



RESEARCH ARTICLE

Semiregularity as a consequence of Goodwillie’s theorem

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Received: 21 October 2024; **Revised:** 7 August 2024; **Accepted:** 2 November 2024

2020 Mathematical Subject Classification: *Primary* – 14B12, 14A30; *Secondary* – 19L10

Abstract

We realise Buchweitz and Flenner’s semiregularity map (and hence *a fortiori* Bloch’s semiregularity map) for a smooth variety X as the tangent of a generalised Abel–Jacobi map on the derived moduli stack of perfect complexes on X . The target of this map is an analogue of Deligne cohomology defined in terms of cyclic homology, and Goodwillie’s theorem on nilpotent ideals ensures that it has the desired tangent space (a truncated de Rham complex).

Immediate consequences are the semiregularity conjectures: that the semiregularity maps annihilate all obstructions, and that if X is deformed, semiregularity measures the failure of the Chern character to remain a Hodge class. This gives rise to reduced obstruction theories of the type featuring in the study of reduced Gromov–Witten and Pandharipande–Thomas invariants. We also give generalisations allowing X to be singular, and even a derived stack.

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Introduction

In [Blo], Bloch defined a semiregularity map

$$\tau: H^1(Z, \mathcal{N}_{Z/X}) \rightarrow H^{p+1}(X, \Omega_X^{p-1})$$

for every local complete intersection Z of codimension p in a smooth proper complex variety X , and showed that curvilinear obstructions lie in the kernel of τ . He also showed that if X is deformed, then

τ measures the curvilinear obstruction to $[Z]$ remaining a Hodge class, and conjectured that these statements should hold for all obstructions, not just curvilinear ones.

In [BF1], Buchweitz and Flenner extended τ to give maps

$$\sigma_q : \text{Ext}_{\mathcal{O}_X}^2(\mathcal{F}, \mathcal{F}) \rightarrow H^{q+2}(X, \Omega_X^q)$$

for any perfect complex \mathcal{F} on X , given (up to scalar) by composing with the q th power of the Atiyah class and then taking the trace. They then showed that curvilinear obstructions to deforming \mathcal{F} lie in the kernel of σ_q . If X is allowed to deform, they showed the same holds provided $\text{ch}_p(\mathcal{F})$ deforms as a Hodge class, with consequences for the variational Hodge conjecture [BF1, §5].

Obstruction spaces feature in the construction of virtual fundamental classes in enumerative geometry, used to construct Gromov–Witten and similar invariants. Often the natural obstruction spaces are too large, killing the naïvely defined invariants, but the semiregularity conjectures yield smaller reduced obstruction theories $\ker(\tau)$ and $\ker(\sigma_q)$ in many cases, paving the way for nontrivial reduced invariants to be defined.

Buchweitz and Flenner [BF1, §1, p. 138] also conjectured that the semiregularity map should arise as a morphism of obstruction spaces associated to a morphism of deformation theories and then speculated that this morphism would most likely take the form of a generalised Abel–Jacobi map from the deformation groupoid to an intermediate Jacobian or to Deligne cohomology. The underlying idea is that for a deformation $\tilde{\mathcal{F}}$ of \mathcal{F} , we must have $\text{ch}(\mathcal{F}) = \text{ch}(\tilde{\mathcal{F}})$ because the cohomological Chern character takes rational values. The homotopy between cycles representing $\text{ch}_{q+1}(\mathcal{F})$ and $\text{ch}_{q+1}(\tilde{\mathcal{F}})$ should then be given by $\sigma_q(\tilde{\mathcal{F}})$.

When seeking functorial obstruction theories, one is naturally drawn to derived deformation theory, which generates obstruction spaces as higher tangent spaces (e.g., see Lemma 1.8) and guarantees functoriality of obstruction maps. In [Man, IM], Manetti and Iacono constructed explicit infinitesimal derived Abel–Jacobi maps by L_∞ methods, proving Bloch’s first semiregularity conjecture in the case where Z is smooth, and then for complete intersections of hypersurfaces.¹ Like [BF1], [IM] identified $H^{2p}(X, \Omega_X^{<p})$ as a more natural target for the semiregularity map than $H^{p+1}(X, \Omega_X^{p-1})$. Other work such as [STV, KT] focuses on the case $p = 1$, where this discrepancy does not arise.²

In this paper, we construct a morphism of derived deformation theories of the form envisaged in [BF1], but to a slightly different target. This leads to the following theorem, which proves and generalises the conjectures of [BF1] and hence [Blo], showing that the semiregularity map measures the failure of the unique horizontal lift of the Chern character to remain in F^p :

Theorem. *Take a local Artinian \mathbb{C} -algebra A , a smooth morphism $X \rightarrow \text{Spec } A$ of Artin stacks and square-zero ideal $I \subset A$ with quotient $B = A/I$. Then for any perfect complex \mathcal{F} over $X' := X \otimes_A B$, with obstruction $o(\mathcal{F}) \in \text{Ext}_{\mathcal{O}_{X'}}^2(\mathcal{F}, \mathcal{F} \otimes_B I)$ to deforming \mathcal{F} to a complex of \mathcal{O}_X -modules, the image of the Chern character $\text{ch}_p(\mathcal{F})$ under the map*

$$H^{2p}(X'(\mathbb{C})_{\text{an}}, \mathbb{Q}) \cong H^{2p}(X(\mathbb{C})_{\text{an}}, \mathbb{Q}) \rightarrow H^{2p}(X(\mathbb{C})_{\text{an}}, A) \cong H^{2p}(X, \Omega_{X/A}^\bullet)$$

lies in $F^p H^{2p}(X, \Omega_{X/A}^\bullet)$ if and only if $o(\mathcal{F})$ maps to zero under the composite map

$$\text{Ext}_{\mathcal{O}_{X'}}^2(\mathcal{F}, \mathcal{F} \otimes_B I) \xrightarrow{\sigma_{p-1}} H^{p+1}(X, I\Omega_{X/A}^{p-1}) \rightarrow H^{2p}(X, \Omega_{X/A}^{<p}).$$

In fact, we establish a more general statement, Corollary 2.25, using derived differential forms to remove the smoothness hypothesis and allowing derived objects. Note that if the family X is constant

¹Nearly a decade after this was first written, Bandiera, Lepri and Manetti [BLM] used Chern–Simons classes to recover the main results of this paper in the absolute case $R = \mathbb{C}$ for smooth proper varieties X , including the first semiregularity conjecture.

²In that special case, the derived Picard stack already provides a suitable target (since the trace σ_0 is an isomorphism for line bundles), with the Abel–Jacobi map simply corresponding to the derived determinant.

over A (i.e., $X \cong X_0 \times \text{Spec } A$), then the condition $\text{ch}_p(\mathcal{F}) \in F^p$ is automatically satisfied and the obstruction $o(\mathcal{F})$ always maps to zero (Remark 2.23).

The theorem produces reduced obstruction theories for the stable pairs and stable curves featuring in the study of Pandharipande–Thomas and Gromov–Witten invariants (Remarks 2.27 and Remark 2.28). The latter follows by considering perfect complexes of the form $\mathbf{R}f_*\mathcal{O}_Z$ to produce reduced obstruction theories for proper morphisms $f: Z \rightarrow X$. As a special case, this establishes Bloch’s semiregularity conjectures in the generality envisaged (Remark 2.29). Since we can apply the theorem to gerbes, it also leads to reduced obstructions for μ -twisted sheaves (Remark 2.26).

The previous theorem is a consequence of the following more general result, which only involves cohomology groups of algebraic, not analytic or topological, origin (cf. Corollary 2.24 for the derived generalisation not requiring smoothness):

Theorem. *Take a smooth morphism $f: X \rightarrow S$ of Artin stacks over \mathbb{Q} , with a closed immersion $S' \hookrightarrow S$ defined by a nilpotent ideal \mathcal{I} . Then for $X' := X \times_S S'$, the Chern character refines to give maps Ξ_p from $K_0(X')$ to the vector spaces*

$$\begin{aligned} & \mathbb{H}^{2p}(X, \Omega_{X/S}^\bullet \times_{\Omega_{X'/S'}} F^p \Omega_{X'/S'}) \\ &= \mathbb{H}^{2p}(X, \mathcal{I}\mathcal{O}_X \xrightarrow{d} \mathcal{I}\Omega_{X/S}^1 \xrightarrow{d} \dots \xrightarrow{d} \mathcal{I}\Omega_{X/S}^{p-1} \xrightarrow{d} \Omega_{X/S}^p \xrightarrow{d} \Omega_{X/S}^{p+1} \xrightarrow{d} \dots). \end{aligned}$$

If $\mathcal{I}^2 = 0$, then for any perfect complex \mathcal{F} over X' , the obstruction to lifting $\Xi_p(\mathcal{F})$ to $\mathbb{H}^{2p}(X, F^p \Omega_{X/S}^\bullet)$ is given by applying the composite map

$$\text{Ext}_{\mathcal{O}_{X'}}^2(\mathcal{F}, \mathcal{F} \otimes_{\mathcal{O}_{S'}} \mathcal{I}) \xrightarrow{\sigma_{p-1}} \mathbb{H}^{p+1}(X, \mathcal{I}\Omega_{X/S}^{p-1}) \rightarrow \mathbb{H}^{2p}(X, \mathcal{I}\Omega_{X/S}^{<p})$$

to the obstruction $o(\mathcal{F})$ to deforming \mathcal{F} to a complex of \mathcal{O}_X -modules.

The crucial observation enabling our construction Ξ is that if we modify Deligne cohomology slightly, replacing rational Betti cohomology with any other cohomology theory which is formally étale (i.e., invariant under nilpotent thickenings), then the obstruction spaces are unchanged. The theory we work with is Hartshorne’s algebraic de Rham cohomology DR^{alg} in the guise of derived de Rham cohomology, which is formally étale by Goodwillie’s theorem on nilpotent ideals [Goo1]. Our map Ξ is then induced from the Goodwillie–Jones Chern character ch^- .

Given a smooth morphism $X \rightarrow \text{Spec } R$ over \mathbb{C} , we set $J^p(X/R, \mathbb{C})[2p]$ to be the cocone (i.e., shifted cone or homotopy fibre) of

$$\text{DR}^{\text{alg}}(X/\mathbb{C}) \rightarrow \mathbf{R}\Gamma(X, \Omega_{X/R}^{<p}),$$

noting that, in general, X will not be smooth over \mathbb{C} . This definition also adapts in the obvious way to any base \mathbb{Q} -algebra k in place of \mathbb{C} and admits further generalisations to derived stacks with no smoothness hypothesis (Definition 2.13).

To establish existence of the Abel–Jacobi map and functoriality, we reformulate in terms of cyclic homology. Derived de Rham cohomology is isomorphic to periodic cyclic homology HP , giving

$$\prod_p J^p(X/R, k) \simeq \text{HP}^k(X) \times_{\text{HP}^R(X)}^h \text{HN}^R(X),$$

for negative cyclic homology HN . The generalised Abel–Jacobi maps

$$\Xi_p: K(X) \rightarrow J^p(X/R, k)$$

are then induced from the Goodwillie–Jones Chern character $\text{ch}^-: K(X) \rightarrow \text{HN}^k(X)$.

Now, Ξ_p restricts from K -theory to a map on the ∞ -groupoid $\text{Perf}(X)$ of perfect complexes on X . As we change the base, setting $X_A := X \otimes_R A$ and $J_X^p(A, k) := J^p(X_A/A, k)$, this gives us morphisms

$$\Xi_p : \text{Perf}(X_A) \rightarrow J_X^p(A, k),$$

functorial in simplicial R -algebras A .

Goodwillie’s Theorem on nilpotent ideals implies that $\text{HP}^k(X_A)$ is formally étale as a functor in A , so $J_X^p(-, k)$ has the same derived tangent space as cyclic homology with a degree shift. On derived tangent spaces, Ξ_p thus induces maps

$$\xi_p : \text{Ext}_{\mathcal{O}_{X_A}}^r(\mathcal{F}, \mathcal{F} \otimes_A M) \rightarrow \mathbb{H}^{2p-2+r}(X, (\mathcal{O}_X \rightarrow \dots \rightarrow \Omega_X^{p-1}) \otimes_R M),$$

for A -modules M ; in Proposition 1.17, we show that ξ_p is just the $(p - 1)$ th component of the Lefschetz map \mathcal{L} of [BNT], and hence (Remark 2.21) equivalent to the semiregularity map σ_{p-1} of [BF1].

The étale hypersheaves Perf_X and $J_X^p(-, k)$ satisfy homotopy-homogeneity,³ a left-exactness property analogous to Schlessinger’s conditions, which in particular gives a functorial identification of higher tangent spaces with obstruction spaces. Since $J_X^p(-, k)$ has the same obstruction space as Deligne cohomology, the map Ξ_p thus fully realises the hope expressed in [BF1, §1].

Given a square-zero extension $e : A \rightarrow B$ of simplicial algebras with kernel I , and a perfect complex \mathcal{F} on X_B , the obstruction $o_e(\mathcal{F})$ to lifting \mathcal{F} to X_A lies in $\text{Ext}_{\mathcal{O}_{X_B}}^2(\mathcal{F}, \mathcal{F} \otimes_B I)$. Derived functoriality and homotopy-homogeneity then ensure (Corollary 2.22) that the obstruction to lifting $\Xi_p(\mathcal{F})$ from $H_0 J_X^p(B, k)$ to $H_0 J_X^p(A, k)$ is just

$$\xi_p(o_e(\mathcal{F})) \in \mathbb{H}^{2p}(X, (\mathcal{O}_X \rightarrow \Omega_{X/R}^1 \rightarrow \dots \rightarrow \Omega_{X/R}^{p-1}) \otimes_R I),$$

leading to the theorems above (Corollaries 2.25 and 2.24), corresponding to the choices $k = \mathbb{C}$ and $k = R = A$, respectively. Our formulation in terms of cyclic homology and the Lefschetz map \mathcal{L} also extends these results to certain noncommutative spaces (Corollary 1.20), where one effectively has to consider all values of p simultaneously.

Notation and conventions

The Dold–Kan correspondence gives an equivalence of categories between simplicial abelian groups and nonnegatively graded chain complexes, with homotopy groups corresponding to homology groups, and we will pass between these categories without further comment.

Given a chain complex V , we will write $V[n]$ for the chain complex given by $V[n]_i = V_{n+i}$; beware that this is effectively opposite to the standard convention for cochain complexes. We also write $\tau_{\geq 0} V$ for the good truncation [Wei2, Truncations 1.2.7] of V in nonnegative chain degrees, which we can then regard as a simplicial abelian group by the Dold–Kan correspondence above.

For a morphism $f : V \rightarrow W$ of chain complexes, $\text{cocone}(f)$ will denote the shifted cone, a model for the homotopy fibre of f , which fits into an exact triangle

$$\text{cocone}(f) \rightarrow V \rightarrow W \rightarrow \text{cocone}(f)[-1].$$

Definition 0.1. Given a commutative ring A and a flat A -algebra E , write $\text{HC}^A(E)$ (resp. $\text{HN}^A(E)$), resp. $\text{HP}^A(E)$, resp. $\text{HH}^A(E)$) for the chain complex associated to cyclic (resp. negative cyclic, resp. periodic cyclic, resp. Hochschild) homology of E over A , as in [Wei2, §9.6].

Given a simplicial commutative ring A and a simplicial A -algebra E with each E_n flat over A_n , together with a homology theory \mathcal{H} as in the previous paragraph, define the complex $\mathcal{H}^A(E)$ by first forming the simplicial chain complex given by $\mathcal{H}^{A_n}(E_n)$ in level n , then taking the product total complex.

³Later terms for this concept are infinitesimal cohesiveness on one factor and (on Artinian input) being a formal moduli problem, although the property neither associates genuine moduli problems to such functors nor is automatically satisfied by such.

Remark 0.2. When working with cyclic homology, it is usual to fix a base ring and to omit it from the notation. Since varying the base will be crucial to our constructions, we have introduced the superscript A above. Also beware that the cohomology theories HN and HP are frequently denoted by HC^- and HC^{per} in the literature and that our complexes are related to cyclic homology groups by

$$\text{HC}_i^A(E, M) := \text{H}_i \text{HC}^A(E, M)$$

etc. In the notation of [Wei2, Ch. 9], the complexes $\text{HH}, \text{HC}, \text{HN}, \text{HP}$ are denoted by $\text{CC}_*^h, \text{Tot CC}_{**}, \text{Tot}^{\Pi} \text{CC}_{**}^N, \text{Tot}^{\Pi} \text{CC}_{**}^P$.

When A is a discrete ring, note that the complexes above are those studied in [Goo2].

Definition 0.3. As in [Wei2, §9.8.2], when the A -algebra E is commutative, each of the homology theories \mathcal{H} above admits a Hodge decomposition, which we denote by

$$\mathcal{H}^A(E) = \prod_{p \in \mathbb{Z}} \mathcal{H}^A(E)^{(p)}.$$

Note that in the case of HC , we have $\text{HC}^A(E)^{(p)} = 0$ for $p < 0$, with degree bounds on the other terms (specifically, $\text{HC}^A(E)^{(p)}$ concentrated in chain degrees $\geq p$) making the infinite product a direct sum in that case only.

For E commutative, recall from [Wei2, §9.6.1] that there are exact triangles (the SBI sequences)

$$\begin{aligned} \text{HN}^A(E)^{(p)} &\xrightarrow{1} \text{HP}^A(E)^{(p)} \xrightarrow{\text{S}} \text{HC}^A(E)^{(p-1)}[-2] \xrightarrow{\text{B}} \text{HN}^A(E)^{(p)}[-1] \\ \text{HH}^A(E)^{(p)} &\xrightarrow{1} \text{HC}^A(E)^{(p)} \xrightarrow{\text{S}} \text{HC}^A(E)^{(p-1)}[-2] \xrightarrow{\text{B}} \text{HH}^A(E)^{(p)}[-1], \end{aligned}$$

compatible with the projection map $\pi_{\text{HH}}: \text{HN}^A(E)^{(p)} \rightarrow \text{HH}^A(E)^{(p)}$ and $\text{S}: \text{HP}^A(E)^{(p)} \rightarrow \text{HC}^A(E)^{(p)}$. For E noncommutative, these sequences still exist once we drop the superscripts (p) , there being no Hodge decomposition in this case. See Proposition 2.4 below for the relation of these sequences with the Hodge filtration on derived de Rham cohomology.

Definition 0.4. Given a simplicial ring E , we follow [Wal] in writing $K(E)$ for the simplicial set constituting the K -theory space of E (the 0th part of the K -theory spectrum). This is an infinite loop space with $\pi_i K(E) = K_i(E)$.

1. The Abel–Jacobi map for rings

Fix a simplicial commutative \mathbb{Q} -algebra R and a simplicial associative R -algebra $O(X)$, which need not be commutative. Assume that each $O(X)_n$ is flat as an R_n -module. We will write $F(X) := F(O(X))$ when F is a functor such as $K, \text{HP}, \text{HC}, \text{HN}, \text{HH}$.

Write $s\text{CAlg}_R$ for the category of simplicial commutative R -algebras. We use its model structure induced from the Kan–Quillen model structure on simplicial sets. We call a functor from $s\text{CAlg}_R$ to a model category homotopy-preserving if it preserves weak equivalences.

1.1. The Abel–Jacobi map

Definition 1.1. Define chain complexes

$$\begin{aligned} J(X/R) &:= \text{cocone}(\text{HP}^{\mathbb{Q}}(X) \xrightarrow{\text{S}} \text{HC}^R(X)[-2]), \\ J^p(X/R) &:= \text{cocone}(\text{HP}^{\mathbb{Q}}(X)^{(p)} \xrightarrow{\text{S}} \text{HC}^R(X)^{(p-1)}[-2]), \end{aligned}$$

the latter only being defined when $O(X)$ is commutative, with $J(X/R) = \prod_{p \in \mathbb{Z}} J^p(X/R)$.

These are functors on the category of arrows $(R \rightarrow O(X))$ in $s\mathcal{CAlg}_{\mathbb{Q}}$, so we may then define functors J_X and J_X^p from $s\mathcal{CAlg}_R$ to chain complexes by setting $J_X(A) := J(X_A/A)$ and $J_X^p(A) := J^p(X_A/A)$, where $O(X_A) := O(X) \otimes_R A$.

Note that the equivalence $\text{cocone}(\text{HP}^R(X) \xrightarrow{S} \text{HC}^R(X)[-2]) \simeq \text{HN}^R(X)$ gives a homotopy fibre product characterisation

$$J(X/R) \simeq \text{HP}^{\mathbb{Q}}(X) \times_{\text{HP}^R(X)}^h \text{HN}^R(X);$$

the motivation for this construction is that $\text{HN}^R(X)$ behaves like the Hodge filtration over R , while $\text{HP}^{\mathbb{Q}}(X)$ behaves in some respects like Betti cohomology.

The Goodwillie–Jones Chern character

$$\text{ch}^- : K(X) \rightarrow \tau_{\geq 0}\text{HN}^{\mathbb{Q}}(X)$$

of [Goo2, Theorem II.3.1] (there denoted α) or [HJ, §5], or rather its promotion to a natural ∞ -transformation as in [Goo2, §III.2.3], then combines with the map

$$\text{HN}^{\mathbb{Q}}(X) \rightarrow \text{HP}^{\mathbb{Q}}(X) \times_{\text{HP}^R(X)}^h \text{HN}^R(X),$$

natural in X and R , to give a natural map

$$\Xi : K(X) \rightarrow \tau_{\geq 0}J(X/R)$$

in the ∞ -category of simplicial sets,⁴ which we call the (generalised) Abel–Jacobi map.

Definition 1.2. Given a simplicial ring S , define $\text{Perf}(S)$ to be the simplicial set given by first forming the core (i.e., the subcategory of quasi-isomorphisms) of the simplicial category $\text{Perf}(S)$ of perfect S -modules in complexes, then taking the nerve; see, for instance, [Pri1, Definitions 2.8 and 2.29] or [TV2, §1.3.7]. This becomes a simplicial semiring with addition given by block sum and multiplication by tensor product.

Definition 1.3. By [Wal, Theorem 2.3.2] and [TT, Theorem 1.9.8], there is a natural map $\text{Perf}(X) \rightarrow K(X)$. Composing this with the Abel–Jacobi map above gives us a map

$$\Xi : \text{Perf}(X) \rightarrow \tau_{\geq 0}J(X/R).$$

1.2. Homogeneity and obstructions

Say that a map $A \rightarrow B$ in $s\mathcal{CAlg}_R$ is a nilpotent extension if it is levelwise surjective, with the kernel I satisfying $I^n = 0$ for some n , where both the kernel and its powers are defined levelwise. In other words, the maps $A_i \rightarrow B_i$ are all nilpotent surjections, with a common bound on the index of nilpotency.

Definition 1.4. We say that a homotopy-preserving functor F from $s\mathcal{CAlg}_R$ to a model category \mathcal{C} is homotopy-homogeneous if for $A \rightarrow B$ a nilpotent extension in $s\mathcal{CAlg}_R$ and $C \rightarrow B$ any morphism, the map

$$F(A \times_B C) \rightarrow F(A) \times_{F(B)}^h F(C)$$

(to the homotopy fibre product) is a weak equivalence. When \mathcal{C} is the category of chain complexes, this is equivalent to saying that we have an exact triangle

$$F(B)[1] \rightarrow F(A \times_B C) \rightarrow F(A) \oplus F(C) \rightarrow F(B).$$

⁴Note that we are here following our stated convention of reinterpreting chain complexes as simplicial abelian groups. The natural ∞ -transformation is given by a zigzag alternating natural transformations with natural weak equivalences, which is thus a natural transformation of the associated simplicial functor on hammock localisations.

Definition 1.5. Define the simplicial set-valued functor Perf_X on $s\text{CAlg}_R$ by $\text{Perf}_X(A) := \text{Perf}(O(X) \otimes_R A)$.

Definition 1.6. Given a homotopy-homogeneous functor F from $s\text{CAlg}_R$ to simplicial sets (with the Kan–Quillen model structure), an element $x \in F(A)_0$ and an A -module M in simplicial abelian groups, define the tangent space

$$T_x(F, M)$$

to be the homotopy fibre of $F(A \oplus M) \rightarrow F(A)$ over x , for the trivial square-zero extension $A \oplus M \rightarrow A$.

Note that homotopy-homogeneity of F ensures that $T_x(F, M)$ has an infinite loop space structure; see, for instance, [Pri2, Lemma 1.12]. We thus define tangent cohomology to be the abelian groups $D_x^{n-i}(F, M) := \pi_i T_x(F, M[-n])$, which are well-defined by that lemma.

Definition 1.7. Define a square-zero extension $I \rightarrow A \rightarrow B$ in $s\text{CAlg}_R$ to be a levelwise surjection $A \rightarrow B$ in $s\text{CAlg}_R$ with kernel I being a square-zero simplicial ideal.

Lemma 1.8. Take a homotopy-preserving and homotopy-homogeneous simplicial set-valued functor F on $s\text{CAlg}_R$ and a square-zero extension $I \rightarrow A \xrightarrow{e} B$ in $s\text{CAlg}_R$. Then there is a naturally associated section $o_e: FB \rightarrow F(B \oplus I[-1])$ in the ∞ -category of simplicial sets, such that for any $x \in (FB)_0$, the homotopy fibre $(FA)_x$ of $F(e)$ over x is naturally homotopic to the space $\{o_e(x)\} \times_{T_x(F, I[-1])}^h \{0\}$ of paths from 0 to $o_e(x)$.

In particular, we have a functorial obstruction

$$o_e(x) \in D_x^1(F, I),$$

which is zero if and only if $[x]$ lies in the image of

$$e_*: \pi_0(FA) \rightarrow \pi_0(FB).$$

Proof. This is contained in [Pri2, Lemma 1.17] and its proof, in which the obstruction maps given here are just one term in a long exact sequence of homotopy groups.

Explicitly, the Čech nerve of A over B gives a bisimplicial R -algebra by sending n to the $(n + 1)$ -fold fibre product of A over B , and taking the diagonal gives us a simplicial R -algebra \tilde{B} . The natural map $\tilde{B} \rightarrow B$ is a square-zero extension with acyclic kernel J , where $J_n \cong (I_n)^{n+1}$ acyclicity follows because J is the diagonal of a levelwise acyclic bisimplicial module. The diagonal embedding gives a natural map $I \hookrightarrow J$, with the square-zero property ensuring that $I \subset \tilde{B}$ is a simplicial ideal. We then have an isomorphism $\tilde{B}/I \cong B \oplus (J/I)$, and since J is acyclic, the B -module J/I is naturally weakly equivalent to $I[-1]$. The obstruction map o_e then comes from the zigzag of simplicial sets

$$F(B) \xleftarrow{\sim} F(\tilde{B}) \rightarrow F(B \oplus (J/I)) \xrightarrow{\sim} F(B \oplus I[-1]).$$

Moreover, for the zero section $B \rightarrow B \oplus J/I$, we have $\tilde{B} \times_{B \oplus (J/I)} B \cong A$, and hence, $F(A) \simeq F(B) \times_{o_e, F(B \oplus I[-1]), 0}^h F(B)$, giving the required fibre sequence on taking homotopy fibres over $x \in F(B)$. □

The following is well-known (see, for instance, [Pri1, Theorem 4.12]; although stated for $O(X)$ commutative, the proof works verbatim in our generality):

Lemma 1.9. The functor Perf_X is homotopy-preserving and homotopy-homogeneous. At $\mathcal{F} \in \text{Perf}_X(A)$, the tangent space $T_{\mathcal{F}}(\text{Perf}_X, M)$ is $\tau_{\geq 0}(\underline{\mathbf{R}}\text{Hom}_{O(X) \otimes_R A}(\mathcal{F}, \mathcal{F} \otimes_A M)[-1])$, so the tangent cohomology groups are

$$D_{\mathcal{F}}^i(\text{Perf}_X, M) \cong \mathbb{E}xt_{O(X) \otimes_R A}^{i+1}(\mathcal{F}, \mathcal{F} \otimes_A M).$$

1.3. Goodwillie’s theorem

The following is [Goo2, Lemma I.3.3], a reformulation of Goodwillie’s theorem on nilpotent ideals ([Goo1, Theorems II.5.1 and IV.2.6]):

Theorem 1.10. *If $S \rightarrow T$ is a map of simplicial \mathbb{Q} -algebras such that $\pi_0 S \rightarrow \pi_0 T$ is a nilpotent extension, then the map*

$$\text{HP}^{\mathbb{Q}}(S) \rightarrow \text{HP}^{\mathbb{Q}}(T)$$

is a quasi-isomorphism of chain complexes.

Proposition 1.11. *The functor J_X from $s\text{CAlg}_R$ to chain complexes is homotopy-homogeneous.*

Proof. The chain complexes $\text{HC}^{R_n}(O(X)_n)$ of flat R_n -modules satisfy $\text{HC}^{R_n}(O(X)_n) \otimes_{R_n} A_n \cong \text{HC}^{A_n}(O(X_A)_n)$ by construction and lie in nonnegative degrees. These tensor products are thus derived tensor products, and since HC is concentrated in nonnegative chain degrees, the product total complex in our definition of $\text{HC}^R(X)$ and $\text{HC}^A(X_A)$ is just a total direct sum. Thus, $\text{HC}^A(X_A) \simeq \text{HC}^R(X) \otimes_R^L A$ for all $A \in s\text{CAlg}_R$, which ensures that the functor $A \mapsto \text{HC}^A(X_A)$ is homotopy-homogeneous by exactness of derived tensor products.

Take a morphism $C \rightarrow B$ and a nilpotent extension $A \rightarrow B$ in $s\text{CAlg}_R$ with kernel I . Now, since $A \times_B C \rightarrow C$ is a nilpotent extension, having kernel $I \times \{0\}$, Theorem 1.10 gives quasi-isomorphisms

$$\text{HP}^{\mathbb{Q}}(X_{A \times_B C}) \rightarrow \text{HP}^{\mathbb{Q}}(X_C), \quad \text{HP}^{\mathbb{Q}}(X_A) \rightarrow \text{HP}^{\mathbb{Q}}(X_B).$$

Thus, $\text{HP}^{\mathbb{Q}}(X_{A \times_B C})$ is trivially quasi-isomorphic to the cocone of

$$\text{HP}^{\mathbb{Q}}(X_C) \oplus \text{HP}^{\mathbb{Q}}(X_A) \rightarrow \text{HP}^{\mathbb{Q}}(X_B),$$

that is, to the homotopy fibre product $\text{HP}^{\mathbb{Q}}(X_A) \times_{\text{HP}^{\mathbb{Q}}(X_B)}^h \text{HP}^{\mathbb{Q}}(X_C)$, so the functor $A \mapsto \text{HP}^{\mathbb{Q}}(X_A)$ is also homotopy-homogeneous. The result for J_X now follows by taking homotopy fibres. \square

Lemma 1.12. *For all $[f] \in H_0(J_X(A))$, the tangent space $T_f(\tau_{\geq 0}(J_X, M))$ is canonically quasi-isomorphic to $\tau_{\geq 0}(\text{HC}^R(X) \otimes_R^L M[-1])$.*

If $O(X)$ is commutative, then the tangent space $T_f(\tau_{\geq 0}(J_X^p, M))$ is canonically quasi-isomorphic to $\tau_{\geq 0}(\text{HC}^R(X)^{(p-1)} \otimes_R^L M[-1])$.

Proof. Since $A \oplus M \rightarrow A$ is a nilpotent extension, substituting in Theorem 1.10 gives

$$\begin{aligned} J_X(A \oplus M) &= \text{cocone}\left(\text{HP}^{\mathbb{Q}}(X_{A \oplus M}) \xrightarrow{\text{S}} \text{HC}^{A \oplus M}(X_{A \oplus M})[-2]\right), \\ &\xrightarrow{\sim} \text{cocone}\left(\text{HP}^{\mathbb{Q}}(X_A) \xrightarrow{\text{S}} \text{HC}^R(X) \otimes_R^L (A \oplus M)[-2]\right), \\ &\simeq J_X(A) \oplus (\text{HC}^R(X) \otimes_R^L M[-1]), \end{aligned}$$

and similarly for J_X^p . \square

1.4. The semiregularity map

Definition 1.13. Given a simplicial A -algebra S with ideal J , such that S and S/J are both levelwise flat over A , write $\text{HC}^A(S \rightarrow S/J) := \text{cocone}(\text{HC}^A(S) \rightarrow \text{HC}^A(S/J))$ and define $\text{HH}^A(S \rightarrow S/J)$ and $\text{HN}^A(S \rightarrow S/J)$ similarly.

Following the convention that weights are additive with respect to tensor products, a grading \mathcal{W} on a ring compatible with the multiplication induces gradings \mathcal{W} on the various cyclic homology complexes as in [Wei2, §9.9], with all natural constructions automatically preserving the gradings.

Take a simplicial commutative \mathbb{Q} -algebra A , a simplicial A -module M , and a (possibly noncommutative) simplicial A -algebra E , with M and E levelwise flat over A . Then we may put a grading on the trivial square-zero extension $C := A \oplus M$ of A by setting A to have weight 0 and M to have weight 1. Also setting E to have weight 0, we then have the following lemma.

Lemma 1.14. *There is a commutative diagram, for $\otimes = \otimes_A$,*

$$\begin{array}{ccccc}
 \mathrm{HC}^A(E) \otimes \mathcal{W}_1 C & \xleftarrow{l_E \otimes \mathrm{id}} & \mathrm{HH}^A(E) \otimes \mathcal{W}_1 \mathrm{HH}^A(C)^{(0)} & \xrightarrow[\sim]{\mathrm{id} \otimes \mathcal{B}_C} & \mathrm{HH}^A(E) \otimes \mathcal{W}_1 \mathrm{HH}^A(C)^{(1)}[1] \\
 \parallel & & \downarrow l_{E \otimes C} \sim & & \uparrow (\mathrm{id} \otimes \mathrm{pr}_{\mathrm{HH}^{(1)}}) \circ \pi_{\mathrm{HH}} \sim \\
 \mathcal{W}_1 \mathrm{HC}^C(E \otimes C) & \xleftarrow{\phi} & \mathcal{W}_1 \mathrm{HC}^A(E \otimes C) & \xrightarrow[\sim]{\mathcal{B}_{E \otimes C}} & \mathcal{W}_1 \mathrm{HN}^A(E \otimes C)[1],
 \end{array}$$

where \sim denotes quasi-isomorphism, ϕ is the map induced by base change $A \rightarrow C$, and the map on the right combines $\pi_{\mathrm{HH}}: \mathrm{HN} \rightarrow \mathrm{HH}$ with the Künneth isomorphism $\mathrm{HH}^A(E \otimes_A C) \cong \mathrm{HH}^A(E) \otimes_A \mathrm{HH}^A(C)$ and projection $\mathrm{pr}_{\mathrm{HH}^{(1)}}: \mathrm{HH}^A(C) \rightarrow \mathrm{HH}^A(C)^{(1)}$.

Proof. The first square commutes by compatibility of l with base change, since $\mathrm{HH}^C(C) = C$. For commutativity of the second square, observe that, by definition,

$$\pi_{\mathrm{HH}} \circ \mathcal{B}_{E \otimes C} \circ l_{E \otimes C} = \mathcal{B}_{E \otimes C}: \mathrm{HH}^A(E \otimes C) \xrightarrow{\mathcal{B}_{E \otimes \mathrm{id} + \mathrm{id} \otimes \mathcal{B}_C}} \mathrm{HH}^A(E \otimes C),$$

with projection to $\mathrm{HH}^A(C)^{(1)}$ then killing the identity term on $\mathrm{HH}^A(C)^{(0)}$.

By Goodwillie’s Theorem over A (in the form of [Wei2, Corollary 9.9.2], based on the proof of [Goo1, Theorem II.5.1]), $\mathcal{W}_i \mathrm{HP}^A(E \otimes C)$ is acyclic for all $i > 0$, so the map $\mathcal{B}_{E \otimes C}: \mathrm{HC}^A(E \otimes C) \rightarrow \mathrm{HN}^A(E \otimes C)[1]$ is a quasi-isomorphism in all nonzero weights by the SBI sequence.

Since we discard higher weights, the quasi-isomorphism $\mathcal{B}_C: \mathcal{W}_1 \mathrm{HH}^A(C)^{(0)} \rightarrow \mathcal{W}_1 \mathrm{HH}^A(C)^{(1)}[1]$ can be seen from the HKR equivalence for $\mathrm{HH}^A(\mathrm{Sym}_A M)$, via which it corresponds to the de Rham derivative $d: \mathcal{W}_1 C \xrightarrow{\sim} \mathcal{W}_1 \Omega_{C/A}^1$.

It thus remains only to show that the middle map $l_{E \otimes C}$ is a quasi-isomorphism. Writing $\mathrm{HC} := \mathrm{HC}^A$, $\mathrm{HH} := \mathrm{HH}^A$, we can characterise it as the composite of the quasi-isomorphisms

$$\begin{aligned}
 \mathrm{HH}(E) \otimes M & \xrightarrow{(l_E, 0) \otimes \mathrm{id}} \mathrm{cocone} \left(\mathrm{HC}(E) \xrightarrow{S_E} \mathrm{HC}(E) \right) \otimes M \\
 & \rightarrow \mathrm{cocone} \left(\mathrm{HC}(E) \otimes \mathcal{W}_1 \mathrm{HC}(C) \xrightarrow{S_E \otimes \mathrm{id} - \mathrm{id} \otimes S_C} \mathrm{HC}(E) \otimes \mathcal{W}_1 \mathrm{HC}(C)[-2] \right) \\
 & \simeq \mathcal{W}_1 \mathrm{HC}(E \otimes C).
 \end{aligned}$$

Here, the first map is a quasi-isomorphism by the SBI sequence. The second map is well-defined because S vanishes on $\mathrm{HC}(C)^{(0)}$, with the natural isomorphism $M \cong \mathcal{W}_1 \mathrm{HC}(C)^{(0)}$ inducing a quasi-isomorphism $M \rightarrow \mathcal{W}_1 \mathrm{HC}(C)$ by [Wei2, Exercise 9.9.1]. The final quasi-isomorphism is from the Künneth formula for cyclic homology [Lod, Corollary 4.3.12]. \square

Lemma 1.15. *In the setting of Lemma 1.14, the obvious map*

$$\begin{aligned}
 \bar{\phi}: \mathrm{HC}^A(E \otimes_A C \rightarrow E) & \rightarrow \mathrm{cocone} \left(\mathrm{HC}^C(E \otimes_A C) \rightarrow \mathrm{HC}^A(E) \right) \\
 & \cong \mathrm{cocone} \left(\mathrm{HC}^A(E) \otimes_A C \rightarrow \mathrm{HC}^A(E) \right) \simeq \mathrm{HC}^A(E) \otimes_A M
 \end{aligned}$$

is naturally homotopic to the composite

$$\begin{aligned} \mathrm{HC}^A(E \otimes_A C \rightarrow E) &\xrightarrow[\sim]{\mathbf{B}} \mathrm{HN}^A(E \otimes_A C \rightarrow E)[1] \\ &\xrightarrow{\pi_{\mathrm{HH}}} \mathrm{HH}^A(E \otimes_A C \rightarrow E)[1] \cong \mathrm{HH}^A(E) \otimes_A \mathrm{HH}^A(C \rightarrow A)[1] \\ &\xrightarrow{|\otimes \delta} \mathrm{HC}^A(E) \otimes_A M, \end{aligned}$$

where $\delta: \mathrm{HH}^A(C \rightarrow A)[1] \rightarrow M$ is the map induced by the canonical derivation $\Omega_{C/A}^1 \cong M$.

Proof. Since $\mathrm{HC}(E) \otimes M$ has weight 1, we may restrict to \mathcal{W}_1 . In particular, $\bar{\phi}$ is given by the natural map ϕ from Lemma 1.14 composed with projection to \mathcal{W}_1

The statement now follows by substitution in Lemma 1.14, once we note that δ is left inverse to the quasi-isomorphism $\mathbf{B}_C: \mathcal{W}_1 \mathrm{HH}^A(C)^{(0)} \rightarrow \mathcal{W}_1 \mathrm{HH}^A(C)^{(1)}[1]$. \square

Definition 1.16. Following [BNT, §4.3], given a levelwise flat simplicial A -algebra E and a perfect E -complex \mathcal{F} , define the Lefschetz map $\mathcal{L}_{E/A}: \mathbf{R}\underline{\mathrm{Hom}}_E(\mathcal{F}, \mathcal{F}) \rightarrow \mathrm{HH}^A(E)$ to be given by

$$\begin{aligned} \mathbf{R}\underline{\mathrm{Hom}}_E(\mathcal{F}, \mathcal{F}) &\simeq \mathcal{F} \otimes_E^{\mathbf{L}} \mathbf{R}\underline{\mathrm{Hom}}_E(\mathcal{F}, E) \xleftarrow[\beta]{\sim} (\mathcal{F} \otimes_A^{\mathbf{L}} \mathbf{R}\underline{\mathrm{Hom}}_E(\mathcal{F}, E)) \otimes_{E \otimes_A^{\mathbf{L}} E^{\mathrm{opp}}}^{\mathbf{L}} E \\ &\xrightarrow{\mathrm{ev} \otimes \mathrm{id}} E \otimes_{E \otimes_A^{\mathbf{L}} E^{\mathrm{opp}}}^{\mathbf{L}} E, \end{aligned}$$

where $\mathrm{ev}: \mathcal{F} \otimes_A^{\mathbf{L}} \mathbf{R}\underline{\mathrm{Hom}}_E(\mathcal{F}, E) \rightarrow E$ is the evaluation map.

Derived Morita invariance of Hochschild homology [Kel2, Theorem 5.2] gives a quasi-isomorphism $\mathrm{HH}^A(E) \rightarrow \mathrm{HH}^A(\mathrm{per}_{\mathrm{dg}}(E))$, where $\mathrm{per}_{\mathrm{dg}}(E)$ is the dg category of cofibrant perfect E -complexes given by applying [Kel2, §4.6] to the dg algebra given by the Dold–Kan normalisation of E equipped with the Eilenberg–Zilber shuffle product. Writing $\mathcal{P}(X, Y) := \underline{\mathrm{Hom}}_E(Y, X)$, the bar construction $\mathrm{HH}^A(\mathrm{per}_{\mathrm{dg}}(E))$ contains, for \mathcal{F} cofibrant, a subcomplex given by the total complex of

$$\begin{aligned} \dots &\rightarrow \left(\mathcal{P}(\mathcal{F}, E) \otimes_A \mathcal{P}(E, E)^{\otimes_A^{n-1}} \otimes_A \mathcal{P}(E, \mathcal{F}) \right) \oplus \mathcal{P}(E, E)^{\otimes_A^{n+1}} \rightarrow \dots \\ \dots &\rightarrow (\mathcal{P}(\mathcal{F}, E) \otimes_A \mathcal{P}(E, \mathcal{F})) \oplus (\mathcal{P}(E, E) \otimes_A \mathcal{P}(E, E)) \rightarrow \mathcal{P}(\mathcal{F}, \mathcal{F}) \oplus \mathcal{P}(E, E), \end{aligned}$$

which is a quasi-isomorphic subcomplex since it is just a copy of

$$\mathrm{cone} \left(\mathrm{HH}^A(E, \underline{\mathrm{Hom}}_E(E, \mathcal{F}) \otimes_A \underline{\mathrm{Hom}}_E(\mathcal{F}, E)) \xrightarrow{(\beta, \mathrm{HH}^A(\mathrm{ev}))} \underline{\mathrm{Hom}}_E(\mathcal{F}, \mathcal{F}) \oplus \mathrm{HH}^A(E) \right)$$

and β is a quasi-isomorphism. It thus follows that the Lefschetz map can alternatively be characterised as the composite $\underline{\mathrm{Hom}}_E(\mathcal{F}, \mathcal{F}) \rightarrow \mathrm{HH}^A(\mathrm{per}_{\mathrm{dg}}(E)) \xleftarrow{\sim} \mathrm{HH}^A(E)$.

Proposition 1.17. Given $\mathcal{F} \in \mathrm{Perf}(X)$, the tangent map

$$\begin{aligned} T_{\mathcal{F}}(\Xi, M): T_{\mathcal{F}}(\mathrm{Perf}_X, M) &\rightarrow T_{\Xi(\mathcal{F})}(\tau_{\geq 0} J_X, M) \\ \tau_{\geq 0}(\mathbf{R}\underline{\mathrm{Hom}}_{O(X)}(\mathcal{F}, \mathcal{F} \otimes_R M)[-1]) &\rightarrow \mathrm{HC}^R(X) \otimes_R^{\mathbf{L}} M[-1] \end{aligned}$$

on simplicial R -modules M is given by the composite map

$$\mathbf{R}\underline{\mathrm{Hom}}_{O(X)}(\mathcal{F}, \mathcal{F} \otimes_R M) \xrightarrow{\mathcal{L}_{O(X)/R}} \mathrm{HH}^R(X) \otimes_R^{\mathbf{L}} M \xrightarrow{!} \mathrm{HC}^R(X) \otimes_R^{\mathbf{L}} M.$$

Proof. Writing $C = R \oplus M$ and $E = O(X)$, with λ the equivalence of Lemma 1.12 and $\theta: \text{HN}^{\mathbb{Q}}(E \otimes_{\mathbb{Q}} -) \rightarrow J_X$ the canonical map, we have a homotopy commutative diagram

$$\begin{array}{ccc}
 \text{HC}^{\mathbb{Q}}(E \otimes_R^{\mathbb{L}} C \rightarrow E)[-1] & \xrightarrow{\sim \text{B}} & \text{HN}^{\mathbb{Q}}(E \otimes_R^{\mathbb{L}} C \rightarrow E) \\
 \downarrow & & \swarrow \\
 \text{HC}^R(E \otimes_R^{\mathbb{L}} C \rightarrow E)[-1] & \xrightarrow{\sim \text{B}} & \text{HN}^R(E \otimes_R^{\mathbb{L}} C \rightarrow E) \\
 \downarrow \bar{\phi} & \swarrow (1 \otimes \delta) \circ \pi_{\text{HH}} & \downarrow \theta \\
 \text{HC}^R(E) \otimes_R^{\mathbb{L}} M[-1] & \xrightarrow{\sim \lambda} & J_X(C \rightarrow R)
 \end{array}$$

for $\bar{\phi}$ and $(1 \otimes \delta) \circ \pi_{\text{HH}}$ as in Lemma 1.15, which gives commutativity of that triangle; the top quadrilateral commutes by naturality of B , and the outer rectangle commutes up to canonical homotopy by the SBI sequence.

Now, $\Xi = \theta \circ \text{ch}^-$, and the composite map $K \xrightarrow{\text{ch}^-} \text{HN} \xrightarrow{\pi_{\text{HH}}} \text{HH}$ is just the Dennis trace ch , so commutativity of the lower triangle in the diagram above gives the tangent map of Ξ as the composite

$$T_{\mathcal{F}}(\text{Perf}, M) \xrightarrow{\text{ch} - \text{ch}(\mathcal{F})} \text{HH}^R(E) \otimes_R^{\mathbb{L}} \text{HH}^R(C \rightarrow R) \xrightarrow{1 \otimes \delta} \text{HC}^R(E) \otimes_R^{\mathbb{L}} M[-1].$$

Since $T_{\mathcal{F}}(\text{Perf}_E, M[-1])$ deloops $T_{\mathcal{F}}(\text{Perf}_E, M)$, the map is determined by its behaviour on morphisms $(\tau_{\geq 0} \text{RHom}_E(\mathcal{F}, \mathcal{F} \otimes_R M))[-1]$.

Take \mathcal{F} to be cofibrant and write E' for the simplicial ring corresponding to the dg algebra $\tau_{\geq 0} \text{Hom}_E(\mathcal{F}, \mathcal{F})$ (taking levelwise flat replacement if necessary). The Dennis trace $\text{ch}_{E'}: \text{GL}_1(E' \otimes_R C) \rightarrow \text{HH}^R(E' \otimes_R C)$ maps g to $g^{-1} \otimes g$, so for $\alpha \in E' \otimes_R M$, we have

$$\delta(\text{ch}_{E'}(1 + \alpha)) = \delta((1 - \alpha) \otimes (1 + \alpha)) = (1 - \alpha)\alpha = \alpha \in \text{HH}^R(E') \otimes_R M,$$

meaning the composite $\delta \circ \text{ch}_{E'}: E' \otimes_R M[-1] \rightarrow \text{HH}^R(E') \otimes_R^{\mathbb{L}} M[-1]$ is just the natural inclusion map.

Morita functoriality allows us to pass from $\text{ch}_{E'}$ to ch_E by composing with the maps $\text{HH}^R(E') \rightarrow \text{HH}^R(\text{per}_{d_g}(E)) \xrightarrow{\sim} \text{HH}^R(E)$, yielding the Lefschetz map $\mathcal{L}_{E/R}$. □

Combining Lemma 1.8 with Proposition 1.17 (after base change $R \rightarrow B$) gives the following:

Corollary 1.18. *Given a square-zero extension $I \rightarrow A \xrightarrow{e} B$ in $s\text{CAlg}_R$ and $\mathcal{F} \in \text{Perf}_X(B)$, the space $\text{Perf}_X(A)_{\mathcal{F}}$ of deformations of \mathcal{F} over A (i.e., the homotopy fibre of $\text{Perf}_X(A) \rightarrow \text{Perf}_X(B)$ over $\{\mathcal{F}\}$) fits into a natural commutative diagram*

$$\begin{array}{ccccc}
 \text{Perf}_X(A)_{\mathcal{F}} & \longrightarrow & \{\mathcal{F}\} & \xrightarrow{o_e} & \tau_{\geq 0}(\text{RHom}_{X_B}(\mathcal{F}, \mathcal{F} \otimes_B^{\mathbb{L}} I)[-2]) \\
 \Xi \downarrow & & \Xi \downarrow & & \downarrow |_{\circ \mathcal{L}_{O(X_B)/B}} \\
 J_X(A) & \longrightarrow & J_X(B) & \xrightarrow{o_e} & \tau_{\geq 0}(\text{HC}^R(X) \otimes_R^{\mathbb{L}} I[-2])
 \end{array}$$

of homotopy fibre sequences, for the Lefschetz map \mathcal{L} of Definition 1.16.

Remark 1.19. At this stage, everything being affine, the corollary implies that deformations of objects are unobstructed, simply because in this affine setting, the obstruction space $\text{HC}_{-2}^R(X, I)$ is 0. Taking derived global sections leads to the constructions for schemes and stacks in §2, where the relevant space becomes nontrivial.

1.5. dg algebras and dg categories

The definitions of J_X and Ξ generalise in the obvious way to dg algebras, and indeed dg categories, in place of simplicial R -algebras $\mathcal{O}(X)$. Although we will not need these results in §2, we now discuss the extent to which the preceding results extend to those settings.

When chain complexes are not bounded below, there are some subtleties in generalising Definition 0.1. Spaltenstein K -flatness [Spa] is the relevant flatness condition. As in [Kel1, §3.2], the Hochschild complex HH has a natural analogue as a double complex, whose direct sum total complex we take to be the Hochschild complex HH . The complexes HC , HN are then obtained from this in the usual way [Kel1, §3.4], as $\mathrm{Tot} \mathrm{CC}_{**}$ and $\mathrm{Tot}^{\Pi} \mathrm{CC}_{**}^N$, while HP has to be taken as a product-sum total complex $\mathrm{Tot}^{\Pi, \oplus} \mathrm{CC}_{**}^P$ to ensure quasi-isomorphism invariance.

Given a dg category \mathcal{C} for which $\mathrm{Perf}_{\mathcal{C}}$ is homotopy-homogeneous, the proof of Corollary 1.18 then automatically applies provided the conclusion of Goodwillie’s Theorem holds for \mathcal{C} (i.e., provided $\mathrm{HP}^{\mathbb{Q}}(\mathcal{C} \otimes_R^{\mathbb{L}} A) \xrightarrow{\sim} \mathrm{HP}^{\mathbb{Q}}(\mathcal{C} \otimes_R^{\mathbb{L}} B)$ for all square-zero extensions $e: A \rightarrow B$ of simplicial R -algebras).

From the long exact sequence of homology, the obstruction $\mathbb{L}\mathcal{L}_{(\mathcal{C} \otimes_R B)/B}(o_e(\mathcal{F}))$ to lifting $\Xi(\mathcal{F})$ from $\pi_0 J_{\mathcal{C}}(B)$ to $\pi_0 J_{\mathcal{C}}(A)$ is then guaranteed to vanish whenever the natural map $\mathrm{HC}_{-2}^R(\mathcal{C}, I) \rightarrow \mathrm{H}_{-1} J_{\mathcal{C}}(A)$ is injective, where $I = \ker e$. Injectivity of $\mathrm{HC}_{-2}^R(\mathcal{C}, I) \rightarrow \mathrm{HC}_{-2}^R(\mathcal{C}, A)$ would imply this; in particular, it holds if R is a field and $\pi_* A \rightarrow \pi_* B$. Thus, for any such dg category \mathcal{C} , the obstruction $o_e(\mathcal{F})$ to lifting a perfect $\mathcal{C} \otimes_R B$ -module \mathcal{F} from the derived category $\mathcal{D}(\mathcal{C} \otimes_R B)$ to $\mathcal{D}(\mathcal{C} \otimes_R A)$ would then lie in

$$\ker\left(\mathbb{L}\mathcal{L}_{(\mathcal{C} \otimes_R B)/B}: \mathrm{Ext}_{\mathcal{C} \otimes_R B}^2(\mathcal{F}, \mathcal{F} \otimes_B I) \rightarrow \mathrm{HC}_{-2}^R(\mathcal{C}, I)\right).$$

Goodwillie’s Theorem and hence Corollary 1.18 do not extend to all dg algebras and dg categories, but they are true for dg algebras concentrated in nonnegative chain degrees, via their Quillen equivalence with simplicial algebras given by the Dold–Kan and Eilenberg–Zilber constructions. All variants of cyclic homology are derived Morita invariant, preserve exact sequences in the Morita homotopy category, and are additive with respect to semi-orthogonal decompositions [Kel2, Theorem 5.2a]. Thus, the class of small \mathbb{Q} -linear dg categories for which the conclusion of Goodwillie’s Theorem holds is closed under those operations. As a special case, we have the following:

Corollary 1.20. *If \mathcal{N} is a geometric noncommutative scheme in the sense of [Orl, Definition 4.3] over a characteristic 0 field k , then for any square-zero extension $e: A \rightarrow B$ of commutative k -algebras with kernel I and for any perfect $\mathcal{N} \otimes_k B$ -complex \mathcal{F} , the obstruction $o_e(\mathcal{F})$ to lifting \mathcal{F} to a perfect $\mathcal{N} \otimes_k A$ -complex lies in*

$$\ker\left(\mathcal{L}_{(\mathcal{N} \otimes_k B)/B}: \mathrm{Ext}_{\mathcal{N} \otimes_k B}^2(\mathcal{F}, \mathcal{F} \otimes_B^{\mathbb{L}} I) \rightarrow \mathrm{HH}_{-2}^k(\mathcal{N}) \otimes_k I\right).$$

Proof. Up to derived Morita equivalence, \mathcal{N} is defined to be an admissible dg subcategory of the dg category $\mathrm{per}_{dg}(\mathcal{O}_Y)$ of perfect complexes on a smooth proper R -scheme Y . By [Orl, §4.4 following Theorem 3.25], \mathcal{N} is thus a smooth proper dg category, so [TV1, Lemma 2.8 and Corollary 3.15] imply that $\mathrm{Perf}_{\mathcal{N}}$ is a derived Artin ∞ -stack, and hence homotopy-homogeneous.

Since \mathcal{N} is a semi-orthogonal summand of $\mathrm{per}_{dg}(\mathcal{O}_Y)$, [Kel2, Theorem 5.2a] implies that it gives rise to a direct summand on all variants of cyclic homology. Because Y is quasi-compact and semi-separated, the Thomason–Trobaugh excision argument of [Kel1, §5] as summarised in [Kel2, Theorem 5.2b,c] means we can calculate cyclic homology of $\mathrm{per}_{dg}(\mathcal{O}_Y) \otimes_k A$ using Čech complexes, so the map $\mathrm{HP}^{\mathbb{Q}}(\mathcal{N} \otimes_k A) \rightarrow \mathrm{HP}^{\mathbb{Q}}(\mathcal{N} \otimes_k B)$ is a quasi-isomorphism as a consequence of Goodwillie’s theorem applied Zariski locally on Y .

Corollary 1.18 thus extends to this setting. Since k is a field, the morphism $A \rightarrow B$ admits a k -linear splitting, so $\mathrm{H}_0 J_{\mathcal{N}}(A) \rightarrow \mathrm{H}_0 J_{\mathcal{N}}(B)$ is surjective and $\mathbb{L}\mathcal{L}(o_e(\mathcal{F})) = 0$.

Finally, as in Remark 2.27, the Lefschetz principle makes \mathbb{L} injective for $\mathrm{per}_{dg}(\mathcal{O}_Y)$. Thus, $\mathbb{L}: \mathrm{HH}_*^k(\mathcal{N}) \rightarrow \mathrm{HC}_*^k(\mathcal{N})$ is also injective, being a restriction to direct summands, so $\mathcal{L}(o_e(\mathcal{F})) = 0$. \square

Remark 1.21. Until the final step, the proof of Corollary 1.20 only requires that \mathcal{N} be a smooth proper dg subcategory of $\text{per}_{dg}(\mathcal{O}_Y)$, for Y quasi-compact and semi-separated. In such cases, we can still say that $l\mathcal{L}(o_e(\mathcal{F})) = 0$.

We can even replace \mathcal{O}_Y with any presheaf \mathcal{A} of simplicial associative R -algebras on the site of affine open subschemes of Y for which $\pi_*\mathcal{A}$ admits a quasi-coherent \mathcal{O}_Y -algebra structure. In particular, we can take \mathcal{A} to be the structure hypersheaf $\mathcal{O}_{Y'}$ of any derived scheme Y' with underived truncation Y , using [Pri3, Theorem 6.14], or to be any quasi-coherent simplicial associative $\mathcal{O}_{Y'}$ -algebra.

The extension to \mathcal{A} follows because the excision argument of [Kell, §5], implying Zariski descent for HP, just relies on the functors $j^* : \text{per}_{dg}(\mathcal{A}(U)) \rightarrow \text{per}_{dg}(\mathcal{A}(V))$ being dg quotient maps whose dg kernels depend only on the complement $U \setminus V$. This follows from [Dri, Proposition 1.4ii] and the isomorphism $H_*(j^*M) \cong \mathcal{O}(V) \otimes_{\mathcal{O}(U)} H_*M$ given by the algebraic Eilenberg–Moore spectral sequence.

2. The Abel–Jacobi map for (derived) schemes and stacks

From now on, all rings will be commutative.

2.1. Derived de Rham cohomology

Definition 2.1. Given $A \in s\text{CAlg}_{\mathbb{Q}}$ and $B \in s\text{CAlg}_A$, define the de Rham complex to be the chain complex

$$\text{DR}(B/A) := \prod_{n \geq 0} \Omega^n(B/A)[n] = \prod_{n \geq 0} (\Lambda_B^n \Omega^1(B/A))[n],$$

with differential given by combining the differentials on the chain complexes $\Omega^n(B/A)$ with the de Rham differential (i.e., the derivation induced by $d : B \rightarrow \Omega^1(B/A)$). This has a Hodge filtration given by $F^p \text{DR}(B/A) := \prod_{n \geq p} \Omega^n(B/A)[n]$.

Beware that the de Rham complex is usually regarded as a cochain complex, so negative homology groups will correspond to positive cohomology groups; we are using chain complexes to facilitate comparison with cyclic homology.

The functors $\Omega^n(-/A)$ clearly preserve weak equivalences between cofibrant (and indeed levelwise ind-smooth) simplicial A -algebras and the product total complex sends column quasi-isomorphisms of second quadrant double complexes to quasi-isomorphisms (see e.g. [Wei2, Acyclic Assembly Lemma 2.7.3]), so $\text{DR}(-/A)$ preserves weak equivalences between such objects.

Definition 2.2. Given $A \in s\text{CAlg}_{\mathbb{Q}}$ and $B \in s\text{CAlg}_A$, define the left-derived de Rham complex $\text{LDR}(B/A)$ by first taking a cofibrant replacement $\tilde{B} \rightarrow B$ over A in the model structure of [Qui, §II.4], and then setting

$$\mathbf{L}\Omega^p(B/A) := \Omega^p(\tilde{B}/A), \quad \mathbf{LDR}(B/A) := \text{DR}(\tilde{B}/A).$$

Note that this is well-defined up to quasi-isomorphism, that such replacements can be chosen functorially in both B and A , and that $\mathbf{L}\Omega^1(B/A)$ is a model for the cotangent complex $\mathbb{L}(B/A)$.

The complex $\text{LDR}(B/A)$ has a Hodge filtration $F^p \text{LDR}(B/A) := F^p \text{DR}(\tilde{B}/A)$, and we write $\text{LDR}(B/A)/F^p := \text{LDR}(B/A)/F^p \text{LDR}(B/A)$.

Remark 2.3. Following the ideas of [Gro] as developed in [Sim, GR], there is a more conceptual interpretation of the derived de Rham complex.

For any functor F on $s\text{CAlg}_A$, we may define $F_{\text{inf}}(C) := F((\pi_0 C)^{\text{red}})$, and then $F_{\text{strat}}(C) := \text{Im}(\pi_0 F(C) \rightarrow F_{\text{inf}}(C))$ whenever images make sense. Note that if F is represented by a smooth algebraic space, then $F_{\text{strat}} = F_{\text{inf}}$.

Now for $A \rightarrow B$ as above, derived Hom in the model category $s\mathcal{CAlg}_A$ gives a simplicial set-valued functor $\mathbf{R}\mathrm{Spec} B = \underline{\mathrm{Hom}}_{s\mathcal{CAlg}_A}(\tilde{B}, -)$ on $s\mathcal{CAlg}_A$, with $(\mathbf{R}\mathrm{Spec} B)_{\mathrm{inf}}(C) \cong \mathrm{Hom}_{\mathcal{CAlg}_{\pi_0 A}}(\pi_0 B, (\pi_0 C)^{\mathrm{red}})$.

Now, $F_{\mathrm{strat}}(C)$ is equivalent to the Čech nerve of $F(C)$ over $F_{\mathrm{inf}}(C)$, which for $F = \mathbf{R}\mathrm{Spec} B$ is represented in level n by formal completions of the diagonal map $\tilde{B}^{\otimes_A(n+1)} \rightarrow \tilde{B}$. Homology of symmetric powers then shows that $\mathbf{LDR}(B/A) \simeq \mathbf{R}\Gamma((\mathbf{R}\mathrm{Spec} B)_{\mathrm{strat}}, \mathcal{O})$, where \mathcal{O} is the hypersheaf given by $\mathcal{O}(C) = C$.

Proposition 2.4. *For a levelwise flat morphism $A \rightarrow B$ in $s\mathcal{CAlg}_{\mathbb{Q}}$ and for all $p \in \mathbb{Z}$, there are canonical quasi-isomorphisms*

$$\begin{aligned} \mathrm{HP}^A(B)^{(p)} &\simeq \mathbf{LDR}(B/A)[-2p], & \mathrm{HC}^A(B)^{(p)} &\simeq (\mathbf{LDR}(B/A)/F^{p+1})[-2p], \\ \mathrm{HN}^A(B)^{(p)} &\simeq F^p \mathbf{LDR}(B/A)[-2p], & \mathrm{HH}^A(B)^{(p)} &\simeq \mathbf{L}\Omega^p(B/A)[-p], \end{aligned}$$

with the SBI sequences corresponding to the short exact sequences $0 \rightarrow F^p \mathbf{LDR} \rightarrow \mathbf{LDR} \rightarrow \mathbf{LDR}/F^p \rightarrow 0$ and $0 \rightarrow \mathbf{L}\Omega^p[-p] \rightarrow \mathbf{LDR}/F^{p+1} \rightarrow \mathbf{LDR}/F^p \rightarrow 0$.

Proof. When A and B are concentrated in degree 0, with B smooth over A , this is a well-known consequence of the Hochschild–Kostant–Rosenberg theorem, as in [Wei2, Theorem 9.8.13], and that proof immediately generalises to filtered colimits of smooth morphisms, including infinite polynomial algebras. As observed in [Maj, §5], the general case for HH and HC then follows by taking a cofibrant replacement for B and passing to the total complex of the resulting bisimplicial diagram. The expressions for HN and HP follow similarly because product total complexes respect the relevant quasi-isomorphisms. \square

Remark 2.5. Note that [Emm, Theorem 2.2] (following [FT, Theorem 5]) shows that for a finitely generated algebra B over a characteristic zero field k , the complex $\mathrm{HP}^k(B)^{(p)}[2p]$ is quasi-isomorphic to the infinitesimal cohomology complex, or equivalently to Hartshorne’s algebraic de Rham cohomology [Har] over k . For an alternative proof, see [Bha].

2.2. Global constructions

As in [Wei1], we now use naturality of the affine constructions for cyclic homology to pass from local to global.

Fixing terminology, we will refer to a simplicial set-valued functor on $s\mathcal{CAlg}_R$ as a derived ∞ -stack over R if it preserves weak equivalences and satisfies étale hyperdescent, for étale morphisms in the sense of [TV2, Theorem 2.2.2.6]; this is a D^- -stack in the terminology of [TV2, Definition 2.2.2.14]. Derived affine schemes of the form $\mathbf{R}\mathrm{Spec} A$ for $A \in s\mathcal{CAlg}_R$ (see Remark 2.3) are known as derived affine schemes.

2.2.1. Perfect complexes

Definition 2.6. Given a derived ∞ -stack X over \mathbb{Q} , define the simplicial set $\mathrm{Perf}(X)$ to be the space $\mathbf{R}\Gamma(X, \mathrm{Perf})$ of maps from X to Perf in the simplicial category of simplicial set-valued functors on $s\mathcal{CAlg}_R$.

Note that this is consistent with Definition 1.2 when X is a derived affine scheme, by the model Yoneda lemma.

Remarks 2.7. Using the explicit hyperdescent formulae of [Pri3, Examples 1.15], when X is a strongly quasi-compact derived Artin n -stack, the simplicial semiring $\mathrm{Perf}(X)$ can be constructed as follows. First, [Pri3, Theorem 4.7] provides the existence of a suitable resolution of the étale hypersheaf X by a derived Artin hypergroupoid X_\bullet , which is a simplicial derived affine scheme satisfying properties analogous to those of a Kan complex [Pri3, Examples 3.5]. We then define a cosimplicial simplicial semiring given by $\mathbf{C}^n(X_\bullet, \mathrm{Perf}(\mathcal{O}_X)) := \mathrm{Perf}(X_n)$, and set

$$\mathrm{Perf}(X) = \mathbf{RTot}_{s\mathrm{Set}} \mathbf{C}^\bullet(X_\bullet, \mathrm{Perf}(\mathcal{O}_X)),$$

where $\mathbf{RTot}_{s\text{Set}}$ is the derived total functor from cosimplicial simplicial sets to simplicial sets, as in [GJ, §VIII.1].

Because Perf forms an étale hypersheaf, the definition of $\text{Perf}(X)$ above agrees with the standard definition for underived schemes, and indeed for algebraic stacks. In the case when X is a quasi-compact semi-separated scheme, X_\bullet can just be constructed by taking the Čech nerve of an affine cover, in which case \mathbf{C}^\bullet is just a Čech complex.

In our main applications in §2.5, X will be of the form $Y \otimes_R A$, for $A \in s\text{CAlg}_R$ and Y a smooth quasi-compact semi-separated scheme over a Noetherian ring R . For such applications, we can regard X as being the derived scheme associated to a dg scheme (or even a dg manifold) in the sense of [CFK] (for instance, by the construction of [Pri3, §6.4]). Then $\text{Perf}(X)$ corresponds to the space of compact objects in the dg derived category of A -modules in complexes of quasi-coherent sheaves on Y , via the equivalences summarised in [EP, Remark 5.32].

2.2.2. Derived de Rham cohomology

We begin with very general definitions; readers interested in nothing more exotic than a derived Artin n -stack can jump straight to Lemma 2.10 and take it as a definition.

Definition 2.8. Given a derived ∞ -stack X over R , we write $F^p\text{LDR}(X/R) := \mathbf{R}\Gamma(X, F^p\text{LDR}(\mathcal{O}/R))$, where \mathcal{O}_X is the étale hypersheaf $\mathbf{R}\text{Spec } A \mapsto A$ on the site of derived affine R -schemes over X . Equivalently, $F^p\text{LDR}(X/R)$ is the homotopy end $\int_{A \in s\text{CAlg}_R}^h F^p\text{LDR}(A/R)^{X(A)}$.

For such X and $A \in s\text{CAlg}_R$, we then write X_A for the derived ∞ -stack X over A given by base change (i.e., composition with the forgetful functor $s\text{CAlg}_A \rightarrow s\text{CAlg}_R$).

More generally, given a morphism $X \rightarrow Y$ of derived ∞ -stacks, we can define the étale hypersheaf $X_{\mathcal{O}_Y}$ (taking values in derived ∞ -stacks over \mathcal{O}_Y) on the site of derived affine R -schemes $V \simeq \mathbf{R}\text{Spec } A$ over Y by letting $X_{\mathcal{O}_Y}(V)$ be the derived ∞ -stack over A sending B to $\mathbf{R}\text{Map}_Y(\mathbf{R}\text{Spec } B, X)$

Definition 2.9. Given a morphism $X \rightarrow Y$ of derived ∞ -stacks over R , set $F^p\text{LDR}(X/Y) := \mathbf{R}\Gamma(Y, F^p\text{LDR}(X_{\mathcal{O}_Y}/\mathcal{O}_Y))$, the $\mathbf{R}\Gamma(Y, \mathcal{O}_Y)$ -module of derived global sections of the \mathcal{O}_Y -module $F^p\text{LDR}(X_{\mathcal{O}_Y}/\mathcal{O}_Y)$ in unbounded complexes.

Equivalently, for the functors $F^p\text{LDR}(-/-)$ and $(X/Y): (A \rightarrow B) \mapsto X(B) \times_{Y(B)}^h Y(A)$ on the arrow category $s\text{CAlg}_R^{[1]}$, we can interpret $F^p\text{LDR}(X/Y)$ as derived global sections of (X/Y) with values in $F^p\text{LDR}$ (i.e., as the homotopy end $\int_{(A \rightarrow B)}^h F^p\text{LDR}(B/A)^{(X/Y)(A \rightarrow B)}$).

For morphisms $X \rightarrow Y$ of strongly quasi-compact derived Artin n -stacks, the following lemma allows us to express $F^p\text{LDR}(X/Y)$ simply in terms of the algebraic complexes from Definition 2.2, since double application of [Pri3, Theorem 4.7] gives resolutions by derived affine schemes of the required form. When Y itself is a derived affine scheme, we can take $\tilde{Y}_\bullet = Y$, and the condition is then just that \tilde{X}_\bullet be a hypercover of X .

We could alternatively take this formula as a definition, observing that since hypercovers yield quasi-isomorphisms of such complexes, it defines a functor on the relative category of strongly quasi-compact derived Artin n -stacks, by [EP, Theorem 6.11].

Lemma 2.10. Given a morphism $X \rightarrow Y$ of derived ∞ -stacks over R , a simplicial hypercover $\tilde{Y}_\bullet \rightarrow Y$ for the étale topology, and a relative simplicial hypercover $\tilde{X}_\bullet \rightarrow X \times_Y^h \tilde{Y}_\bullet$, the complex $F^p\text{LDR}(X/Y)$ is quasi-isomorphic to the product total complex of the double complex $n \mapsto F^p\text{LDR}(\tilde{X}_n/\tilde{Y}_n)$.

Proof. Given a morphism from $(A \rightarrow B)$ to $(A' \rightarrow B')$ in $s\text{CAlg}_R^{[1]}$ with $A \rightarrow A'$ and $B \rightarrow B'$ étale, we have a natural quasi-isomorphism $\mathbf{L}\Omega^p(B/A) \otimes_B^{\mathbf{L}} B' \rightarrow \mathbf{L}\Omega^p(B'/A')$. Thus, the functor $\mathbf{L}\Omega^p$ from the arrow category $s\text{CAlg}_R^{[1]}$ to the category of R -linear unbounded complexes satisfies hyperdescent with respect to the topology τ generated by finite families $\{(A \rightarrow B) \rightarrow (A'(i) \rightarrow B'(i))\}_i$ for which the maps $A \rightarrow A'(i)$ and $B \rightarrow B'(i)$ are étale and $\coprod_i \text{Spec } \pi_0 B'(i) \rightarrow \text{Spec } \pi_0 B$ is surjective. By taking

homotopy limits of finite extensions, it follows that the functor **LDR** on the arrow category also satisfies τ -hyperdescent.

The question thus reduces to showing that the simplicial diagram $n \mapsto (\tilde{X}_n/\tilde{Y}_n)$ is a τ -hypercover of (X/Y) . This in turn reduces to showing that if $Y' \rightarrow Y$ and $X' \rightarrow X \times_Y Y'$ are étale local epimorphisms, then $(X'/Y') \rightarrow (X/Y)$ is a τ -local epimorphism, but this follows easily from the factorisation

$$\begin{aligned} X'(B) \times_{Y'(B)}^h Y'(A) &\rightarrow (X \times_Y^h Y')(B) \times_{Y'(B)}^h Y'(A) \\ &\simeq X(B) \times_{Y(B)}^h Y'(A) \rightarrow X(B) \times_{Y(B)}^h Y(A). \end{aligned} \quad \square$$

Definition 2.11. Given an Artin n -stack X locally of finite presentation over \mathbb{C} , we can form a resolution \tilde{X}_\bullet of X by l.f.p. \mathbb{C} -schemes as in [Pri3, Theorem 4.7], giving a simplicial topological space $\tilde{X}_\bullet(\mathbb{C})_{\text{an}}$ on taking the analytic topology. Since \tilde{X}_\bullet is unique up to smooth hypercovers and since hypercovers induce equivalences on categories of local systems and their cohomology, the homotopy type of the homotopy colimit is independent of choices made. It can be realised by the fat geometric realisation $|\tilde{X}_\bullet(\mathbb{C})_{\text{an}}|$, and we denote it simply by $|X(\mathbb{C})_{\text{an}}|$. When X is an algebraic space, note that this just recovers $X(\mathbb{C})_{\text{an}}$.

Given a derived Artin n -stack X over \mathbb{C} whose underived truncation $\pi^0 X$ is locally of finite type, we write $|X(\mathbb{C})_{\text{an}}| := |(\pi^0 X)(\mathbb{C})_{\text{an}}|$; the notation is justified because X and $\pi^0 X$ have the same \mathbb{C} -points.

When working over Artinian simplicial \mathbb{C} -algebras (i.e., simplicial \mathbb{C} -algebras which are levelwise Artinian with bounded Dold–Kan normalisation), we can compare derived de Rham cohomology with Betti cohomology of this analytic space of \mathbb{C} -points:

Lemma 2.12. *For X a derived Artin n -stack over a local Artinian simplicial \mathbb{C} -algebra A , with underived truncation locally of finite type, there is a canonical zigzag of quasi-isomorphisms $\mathbf{LDR}(X/A) \simeq \mathbf{R}\Gamma(|X(\mathbb{C})_{\text{an}}|, A)$.*

Proof. Given a cofibrant simplicial A -algebra B with $\pi_0 B$ finitely generated, we can write B as the filtered colimit of its levelwise finitely generated cofibrant A -subalgebras B' with $\pi_0 B' \cong \pi_0 B$. We then have zigzags

$$\mathbf{R}\Gamma((\text{Spec } \pi_0 B)_{\text{an}}, A_n) \rightarrow \mathbf{R}\Gamma(\widehat{(\text{Spec } B'_n)}_{\text{an}}, \Omega_{\mathcal{O}/A_n}^{\bullet, \text{an}}) \leftarrow \hat{\Omega}_{B'_n/A_n}^{\bullet},$$

where completions are with respect to the map $B'_n \rightarrow \pi_0 B$, with $\widehat{(\text{Spec } B'_n)}_{\text{an}}$ being the formal Stein space of [Har, §I.6] and $\Omega_{\mathcal{O}/A_n}^{\bullet, \text{an}}$ the analytic de Rham cohomology complex; note that $\hat{\Omega}_{B'_n/A_n}^{\bullet}$ is Hartshorne’s algebraic de Rham cohomology of $\pi_0 B$ over A_n . Since everything in sight is flat over the local Artinian ring A_n , to see that these maps are quasi-isomorphisms, it suffices to know that they are so after base change along $A_n \rightarrow \mathbb{C}$, which they are by [Har, Theorem IV.1.1].

Now consider the zigzag $\mathbf{DR}(B/A) \leftarrow \varinjlim_{B'} \mathbf{DR}(B'/A) \rightarrow \varinjlim_{B'} \widehat{\mathbf{DR}(B'/A)}$, where we write $\widehat{\mathbf{DR}(B'/A)} := \text{Tot}^{\Pi} \hat{\Omega}_{B'_n/A_n}^{\bullet}$. Again, these maps are all quasi-isomorphisms because after base change along $A \rightarrow \mathbb{C}$, they are all quasi-isomorphic to $\mathbf{HP}^{\mathbb{C}}(\pi_0 B \otimes_{\pi_0 A} \mathbb{C})$: the first two cases follow from Proposition 2.4 combined with Theorem 1.10 (replacing \mathbb{Q} with \mathbb{C} in Goodwillie’s proof), and the third because [Emm, Theorem 2.2] (following [FT] and Goodwillie’s theorem) gives $\hat{\Omega}_{B'_n/A_n}^{\bullet} \otimes_{A_n} \mathbb{C} \simeq \mathbf{HP}^{\mathbb{C}}(\pi_0 B \otimes_{\pi_0 A} \mathbb{C})$ for all n and B' .

Putting everything together and taking filtered colimits, we have a canonical zigzag $\mathbf{DR}(B/A) \simeq \mathbf{R}\Gamma((\text{Spec } \pi_0 B)_{\text{an}}, A)$. Applying this locally to a cofibrant simplicial A -algebra resolution $\tilde{\mathcal{O}}_X$ for \mathcal{O}_X and taking derived global sections then gives us the required zigzag $\mathbf{LDR}(X/A) \simeq \mathbf{R}\Gamma(|X(\mathbb{C})_{\text{an}}|, A)$. \square

2.3. Generalised Abel–Jacobi maps

Definition 2.13. Given a morphism $X \xrightarrow{f} Y$ of derived ∞ -stacks over a simplicial \mathbb{Q} -algebra k , define

$$J^P(X/Y, k) := \text{cocone}(\mathbf{LDR}(X/k) \rightarrow \mathbf{LDR}(X/Y)/F^P)[-2p].$$

Note that when X is affine, Proposition 2.4 gives an equivalence $J^p(X/R, \mathbb{Q}) \simeq J^p(X/R)$ with the construction of Definition 1.1.

Example 2.14. If $C \rightarrow A$ is a morphism of Artinian local \mathbb{C} -algebras and X a smooth proper scheme over A , then the complexes $J^p(X/A, C)$ admit an underived analytic description as follows.

Applying GAGA to graded pieces as in [Blo, Proposition 3.8] gives a quasi-isomorphism $\mathbf{R}\Gamma(X, \Omega_{X/A}^\bullet/F^p) \rightarrow \mathbf{R}\Gamma(X(\mathbb{C})_{\text{an}}, \Omega_{X_{\text{an}}/A}^\bullet/F^p)$, and Lemma 2.12 gives a compatible quasi-isomorphism $\mathbf{R}\Gamma(X, \mathbf{LDR}(\mathcal{O}_X/C)) \simeq \mathbf{R}\Gamma(X(\mathbb{C})_{\text{an}}, C)$. Combining these, we have

$$J^p(X/A, C) \simeq \mathbf{R}\Gamma(X(\mathbb{C})_{\text{an}}, C \rightarrow \mathcal{O}_{X/A}^{\text{an}} \xrightarrow{d} \Omega_{X/A}^{1,\text{an}} \xrightarrow{d} \dots \xrightarrow{d} \Omega_{X/A}^{p-1,\text{an}})[-2p],$$

giving our space an interpretation as a form of Deligne cohomology.

Definition 2.15. Given a simplicial commutative \mathbb{Q} -algebra R and a derived ∞ -stack X over R , define J_X^p to be the functor on $s\text{CAlg}_R$ given by $J_X^p(A) := J^p(X_A/A, \mathbb{Q})$.

Note that Proposition 2.4 ensures this is equivalent to Definition 1.1 when X is a derived affine scheme represented by a simplicial R -algebra $O(X)$ chosen to be levelwise flat.

Since Lemma 1.12 and Proposition 2.4 give a canonical zigzag of quasi-isomorphisms between the corresponding affine constructions, passing to homotopy limits gives the following:

Lemma 2.16. For $f \in H_0(J_X^p(A))$, the tangent space $T_f(\tau_{\geq 0}(J_X^p, M))$ is canonically quasi-isomorphic to $\tau_{\geq 0}((\mathbf{LDR}(X/R)/F^p) \otimes_R^L M)[2p - 1]$.

Remark 2.17. Although not needed for our applications, if we replace \mathbb{Q} with any simplicial \mathbb{Q} -algebra k throughout Definition 2.15, the description of the tangent space in Lemma 2.16 remains valid.

Definition 2.18. Globalising definition 1.3, define the Abel–Jacobi map

$$\Xi_k: \text{Perf}(X) \rightarrow \tau_{\geq 0} \prod_{p \geq 0} J^p(X/Y, k)$$

(which we simply denote as Ξ when $k = \mathbb{Q}$) as follows. Via the equivalences of Proposition 2.4, the Goodwillie–Jones Chern character gives us maps

$$\text{ch}_p^-: \text{Perf}(\mathcal{O}_X) \rightarrow \tau_{\geq 0}(F^p \mathbf{LDR}(X/k)[-2p]).$$

We then take derived global sections $\mathbf{R}\Gamma(X, -)$ and compose with the natural maps

$$\mathbf{R}\Gamma(X, \tau_{\geq 0}(F^p \mathbf{LDR}(X/k)[-2p])) \rightarrow \tau_{\geq 0}(\mathbf{R}\Gamma(X, F^p \mathbf{LDR}(X/k)))[-2p] \rightarrow \tau_{\geq 0} J^p(X/Y, k).$$

Definition 2.19. For $\mathcal{F} \in \text{Perf}_X(A)$ and a simplicial A -module M , write

$$\xi^i: \text{Ext}_{\mathcal{O}_{X_A}}^{i+1}(\mathcal{F}, \mathcal{F} \otimes_A^L M) \rightarrow \prod_{p \geq 0} H^{2p+i-1}((\mathbf{LDR}(X/R)/F^p) \otimes_R^L M)$$

for the tangent map

$$D_{\mathcal{F}}^i(\Xi, M): D_{\mathcal{F}}^i(\text{Perf}_X, M) \rightarrow \prod_{p \geq 0} D_{\Xi_p(\mathcal{F})}^i(\tau_{\geq 0} J_X^p, M).$$

Substituting our hypersheaves in Proposition 1.17 and taking derived global sections yields the following:

Proposition 2.20. The tangent map ξ_p^i is given by composing the Lefschetz map $\mathcal{L}_{X/R}: \text{Ext}_{\mathcal{O}_{X_A}}^{i+1}(\mathcal{F}, \mathcal{F} \otimes_A^L M) \rightarrow H^{p+i}(X, \mathbf{L}\Omega_{X/R}^{p-1} \otimes_R^L M)$ with the canonical map $!: H^{p+i}(X, \mathbf{L}\Omega_{X/R}^{p-1} \otimes_R^L M) \rightarrow H^{2p+i-1}(X, (\mathbf{LDR}(\mathcal{O}_X/R)/F^p) \otimes_R^L M)$.

Remark 2.21. The construction of the Atiyah–Hochschild character $\text{AH}(\mathcal{F})$ of [BF2, §5] just makes it the dual of the Lefschetz map, in the sense that $\mathcal{L}(\alpha) = \text{tr}(\text{AH}(\mathcal{F}) \circ \alpha)$. Thus, [BF2, Theorem 5.1.3 and Proposition 6.2.1] ensure that \mathcal{L} is the same as the semiregularity map σ of [BF1], given by applying the exponential of the Atiyah class then taking the trace.

Thus, the semiregularity map σ is induced by our Abel–Jacobi map $\Xi_{p,C}$, which Example 2.14 interprets as a Chern character taking values in a form of Deligne cohomology, exactly as anticipated in [BF1, §1].

Combining Lemma 1.8 with Proposition 2.20 and Remark 2.21, or just substituting in Corollary 1.18 and taking derived global sections, gives the following:

Corollary 2.22. *Given a square-zero extension $I \rightarrow A \xrightarrow{e} B$ in $s\text{CAlg}_R$ and an object $\mathcal{F} \in \text{Perf}_X(B)$, the space $\text{Perf}_X(A)_{\mathcal{F}}$ of deformations of \mathcal{F} over A fits into a natural commutative diagram*

$$\begin{CD} \text{Perf}_X(A)_{\mathcal{F}} @>>> \{\mathcal{F}\} @>{o_e}>> \tau_{\geq 0}(\mathbf{R}\underline{\text{Hom}}_{\mathcal{O}_{XB}}(\mathcal{F}, \mathcal{F} \otimes_B^{\mathbf{L}} I)[-2]) \\ @V{\Xi_p}VV @V{\Xi_p}VV @V{\mathcal{L}_{p-1} = \sigma_{p-1}}VV \\ J_X^p(A) @>>> J_X^p(B) @>{o_e}>> \tau_{\geq 0}\mathbf{R}\Gamma(X, (\mathbf{LDR}(\mathcal{O}_X/R)/F^p) \otimes_R^{\mathbf{L}} I[-2p]) \end{CD}$$

of homotopy fibre sequences, for the Lefschetz map \mathcal{L} of [BNT] and semiregularity map σ of [BF1].

Thus, the image of $\mathcal{L}_{p-1}(o_e(\mathcal{F}))$ in $\mathbb{H}^{2p}(X, (\mathbf{LDR}(\mathcal{O}_X/R)/F^p) \otimes_R^{\mathbf{L}} I)$ is the obstruction to lifting $\Xi(\mathcal{F})$ from $\text{H}_0 J_X(B)$ to $\text{H}_0 J_X(A)$.

Remark 2.23. In particular, if $A \rightarrow B$ admits a section in the derived category of R -modules (automatic if R is a field and $\pi_* A \rightarrow \pi_* B$), then $J_X^p(A) \simeq J_X^p(B) \oplus (\mathbf{LDR}(X/R)/F^p) \otimes_R I$, so $\text{H}_0 J_X(A) \rightarrow \text{H}_0 J_X(B)$ is surjective and Corollary 2.22 implies that

$$o_e(\mathcal{F}) \in \ker(\mathcal{L}_{p-1}: \text{Ext}_{\mathcal{O}_{XA}}^2(\mathcal{F}, \mathcal{F} \otimes_A^{\mathbf{L}} I) \rightarrow \mathbb{H}^{2p}(X, (\mathbf{LDR}(\mathcal{O}_X/R)/F^p) \otimes_R^{\mathbf{L}} I)).$$

2.4. Horizontal sections

We now introduce variants of the obstruction $\mathcal{L}_{p-1}(o(\mathcal{F}))$ which are potentially weaker but tend to be more tractable. Our most generally applicable result is the following, writing $\mathbf{L}\Omega^j(X/S) := \mathbf{R}\Gamma(S, \mathbf{R}\Gamma(X_{\mathcal{O}_S}, \mathbf{L}\Omega_{X_{\mathcal{O}_S}/\mathcal{O}_S}^j))$ and $\mathbf{L}\mathcal{F}\Omega^j(X/S) := \mathbf{R}\Gamma(S, \mathbf{R}\Gamma(X_{\mathcal{O}_S}, \mathcal{F} \otimes_{\mathcal{O}_S}^{\mathbf{L}} \mathbf{L}\Omega_{X_{\mathcal{O}_S}/\mathcal{O}_S}^j))$:

Corollary 2.24. *Take a morphism $f: X \rightarrow S$ of derived ∞ -stacks over \mathbb{Q} and a closed immersion $e: S' \hookrightarrow S$ [TV2, Definition 2.2.3.5] defined by an ideal \mathcal{F} with $\pi_0 \mathcal{F}$ nilpotent in $\mathcal{O}_{\pi_0 S} = \pi_0 \mathcal{O}_S$.*

Then for $X' := X \times_S^h S'$, the Goodwillie–Jones Chern character ch_p^- for $\mathcal{O}_{X'}$ over \mathcal{O}_S induces a form of Abel–Jacobi map $\Xi_{p,S}$ from $K_0(X')$ to the cohomology group

$$\mathbb{H}^{2p} \text{Tot}^{\Pi}(\mathbf{L}\mathcal{F}\Omega^0 \xrightarrow{d} \mathbf{L}\mathcal{F}\Omega^1 \xrightarrow{d} \dots \xrightarrow{d} \mathbf{L}\mathcal{F}\Omega^{p-1} \xrightarrow{d} \mathbf{L}\Omega^p \xrightarrow{d} \mathbf{L}\Omega^{p+1} \xrightarrow{d} \dots)(X/S).$$

If \mathcal{F} is square-zero, then for any perfect complex \mathcal{F} over X' , the image of $\Xi_{p,S}(\mathcal{F})$ in $\mathbb{H}^{2p} \mathbf{L}\mathcal{F}\Omega^{<p}(X/S)$ is given by applying the composite map

$$\text{Ext}_{\mathcal{O}_{X'}}^2(\mathcal{F}, \mathcal{F} \otimes_{\mathcal{O}_S}^{\mathbf{L}} \mathcal{F}) \xrightarrow[=\sigma_{p-1}]{\mathcal{L}_{p-1}} \mathbb{H}^{p+1} \mathbf{L}\mathcal{F}\Omega^{p-1}(X/S) \rightarrow \mathbb{H}^{2p} \mathbf{L}\mathcal{F}\Omega^{<p}(X/S)$$

to the obstruction $o_e(\mathcal{F})$ to deforming \mathcal{F} to an \mathcal{O}_X -module in complexes.

Proof. The map $\Xi_{p,S}$ is defined as the composite of Ξ_p with the natural map

$$\begin{aligned} J^p(X'/S', \mathbb{Q}) &= \text{cocone}(\mathbf{LDR}(X'/\mathbb{Q}) \rightarrow \mathbf{LDR}(X'/S')/F^p)[-2p] \\ &\xrightarrow{\sim} \text{cocone}(\mathbf{LDR}(X/\mathbb{Q}) \rightarrow \mathbf{LDR}(X'/S')/F^p)[-2p] \\ &\rightarrow \text{cocone}(\mathbf{LDR}(X/S) \rightarrow \mathbf{LDR}(X'/S')/F^p)[-2p] =: J^p(X'/S', S), \end{aligned}$$

where the equivalence of the second line follows from Theorem 1.10 because $\pi^0 X' \rightarrow \pi^0 X$ is a nilpotent thickening. Since $\mathbf{L}\mathcal{F}\Omega^j(X/S) \simeq \text{cocone}(\mathbf{L}\Omega^j(X/S) \rightarrow \mathbf{L}\Omega^j(X'/S'))$, this gives the target in the required form.

When \mathcal{F} is square-zero, substituting in Corollary 2.22 and taking derived global sections over S then gives us the first two rows of the commutative diagram

$$\begin{array}{ccccc} \text{Perf}(X)_{\mathcal{F}} & \longrightarrow & \{\mathcal{F}\} & \xrightarrow{o_e} & \mathbf{R}\underline{\text{Hom}}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F} \otimes_{f^{-1}\mathcal{O}_S}^{\mathbf{L}} f^{-1}\mathcal{F})[-2] \\ \Xi_p \downarrow & & \Xi_p \downarrow & & \mathcal{L}_{p-1} \downarrow = \sigma_{p-1} \\ J^p(X/S, \mathbb{Q}) & \longrightarrow & J^p(X'/S', \mathbb{Q}) & \xrightarrow{o_e} & \mathbf{L}\mathcal{F}\Omega^{<p}(X/S)[-2p] \\ \downarrow & & \downarrow & & \parallel \\ F^p\mathbf{LDR}(X/S)[-2p] & \longrightarrow & J^p(X'/S', S) & \xrightarrow{o_e} & \mathbf{L}\mathcal{F}\Omega^{<p}(X/S)[-2p] \end{array}$$

of homotopy fibre sequences, where we have omitted $\tau_{\geq 0}$ from all complexes to lighten the notation; the third sequence follows from the simple calculation that $\text{cone}(F^p\mathbf{LDR}(X/S) \rightarrow J^p(X'/S', S)[2p]) \simeq \mathbf{L}\mathcal{F}\Omega^{<p}(X/S)$. This diagram yields the required description since the first two composite vertical maps are ch_p^- and $\Xi_{p,S}$. \square

Corollary 2.24 relates the semiregularity map to the Goodwillie–Jones Chern character. Over \mathbb{C} , we now give a slightly weaker, but more accessible, statement in terms of the topological Chern character.

When applied to the case where X is a smooth proper scheme over a local Artinian \mathbb{C} -algebra A , the following corollary establishes the conjectures of [BF1] and hence those of [Blo]. By analogy with [Blo, 3.9], we can regard it as saying that the semiregularity map constrains deformations of the Chern character $\text{ch}_p(\mathcal{F})$ as a horizontal section in F^p .

Writing $F^p\mathbf{H}^*(\mathbf{LDR}(X/A)) := \text{Im}(\mathbf{H}^*(F^p\mathbf{LDR}(X/A)) \rightarrow \mathbf{H}^*(\mathbf{LDR}(X/A)))$, we have the following:

Corollary 2.25. *Take a local Artinian simplicial \mathbb{C} -algebra A , a derived Artin n -stack X over A whose underived truncation is locally of finite type, and a square-zero simplicial ideal $I \subset A$ with quotient $e: A \rightarrow B$.*

Then for any perfect complex \mathcal{F} over $X' := X \otimes_A^{\mathbf{L}} B$, with $o_e(\mathcal{F}) \in \text{Ext}_{\mathcal{O}_{X'}}^2(\mathcal{F}, \mathcal{F} \otimes_B I)$ the obstruction to deforming \mathcal{F} to an \mathcal{O}_X -module in complexes, the image of the Chern character $\text{ch}_p(\mathcal{F})$ under the map

$$\mathbf{H}^{2p}(|X'(\mathbb{C})_{\text{an}}|, \mathbb{Q}) \cong \mathbf{H}^{2p}(|X(\mathbb{C})_{\text{an}}|, \mathbb{Q}) \rightarrow \mathbf{H}^{2p}(|X(\mathbb{C})_{\text{an}}|, A) \cong \mathbf{H}^{2p}(\mathbf{LDR}(X/A))$$

lies in $F^p\mathbf{H}^{2p}(\mathbf{LDR}(X/A))$ if and only if the image of $\mathcal{L}_{p-1}(o_e(\mathcal{F}))$ under the map $\mathbb{H}^{p+1}(X, \mathbf{L}\Omega_{X/A}^{p-1} \otimes_A^{\mathbf{L}} I) \rightarrow \mathbb{H}^{2p}(X, \text{Tot}^{\Pi}(\mathbf{L}\Omega_{X/A}^{<p}, d))$, coming from the inclusion $I \hookrightarrow A$, is zero.

Proof. The obstruction κ is given by the image of $\text{ch}_p(\mathcal{F})$ in $\mathbf{H}^{2p}(\mathbf{LDR}(X/A)/F^p)$, which Lemma 2.12 tells us is the image of the Chern character $\text{ch}_p^{\text{dR}}(\mathcal{F}) \in \mathbf{H}^{2p}(\mathbf{LDR}(X'/\mathbb{C})) \cong \mathbf{H}^{2p}(\mathbf{LDR}(X/\mathbb{C}))$ under the morphism

$$\mathbf{H}^{2p}(\mathbf{LDR}(X/\mathbb{C})) \rightarrow \mathbf{H}^{2p}(\mathbf{LDR}(X/A)/F^p).$$

By compatibility of the Goodwillie–Jones Chern character with more classical Chern characters as in [HJ] and references therein, $\text{ch}_p^{\text{DR}}(\mathcal{F})$ is the image of $\Xi_p(\mathcal{F})$ under the natural map $J^p(X'/B) \rightarrow \text{LDR}(X'/\mathbb{C})[-2p]$. It thus follows that κ is the image of $\Xi_{p,A}(\mathcal{F})$ under the composite map

$$\begin{aligned} \mathbb{H}_0 J^p(X'/B, A) &= \mathbb{H}^{2p} \text{cone}(\text{LDR}(X/A) \rightarrow \text{LDR}(X'/B)/F^p) \\ &\rightarrow \mathbb{H}^{2p} \text{LDR}(X/A) \rightarrow \mathbb{H}^{2p}(\text{LDR}(X/A)/F^p), \end{aligned}$$

which is the same as the composite map

$$\begin{aligned} \mathbb{H}_0 J^p(X'/B, A) &\xrightarrow{o_e} \mathbb{H}^{2p} \text{cone}(\text{LDR}(X/A)/F^p \rightarrow \text{LDR}(X'/B)/F^p) \\ &\cong \mathbb{H}^{2p}(X, \mathbf{L}\Omega_{X/A}^{<p} \otimes_A^{\mathbf{L}} I) \rightarrow \mathbb{H}^{2p}(X, \mathbf{L}\Omega_{X/A}^{<p}), \end{aligned}$$

so the description of $o_e(\Xi_{p,A}(\mathcal{F}))$ from Corollary 2.24 completes the proof. □

Remark 2.26 (μ -twisted sheaves). Because the group scheme μ_r of r th roots of unity is étale over \mathbb{Q} and μ_r is reductive, $\mathbb{H}^*(B\mu_r, \mathbf{L}\Omega_{B\mu_r/\mathbb{Q}}^p)$ vanishes for $p > 0$ and is \mathbb{Q} for $p = 0$, so the Künneth isomorphism applied to the Čech nerve of $\tilde{X}_\alpha \rightarrow X$ gives $\mathbb{H}^*(\tilde{X}_\alpha, \mathbf{L}\Omega^p) \cong \mathbb{H}^*(X, \mathbf{L}\Omega^p)$ for all μ_r -gerbes \tilde{X}_α over X (i.e., homotopy fibres of maps $\alpha: X \rightarrow B^2\mu_r$). Since μ_r -twisted perfect complexes on X (i.e., maps $X \rightarrow [\text{Perf}/B\mu_r]$) give rise to perfect complexes on such gerbes, replacing X with \tilde{X}_α in Corollaries 2.24 and 2.25 immediately extends their conclusions to μ_r -twisted perfect complexes \mathcal{F} on X' with twist $[\alpha] \in \mathbb{H}_{\text{ét}}^2(X', \mu_r) \cong \mathbb{H}_{\text{ét}}^2(X, \mu_r)$.

2.5. Reduced obstructions

Remark 2.27. Assume that R is a Noetherian \mathbb{Q} -algebra, with X a smooth proper scheme over $\text{Spec } R$. Then the Lefschetz principle and degeneration of the Hodge–de Rham spectral sequence [Del, §5–6] imply there exists an R -linear quasi-isomorphism $\text{DR}(X/R)/F^p \simeq \bigoplus_{i=0}^{p-1} \mathbf{R}\Gamma(X, \Omega_{X/R}^i)[i]$, and in particular that the morphism $\mathbf{R}\Gamma(X, \Omega_{X/R}^{p-1})[p-1] \rightarrow \text{DR}(X/R)/F^p$ admits an R -linear retraction.

For any simplicial ideal I in a simplicial R -algebra A for which $\pi_* I \rightarrow \pi_* A$ is injective, the map

$$\mathbb{H}^{p+1}(X, \Omega_{X/R}^{p-1} \otimes_R I) \rightarrow \mathbb{H}^{2p}(X, (\text{DR}(\mathcal{O}_X/R)/F^p) \otimes_R A)$$

is therefore injective. In particular, this means that the element $\mathcal{L}_{p-1}(o_e(\mathcal{F}))$ in $\mathbb{H}^{p+1}(X, \Omega_{X/R}^{p-1} \otimes_R I)$ will vanish provided its image in $\mathbb{H}^{2p}(X, \text{DR}(\mathcal{O}_{X_A}/A)/F^p)$ does so.

In the case where X is a smooth proper scheme over an Artinian \mathbb{C} -algebra, vanishing of $\mathcal{L}_{p-1}(o_e(\mathcal{F}))$ itself is thus equivalent to the conditions of Corollary 2.25 (taking $R = A$), which we can paraphrase as saying that $\mathcal{L}_{p-1}(o_e(\mathcal{F}))$ is the obstruction to the unique horizontal lift of $\text{ch}_p^{\text{DR}}(\mathcal{F})$ still lying in $F^p \mathbb{H}^{2p} \text{DR}(X/A)$, or equivalently remaining of pure Hodge type (p, p) .

Taking an open substack $\mathfrak{M} \subset \text{Perf}_X$ for which that obstruction vanishes at all points $[\mathcal{F}]$ (for instance, by restricting to the Hodge locus of [Voi]), we then have a functorial obstruction theory

$$([\mathcal{F}] \in \mathfrak{M}(B)) \mapsto \ker\left(\mathcal{L}_{p-1}: \mathbb{E}\text{xt}_{\mathcal{O}_{X_B}}^2(\mathcal{F}, \mathcal{F} \otimes_B^{\mathbf{L}} -) \rightarrow \mathbb{H}^{p+1}(X, \Omega_{X/R}^{p-1} \otimes_R^{\mathbf{L}} -)\right)$$

for \mathfrak{M} as a subspace of the standard obstruction theory.

Such reduced obstruction theories for pairs (X, \mathcal{F}) are required in the study of Pandharipande–Thomas invariants.

Remark 2.28. There is a morphism from the derived moduli stack of proper schemes over X ([Pri1, Theorem 3.32]) to the derived stack Perf_X , given by sending $(f: Z \rightarrow X_B)$ to $\mathbf{R}f_* \mathcal{O}_Z$. Since the

obstruction maps of Lemma 1.8 are functorial, this gives rise to a morphism

$$\psi : \mathrm{Ext}_{\mathcal{O}_Z}^2(\mathbb{L}^{Z/X_B}, \mathcal{O}_Z \otimes_B^{\mathbb{L}} -) \rightarrow \mathrm{Ext}_{\mathcal{O}_{X_B}}^2(\mathbf{R}f_*\mathcal{O}_Z, \mathbf{R}f_*\mathcal{O}_Z \otimes_B^{\mathbb{L}} -)$$

of obstruction theories.

For X smooth and proper over R , Remark 2.27 then implies that $\mathcal{L}_{p-1} \circ \psi$ annihilates obstructions to deforming Z over X provided the unique horizontal lift of $\mathrm{ch}_p(\mathbf{R}f_*\mathcal{O}_Z)$ remains of Hodge type. For any open substack \mathfrak{N} of the moduli stack of proper schemes over X which parametrises schemes satisfying that condition, this gives rise to a reduced global obstruction theory, sending $[Z \rightarrow X_B] \in \mathfrak{N}(B)$ to

$$\ker\left(\mathcal{L}_{p-1} \circ \psi : \mathrm{Ext}_{\mathcal{O}_Z}^2(\mathbb{L}^{Z/X_B}, \mathcal{O}_Z \otimes_B^{\mathbb{L}} -) \rightarrow \mathrm{H}^{p+1}(X, \Omega_{X/R}^{p-1} \otimes_R^{\mathbb{L}} -)\right);$$

in particular, this applies to stable curves Z over X , as required in the study of Gromov–Witten invariants (see for instance [KT, §2.2]).

Remark 2.29. The proof of Proposition 1.17 characterises the Lefschetz map \mathcal{L} as a deformation of the Dennis trace. This means that for any proper LCI morphism $f : Z \rightarrow X_B$, the Grothendieck–Riemann–Roch theorem allows us to interpret the semiregularity map $\mathcal{L} \circ \psi$ of Remark 2.28 as the deformation of the Todd class $f_*(\mathrm{Td}(T_f))$ as f varies.

When $Z \subset X$ is a codimension p LCI subscheme of a smooth proper scheme, [BF1, Proposition 8.2] combines with Remark 2.21 to show that the map

$$\mathcal{L}_{p-1} \circ \psi : \mathrm{H}^1(Z, \mathcal{N}_{Z/X}) \rightarrow \mathrm{H}^{p+1}(X, \Omega_X^{p-1})$$

is just Bloch’s semiregularity map from [Blo], which is defined in a relatively elementary way in terms of Grothendieck–Verdier duality.

Acknowledgements. I am indebted to Timo Schürg for bringing [BF1] to my attention. I would also like to thank Barbara Fantechi, Richard Thomas and Daniel Huybrechts for helpful comments, and the anonymous referee for diligently suggesting improvements.

Competing interest. The author has no competing interest to declare.

Funding statement. This work was supported by the Engineering and Physical Sciences Research Council [grant number EP/I004130/1].

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