Can. J. Math., Vol. XXIV, No. 1, 1972, pp. 5–16

COMPACT INDUCED REPRESENTATIONS

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1. Introduction. In [**15**; **16**; **17**], Horst Leptin introduced what he called generalized group algebras. These Banach *-algebras are formed by letting a locally compact group G act on a Banach *-algebra A both by *-automorphisms and by a cocycle with values in the multiplier algebra, $M(A)$, of A. We will review the precise construction later, but for now we remark that examples include the group algebra of a group extension, the covariance algebras of quantum field theory, the "projective group algebras" of a group G (that is, for each complex-valued cocycle λ, called a multiplier in the literature, the Banach *-algebra whose nondegenerate *-representations are in bijective correspondence with the λ-projective representations of G), and the twisted group algebras of Edwards and Lewis [**8**; **9**].

Essentially the same algebras (some minor technical differences are involved) were studied in [**4**] and called twisted group algebras. We will follow this construction throughout this paper. Representations of a twisted group algebra induced from the object algebra A were defined in [**4**], and in the group extension example such representations correspond to those induced (in the sense of Mackey) from a closed, normal subgroup. In [**5**], the authors and H. A. Smith gave necessary and sufficient conditions for an induced representation, in the above sense, to be compact; that is, to consist entirely of compact operators. Irreducibility was not assumed either for the induced representation, or the representation from which we induced. In this paper, we show (§ 3) that if irreducibility is assumed for the induced representation, then what appear to be stronger conditions than those in [**5**] are in fact necessary and sufficient for the compactness of that induced representation. Specifically, if the representation from which we induce is denoted by π (π must be irreducible if the representation induced from it is), and if the group G is allowed to act on the dual space $\hat{A}$ of A in the natural way, our conditions say that the orbit of π in $\hat{A}$ (under the action of G) should be a closed set, and the natural mapping of G onto this orbit should be a homeomorphism.

In § 4 of the paper, we consider the case when the induced representation is no longer irreducible. In this case, there may be a non-trivial stability subgroup of G for π (this is the set of all x in G which leave π fixed). We show that if the dual space of A is Hausdorff, then necessary and sufficient conditions will consist of those previously given together with compactness of the stability subgroup.

In § 5 we interpret all these results for group extensions.

Received January 8, 1971 and in revised form, September 7, 1971.

2. Review of needed background material. In the course of proving the main theorem of this paper, we will need to use a number of results concerning direct integrals of Hilbert spaces and direct integral decomposition of representations. These results together with other miscellaneous definitions and results are collected together in this section. In the rest of this paper, group will always mean locally compact second countable group, algebra will mean separable Banach *-algebra with bounded two-sided approximate identity, and representation of an algebra A will always mean non-degenerate *-representation of A on a separable Hilbert space.

We now review some basic facts and needed theorems from direct integral theory. As a general reference, see [**6**, Ch. 8; **7**, Ch. 2]. For the theory of standard and analytic measure spaces, see [**18**]. Let μ be a Borel measure on an analytic Borel space X. Suppose that $[(H_x), x \in X]$ is a Borel family of Hilbert spaces and

$$\mathscr{H} = \int_X^{\oplus} H_x d\mu(x)$$

is the direct integral of this family (see [**7**, Ch. 2]). Crudely speaking, $\mathscr{H}$ is a Hilbert space composed of sections $x \to \xi_x$ ($\xi_x \in H_x$) which are measurable and norm square integrable with respect to μ, where sufficiently many such sections are chosen so that their values over each x are dense in H_x, and they form a maximal set of sections with these properties. An example would be $L^2(X, H, \mu)$, the μ-square integrable functions from X to a Hilbert space H. Now suppose also that we have a set $[(\pi_x), x \in X]$ of representations of an algebra A, such that each π_x represents A on H_x and such that, for each a in A, the family $[(\pi_x(a)), x \in X]$ is a Borel measurable family of operators in the sense of [**7**, Ch. 2]. Pointwise multiplication of this field of operators with the vector fields of $\mathscr{H}$ yields an operator on $\mathscr{H}$ which is written

$$\int_X^{\oplus} \pi_x(a) d\mu(x).$$

The collection of such operators with operations defined in the obvious way makes up a representation of A on $\mathscr{H}$ which is denoted

$$\pi = \int_X^{\oplus} \pi_x d\mu(x),$$

and called a direct integral decomposition of π. In a natural way (by pointwise multiplication), the complex algebra $L^\infty(X, \mu)$ acts on $\mathscr{H}$ as an algebra of operators. Since this correspondence between functions and operators is an isometric isomorphism, we shall identify the two sets and think of $L^\infty(X, \mu)$ as a set of operators on $\mathscr{H}$. The following facts are known (we retain the above notations):

Remark 1. Let $\mathscr{A}$ be the weak operator closure of the range of π. Then $\mathscr{A}$ is a von Neumann algebra whose commutant will be denoted $\mathscr{A}'$.

(*a*) $L^\infty(X, \mu)$ is always in $\mathscr{A}'$, and $L^\infty(X, \mu)$ is a maximal abelian subalgebra of $\mathscr{A}'$ if and only if π_x is an irreducible representation for μ-almost all x in X. [**6**; Lemma 8.5.1].

(*b*) $L^\infty(X, \mu)$ contains the center of $\mathscr{A}$ (that is, $\mathscr{A} \cap \mathscr{A}'$) if and only if π_x is a factor representation (its image generates a von Neumann algebra which is a factor) for μ-almost all x in X. [**6**, Lemma 8.4.1]. If $L^\infty(X, \mu) = \mathscr{A} \cap \mathscr{A}'$, then we have a central direct integral decomposition.

Definition 1. If $\mathscr{A}'$ is abelian then we say that π is multiplicity free.

Remark 2. Suppose that

$$\pi = \int_X^\oplus \pi_x d\mu(x)$$

is a type I representation and this is the central direct integral decomposition. Then μ-almost all π_x are type I factor representations [**6**, Proposition 8.4.8].

PROPOSITION 1. *Suppose that $\mathscr{H}$ is a direct integral*

$$\int_X^\oplus H_x d\mu(x)$$

on an analytic Borel space X with Borel measure μ. Let

$$\pi = \int_X^\oplus \pi_x d\mu(x)$$

be a corresponding direct integral decomposition on X, and suppose that π is type I *and μ-almost all π_x are irreducible. Then there exists an analytic Borel space Y, a Borel measure ν on Y, and direct integrals*

$$\mathscr{H}' = \int_Y^\oplus H_y' d\nu(y)$$

and

$$\rho = \int_Y^\oplus \rho_y d\nu(y)$$

such that:

(1) $\rho \cong \pi$, *and the decomposition over Y is central* (*see Remark* 1);

(2) *For ν-almost all y in Y, there is a point $\beta(y)$ in X and a* (*possibly infinite*) *cardinal $n(y)$ such that ρ_y is unitarily equivalent with $n(y)\pi_{\beta(y)}$.*

Proof. The proof of this proposition is essentially Mackey's. It is a very slightly altered version of the proof of [**18**, Theorem 10.5].

As Mackey shows in the above reference, there is a measurable equivalence relation in X, a measure ν in the set Y of equivalence classes, and for each y in Y a measure ν_y in the class y such that

$$\int_X^\oplus \pi_x d\mu(x) = \int_Y^\oplus \int_X^\oplus \pi_x d\nu_y(x) d\nu(y)$$

https://doi.org/10.4153/CJM-1972-002-3 Published online by Cambridge University Press

(each ν_y is considered to be an analytic measure on X), and the outside decomposition is central. This is done by noting that $L^\infty(X, \mu)$ is maximal abelian in the commutant of π (Remark 1) and so contains the center. Then a point in Y is taken to be the equivalence class obtained by saying $x_1 \cong x_2$ if x_1 and x_2 are not separated by functions in the center. In a natural way,the functions in the center correspond to $L^\infty(Y, \nu)$ for a natural measure ν. Now, Remarks 1 and 2 above show that for ν-almost all y in Y the integral

$$\rho_y = \int_X^\oplus \pi_x d\nu_y(x)$$

is a type I factor and so is a multiple of a unique (up to equivalence) irreducible representation which must be one of the π_x in the support of ν_y. Thus, $\rho_y \cong n(y)\pi_{\beta(y)}$ for some $\beta(y) \in X$, for ν_y-almost all $y \in Y$.

Remark 3. Suppose that (X_1, μ_1) and (X_2, μ_2) are two analytic Borel spaces with Borel measures. Suppose that

$$\mathscr{H}_i = \int_{X_i}^\oplus H_x{}^i d\mu_i(x)$$

are direct integrals of Hilbert spaces over these Borel spaces and that

$$\pi_i = \int_{X_i}^\oplus \pi_x{}^i d\mu_i(x)$$

are direct integrals of representations of A, $i = 1, 2$. Finally, suppose that there is an isomorphism U of $\mathscr{H}_1$ onto $\mathscr{H}_2$ carrying π_1 onto π_2 and $L^\infty(X_1, \mu_1)$ onto $L^\infty(X_2, \mu_2)$. Then there must exist:

(i) Borel sets N_i of μ_i measure zero ($i = 1, 2$),
(ii) A Borel isomorphism η of $X_1 - N_1$ onto $X_2 - N_2$ which transforms μ_1 into a measure $\tilde{\mu}_2$ equivalent with μ_2,
(iii) An isomorphism U_x from $H_x{}^1$ to $H^2{}_{\eta(x)}$, for each x in $X_1 - N_1$, which transforms $\pi_x{}^1$ into $\pi^2{}_{\eta(x)}$,

such that U is the composition of the isomorphism

$$\int_{X_1}^\oplus U_x d\mu(x)$$

(with the expected meaning, see [**7**, Ch. 2]) with the natural isomorphism of

$$\int_{X_2}^\oplus H_x{}^2 d\tilde{\mu}_2(x)$$

on $\mathscr{H}_2$.

This important theorem was proved by von Neumann and is stated and proved in [**6**, Proposition 8.2.4], with appropriate references.

A left centralizer on an algebra A is a bounded linear mapping m from A to A such that for all a, b in A, $m(ab) = (ma)b$. A double centralizer on A is a pair (m_1, m_2) of bounded linear mappings from A to A such that for all a, b in A,

https://doi.org/10.4153/CJM-1972-002-3 Published online by Cambridge University Press

$a(m_1b) = (m_2a)b$. The double centralizers form an algebra $M(A)$ "containing" A as a closed two-sided ideal. If $m = (m_1, m_2)$ is in $M(A)$, then m_1 (respectively, m_2) is a left (respectively, right) centralizer (given our assumptions on A) and we write $m_1(a) = ma$ and $m_2(a) = am$. The left centralizers also form an algebra $M_L(A)$. If π is a representation of A on H, it extends uniquely to a representation of $M_L(A)$ on H and (if we also denote the extended representation by π) $\pi(M_L(A))$ is contained in the weak operator closure of $\pi(A)$. If ϕ is an isometric isomorphism of an algebra A_1 with an algebra A_2, ϕ extends uniquely to an isometric isomorphism (also denoted by ϕ) of $M(A_1)$ on $M(A_2)$. For all these facts, see [**14**]. The strict topology on $M(A)$ is that given by the seminorms $m \to ||ma||$, and $m \to ||am||$, a in A (see [**2**]).

We now briefly review the construction of twisted group algebras and induced representations. Let G be a group, A an algebra, and μ and Δ a left Haar measure for G and the corresponding modular function, respectively. Let $L^1(A, G)$ be the Banach space of Bochner integrable A-valued functions on G. Let T be a Borel map from G to the set $\mathrm{Aut}^1(A)$ of isometric *-automorphisms of A ($\mathrm{Aut}^1(A)$ has the pointwise convergence topology), and let α be a Borel map from $G \times G$ to $U(A)$ (with the strict topology) such that:

(1) T and α are continuous in a neighborhood of the identity in G and $G \times G$, respectively.

(2) (T, α) is a twisting pair for (G, A) (see [**4**] for the definition). Note that the smoothness requirements imposed on (T, α) are stronger than those imposed in [**4**] and [**5**]. This will be explained in the proof of Theorem 1. We then define multiplication and involution on $L^1(A, G)$ as follows:

$$(3) \qquad (f \circ g)(x) = \int_G f(y)(T(y)g(y^{-1}x))\alpha(y, y^{-1}x)d\mu(y).$$

$$(4) \qquad f^*(x) = \alpha(x, x^{-1})^*(T(x)f(x^{-1})^*)\Delta(x^{-1}).$$

Here, x, y are in G and f, g are in $L^1(A, G)$. The resulting algebra is denoted $L^1(A, G; T, \alpha)$.

Now let π be a representation of A on H, and let $\mathscr{H} = L^2(G, H, \mu)$. We then define, for each x in G and a in A, operators $U^\pi(x)$ and $\tilde{\pi}(a)$ on $\mathscr{H}$ as follows:

$$(5) \qquad (U^\pi(x)h)(y) = \pi(\alpha(y, x))h(yx)\Delta(x)^{1/2}.$$

$$(6) \qquad (\tilde{\pi}(a)h)(x) = \pi(T(x)a)h(x).$$

Here, h is in $\mathscr{H}$, a in A, x, y in G. We can apply π to α by previous remarks on extending representations. Now, if f is in $L^1(A, G; T, \alpha)$, we define the representation Π induced from π by:

$$\Pi(f) = \int_G \tilde{\pi}(f(x))U^\pi(x)d\mu(x).$$

(For complete details see [**4**].) We will need the following remarks:

Remark 4. The representation $\tilde{\pi}$ of A on $\mathscr{H}$ defined by equation (6) above (it is trivial to verify that this is a representation) is the direct integral

$$\int_G^{\oplus} \pi_x d\mu(x),$$

where for all x in G, $\pi_x(a) = \pi(T(x)a)$. This is immediate from the construction of $\tilde{\pi}$.

Remark 5. (We use notation analogous to that used above.) Let π_1 and π_2 be representations of A on H_1 and H_2 respectively, and let Π_1 and Π_2 be the corresponding induced representations of $L^1(A, G; T, \alpha)$ on $\mathscr{H}_1$ and $\mathscr{H}_2$ respectively. Then if Π_1 and Π_2 are unitarily equivalent, so are the representations $\tilde{\pi}_1$ and $\tilde{\pi}_2$ of A.

Proof. It is shown in [**4**] that A can be embedded in the left centralizer algebra of $L^1(A, G; T, \alpha)$ and that if the induced representation Π_i is extended to A by first extending to $M_L(L^1(A, G; T, \alpha)$ and then restricting to A, the resulting representation is precisely $\tilde{\pi}_i$ $(i = 1, 2)$. It is easily seen that if two representations of an algebra are unitarily equivalent, so are the extensions of these representations to the left centralizer algebras (for example, one can examine the construction of the extension; see [**14**, § 9]). The result follows from this.

3. The main result. Suppose that (T,α) is a twisting pair for (A, G) as above. If $B \subseteq \mathrm{Aut}^1(A)$ is the set of inner automorphisms of A by unitaries in $M(A)$, and p is the natural projection of $\mathrm{Aut}^1(A)$ on the quotient group $C = \mathrm{Aut}^1(A)/B$, then pT is a group homomorphism. Assume that pT is continuous. This is true of all the examples of [**4**], including the group extension example. Since C acts in a natural way on the dual space $\hat{A}$ of A (with the hull-kernel topology) as a topological transformation group, it follows that G also acts on $\hat{A}$ in this way (x acting on π, written $x \cdot \pi$, equals $\pi_{x^{-1}}$). We will not distinguish between an irreducible representation and its unitary equivalence class as long as no confusion results. We let π be in $\hat{A}$, and denote by G_π and $\theta(\pi)$ respectively, the stability subgroup of π in G and the orbit of π in $\hat{A}$ under the action of G. Finally, we define a map ψ_π from G to $\hat{A}$ by:

$$\psi_\pi(x) = \pi_{x^{-1}} = x \cdot \pi.$$

THEOREM 1. *Suppose that the representation* Π *of* $L^1(A, G; T, \alpha)$ *induced from* π *is irreducible. Then the following statements are equivalent:*

(1) Π *is compact.*

(2) π *is compact and for all* a *in* A, *the function* γ_a *from* G *to* $\mathbf{R}^+$, *given by* $\gamma_a(x) = ||\pi_{x^{-1}}(a)||$, *vanishes at infinity on* G.

(3) $\theta(\pi)$ *is closed in* $\hat{A}$ *and* ψ_π *is a homeomorphism of* G *onto* $\theta(\pi)$.

(4) ψ_π *is a proper (equivalently closed) injection map from* G *to* $\hat{A}$ *in the sense of Bourbaki* (see [**1**, Ch. 1, § 10]).

Proof. ψ_π is one-to-one because the irreducibility of Π implies that G_π trivial (see [**4**, Proposition 4.8]). Thus, (3) $\Leftrightarrow$ (4) follows from [**1**, Ch. 1, § 10, no. 1, Proposition 2].

(3) $\Rightarrow$ (2). First of all, (3) implies that $\{\pi\}$ is closed in $\hat{A}$, so π is compact. Now, for a given a in A and $\epsilon > 0$, the set $K(a, \epsilon)$ of all τ in $\hat{A}$ for which $||\tau(a)|| \geqq \epsilon$ is a compact (not necessarily Hausdorff) set in $\hat{A}$ (see [**6**, Proposition 3.3.7]). Now, $\{x|\gamma_a(x) \geqq \epsilon\}$ is just $\psi_\pi^{-1}(K(a, \epsilon))$, and so is compact [**1**, Ch. 1, § 10, No. 2, Proposition 6]. Since a and ϵ are arbitrary, we get (2).

(2) $\Rightarrow$ (1) is a special case of [**5**, Theorem 4.1].

(1) $\Rightarrow$ (3). We first show that $\theta(\pi)$ is closed. We are given that Π is irreducible and compact. Suppose that σ is in $\hat{A}$, and $\sigma \in (\theta(\pi))^-$ (closure of $\theta(\pi)$). Let $\tilde{\sigma}$ and Σ have the same relationship with σ as $\tilde{\pi}$ and Π have with π. Now, Fell has investigated weak containment for induced representations of groups in [**10**] and [**11**], and in [**3**] it is shown that his main result in [**10**] goes over to twisted group algebras. (Specifically, inducing preserves weak containment.) In particular, $\{\Sigma\}$ is weakly contained in the set of representations induced from representations in $\theta(\pi)$ and (by [**4**, Theorem 4.4.a]) the latter set is (up to unitary equivalence) just $\{\Pi\}$. On the other hand, since Π is assumed to be compact and irreducible, it follows from [**13**, Theorem **4**] that the set $\{\Pi\}$ is closed in the dual of $L^1(A, G; T, \alpha)$. By [**6**, Theorem 8.5.2], we can write Σ as the direct integral of irreducible representations of $L^1(A, G; T, \alpha)$, and by [**10**, Theorem 3.1] the set of irreducibles essentially involved in this decomposition is weakly equivalent with $\{\Sigma\}$ and so weakly contained in $\{\Pi\}$. The fact that the latter set is closed then shows that Σ is unitarily equivalent with a (possibly infinite) multiple $n\,\Pi$ of Π.

We now know that Σ is unitarily equivalent with the representation induced from $n\pi$, and Remark 5 of § 2 then shows us that $\tilde{\sigma}$ is unitarily equivalent with $\widetilde{n\pi} = n\tilde{\pi}$. (Inducing preserves direct sums.) The fact that π is type I and G_π is trivial, coupled with a technique of Blattner (explicitly given in the twisted group algebra setting in the proof of [**4**, Theorem **4**.14]) shows that $\tilde{\pi}$ is multiplicitly free. Thus, $\tilde{\sigma}$ (being equivalent with a multiple of $\tilde{\pi}$) is type I and, applying Proposition 1 of § 2, we get a central decomposition

$$\tilde{\sigma} = \int_Y^{\oplus} n(y)\sigma_{\beta(y)}d\nu(y),$$

with respect to a Borel measure on an analytic Borel space. Our conditions on G imply that G with Haar measure is an analytic (in fact, standard) Borel space. We note that, since

$$\tilde{\sigma} = \int_Y^{\oplus} n(y)\sigma_{\beta(y)}d\nu(y)$$

and

$$n\tilde{\pi} = \int_G^{\oplus} n\pi_x d\mu(x)$$

are central decompositions (the latter since $\tilde{\pi}$ is multiplicity free), the unitary equivalence between them takes $L^\infty(Y, \nu)$ onto $L^\infty(G, \mu)$ (it must take the center of the algebra generated by $\tilde{\sigma}$ onto the center of the algebra generated by $n\tilde{\pi}$). Thus, all the conditions necessary to apply Remark 3 of § 2 hold, and that remark implies that for some x and some y, $n\pi_x$ and $n(y)\sigma_{\beta(y)}$ are unitarily equivalent. Since π and σ are irreducible, σ is equivalent to some π_x, and so σ is in $\theta(\pi)$. Thus, $\theta(\pi)$ is closed.

To prove that the map ψ_π is a homeomorphism onto $\theta(\pi)$, we first notice that since $\theta(\pi)$ is closed it is locally compact, and G acts on it as a transitive topological transformation group. Since $\theta(\pi)$ is the dual space of a C^*-algebra (a quotient algebra of the enveloping C^*-algebra of A), all of whose irreducible representations are compact, the conditions necessary to apply [**12**, Theorem 1] are satisfied, and this theorem then shows that ψ_π is a homeomorphism.

4. The case when A has Hausdorff dual. We now suppose that $\hat{A}$ is Hausdorff and $\pi \in \hat{A}$. Continuing the notation of § 3, we will form $L^1(A, G; T, \alpha)$ and the induced representation Π of this algebra, but we will not assume that Π is irreducible. In this case, there may be a non trivial stability subgroup G_π of G, which, since $\hat{A}$ is Hausdorff, must be closed. As before, $\theta(\pi)$ represents the orbit of π in $\hat{A}$, and for all a in A, $\gamma_a(x)$ is defined to be $||\pi_{x^{-1}}(a)||$. The authors are indebted to E. Effros for valuable conversations concerning the next proposition.

PROPOSITION 2. *For every a in A, the function γ_a is uniformly continuous on G.*

Proof. Since $\hat{A}$ is Hausdorff, each of the functions $\sigma \to ||\sigma(a)||$ is continuous and vanishes at infinity on $\hat{A}$ (see [**6**, Proposition 3.3.7 and Corollary 3.3.9]). Now [**1**, Ch. 3, § 4, No. 5, Theorem 1(a)] tells us that if K is compact in $\hat{A}$ and U is open in $\hat{A}$, then the set $(U:K)$ of all x in G such that $x \cdot K \subseteq U$ is open in G. Choose $\epsilon > 0$, let $U_0 = \{\sigma \in \hat{A} | \, ||\sigma(a)|| > \epsilon/3\}$, $K_0 = K(a, \epsilon/2)$ (notation as in the proof of Theorem 1, § 3), $K = K(a, \epsilon/3)$. For each σ in K, there is an open neighborhood U_σ of σ such that if $\delta \in U_\sigma$, then

$$| \, ||\sigma(a)|| - ||\delta(a)|| \, | < \epsilon/2.$$

It follows that if δ_1, δ_2 are in U_σ then

$$| \, ||\delta_1(a)|| - ||\delta_2(a)|| \, | < \epsilon.$$

For each σ, choose an open neighborhood W_σ of σ such that $\bar{W}_\sigma$ is compact and contained in U_σ, and cover K with a finite number $W_1, \ldots, W_n$ of the W_σ. Denote the corresponding sets U_σ by $U_1, \ldots, U_n$, and denote $\bar{W}_i$ by K_i. Finally, let V be the (non-empty) open neighborhood of the identity in G given by

$$V = \bigcap_{i=0}^{n} ((U_i : K_i) \cap (U_i : K_i)^{-1}).$$

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Now, suppose that s and t are in G and st^{-1} is in V (then, also, ts^{-1} is in V). If $t\pi \in K_0$, then $s\pi = (st^{-1})t\pi$ is in U_0 and so both $t\pi$ and $s\pi$ are in K. The same is true if $s\pi \in K_0$. Thus, in either case, $t\pi \in K$ and so for some i, $t\pi \in K_i \subset U_i$. Since st^{-1} is in $(U_i : K_i)$, we also have $s\pi \in U_i$. It follows that $|\gamma_a(t) - \gamma_a(s)| < s$. On the other hand, if, neither $t\pi$ nor $s\pi$ is in K_0 then

$$|\gamma_a(t) - \gamma_a(s)| \leqq ||t\pi(a)|| + ||s\pi(a)|| < \epsilon/2 + \epsilon/2 = \epsilon.$$

This completes the proof.

We recall from [**5**] that a real valued function h on G is said to almost vanish at infinity with respect to μ if for every $\epsilon > 0$, the set S_ϵ, consisting of all x such that $|h(x)| \geqq \epsilon$, is thin at infinity. Thin at infinity is a measure theoretic generalization of compactness which means that for every compact neighborhood I of the identity in G, the function $\mu(Ix \cap S_\epsilon)/\Delta(x)$ vanishes at infinity on G (recall that μ is a left Haar measure). It follows from [**5**, Theorem 4.1], that in the above situation the induced representation Π is compact (whether or not it is irreducible) if and only if π is compact, and each of the functions $\gamma_a(x)$ almost vanishes at infinity.

PROPOSITION 3. *Let G be a group with left Haar measure μ and modular function Δ. Let h be a real valued function on G which is uniformly continuous and almost vanishes at infinity with respect to μ. Then h vanishes at infinity.*

Proof. Choose $\epsilon > 0$, let $S_\epsilon = \{x \in G |\, |h(x)| \geqq \epsilon/2\}$, let V be a compact neighborhood of the identity in G such that if $yx^{-1} \in V$, $|h(x) - h(y)| < \epsilon/2$, and let K be a compact set in G such that if x is not in K,

$$\mu(Vx \cap S_\epsilon)/\Delta(x) < \mu(V).$$

Then if $x \notin K$, $Vx \not\subset S_\epsilon$ since $\mu(Vx)/\Delta(x) = \mu(V)$. We can then find y in Vx such that $|h(y)| < \epsilon/2$. Since $yx^{-1} \in V$ we know that $|h(y) - h(x)| < \epsilon/2$ and so $|h(x)| < \epsilon$ whenever $x \notin K$. Since ϵ is arbitrary, h vanishes at infinity.

We are now in a position to prove the main result of this section.

THEOREM 2. *Suppose that $\hat{A}$ is Hausdorff, $\pi \in \hat{A}$, and Π is the representation of $L^1(A, G; T, \alpha)$ induced from π. Then the following are equivalent:*

(1) *Π is compact.*

(2) *For all a in A, the function γ_a vanishes at infinity.*

(3) *$\theta(\pi)$ is closed in $\hat{A}$, G_π is compact, and the natural map of G/G_π onto $\theta(\pi)$ induced by ψ_π is a homeomorphism.*

(4) *ψ_π is a proper map from G into $\hat{A}$.*

Proof. (1) $\Leftrightarrow$ (2) follows from [**5**, Theorem 4.1] and Propositions 2 and 3 above.

(2) $\Leftrightarrow$ (3). Notice first that each of the functions γ_a is constant on G_π, and since they vanish at infinity, G_π must be compact. We now show that $\theta(\pi)$ is closed. Let δ be in the closure of $\theta(\pi)$. A typical basic neighborhood for δ is

the set $K(a, \epsilon)$ [**6**, Ch. 3], where for instance ϵ may be chosen to be $\frac{1}{2}||\delta(a)||$. (There exists a in A for which $\delta(a) \neq 0$.) δ must be in the closure of the set $S = K(a, \epsilon) \cap \theta(\pi)$. Now, $\gamma_a^{-1}([\epsilon, +\infty)) \equiv M$ is compact by hypothesis. It is easy to see that $S = \psi_\pi(M)$ and so S is the continuous image of a compact set and so is compact. Since $\hat{A}$ is Hausdorff, S is closed and so $\delta \in S \subset \theta(\pi)$. This proves that $\theta(\pi)$ is closed. The final statement of (3) follows as it did in Theorem 1 of § 3.

(3) $\Rightarrow$ (4). It follows from the compactness of G_π that the natural projection p of G onto G/G_π is a closed map. This, together with the fact that G/G_π is homeomorphic with $\theta(\pi)$, shows that ψ_π is a closed map. The inverse image under ψ_π of any point in $\hat{A}$ is vacuous or a coset of G_π, and so is compact. It then follows from [**1**, Ch. 1, § 10, No. 2, Theorem 1] that ψ_π is proper.

(4) $\Rightarrow$ (3). The theorem quoted immediately above shows that $\theta(\pi)$ is closed, G_π is compact, and ψ_π is closed. The natural one-to-one map from G/G_π to $\theta(\pi)$ is then a closed map. On the other hand, since ψ_π is continuous and the projection from G to G/G_π is open, the correspondence between G/G_π and $\theta(\pi)$ is a homeomorphism.

(3) $\Rightarrow$ (2). If we denote by $\bar{\psi}_\pi$ the mapping of G/G_π onto $\theta(\pi)$ induced by ψ_π, and let p be the projection of G onto G/G_π, then

$$\{x|\gamma_a(x) \geqq \epsilon\} = p^{-1}(\bar{\psi}_\pi^{-1}(K(a, \epsilon))).$$

Since $K(a, \epsilon)$ is compact, (3) tells us that $\bar{\psi}_\pi^{-1}(K(a, \epsilon)) = C$ is compact in G/G_π. Since G_π is compact, it is easy to see that $p^{-1}(C)$ is compact and the theorem is proved.

5. Induced representations of groups. The usual notations and terminology for induced representations of groups differ a little from those we have used. Therefore, we give in this section the translations of Theorems 1 and 2 for the case of group extensions.

Let $e \rightarrow H \rightarrow G \rightarrow K \rightarrow e$ be a group extension (always second countable, locally compact groups). G acts on H by inner automorphisms and thus, by composition, G acts on the dual $\hat{H}$ of H. Then, using the results of [**4**] to translate into the language of twisted group algebras, we get the following interpretations of Theorems 1 and 2 (see [**5**, § 5]).

THEOREM 1′. *Let s be a strongly continuous unitary representation of H on a Hilbert space, and let U^s be the irreducible induced representation of G in Mackey's sense. Then the following statements are equivalent:*

(1) *U^s is compact (which, by definition, means that the natural extension to $L^1(G)$ is compact).*

(2) *s is compact, and for all f in $L^1(H)$, the function*

$$g \rightarrow \left\| \int_H f(h)s(g^{-1}hg)dh \right\|$$

(which is constant on H cosets and so may be considered as a function on K) vanishes at infinity on K.

(3) *$\theta(s)$ is closed in $\hat{H}$ and ψ_s induces a homeomorphism from $G/H \equiv K$ onto $\theta(s)$ (the notation is as in Theorem 1. Here, ψ_s is defined on G).*

(4) *ψ_s induces a proper injection map from K to $\hat{H}$.*

Remark. The twisting pair (T, α) in the group extension case need not be continuous in a neighborhood of the identity, although it is continuous at the identity [**5**]. The extra smoothness was needed solely to conclude that inducing representations of twisted group algebras preserves weak containment [**3**]. However, this is well known to be true for inducing representations of groups [**10**; **11**]. Thus, the proof of Theorem 1 is valid here.

THEOREM 2′. *Suppose that $\hat{H}$ is Hausdorff and $s \in \hat{H}$. The stability subgroup G_s of s in G is a subgroup which contains H (notice that this usage differs from that of the previous section. In the previous sense, the stability group would be G_s/H). Then the following are equivalent:*

(1) *U^s is compact.*

(2) *For all f in $L^1(H)$, the function*

$$g \to \left\| \int_H f(h)s(g^{-1}hg)dh \right\|$$

vanishes at infinity modulo H (i.e., as a function on K).

(3) *$\theta(s)$ is closed in $\hat{H}$, G_s/H is compact in K, and the map of G/G_s (which equals $K/(G_s/H)$) onto $\theta(s)$ induced by ψ_s is a homeomorphism.*

(4) *ψ_s induces a proper map from K into $\hat{H}$.*

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https://doi.org/10.4153/CJM-1972-002-3 Published online by Cambridge University Press