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(STRONGLY) GORENSTEIN FLAT MODULES OVER GROUP RINGS

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Abstract

Let Γ be a group and Γ' be a subgroup of Γ of finite index. Let M be a Γ-module. It is shown that M is (strongly) Gorenstein flat if and only if it is (strongly) Gorenstein flat as a Γ'-module. We also provide some criteria in which the classes of Gorenstein projective and strongly Gorenstein flat Γ-modules are the same.

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1. Introduction

As a nice generalisation of the notion of finitely generated projective modules, Auslander and Bridger introduced in [3] the notion of modules in Auslander's G-class (modules of Gorenstein dimension zero) over left and right Noetherian rings. Several decades later, Enochs and Jenda [12] extended the ideas of Auslander and Bridger and introduced the notion of Gorenstein projective modules. These modules generalise the notion of a module of G-dimension zero to the class of all modules (so not necessarily finitely generated ones). An R-module M is said to be Gorenstein projective, if M is a syzygy of a complete projective resolution, that is, if there exists an acyclic complex of projective (left) R-modules

$$\mathbf{P}_\bullet : \cdots \longrightarrow P_{n+1} \xrightarrow{\delta_{n+1}} P_n \xrightarrow{\delta_n} P_{n-1} \longrightarrow \cdots ,$$

which remains acyclic when applying the functor $\mathrm{Hom}_R(-, P)$ for any projective R-module P, such that $M = \mathrm{Im}\, \delta_0$. This definition can be dualised and allows one to define the class of Gorenstein injective modules. These notions generalise the classical notions of projective and injective modules. To complete the analogy between classical homological (dimension) theory and Gorenstein homological (dimension) theory,

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Enochs *et al.* [13] introduced the notion of Gorenstein flat modules as a generalisation of the notion of flat modules. These Gorenstein homological modules have been studied extensively by many authors (see [4, 12, 15], for instance). In particular, it is known that Gorenstein homological modules share many nice properties of the classical homological modules: projective, injective and flat modules, respectively.

A (left) R-module M is called Gorenstein flat if there exists an exact sequence of flat (left) R-modules

$$\mathbf{F}_\bullet : \cdots \longrightarrow F_{n+1} \xrightarrow{\delta_{n+1}} F_n \xrightarrow{\delta_n} F_{n-1} \longrightarrow \cdots ,$$

such that $I \otimes_R \mathbf{F}_\bullet$ is exact for any right injective R-module I and $M = \operatorname{Im} \delta_0$.

The notion of strongly Gorenstein flat modules is introduced and studied by Ding *et al.* [11] and for this reason these modules are renamed Ding projective modules by Gillespie; see [14]. A left R-module M is said to be strongly Gorenstein flat if there exists an exact sequence of projective R-modules

$$\mathbf{P}_\bullet : \cdots \longrightarrow P_{n+1} \xrightarrow{\delta_{n+1}} P_n \xrightarrow{\delta_n} P_{n-1} \longrightarrow \cdots ,$$

with $M = \operatorname{Im} \delta_0$ and such that the functor $\operatorname{Hom}_R(-, F)$, where F is a flat R-module, leaves the sequence exact. It is worth noting that this definition is different from the concept of strongly Gorenstein flat modules studied in [8]. The strongly Gorenstein flat dimension is defined via resolutions by strongly Gorenstein flat modules. It follows from the definition that every strongly Gorenstein flat module is Gorenstein projective. On the other hand, there are examples of the strongly Gorenstein flat module which is not projective (flat), and (Gorenstein) flat module which is not strongly Gorenstein flat; see [11]. In addition, it is proved in [11] that over coherent rings, strongly Gorenstein flat modules lie strictly between projective and Gorenstein flat modules. Strongly Gorenstein flat modules and dimensions were studied further in [19, 20]. In this note, we study (strongly) Gorenstein flat modules over group rings. Actually, our main result in this direction reads as follows.

Main Theorem. *Let Γ be a group and Γ' a subgroup of Γ of finite index. Let M be a Γ-module. Then, M is (strongly) Gorenstein flat if and only if it is (strongly) Gorenstein flat as a Γ'-module.*

We end this paper by presenting some criteria which guarantee that the class of Gorenstein projective Γ-modules coincides with the strongly Gorenstein flat ones.

Throughout the paper, Γ is a group and $\mathbb{Z}\Gamma$ is its integral group ring. By a Γ-module, we mean a $\mathbb{Z}\Gamma$-module. We use this abbreviation throughout our notation. For example, for a Γ-module M, the flat (respectively, Gorenstein flat) dimension of M over $\mathbb{Z}\Gamma$ is denoted by $\operatorname{fd}_\Gamma M$, (respectively, $\operatorname{Gfd}_\Gamma M$). The tensor product and Hom functor over $\mathbb{Z}\Gamma$ are denoted by $- \otimes_\Gamma -$ and $\operatorname{Hom}_\Gamma(-, -)$. All modules are considered as left modules unless specified.

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2. Gorenstein flat modules

DEFINITION 2.1. Let Γ be a group and M be a Γ-module. We call M Gorenstein flat if there is an exact sequence of flat Γ-modules

$$\mathbf{F}_\bullet : \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow F_{-1} \longrightarrow \cdots ,$$

such that $M \cong \operatorname{Im}(F_0 \longrightarrow F_{-1})$ and $I \otimes_\Gamma -$ leaves the sequence $\mathbf{F}_\bullet$ exact, whenever I is a right injective Γ-module.

EXAMPLE 2.2. The following statements are well known.

(1) Every flat Γ-module is Gorenstein flat.
(2) Let Γ be a finite group. Then the trivial Γ-module $\mathbb{Z}$ is a Gorenstein flat module; see [1, Example 4.2].
(3) Let Γ be a group and Γ' be a subgroup of finite index. Let M be a Gorenstein flat Γ'-module. Then $\mathbb{Z}\Gamma \otimes_{\Gamma'} M$ is a Gorenstein flat Γ-module.

REMARK 2.3. Using the facts that injectives are summands of co-frees and that co-frees are injectives, one can replace 'injective' by 'co-free' in Definition 2.1. In addition, due to the fact that the injective dimension of any coinduced module is at most one, 'co-free' may be replaced by 'coinduced'; see [16].

Recall that a Γ-module is coinduced if it has the form $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}\Gamma, A)$ where A is an abelian group, and a Γ-module is co-free if it is a direct product of copies of $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}\Gamma, \mathbb{Q}/\mathbb{Z})$.

PROPOSITION 2.4. *Let Γ be a group and Γ' be a subgroup of Γ of finite index. Let M be a Γ-module which is flat over Γ'. Then M is a Gorenstein flat Γ-module.*

PROOF. Let M be a Γ-module. It is known that there exist a Γ-epimorphism $\mathbb{Z}\Gamma \otimes_{\Gamma'} M \longrightarrow M$ and a Γ-monomorphism $M \longrightarrow \operatorname{Hom}_{\Gamma'}(\mathbb{Z}\Gamma, M)$ which are split over Γ'. The hypothesis imposed on M yields that $\mathbb{Z}\Gamma \otimes_{\Gamma'} M \cong \operatorname{Hom}_{\Gamma'}(\mathbb{Z}\Gamma, M)$ is a flat Γ-module. By repeating this argument for the kernel and the cokernel of these maps respectively, and continuing this way, one obtains an acyclic complex of flat Γ-modules which splits over Γ',

$$\mathbf{X}_\bullet : \cdots \longrightarrow X_2 \longrightarrow X_1 \longrightarrow X_0 \longrightarrow X_{-1} \longrightarrow X_{-2} \longrightarrow \cdots ,$$

such that $M = \operatorname{Im}(X_0 \longrightarrow X_{-1})$. So, in order to complete the proof one only needs to show that $J \otimes_\Gamma \mathbf{X}_\bullet$ is acyclic, where J is a coinduced Γ-module. To do this, one should note that it is not hard to see that there is a coinduced Γ'-module J' such that $J \cong J' \otimes_{\Gamma'} \mathbb{Z}\Gamma$ (as Γ-isomorphism). Therefore, we obtain the isomorphism $J \otimes_\Gamma \mathbf{X}_\bullet \cong J' \otimes_{\Gamma'} \mathbf{X}_\bullet$. According to the construction of $\mathbf{X}_\bullet$, one deduces that the complex $J' \otimes_{\Gamma'} \mathbf{X}_\bullet$ is exact. So the claim follows. □

As an application of the above proposition, we include the following example.

EXAMPLE 2.5. Let Γ be a finite group and Γ' be its trivial subgroup. It is known that the Γ-module $\mathbb{Q}$, with trivial action, is flat over Γ'. So the above proposition yields that $\mathbb{Q}$ is a Gorenstein flat Γ-module.

Theorem 2.6. *Let Γ' be a subgroup of Γ of finite index and M be a Γ-module. Then M is Gorenstein flat if and only if it is Gorenstein flat as a Γ'-module.*

Proof. According to [1, Lemma 4.10], we only need to show the 'if' part. So, assume that M is a Γ-module which is Gorenstein flat over Γ'. Therefore, it follows from the definition that there is a short exact sequence of Γ'-modules $0 \longrightarrow M \xrightarrow{d} F' \longrightarrow L' \longrightarrow 0$, where F' is flat and L' is Gorenstein flat. Consider the following commutative diagram of Γ-modules and Γ-homomorphisms with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & \mathrm{Hom}_{\Gamma'}(\mathbb{Z}\Gamma, M) & \longrightarrow & L & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \Theta} & & \downarrow{\scriptstyle \varphi} & & \\ 0 & \longrightarrow & M & \longrightarrow & \mathrm{Hom}_{\Gamma'}(\mathbb{Z}\Gamma, F') & \longrightarrow & K & \longrightarrow & 0 \end{array}$$

where $\Theta = \mathrm{Hom}(\mathrm{id}, d)$. Since the Γ'-homomorphism $F' \longrightarrow \mathrm{Hom}_{\Gamma'}(\mathbb{Z}\Gamma, F')$ is a split monomorphism, there is a flat Γ'-module X in which $\mathrm{Hom}_{\Gamma'}(\mathbb{Z}\Gamma, F') \cong F' \oplus X$, as Γ'-modules. Consequently, one has the Γ'-isomorphism $K \cong L' \oplus X$, implying that K is Gorenstein flat as a Γ'-module. We may therefore repeat the argument with K replacing M and in this way we produce a right Γ-flat resolution of M,

$$0 \longrightarrow M \longrightarrow F_{-1} \longrightarrow F_{-2} \longrightarrow \cdots , \tag{2.1}$$

such that every kernel of its morphisms is Gorenstein flat as a Γ'-module. Consider the flat resolution of Γ-module M,

$$\cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0. \tag{2.2}$$

Obviously every syzygy of this flat resolution is Gorenstein flat when viewed as a Γ'-module. Splicing complexes (2.1) and (2.2) yields the acyclic complex of flat Γ-modules

$$\mathbf{F}_\bullet, \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow F_{-1} \longrightarrow F_{-2} \longrightarrow \cdots ,$$

such that $M = \mathrm{Im}\,(F_0 \longrightarrow F_{-1})$ and every syzygy of $\mathbf{F}_\bullet$ is Gorenstein flat as a Γ'-module. Hence, to complete the proof, one needs to show that $J \otimes_\Gamma \mathbf{F}_\bullet$ is exact where J is an arbitrary coinduced Γ-module, thanks to Remark 2.3. But this also follows from the fact that every syzygy of $\mathbf{F}_\bullet$ is Gorenstein flat over Γ' in conjunction with the argument which has been used in the above lemma. The proof is then complete. □

Remark 2.7. Let Γ be a finite group and Γ' be the trivial subgroup of Γ. Then the Γ-module $\mathbb{Z}$, with the trivial action, is not flat; however, $\mathbb{Z}$ is a free (and hence flat) Γ'-module. Therefore, the above theorem does not hold true for flat modules.

Combining the above theorem with [15, Theorem 3.14] gives rise to the following proposition.

Proposition 2.8. *Let Γ be a group such that $\mathbb{Z}\Gamma$ is a coherent ring. Let Γ' be a subgroup of Γ of finite index. Then for a given Γ-module M,* $\mathrm{Gfd}_\Gamma M = \mathrm{Gfd}_{\Gamma'} M$.

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3. Strongly Gorenstein flat modules

Definition 3.1. Let Γ be a group and M be a Γ-module. As we have mentioned in the introduction, M is called *strongly Gorenstein flat* if there exists an acyclic complex of projective Γ-modules

$$\mathbf{P}_\bullet : \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow P_{-1} \longrightarrow P_{-2} \longrightarrow \cdots ,$$

with $M = \mathrm{Im}\,(P_0 \longrightarrow P_{-1})$ and such that the functor $\mathrm{Hom}_\Gamma(-, F)$ leaves the sequence exact, where F is a flat Γ-module. It follows from the definition that every strongly Gorenstein flat module is Gorenstein projective.

Remark 3.2. We should remark that, as in [11, 19], we cannot find an example of a Gorenstein projective module which is not strongly Gorenstein flat. It follows from [9, Theorem 4.2] that the projective dimension of any flat Γ-module is less than or equal to one, whenever Γ is a finite group. Hence, in this case every Gorenstein projective module is a strongly Gorenstein flat Γ-module and so Gorenstein projective modules coincide with strongly Gorenstein flat modules.

Definition 3.3. Let M be a nonzero Γ-module. We say that the strongly Gorenstein flat dimension of M is $n \geq 0$, denoted by $\mathrm{SGfd}_\Gamma M = n$, if n is the least integer for which there exists an exact sequence

$$0 \longrightarrow F_n \longrightarrow \cdots \longrightarrow F_0 \longrightarrow M \longrightarrow 0,$$

where all F_i are strongly Gorenstein flat Γ-modules. If no such n exists, then we shall write $\mathrm{SGfd}_\Gamma M = \infty$. By convention, $\mathrm{SGfd}_\Gamma 0 = -\infty$.

Theorem 3.4. *Let Γ' be a subgroup of Γ of finite index and M be a Γ-module. Then M is strongly Gorenstein flat if and only if it is strongly Gorenstein flat as a Γ'-module.*

Proof. The 'only if' part follows from the fact that every projective Γ-module is also projective over Γ' and for any flat Γ'-module F', the Γ-module $\mathbb{Z}\Gamma \otimes_{\Gamma'} F'$ is flat and $\mathrm{Hom}_{\Gamma'}(\mathbb{Z}\Gamma, F') \cong \mathbb{Z}\Gamma \otimes_{\Gamma'} F'$. So let us prove the 'if' part. Assume that a given Γ-module M is a Γ-module which is strongly Gorenstein flat when viewed as a Γ'-module. It follows from the definition that M is a Gorenstein projective module over Γ'. According to [6, Theorem 2.8], M is indeed a Gorenstein projective Γ-module and hence there is a complete projective resolution $\mathbf{P}_\bullet$ of Γ-modules such that M is its zeroth syzygy and every syzygy of this resolution is a Gorenstein projective Γ'-module. Assume that F is an arbitrary flat Γ-module. Due to Lazard's theorem, there is a direct system $(P_i)_{i\in I}$ of finitely generated projective Γ-modules such that $F \cong \varinjlim_i P_i$. Take, for any i, a projective Γ'-module P'_i such that $P_i \cong \mathbb{Z}\Gamma \otimes_{\Gamma'} P'_i$, as Γ-modules. Letting $F' = \varinjlim_i P'_i$, one infers that F' is a flat Γ'-module and $F \cong \mathbb{Z}\Gamma \otimes_{\Gamma'} F'$. This, in turn, implies the isomorphism $\mathrm{Hom}_\Gamma(\mathbf{P}_\bullet, F) \cong \mathrm{Hom}_{\Gamma'}(\mathbf{P}_\bullet, F')$. So the claim follows. □

The above result, in conjunction with the fact that the class of strongly Gorenstein flat modules is projectively resolving [19, Theorem 2.1], yields the following result.

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Corollary 3.5. *Let* Γ *be a group and* Γ' *be a subgroup of* Γ *of finite index. Then for a given* Γ*-module* M*,* $\mathrm{SGfd}_{\Gamma'}M = \mathrm{SGfd}_{\Gamma}M$.

Remark 3.6. Let R be an associative ring with identity. The left (respectively, right) strongly Gorenstein flat dimension of R, denoted $l\mathrm{SGFD}(R)$ (respectively, $r\mathrm{SGFD}(R)$) is defined as the supremum of the strongly Gorenstein flat dimensions attained on the category of left (respectively, right) R-modules. If $R = \mathbb{Z}\Gamma$, then R is isomorphic with the opposite ring R^{op} and so the distinction between left and right module is redundant. In this case we drop the superfluous letters l and r.

Proposition 3.7. *Let* Γ *be a group and* Γ' *be its subgroup of finite index. Then* $\mathrm{SGFD}(\mathbb{Z}\Gamma) = \mathrm{SGFD}(\mathbb{Z}\Gamma')$.

Proof. First we show the inequality $\mathrm{SGFD}(\mathbb{Z}\Gamma') \leq \mathrm{SGFD}(\mathbb{Z}\Gamma)$. If $\mathrm{SGFD}(\mathbb{Z}\Gamma) = \infty$, there is nothing to prove. So assume that $\mathrm{SGFD}(\mathbb{Z}\Gamma)$ is finite, say n. Take an arbitrary Γ'-module M. Then the Γ-module $\mathbb{Z}\Gamma \otimes_{\Gamma'} M$ has strongly Gorenstein flat dimension at most n and hence the above corollary gives rise to $\mathrm{SGfd}_{\Gamma'}(\mathbb{Z}\Gamma \otimes_{\Gamma'} M) \leq n$, implying that $\mathrm{SGfd}_{\Gamma'}M \leq n$, since M is a Γ'-direct summand of $\mathbb{Z}\Gamma \otimes_{\Gamma'} M$. Consequently, $\mathrm{SGFD}(\mathbb{Z}\Gamma') \leq n$. On the other hand, in order to verify the inequality $\mathrm{SGFD}(\mathbb{Z}\Gamma) \leq \mathrm{SGFD}(\mathbb{Z}\Gamma')$, we may assume that $\mathrm{SGFD}(\mathbb{Z}\Gamma')$ is finite and apply the above corollary again. The proof is complete. □

Proposition 3.8. *Let* Γ *be a group. Then* $\mathrm{SGfd}_{\Gamma}\mathbb{Z} = 0$ *if and only if* Γ *is a finite group.*

Proof. Let $\mathrm{SGfd}_{\Gamma}\mathbb{Z} = 0$. Then $\mathbb{Z}$ is a strongly Gorenstein flat Γ-module and, in particular, it is Gorenstein projective. Now, one may apply [2, Proposition 2.19] to deduce that Γ is a finite group. Conversely, assume that Γ is a finite group. Making use of [2, Proposition 2.19] again one deduces that $\mathbb{Z}$ is a Gorenstein projective Γ-module. Hence, the result follows from the fact that $\mathbb{Z}$ is a finitely presented Γ-module in conjunction with [19, Proposition 2.4]. □

Remark 3.9. Because of [19, Proposition 2.4], a finitely presented module M is Gorenstein projective if and only if it is strongly Gorenstein flat. Moreover, it is shown by Bennis [7, Theorem 3.3] that an infinitely presented module M is Gorenstein projective if and only if it is Gorenstein flat. So every infinitely presented module is Gorenstein projective if and only if it is Gorenstein flat if and only if it is strongly Gorenstein flat. In particular, if Γ is a finite group, then a finitely generated Γ-module M is Gorenstein projective if and only if it is Gorenstein flat if and only if it is a strongly Gorenstein flat Γ-module.

We recall that a (not necessarily nonzero) Γ-module M is said to be infinitely presented, if it admits a free resolution $\cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$, such that, for each i, F_i is a finitely generated free Γ-module.

In the following we show that if the trivial Γ-module $\mathbb{Z}$ has finite strongly Gorenstein flat dimension, then the same is true for all Γ-modules M.

Lemma 3.10. *Let* Γ *be a group. Then for any* Γ*-module* M*,* $\mathrm{SGfd}_{\Gamma}M \leq \mathrm{SGfd}_{\Gamma}\mathbb{Z} + 1$.

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PROOF. Clearly we may assume that $\mathrm{SGfd}_\Gamma\mathbb{Z}$ is finite, say n. Hence, $\mathrm{Gpd}_\Gamma\mathbb{Z} \le n$, and so, by using [1, Proposition 3.2], we infer that every Gorenstein projective Γ-module is strongly Gorenstein flat. In particular, for every Γ-module M, $\mathrm{Gpd}_\Gamma M = \mathrm{SGfd}_\Gamma M$. Now [5, Proposition 2.4(c)] completes the proof. □

Before stating the last result of this paper, we need to recall some terminology. Let Γ be a group. Recall that the finitistic dimension of $\mathbb{Z}\Gamma$, denoted by findim $\mathbb{Z}\Gamma$, is the supremum of the projective dimension of those Γ-modules of finite projective dimensions. We say that Γ is of type FP_∞ provided the trivial Γ-module $\mathbb{Z}$ is infinitely presented. The class $\mathbf{H}\mathfrak{F}$ was defined by Kropholler in [18] as the smallest class of groups which contains the class of finite groups, and whenever a group Γ admits a finite-dimensional contractible Γ-CW-complex with stabilisers in $\mathbf{H}\mathfrak{F}$, then Γ is in $\mathbf{H}\mathfrak{F}$.

THEOREM 3.11. *Let Γ be a group which satisfies one of the following:*

(1) findim $\mathbb{Z}\Gamma < \infty$*;*
(2) $\mathrm{Gpd}_\Gamma\mathbb{Z} < \infty$*;*
(3) Γ *is an* $\mathbf{H}\mathfrak{F}$*-group of type* FP_∞*;*
(4) silf $\Gamma < \infty$*, where* silf Γ *is the supremum of injective lengths of flat* Γ*-modules.*

Then the classes of Gorenstein projective Γ*-modules and strongly Gorenstein flat* Γ*-modules are the same.*

PROOF. (1) Assume that findim $\mathbb{Z}\Gamma$ is finite. Then, as shown by Jensen [17, Proposition 6], any flat Γ-module has finite projective dimension and so the claim follows.

(2) In order to obtain the desired result, one only needs to apply [1, Proposition 3.2] and [2, Proposition 2.8].

(3) If Γ is an $\mathbf{H}\mathfrak{F}$-group of type FP_∞, then, by [10, Corollary C] in conjunction with [2, Remark 2.10], $\mathrm{Gpd}_\Gamma\mathbb{Z} < \infty$. Hence the result follows from part (2).

(4) According to [1, Theorem 2.2] for any group Γ, one has the equality silf Γ = silp Γ, where silp Γ is the supremum of injective lengths of projective Γ-modules. On the other hand, we always have findim $\mathbb{Z}\Gamma \le$ silp Γ, implying that findim $\mathbb{Z}\Gamma$ is finite. Now we get the result by part (1). □

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