I}} \frac{|z(x,t) - z(x,t')|}{|t - t'|^{\lambda/2}}.$$

Finally, let $H^{\lambda,\lambda/2}(Q)$ be the set of all functions belonging to $H^{\lambda,\lambda/2}(\bar{Q}_1)$ for any closed subdomain $\bar{Q}_1 \subset Q$.

3. Preliminaries

In this section, an existence theorem for solutions is given in Theorem 3.1. Further, it will be shown that whenever the controls converge in the weak * topology of $L^\infty(Q; R^m)$ then the corresponding solutions converge uniformly on \bar{Q} . This result is presented in Lemma 3.3.

Let Ω be an open bounded connected subset of R^n . Let $T > 0$ be a fixed constant and let $Q = \Omega \times (0, T)$ and $\Gamma = \{(x, 0) : x \in \Omega\} \cup \partial\Omega \times [0, T]$.

Let U_K denote that set of all bounded measurable functions $u : \bar{Q} \rightarrow R^m$ such that $|u(x, t)| \leq K$ for all $(x, t) \in \bar{Q}$. In the following $K > 0$ denotes a positive fixed constant.

The following conditions will be referred to collectively as *conditions* (B_1) :

- (i) The functions $a_{ij}(x, t, \phi)$, $i, j = 1, \dots, n$, and $a_i(x, t, \phi)$, $i = 1, \dots, n$, are defined on $\bar{Q} \times R^1$ and take finite values for any finite ϕ , and $(x, t) \in \bar{Q}$;
- (ii) For all $(x, t) \in \bar{Q}$ and arbitrary $\phi \in R^1$, $\sum_{i,j=1}^n a_{ij}(x, t, \phi) \xi_i \xi_j \geq 0$ for all $\xi \in R^n$;
- (iii) There exist non-negative constants b_1 and b_2 such that

$$|\phi| |a(x, t; \phi)| \leq b_1 \phi^2 + b_2$$

for all $(x, t) \in \bar{Q}$; and

- (iv) $\psi \in H^\beta(\bar{Q})$ for some β , $0 < \beta < 1$.

Let M_K denote the positive constant defined by

$$M_K = \inf_{\lambda > b_1 K} \left(e^{\lambda T} \max_{\Gamma} \left\{ |\psi|, \sqrt{\frac{b_2 K}{\lambda - b_1 K}} \right\} \right). \tag{3.1}$$

DEFINITION 3.1. *The boundary $\partial\Omega$ of Ω is said to satisfy condition (A) if there exist constants $\alpha_0, \theta_0 > 0$ such that the inequality*

$$|\Omega_\rho^\xi| \leq (1 - \theta_0) |B_\rho|$$

holds for any ball B_ρ with centre on $\partial\Omega$ of radius $\rho \leq \alpha_0$ and any connected component Ω_ρ^ξ of Ω_ρ , where $\Omega_\rho = B_\rho \cap \Omega$.

The following conditions will be referred to as *conditions* (B_2) : For $(x, t) \in \bar{Q}$ and $|\phi| \leq M_K$,

- (i) The functions $a_{ij}(x, t, \phi)$, $i, j = 1, 2, \dots, n$, are continuous in ϕ and differentiable with respect to x and ϕ ; and the function $a(x, t, \phi)$ is continuous in ϕ ;

- (ii)
$$\nu |\xi|^2 \leq \sum_{i,j=1}^n a_{ij}(x, t, \phi) \xi_i \xi_j \leq \mu |\xi|^2$$

for all $\xi \in R^n$ and some constant $\nu, \mu > 0$;

$$|a_{ij}(x, t, \phi)| + \left| \frac{\partial a_{ij}(x, t, \phi)}{\partial \phi} \right| \leq \mu_2,$$

$$\left| \frac{\partial a_{ij}(x, t, \phi)}{\partial x_i} \right| \leq \psi_2(x, t),$$

$$|a[x, t, \phi]| \leq \psi_3(x, t),$$

in which $\|\psi_2\|_{4q,4r,Q} \leq \sqrt{\mu_1}$, $\|\psi_3\|_{2q,2r,Q} \leq \mu_1$ for some constants $\mu_1, \mu_2 > 0$ with

$$\frac{1}{r} + \frac{n}{2q} = 1 - \chi_1,$$

where

$$q \in \left[\frac{n}{2(1-\chi_1)}, \infty \right], \quad r \in \left[\frac{1}{1-\chi_1}, \infty \right], \quad 0 < \chi_1 < 1$$

for $n \geq 2$,

$$q \in [1, \infty], r \in \left[\frac{1}{1-\chi_1}, \frac{2}{1-2\chi_1} \right], \quad 0 < \chi_1 < \frac{1}{2}$$

for $n = 1$; and

(iii) $\partial\Omega$ satisfies the condition (A).

The existence of solutions for system (1.1) follows from Theorem 3.2 of Noussair *et al.* [9]. This theorem is quoted below without proof.

THEOREM 3.2. *Let the assumptions (B_1) and (B_2) be satisfied. Then for each $u \in U_K$, the system (1.1) has a solution $\phi^u \in H^{\alpha,\alpha/2}(\bar{Q}) \cap W^{2,1}_2(Q')$ with $\phi^u_x \in H^{\gamma,\gamma/2}(Q)$, where Q' is any interior subdomain of Q that is separated from Γ by a distance $d > 0$, and $0 < \gamma, \alpha < 1$ are constants determined by $K, n, \nu, \mu, \mu_1, \mu_2, q, M_K$ and r from condition $B_2(ii)$. Furthermore, there exist positive constants c and $c(d)$ such that for all $u \in U_K$, and for any subdomain Q' separated from Γ by a distance $d > 0$, the following estimates are valid*

$$|\phi^u|^{(\alpha)}_Q \leq c \tag{3.2}$$

and

$$|\phi^u_x|^{(\gamma)}_{Q'} + \|\phi^u_t\|_{2,Q'} + \|\phi^u_{xx}\|_{2,Q'} \leq c(d), \tag{3.3}$$

where the constants c and $c(d)$ are determined only by $n, M_K, \nu, \mu_1, \mu_2, \mu, q, r, K$, and the distance d .

LEMMA 3.3. *Let the hypotheses of Theorem 3.2 be satisfied. Let $\{u_n\}$ be a sequence in U_K such that $u_n \rightarrow u_0$ in the weak $*$ topology of $L^\infty(Q; R^m)$. Let $\{\phi^n\}$ denote the*

sequence of solutions of system (1.1) corresponding to u_n . Then there exist a subsequence of $\{\phi^n\}$ which we shall denote by the original sequence and a function $\phi^0 \in H^{\alpha, \alpha/2}(\bar{Q})$ such that

- (i) $\phi^n \rightarrow \phi^0$ uniformly on \bar{Q} ,
- (ii) ϕ^0 is a solution of system (1.1) corresponding to u_0 .

PROOF. It follows from Theorem 3.2 that for any subdomain Q' separated from Γ by a distance $d > 0$,

$$|\phi^n|_Q^{(\alpha)} \leq c, \tag{3.4}$$

$$|\phi^n|_x^{(\gamma)} + \|\phi_t^n\|_{2, Q'} + \|\phi_{xx}^n\|_{2, Q'} \leq c(d), \tag{3.5}$$

where the constants c and $c(d)$ depend only on $n, M_K, \mu, \mu_1, \mu_2, \nu, q, r$ and d .

By using the diagonalization process given in [9], it can be easily shown that there exist a function $\phi^0 \in H^{\alpha, \alpha/2}(\bar{Q}) \cap W_2^{2,1}(Q')$ with $\phi_x^0 \in H^{\gamma, \gamma/2}(Q)$ and a subsequence of $\{\phi^n\}$ such that

$$\left. \begin{aligned} \phi^n &\rightarrow \phi^0 \text{ uniformly on } \bar{Q}, \\ \phi_x^n &\rightarrow \phi_x^0 \text{ uniformly on } Q', \\ \phi_t^n &\rightarrow \phi_t^0 \text{ weakly in } L^2(Q'), \\ \phi_{xx}^n &\rightarrow \phi_{xx}^0 \text{ weakly in } L^2(Q'), \end{aligned} \right\} \tag{3.6}$$

where Q' is any closed subdomain of Q .

Let us first show that there exists a subsequence of $\{(\phi^n, u_n)\}$ which will be denoted by the original sequence such that

$$\int \int_Q a(x, t, \phi^n(x, t)) \cdot u_n(x, t) \cdot z(x, t) \, dx \, dt \rightarrow \int \int_Q a(x, t, \phi^0(x, t)) \cdot u_0(x, t) \cdot z(x, t) \, dx \, dt \tag{3.7}$$

for any $z \in L^2(Q)$. Since $\phi^n \rightarrow \phi^0$ uniformly in \bar{Q} , it is clear that $(\phi^n, u_n) - (\phi^0, u_n) \rightarrow 0$ in measure in Q . Further, (ϕ^n, u_n) and (ϕ^0, u_n) are bounded in the norm of the space $L^{\sigma_1}(Q) \times L^{\sigma_2}(Q)$, $\sigma_1, \sigma_2 \geq 1$. Hence, it follows from Lemma 3.1 of Berkovitz [1, p. 524] that

$$(a(x, t, \phi^n(x, t)) - a(x, t, \phi^0(x, t))) \cdot u_n(x, t) \rightarrow 0$$

in measure in Q . Therefore, there exists a subsequence of $\{(\phi^n, u_n)\}$ such that

$$\Psi^n(x, t) = (a(x, t, \phi^n(x, t)) - a(x, t, \phi^0(x, t))) \cdot u_n(x, t) \rightarrow 0 \tag{3.8}$$

a.e. in Q . From the condition $B_2(ii)$, it follows that for $(x, t) \in \bar{Q}$,

$$|\Psi^n(x, t)| \leq 2K\psi_3(x, t),$$

where K is the bound on u_n and $\psi_3 \in L^1(Q)$. Thus, it follows from the Lebesgue dominated convergence theorem that

$$\lim_{n \rightarrow \infty} \iint_Q \Psi^n(x, t) \cdot z(x, t) \, dx \, dt = 0. \tag{3.9}$$

Since $u_n \rightarrow u_0$ in the weak * topology of $L^\infty(Q; R^m)$, it follows that

$$\lim_{n \rightarrow \infty} \iint_Q a(x, t, \phi^0(x, t)) \cdot z(x, t) \cdot (u_n(x, t) - u_0(x, t)) \, dx \, dt = 0. \tag{3.10}$$

Hence, (3.7) follows from (3.9) and (3.10). Furthermore, by virtue of (3.6) and (3.7) it is easy to see that ϕ^0 is a solution of the system (1.1) corresponding to u_0 . This completes the proof.

4. Existence of optimal controls

In this section, we consider the problem of existence of an optimal control. The exact formulation of the problem is given below and existence proofs in Theorems 4.1 and 4.3.

Let B_K denote the m -dimensional sphere in R^m with diameter K , where K is a fixed positive constant. For each $(x, t, \psi) \in \bar{Q} \times R^1$, let $U(x, t, \psi)$ be a non-empty subset of B_K .

A bounded measurable function $u: Q \rightarrow B_K$ is called an *admissible control* if

- (i) there exists a solution ϕ^u of system (1.1) corresponding to u ; and
- (ii) $u(x, t) \in U(x, t, \phi^u(x, t))$ a.e. in Q .

Let D denote the set of all admissible controls, and let Δ denote the set of all pairs (ϕ^u, u) , where $u \in D$ and ϕ^u is a corresponding solution of system (1.1).

Let the cost functional J be defined on Δ by

$$J[\phi^u, u] = \iint_Q g(x, t, \phi^u(x, t), u(x, t)) \, dx \, dt, \tag{4.1}$$

where $g(t, x, \psi, u)$ is a real-valued function on $Q \times R^1 \times B_K$ and satisfies the following conditions which will be referred to as *conditions (H)*:

- (i) $g(x, t, \psi, u)$ is continuous in (ψ, u) for a.e. $(x, t) \in Q$, measurable in (x, t) for each $(\psi, u) \in R^1 \times B_K$ and convex in u for each $(x, t, \psi) \in Q \times R^1$; and
- (ii) there exists a function $z \in L^1(Q)$ such that

$$g(x, t, \psi, u) \geq z(x, t)$$

for a.e. $(x, t) \in Q$ and for all $(\psi, u) \in R^1 \times B_K$.

We now state our control problem “P”: subject to system (1.1), find an admissible pair $(\phi^{u_0}, u_0) \in \Delta$ such that

$$J[\phi^{u_0}, u_0] \leq J[\phi^u, u] \tag{4.2}$$

for all $(\phi^u, u) \in \Delta$. Such an admissible pair will be called an *optimal pair*.

THEOREM 4.1. *Consider the problem P. Suppose that the following assumptions are satisfied.*

- (i) *The conditions (B_1) , (B_2) and (H) hold;*
- (ii) *for each $(x, t, \psi) \in \bar{Q} \times R^1$, $U(x, t, \psi)$ is closed and convex; and*
- (iii) *the mapping U is upper semicontinuous with respect to inclusion (u.s.c.i.) on $\bar{Q} \times R^1$.*

Then there exists an optimal pair $(\phi^{u_0}, u_0) \in \Delta$ for the problem P.

PROOF. From condition H(ii), it follows that $\inf\{J[\phi^u, u]: (\phi^u, u) \in \Delta\} = \sigma > -\infty$. Let $\{(\phi^{u_k}, u_k)\}_{k=1}^\infty$ be a minimizing sequence such that

$$\lim_{k \rightarrow \infty} J[\phi^{u_k}, u_k] = \sigma. \tag{4.3}$$

Since $u_k(x, t) \in B_{\mathbb{R}^m}$, $k = 1, 2, \dots$, there exist a subsequence of $\{u_k\}$ and a function $u_0 \in L^\infty(Q; R^m)$ such that

$$u_k \rightarrow u_0 \text{ in the weak } * \text{ topology of } L^\infty(Q; R^m). \tag{4.4}$$

By Lemma 3.3, there exists a solution ϕ^{u_0} of system (1.1) corresponding to u_0 and a subsequence $\{\phi^{u_k}\}$ such that

$$\phi^{u_k} \rightarrow \phi^{u_0} \text{ uniformly in } \bar{Q}. \tag{4.5}$$

We show that $u_0(x, t) \in U(x, t, \phi^{u_0}(x, t))$ a.e. in Q . For this, let

$$Q_1^k = \{(x, t) \in Q: u_k(x, t) \notin U(x, t, \phi^{u_k}(x, t))\}.$$

Let $Q_1 = \bigcup_{k=1}^\infty Q_1^k$. Clearly $|Q_1| = 0$. Let $Q_0 = Q \setminus Q_1$. Let $(x_0, t_0) \in \bar{Q}_0$ be any regular point of the function u_0 . We first show that

$$u_0(x_0, t_0) \in U(x_0, t_0, \phi^{u_0}(x_0, t_0)). \tag{4.6}$$

Since U is u.s.c.i., it follows that for a given $\varepsilon > 0$, there exists a $\delta = \delta(\varepsilon) > 0$ such that

$$U(x, t, \phi) \subset U^\varepsilon(x_0, t_0, \phi^{u_0}(x_0, t_0)) \tag{4.7}$$

whenever $|(x, t) - (x_0, t_0)| < \delta$ and $|\phi - \phi^{u_0}(x_0, t_0)| < \delta$, where U^ε denotes the closed ε -neighborhood of U .

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From (4.5), it follows that there exists a positive integer k_0 such that for all $k > k_0$ and $(x, t) \in \bar{Q}$,

$$|\phi^{u_k}(x, t) - \phi^{u_0}(x, t)| < \delta/2. \tag{4.8}$$

Further, since ϕ^{u_0} is continuous on \bar{Q} , there exists a $\delta_1 > 0$ such that

$$|\phi^{u_0}(x, t) - \phi^{u_0}(x_0, t_0)| < \delta/2 \tag{4.9}$$

whenever $|(x, t) - (x_0, t_0)| < \delta_1$.

Let $\delta_2 = \min \delta, \delta_1$. Then, it follows from (4.7), (4.8) and (4.9) that for all $k > k_0$

$$U(x, t, \phi^{u_k}(x, t)) \subset U^\varepsilon(x_0, t_0, \phi^{u_0}(x_0, t_0)) \tag{4.10}$$

whenever $|(x, t) - (x_0, t_0)| < \delta_2$.

Let E be any measurable subset of Q such that $(x_0, t_0) \in E$ and $|E| < \delta_2$. Then, since $U^\varepsilon(t, x, \phi)$ is convex and closed, it follows from (4.10) that for all $k > k_0$,

$$\frac{1}{|E|} \int \int_E u_k(x, t) dx dt \in U^\varepsilon(x_0, t_0, \phi^{u_0}(x_0, t_0)).$$

Since $U^\varepsilon(x, t, \phi)$ is closed, it follows from (4.4) that

$$\frac{1}{|E|} \int \int_E u_0(x, t) dx dt \in U^\varepsilon(x_0, t_0, \phi^{u_0}(x_0, t_0)),$$

and consequently,

$$u_0(x_0, t_0) = \lim_{|E| \rightarrow 0} \left(\frac{1}{|E|} \int \int_E u_0(x, t) dx dt \right) \in U^\varepsilon(x_0, t_0, \phi^{u_0}(x_0, t_0)).$$

Since ε was arbitrary and $U(x_0, t_0, \phi^{u_0}(x_0, t_0))$ is closed, it follows that (4.6) holds. Further, since almost every point in Q is a regular point of u_0 , we obtain that

$$u_0(x, t) \in U(x, t, \phi^{u_0}(x, t)) \text{ a.e. in } Q.$$

Hence, $(\phi^{u_0}, u_0) \in \Delta$. Further, it follows from (4.4), (4.5), assumptions (H) and Theorem 1 of [2, p. 53] that

$$\liminf_{k \rightarrow \infty} J[\phi^{u_k}, u_k] \geq J[\phi^{u_0}, u_0]. \tag{4.11}$$

Therefore, it follows from (4.3) and (4.11) that

$$J[\phi^{u_0}, u_0] = \sigma,$$

That is (ϕ^{u_0}, u_0) is an optimal pair. This completes the proof.

REMARK 4.2. Theorem 4.1 remains valid when the conditions (H) on g is replaced by the weaker conditions (H') (6, p. 524]:

- (i) $g(x, t, \psi, u)$ is lower semicontinuous in (ψ, u) and convex in u for each $(x, t, \psi) \in Q \times R^1$; and
- (ii) there exist functions $\gamma, \lambda \in L^1(Q)$ such that for all $\psi \in R^1, u \in B_K$ and $(x, t) \in Q$,

$$g(x, t, \psi, u) \geq \gamma(x, t)|u| + \lambda(x, t).$$

We now consider the case when the cost integrand g and the control restraint set U satisfy the following conditions which will be referred to as *conditions (G)*:

- (i) $g(x, t, \psi, u)$ is continuous in all its variables and convex in u for each $(x, t, \psi) \in Q \times R^1$;
- (ii) for each $(x, t, u) \in Q \times B_K$ and $\psi, \psi' \in R^1$,

$$|g(x, t, \psi, u) - g(x, t, \psi', u)| \leq \beta(|\psi - \psi'|),$$

where β is a non-negative function defined on $[0, \infty)$ satisfying

$$\lim_{\delta \rightarrow 0^+} \beta(\delta) = 0;$$

- (iii) U is independent of ψ and for each $(x, t) \in \bar{Q}$, $U(x, t)$ is closed and convex.

THEOREM 4.3. Consider the problem P . Suppose that the conditions $(B_1), (B_2)$ and (G) hold. Then, there exists an optimal pair $(\phi^{u_0}, u_0) \in \Delta$ for the problem P .

PROOF. From estimate (3.2) and the condition $G(i)$, we can easily show that $\inf\{J[\phi^u]: (\phi^u, u) \in \Delta\} = \sigma_0 > -\infty$. As in the proof of Theorem 4.1, let $\{(\phi^{u_k}, u_k)\}$, u_0 and ϕ^{u_0} be, respectively, a minimizing sequence, a function in $L^\infty(Q; R^m)$ and a solution of system (1.1) corresponding to u_0 such that

$$\lim_{k \rightarrow \infty} J[\phi^{u_k}, u_k] = \sigma_0, \tag{4.12}$$

$$u_k \rightarrow u_0 \text{ in the weak * topology of } L^\infty(Q; R^m); \tag{4.13}$$

and

$$\phi^{u_k} \rightarrow \phi^{u_0} \text{ uniformly on } \bar{Q}. \tag{4.14}$$

Using condition $G(iii)$ and an argument similar to that given in the proof of Theorem 4.1, we can easily show that $(\phi^{u_0}, u_0) \in \Delta$.

From (4.1), it follows that

$$\begin{aligned} J[\phi^{u_k}, u_k] &= \int \int_Q g(x, t, \phi^{u_k}(x, t), u_k(x, t)) dx dt \\ &= \int \int_Q (g(x, t, \phi^{u_k}(x, t), u_k(x, t)) - g(x, t, \phi^{u_0}(x, t), u_k(x, t))) dx dt \\ &\quad + \int \int_Q g(x, t, \phi^{u_0}(x, t), u_k(x, t)) dx dt. \end{aligned} \quad (4.15)$$

Using the condition G(ii), it follows from (4.14) that

$$\lim_{k \rightarrow \infty} \int \int_Q (g(x, t, \phi^{u_k}(x, t), u_k(x, t)) - g(x, t, \phi^{u_0}(x, t), u_k(x, t))) dx dt = 0. \quad (4.16)$$

Since g is continuous in its all variables and convex in u , it follows from (4.13) and the theorem of [3, p. 213] that

$$\liminf_{k \rightarrow \infty} \int \int_Q g(x, t, \phi^{u_0}(x, t), u_k(x, t)) dx dt \geq \int \int_Q g(x, t, \phi^{u_0}(x, t), u_0(x, t)) dx dt. \quad (4.17)$$

Hence, it follows from (4.15), (4.16) and (4.17) that

$$\liminf_{k \rightarrow \infty} J[\phi^{u_k}, u_k] \geq J[\phi^{u_0}, u_0].$$

Therefore, $J[\phi^{u_0}, u_0] = \sigma_0$, that is (ϕ^{u_0}, u_0) is an optimal pair. This completes the proof.

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