THE SOLUTION OF NON-HOMOGENEOUS SYSTEMS OF DIFFERENTIAL EQUATIONS BY UNDETERMINED COEFFICIENTS

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1. The solution of a linear non-homogeneous differential equation whose non-homogeneous term is of the form $t^k e^{\alpha t}$ can be obtained by what is usually called the method of undetermined coefficients. The application of this method may be justified in several different ways (see for example [1, pp. 114-117], [2, pp. 94-99], [3]).

We shall consider the analogous problem for a system of differential equations. It turns out that we can solve this problem using only elementary techniques of linear algebra. The solution has essentially the same form as in the case of a single equation, but may contain terms which would not be expected and may lack terms which would be expected in a straightforward extension of the theory to systems. Our method of obtaining the solution is constructive, in the sense that while our results give only the form of the solution, the solution itself may be found by substitution of this form into the system of differential equations.

2. We wish to find a particular solution of the linear non-homogeneous system

$$(1) \qquad \underline{y}' = A\underline{y} + t^k e^{\alpha t} \underline{c} .$$

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Here A is a constant $n \times n$ matrix, $\underline{c}$ a constant n-dimensional column vector, k a non-negative integer, and α a complex constant. The unknown function $\underline{y}$ is an n-dimensional column vector.

We consider first the case that α is not an eigenvalue of the matrix A. In this case, we attempt to find a solution of (1) of the form

$$(2) \qquad \underline{\Phi}(t) = e^{\alpha t} \sum_{j=0}^{k} \underline{p}_j t^j ,$$

where $\underline{p}_j$ $(j = 0, 1, \ldots, k)$ is a constant n-dimensional column vector to be determined. Differentiation of (2) gives

$$\underline{\Phi}'(t) = \alpha e^{\alpha t} \sum_{j=0}^{k} \underline{p}_j t^j + e^{\alpha t} \sum_{j=0}^{k} j \underline{p}_j t^{j-1}$$

$$= e^{\alpha t} \left[\sum_{j=0}^{k} \alpha \underline{p}_j t^j + \sum_{j=0}^{k-1} (j+1) \underline{p}_{j+1} t^j \right].$$

Now substitution into (1) gives the condition that $\underline{\Phi}$, as represented by (2), be a solution of (1), namely

$$(3) \qquad \sum_{j=0}^{k} \alpha \underline{p}_j t^j + \sum_{j=0}^{k-1} (j+1) \underline{p}_{j+1} t^j = \sum_{j=0}^{k} A \underline{p}_j t^j + \underline{c}\, t^k .$$

Equating coefficients of corresponding powers of t, we obtain the sequence of systems of linear non-homogeneous algebraic systems

$$(4) \qquad \begin{aligned} A \underline{p}_k - \alpha \underline{p}_k &= \underline{c} \\ A \underline{p}_{k-1} - \alpha \underline{p}_{k-1} &= k \underline{p}_k \\ \vdots \qquad \vdots \qquad &\quad \vdots \\ A \underline{p}_1 - \alpha \underline{p}_1 &= \alpha \underline{p}_2 \\ A \underline{p}_0 - \alpha \underline{p}_0 &= \underline{p}_1 \end{aligned}$$

Since it is assumed that α is not an eigenvalue of A, each of these systems can be solved in turn, once the preceding system has been solved to determine the non-homogeneous term. We can write the solutions explicitly as

$$\begin{aligned} \underline{p}_k &= (A - \alpha I)^{-1} \underline{c} \\ \underline{p}_{k-1} &= (A - \alpha I)^{-1} k \underline{p}_k = k (A - \alpha I)^{-2} \underline{c} \\ &\vdots \\ \underline{p}_0 &= k! (A - AI)^{-k-1} \underline{c} \end{aligned}$$

or

$$\underline{p}_j = \frac{k!}{j!} (A - \alpha I)^{-(k-j+1)} \underline{c} \quad [j = 0, 1, \ldots, k]. \tag{5}$$

Here I denotes the $n \times n$ identity matrix. It is easy to verify that with $\underline{p}_j$ given by (5) $(j = 0, 1, \ldots, k)$, $\underline{\phi}$ is a solution of (1), and we have obtained the desired solution. This solution may be written explicitly in the form

$$\underline{\phi}(t) = e^{\alpha t} \sum_{j=0}^{k} \left\{ \frac{k!}{j!} (A - \alpha I)^{-(k-j+1)} \underline{c} \right\} t^j . \tag{6}$$

We have thus proved the following result.

THEOREM 1. If α is not an eigenvalue of the matrix A, then a particular solution of (1) is given by (6).

If α is an eigenvalue of A, the above procedure cannot be used, since $(A - \alpha I)$ is not invertible. In this case we must seek a different form for the solution, as we would expect from the corresponding problem for a single differential equation.

Now let us assume that α is an eigenvalue of A of multiplicity m, and let us assume that the degree of the corresponding elementary divisor is $d (d \leq m)$. By making a linear transformation of the dependent variable in (1), we may replace A by any matrix similar to A. Thus we may assume that A has the form

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}, \tag{7}$$

where α is not an eigenvalue of the $(n-m) \times (n-m)$ matrix A_1, and where α is an eigenvalue of multiplicity m of the $m \times m$

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matrix A_2. The matrix $(A_2 - \alpha I)$ is nilpotent, with $(A_2 - \alpha I)^d = 0$, but $(A_2 - \alpha I)^{d-1} \neq 0$. Corresponding to this block decomposition of A, we have a decomposition of the system (1) into uncoupled sub-systems

$$\underline{y}_1' = A_1 \underline{y}_1 + t^k e^{\alpha t} \underline{c}_1 \tag{8}$$

$$\underline{y}_2' = A_2 \underline{y}_2 + t^k e^{\alpha t} \underline{c}_2 , \tag{9}$$

where $\underline{y}_1$, $\underline{c}_1$ are (n-m)-dimensional column vectors, and $\underline{y}_2$, $\underline{c}_2$ are m-dimensional column vectors. As the sub-system (8) can be treated by Theorem 1, we need only examine the sub-system (9) to complete the solution of the problem. Thus we may assume that the matrix A in (1) has only a single eigenvalue α of multiplicity n, and that the minimal polynomial of A is $m(z) = (z-a)^d$, i.e., that the corresponding elementary divisor has degree d and $(A - \alpha I)^d = 0$, $(A - \alpha I)^{d-1} \neq 0$. We now attempt to find a solution of (1) of the form

$$\underline{\phi}(t) = e^{\alpha t} \sum_{j=k+1}^{k+d} \underline{p}_j t^j , \tag{10}$$

where $\underline{p}_j$ $(j = k+1, \ldots, k+d)$ is a constant n-dimensional column vector to be determined. Differentiation of (10) gives

$$\underline{\phi}'(t) = \alpha e^{\alpha t} \sum_{j=k+1}^{k+d} \underline{p}_j t^j + e^{\alpha t} \sum_{j=k+1}^{k+d} j \underline{p}_j t^{j-1}$$

$$= e^{\alpha t} \left[\sum_{j=k+1}^{k+d} \alpha \underline{p}_j t^j + \sum_{j=k}^{k+d-1} (j+1) \underline{p}_{j+1} t^j \right].$$

Substitution into (1) gives the condition that $\underline{\phi}$, as represented by (10), is a solution of (1), namely

$$\sum_{j=k+1}^{k+d} \alpha \underline{p}_j t^j + \sum_{j=k}^{k+d-1} (j+1) \underline{p}_{j+1} t^j = \sum_{j=k+1}^{k+d} A \underline{p}_j t^j + \underline{c}\, t^k .$$

Equating coefficients of corresponding powers of t, we obtain the sequence of systems of equations

$$
(11) \qquad
\begin{aligned}
&(k+1)\, \underline{p}_{k+1} = \underline{c} \\
&A\, \underline{p}_{k+1} - \alpha \underline{p}_{k+1} = (k+2)\, \underline{p}_{k+2} \\
&A\, \underline{p}_{k+2} - \alpha \underline{p}_{k+2} = (k+3)\, \underline{p}_{k+3} \\
&\qquad \vdots \\
&A\, \underline{p}_{k+d-1} - \alpha \underline{p}_{k+d-1} = (k+d)\, \underline{p}_{k+d} \\
&A\, \underline{p}_{k+d} - \alpha \underline{p}_{k+d} = 0 .
\end{aligned}
$$

Each of these systems can be solved in turn, once the preceding system has been solved, and we can write the solutions explicitly as

$$
\underline{p}_{k+1} = \frac{1}{k+1}\, \underline{c}\, , \quad \underline{p}_{k+2} = \frac{1}{k+2}\, (A-\alpha I)\, \underline{p}_{k+1} = \frac{1}{(k+1)(k+2)}\, (A-\alpha I)\, \underline{c}\, ,
$$

$$
\ldots, \quad \underline{p}_{k+d} = \frac{1}{(k+1)\ \ldots\ (k+d)}\, (A-\alpha I)^{d-1}\, \underline{c}\, ,
$$

or

$$
(12) \qquad
\begin{aligned}
\underline{p}_j &= \frac{1}{(k+1)(k+2)\ldots j}\, (A-\alpha I)^{j-k-1}\, \underline{c} \\
&= \frac{k!}{j!}\, (A-\alpha I)^{j-k-1}\, \underline{c} \quad (j=k+1, \ldots, k+d).
\end{aligned}
$$

The last system in (11) is then satisfied since $(A-\alpha I)\, \underline{p}_{k+d} = \frac{1}{(k+1)\ldots(k+d)}\, (A-\alpha I)^d\, \underline{c} = 0$. It is easy to verify that with $\underline{p}_j$ given by (12) $(j=k+1, \ldots, k+d)$, $\underline{\phi}$ is a solution of (1). We may now write this solution explicitly in the form

$$
(13) \qquad \underline{\phi}(t) = e^{\alpha t} \sum_{j=k+1}^{k+d} \left\{ \frac{k!}{j!}\, (A-\alpha I)^{j-k-1}\, \underline{c} \right\} t^j .
$$

Thus we have proved the following complement to theorem 1.

THEOREM 2. If the matrix A has only a single eigenvalue α, and if the degree of the corresponding elementary divisor is d, then a particular solution of (1) is given by (13).

As we have already remarked, we can now obtain a particular solution of (1) in all cases, using the decomposition (7) and theorems 1 and 2.

3. The solution of a single differential equation of arbitrary order in the case covered by theorem 2 in general contains terms $t^j e^{\alpha t}$ $(j = d, d+1, \ldots, k+d)$. It might appear, therefore, that if $k + 1 < d$, theorem 2 is not the best possible result. Consider, however, the example

$$y_1' = y_2 + t^k$$

$$y_2' = t^k ,$$

which is of the form (1) with $\alpha = 0$, $d = 2$. Integrating this system directly, we obtain the solution

$$\phi_1(t) = \frac{t^{k+1}}{k+1}, \quad \phi_2(t) = \frac{t^{k+1}}{k+1} + \frac{t^{k+2}}{(k+1)(k+2)} .$$

If $k = 0$, both components of this solution contain a term in t, and since a fundamental matrix for the corresponding homogeneous system $y_1' = y_2$, $y_2' = 0$ is

$$\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix},$$

it is easy to verify that these terms can not be transferred to the homogeneous solution. Thus we must expect, in general, to obtain all the terms indicated in (13), even though they may not enter into the solution of a single equation.

Another difference between systems and single equations is that for a system the solution may not include a term of the highest possible degree $(k+d)$. An example of this situation is given by the system

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$$y_1' = y_2 + t^k$$

$$y_2' = 0 ,$$

which is of the form (1) with $\alpha = 0$, $d = 2$, $\underline{c} = \binom{1}{0}$. The formula (13) gives the particular solution $\frac{t^{k+1}}{k+1}\,\underline{c}$, the term in t^{k+2} having coefficient $\frac{k!}{(k+2)!}\, A\,\underline{c} = 0$, since $A\,\underline{c} = 0$.

For a single differential equation, the multiplicity of a root of the characteristic equation is always equal to the degree of the corresponding elementary divisor, i.e., $m = d$, but beyond this there appears to be no convenient way of specializing theorems 1 and 2 to obtain the precise results for a single equation. The possible forms of the solution for a system are more general than those for a single equation, even though the actual determination of the solution is no more difficult.

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