

# Asymptotic behavior of nonoscillatory solutions of second order functional differential equations

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The asymptotic behavior of nonoscillatory solutions of the second order functional differential equation

$$(*) \quad (r(t)y'(t))' + a(t)f(y(g(t))) = b(t)$$

is studied. First, in the case when  $a(t)$  is oscillatory, sufficient conditions are given in order that all bounded nonoscillatory solutions of (\*) approach zero as  $t \rightarrow \infty$ . Secondly, in the case when  $a(t)$  is nonnegative, conditions are provided under which all nonoscillatory solutions of (\*) tend to zero as  $t \rightarrow \infty$ .

## 1. Introduction

We consider the differential equation

$$(1) \quad (r(t)y'(t))' + a(t)f(y(g(t))) = b(t),$$

where  $a(t)$ ,  $b(t)$ ,  $g(t)$ ,  $r(t)$  are continuous on  $[\tau, \infty)$ , and  $f(y)$  is continuous on  $(-\infty, \infty)$ . The following conditions are assumed to hold throughout the paper:

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- (a)  $\lim_{t \rightarrow \infty} g(t) = \infty$  ;
- (2) (b)  $r(t) > 0$  and  $\lim_{t \rightarrow \infty} R(t) = \infty$  , where  $R(t) = \int_T^t \frac{ds}{r(s)}$  ;
- (c)  $yf(y) > 0$  for  $y \neq 0$  ,  $\liminf_{y \rightarrow \infty} f(y) > 0$  and  $\limsup_{y \rightarrow -\infty} f(y) < 0$  .

We restrict our consideration to those solutions of (1) which exist on some half-line  $[T, \infty)$  , where  $T$  may depend on the particular solution, and are nontrivial in any neighborhood of infinity. Such a solution is said to be nonoscillatory if it is eventually positive or eventually negative. Otherwise the solution is said to be oscillatory.

The problem is to provide sufficient conditions which ensure that all (bounded) nonoscillatory solutions of (1) tend to zero as  $t \rightarrow \infty$  . Since the work of Hammett [2] this problem has received a considerable amount of attention and a number of results have been obtained; see, for example, Grimmer [1], Londen [3], Singh [4], [5], and Singh and Dahiya [6]. The purpose of this paper is to add new results to this problem. First, we consider the case where  $a(t)$  is oscillatory and give conditions under which all bounded nonoscillatory solutions of (1) tend to zero as  $t \rightarrow \infty$  . Secondly, we examine the case where  $a(t)$  is nonnegative and provide sufficient conditions in order that all nonoscillatory solutions of (1) approach zero as  $t \rightarrow \infty$  .

## 2. The case where $a(t)$ is oscillatory

We need the following lemma which is a generalization of a lemma of Staikos and Sficas [7].

LEMMA. Consider the differential equation

$$(3) \quad u' - \frac{\rho'(t)}{\rho(t)} u + \frac{\rho'(t)}{\rho(t)} \phi(t) = 0 ,$$

where  $\phi(t)$  is continuous on  $[T, \infty)$  ,  $\rho(t)$  is continuously differentiable on  $[T, \infty)$  , and

$$\rho(t) > 0 , \quad \rho'(t) > 0 , \quad \lim_{t \rightarrow \infty} \rho(t) = \infty .$$

Let  $u(t)$  be the solution of (3) on  $[T, \infty)$  satisfying  $u(T) = 0$ .

If  $\lim_{t \rightarrow \infty} |\phi(t)| = \phi^*$  exists in the extended real line  $R^\#$ , then

$\lim_{t \rightarrow \infty} |u(t)| = u^*$  exists in  $R^\#$ . In particular,  $\phi^* = \infty$  implies  $u^* = \infty$ .

Proof. The solution  $u(t)$  is given explicitly by the formula

$$u(t) = -\rho(t) \int_T^t \frac{\rho'(s)}{\rho^2(s)} \phi(s) ds, \quad t \geq T.$$

The existence of  $\phi^*$  in  $R^\#$  implies that the improper integral

$$\lim_{t \rightarrow \infty} \int_T^t \frac{\rho'(s)}{\rho^2(s)} \phi(s) ds = \phi^*$$

exists in  $R^\#$ . If  $\phi^* \neq 0$ , then clearly  $\lim_{t \rightarrow \infty} |u(t)| = \infty$ . If  $\phi^* = 0$ ,

then by l'Hospital's Rule,

$$\lim_{t \rightarrow \infty} |u(t)| = \lim_{t \rightarrow \infty} \left| \left[ - \int_T^t \frac{\rho'(s)}{\rho^2(s)} \phi(s) ds \right]' / \left[ \frac{1}{\rho(t)} \right]' \right| = \phi^*.$$

In what follows we use the notation  $a^+(t) = \max\{a(t), 0\}$ ,  $a^-(t) = \max\{-a(t), 0\}$ .

**THEOREM 1.** Every bounded nonoscillatory solution of (1) tends to zero as  $t \rightarrow \infty$  if the following assumptions are satisfied:

(4)  $\int_0^\infty R(t)a^+(t)dt = \infty,$

(5)  $\int_0^\infty R(t)a^-(t)dt < \infty,$

(6)  $\int_0^\infty R(t)|b(t)|dt < \infty.$

Proof. Let  $y(t)$  be a bounded nonoscillatory solution of (1). Suppose that  $y(t) > 0$  for  $t \geq t_0$ . The case  $y(t) < 0$  can be handled similarly. By (2) (a) there exists  $t_1 \geq t_0$  such that  $g(t) \geq t_0$  for  $t \geq t_1$ . Thus,  $y(g(t)) > 0$  for  $t \geq t_1$ . Multiplying (1) by  $R(t)$  and

integrating from  $t_1$  to  $t$ , we obtain

$$(7) \quad \int_{t_1}^t R(s)(r(s)y'(s))'ds + \int_{t_1}^t R(s)a^+(s)f(y(g(s)))ds \\ = \int_{t_1}^t R(s)b(s)ds + \int_{t_1}^t R(s)a^-(s)f(y(g(s)))ds .$$

An integration by parts yields

$$\int_{t_1}^t R(s)(r(s)y'(s))'ds = R(t)r(t)y'(t) - R(t_1)r(t_1)y'(t_1) - \int_{t_1}^t y'(s)ds .$$

By a mere reinterpretation this says that the function  $u(t) = \int_{t_1}^t y'(s)ds$

satisfies the differential equation

$$(8) \quad u' - \frac{R'(t)}{R(t)}u + \frac{R'(t)}{R(t)}\Phi(t) = 0 ,$$

where

$$(9) \quad \Phi(t) = - \int_{t_1}^t R(s)(r(s)y'(s))'ds - R(t_1)r(t_1)y'(t_1) .$$

Since  $u(t_1) = 0$  and since  $R(t) > 0$ ,  $R'(t) > 0$ ,  $\lim_{t \rightarrow \infty} R(t) = \infty$  by

(2) (b), we can apply the above lemma to (8) to conclude that  $\lim_{t \rightarrow \infty} |u(t)|$

exists in  $R^\#$  whenever  $\lim_{t \rightarrow \infty} |\Phi(t)|$  exists in  $R^\#$ .

We examine the following two possible cases:

$$(10) \quad \int_{t_1}^\infty R(t)a^+(t)f(y(g(t)))dt = \infty ,$$

$$(11) \quad \int_{t_1}^\infty R(t)a^+(t)f(y(g(t)))dt < \infty .$$

Suppose (10) holds. Since the right-hand side of (7) remains bounded as  $t \rightarrow \infty$  on account of (5), (6), and the boundedness of  $y(t)$ , it follows

from (7) and (10) that

$$(12) \quad \lim_{t \rightarrow \infty} \int_{t_1}^t R(s)(r(s)y'(s))' ds = -\infty,$$

which implies that the function  $\Phi(t)$  defined by (9) tends to  $\infty$  as  $t \rightarrow \infty$ . Hence, by the lemma,  $\lim_{t \rightarrow \infty} |u(t)| = \infty$ , and consequently

$\lim_{t \rightarrow \infty} y(t) = \infty$ . This, however, contradicts the boundedness of  $y(t)$  and

thus (10) is impossible.

Now it follows from (7) and (11) that the limit on the left-hand side of (12) exists as a finite number; that is,  $\lim_{t \rightarrow \infty} \Phi(t)$  is finite. Applying

the lemma again, we see that  $\lim_{t \rightarrow \infty} |u(t)|$  exists in  $R^\#$ . Obviously this

limit must be finite. Therefore,  $y(t)$  must tend to a finite limit as  $t \rightarrow \infty$ . On the other hand, using (2) (c), (4) and (11), we have

$$\liminf_{t \rightarrow \infty} y(g(t)) = \liminf_{t \rightarrow \infty} y(t) = 0.$$

Consequently we conclude that  $\lim_{t \rightarrow \infty} y(t) = 0$ , completing the proof.

Retracing the above proof it is easy to see that the conclusion of Theorem 1 still holds if the roles of  $a^+(t)$  and  $a^-(t)$  are interchanged. Accordingly we have the following theorem.

**THEOREM 2.** *Every bounded nonoscillatory solution of (1) tends to zero as  $t \rightarrow \infty$  if the following conditions are satisfied:*

$$\begin{aligned} \int_0^\infty R(t)a^+(t)dt &< \infty, \\ \int_0^\infty R(t)a^-(t)dt &= \infty, \\ \int_0^\infty R(t)|b(t)|dt &< \infty. \end{aligned}$$

**COROLLARY 1.** *Every bounded nonoscillatory solution of the equation*

$$y''(t) + a(t)f(y(g(t))) = b(t)$$

tends to zero as  $t \rightarrow \infty$  if either

$$\int^{\infty} ta^+(t)dt = \infty, \int^{\infty} ta^-(t)dt < \infty, \int^{\infty} t|b(t)|dt < \infty, .$$

or

$$\int^{\infty} ta^+(t)dt < \infty, \int^{\infty} ta^-(t)dt = \infty, \int^{\infty} t|b(t)|dt < \infty .$$

EXAMPLES (1) Consider the differential equation

$$(13) \quad (t^{-1}y'(t))' + t^{-2}y^3(\gamma t) = 3t^{-4} + \gamma^{-3}t^{-5}$$

for  $t \geq 1$ , where  $\gamma$  is a positive constant (possibly greater than 1).

Here  $f(y) = y^3$ ,  $g(t) = \gamma t$ ,

$$r(t) = t^{-1}, \quad a^+(t) = t^{-2}, \quad a^-(t) = 0, \quad \text{and} \quad b(t) = 3t^{-4} + \gamma^{-3}t^{-5} .$$

We can take  $R(t) = t^2$  and see that all conditions of Theorem 1 are satisfied. Hence all bounded nonoscillatory solutions of (13) approach zero as  $t \rightarrow \infty$ . In fact,  $y(t) = t^{-1}$  is a bounded nonoscillatory solution of (13).

(2) Consider the equation

$$(14) \quad (ty'(t))' - t^{-1}(\log t)^{-1}y(\log t) = (1-3t+t^2)e^{-t} - t^{-2}$$

for  $t \geq 1$ , which has  $y(t) = te^{-t}$  as a bounded nonoscillatory solution.

Here  $f(y) = y$ ,  $g(t) = \log t$ ,  $r(t) = t$ ,

$$a^+(t) = 0, \quad a^-(t) = t^{-1}(\log t)^{-1}, \quad \text{and} \quad b(t) = (1-3t+t^2)e^{-t} - t^{-2} .$$

All the conditions of Theorem 2 are satisfied with  $R(t) = \log t$ . It follows that all bounded nonoscillatory solutions of (14) tend to zero as  $t \rightarrow \infty$ .

### 3. The case where $a(t)$ is nonnegative

In this section we examine the equation (1) in which  $a(t)$  is nonnegative and provide conditions under which all nonoscillatory solutions are necessarily bounded and tend to zero as  $t \rightarrow \infty$ .

**THEOREM 3.** Assume that  $a(t) \geq 0$  and

$$(15) \quad \int^{\infty} a(t)dt = \infty ,$$

$$(16) \quad \int^{\infty} R(t)|b(t)|dt < \infty .$$

Then all nonoscillatory solutions of (1) tend to zero as  $t \rightarrow \infty$ .

Proof. Let  $y(t)$  be a nonoscillatory solution of (1) such that  $y(g(t)) > 0$  for  $t \geq t_1$ . A parallel argument holds if  $y(g(t)) < 0$  for  $t \geq t_1$ . From (1) we obtain

$$(17) \quad r(t)y'(t) - r(t_1)y'(t_1) + \int_{t_1}^t a(s)f(y(g(s)))ds = \int_{t_1}^t b(s)ds ,$$

$$(18) \quad \int_{t_1}^t R(s)\{r(s)y'(s)\}'ds + \int_{t_1}^t R(s)a(s)f(y(g(s)))ds = \int_{t_1}^t R(s)b(s)ds .$$

Suppose that

$$(19) \quad \int_{t_1}^{\infty} R(t)a(t)f(y(g(t)))dt = \infty .$$

Then, letting  $t \rightarrow \infty$  and using (16), we see that the first integral in (18) tends to  $-\infty$  as  $t \rightarrow \infty$ , so that  $\lim_{t \rightarrow \infty} \Phi(t) = \infty$ , where  $\Phi(t)$  is

defined by (9). From the lemma applied to (8) it follows that

$u(t) = \int_{t_1}^t y'(s)ds$  tends to  $\infty$  as  $t \rightarrow \infty$ . Consequently, we have

$\lim_{t \rightarrow \infty} y(t) = \infty$ . Combining this with (2) (c) and (15), we find

$$(20) \quad \int_{t_1}^{\infty} a(t)f(y(g(t)))dt = \infty .$$

Since (16) implies that the right-hand side of (17) is bounded, we have from (17) and (20),

$$\lim_{t \rightarrow \infty} r(t)y'(t) = -\infty ,$$

from which we conclude with the use of (2) (b) that  $\lim_{t \rightarrow \infty} y(t) = -\infty$ . But this contradicts the fact that  $y(t)$  is positive. Hence (19) is impossible and we must have

$$(21) \quad \int_{t_1}^{\infty} R(t)a(t)f(y(g(t)))dt < \infty .$$

Letting  $t \rightarrow \infty$  again in (18), we see that  $\lim_{t \rightarrow \infty} \Phi(t)$  exists and is finite.

According to the lemma it follows that  $\lim_{t \rightarrow \infty} u(t)$  exists in  $R^\#$ . This

limit must be finite, since otherwise we would be led to a contradiction to the positivity of  $y(t)$  as before. Therefore, there exists a finite limit  $\lim_{t \rightarrow \infty} y(t)$ . On the other hand, (21) implies

$$(22) \quad \int_{t_1}^{\infty} a(t)f(y(g(t)))dt < \infty ,$$

and from (2) (c), (15), and (22) it is easily verified that

$$\liminf_{t \rightarrow \infty} y(g(t)) = \liminf_{t \rightarrow \infty} y(t) = 0 .$$

Thus it follows that  $\lim_{t \rightarrow \infty} y(t) = 0$ , and the proof is complete.

**COROLLARY 2.** *All nonoscillatory solutions of the equation*

$$y''(t) + a(t)f(y(g(t))) = b(t)$$

*approach zero as  $t \rightarrow \infty$  if  $a(t) \geq 0$ ,*

$$\int_{t_1}^{\infty} a(t)dt = \infty \quad \text{and} \quad \int_{t_1}^{\infty} t|b(t)|dt < \infty .$$

**EXAMPLES (3).** Consider the equation

$$(23) \quad (t^{-1}y'(t))' + t^{-1}y^3(\gamma t) = (3+\gamma^{-3})t^{-4}$$

for  $t \geq 1$ , where  $\gamma$  is a positive constant. This equation possesses a nonoscillatory solution  $y(t) = t^{-1}$  tending to zero as  $t \rightarrow \infty$ . Since all conditions of Theorem 3 are satisfied, all other nonoscillatory solutions of (23) also approach zero as  $t \rightarrow \infty$ .



(4) Consider the equation

$$(24) \quad (ty'(t))' + t^{-1}(\log t)^{-1}y(\log t) = (1-3t+t^2)e^{-t} + t^{-2}$$

for  $t \geq 1$ . Clearly, this equation satisfies the assumptions of Theorem 3. Therefore, all of its nonoscillatory solutions tend to zero as  $t \rightarrow \infty$ .

In fact,  $y(t) = te^{-t}$  is a nonoscillatory solution of (24) having this property.

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