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An extension of Pontryagin duality

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Let V denote the symmetric monoidal closed category of limit-space abelian groups and let L denote the full subcategory of locally compact Hausdorff abelian groups. Results of Samuel Kaplan on extension of characters to products of L-groups are used to show that each closed subgroup of a product of L-groups is a limit of L-groups. From this we deduce that the limit closure of L in V is reflective in V and has every group Pontryagin reflexive with respect to the structure of continuous convergence on the character groups. The basic duality $L \simeq L^{\text{op}}$ is then extended.

Introduction

Amongst the cartesian closed extensions of the category T of all topological spaces and continuous maps there is the quasi-topos $C = (C, 1, \times, \{-, -\}, \ldots)$ of limitspaces (also called convergence spaces). We choose to work with this extension because it is perhaps the best known (see Binz [2], [3] and Binz and Keller [4]). Other candidates would include Choquet pseudotopologies [7], [20] or Antoine spaces [1], [20]. However, because we only work with internal-homs of the form $\{X, T\}$, T a topological group, it can be shown (see [9], [20]) that our choice is basically irrelevant since $\{X, T\}$ is always an Antoine space; that is, $\{X, T\}$ always lies in the minimum reflexive concrete cartesian closed extension of T.

We also choose to work with C, an extension of T because, as will become clear in Section 2, any convenient cartesian closed restriction of

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T , such as k-spaces, would require a stronger lifting-of-characters property than that which Kaplan [17] supplies.

In Section 1 we introduce the symmetric monoidal closed category $V = (V, Z, \otimes, [-, -], \ldots)$ of abelian group objects in C prior to studying, in Section 2, the limit closure of L in V (where L denotes the category of locally compact Hausdorff abelian groups). Pontryagin duality of topological groups A , with respect to the structure $[A, R/Z]$ of continuous convergence on the character groups, is considered in Section 3. The results are somewhat analogous to those of Lambek and Rattray [19] except that we seek to localise at the whole category L rather than just at $T = R/Z$.

It is known (see, for example, Hofmann [15], [16]) that every L-group is the inverse limit of elementary quotients, an elementary group being one of the form $T^a \oplus R^b \oplus G$, $a, b \in N$, and G discrete. Thus we could equally well work with elementary groups as models.

For references to the basic category theory we use Day and Kelly [12], Eilenberg and Kelly [14], Mac Lane [21], and Schubert [22].

1. Preliminaries

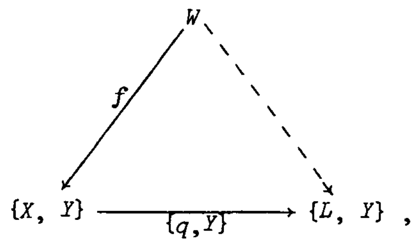
In introducing the symmetric monoidal closed category $V = (V, Z, \otimes, [-, -], \ldots)$ of abelian group objects in C we point out that it is well known that the category of abelian group objects in *any* complete and cocomplete cartesian closed category forms a symmetric monoidal closed category which is itself complete and cocomplete (see, for example, Borceux and Day [5], [6], and Day [8]). In the case of limit-spaces the tensor product structure is just an appropriate limitspace structure on the ordinary tensor product of the underlying abelian groups.

In the following section we shall, however, be more concerned with the internal-hom $[A, B]$ in V which is the group of homomorphisms in $\{A, B\}$ with the subspace limitstructure (often referred to as the structure of continuous convergence). Even if B is a topological group (*qua* limit-space), the hom $[A, B]$ is generally only a limitspace (see Binz [2], [3]). Some instances where $\{X, B\}$ is a topological group may be obtained as follows.

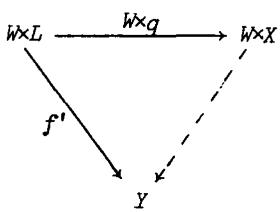
Call a quotient map $q : X \to Y$ in $\mathcal{T}$ *productive* if $q \times W$ is a quotient map in $\mathcal{T}$ for all $W \in \mathcal{T}$; such maps are completely characterised in Day and Kelly [13], Theorem 2.

PROPOSITION 1.1. *The limitspace* $\{X, Y\}$ *of all continuous maps from* X *to* Y *in* $\mathcal{T}$ *is again a topological space if* X *is a productive quotient of a locally compact Hausdorff space.*

Proof. Suppose $q : L \to X$ is a productive quotient of L , a locally compact Hausdorff space. Then $\{q, Y\} : \{X, Y\} \to \{L, Y\}$ is a subspace mapping in $\mathcal{C}$; to establish this note that, for each $W \in \mathcal{T}$ and map $f : W \to \{X, Y\}$, f is continuous in $\mathcal{C}$ if and only if $\{q, Y\} \cdot f$ is continuous in $\mathcal{C}$, since the diagram



in $\mathcal{C}$, transforms to the following diagram in $\mathcal{T}$:



Thus it remains to prove $\{L, Y\}$ is a topological space. Let $\{X, Y\}'$ be $\mathcal{T}(X, Y)$ with the compact-open topology. In order to establish that $\{L, Y\}' \cong \{L, Y\}$ we need only establish that they both admit the same maps from spaces $W \in \mathcal{T}$ since $\mathcal{T}$ is dense in $\mathcal{C}$. But, while

$$\mathcal{C}(W, \{L, Y\}) \cong \mathcal{C}(W \times L, Y) = \mathcal{T}(W \times L, Y) ,$$

we also have

$$\mathcal{T}(W, \{L, Y\}') \cong \mathcal{T}(W \times L, Y) ,$$

and the result follows. //

There is a canonical embedding $TAb \subset V$ and it is epireflexive by the

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special adjoint-functor theorem. The actual embedding will be omitted from the notation, a topological abelian group being regarded as a special instance of a limitspace abelian group. In view of this identification we see that the epireflexive hull H of L in TAb coincides with the epireflexive hull of L in V.

In order to give a neat description of the reflexion functors constructed we introduce what might be called the "standard presentation" of $A \in V$.

PROPOSITION 1.2. *The end* $\int_L L^{V(A,L)}$ *exists in* V *for each* $A \in V$ *and lies in the limit closure of* L *in* V.

Proof. Since the end $\int_L L^{V(A,L)}$ is computed over a large class L, we have to find a representation of it which is small. In order to do this, note that there exists only a small set of continuous maps $f : A \to X$, X Hausdorff, and $\mathrm{im}\, f$ dense in X. For each $L \in L$ let $\mathcal{D}(A, L)$ denote the set of dense continuous abelian group homomorphisms from A to L. Factor any $f : A \to L$ in V into a map $d : A \to M$, $\overline{d(A)} = M$, followed by an inclusion $M \leq L$, where $M \in L$ again. This process gives us a canonical coequaliser diagram in Ens:

$$\sum_{DA \times DA} \mathcal{D}(M, N) \times \mathcal{D}(A, M) \times L(N, L) \rightrightarrows \sum_{DA} \mathcal{D}(A, M) \times L(M, L) \to V(A, L) ,$$

which is natural in $L \in L$ and where DA denotes the small set of dense images of A. Thus we obtain an equaliser diagram in V:

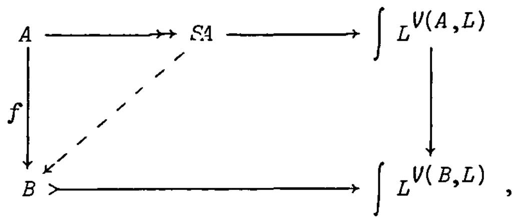
$$\int_L L^{V(A,L)} \to \prod_{DA} L^{\mathcal{D}(A,L)} \rightrightarrows \prod_{DA \times DA} L^{\mathcal{D}(M,L) \times \mathcal{D}(A,M)} ,$$

by the Yoneda Lemma applied to $L \in L$ (Day and Kelly [12] and Mac Lane [21]). //

There is a canonical map $\rho_A : A \to \int_L L^{V(A,L)}$.

PROPOSITION 1.3. *The process of factoring* ρ_A *into a continuous surjection followed by a subspace inclusion constitutes reflexion of* V *into the epireflexive hull* H *of* L *in* V.

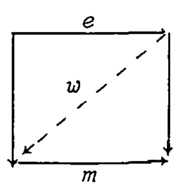
Proof. Let $A \to SA \to \int L^{V(A,L)}$ be the described factorisation of ρ_A. If $B \in H$, then there is a subspace inclusion $B \subset \prod L_\lambda$. This means that $B \to \int L^{V(B,L)}$ is a subspace inclusion. Now any map $f \in V(A, B)$ gives:



as required. //

2. The limit closure of L in V

We recall from Kelly [18] that a monic m in H is called *strong* if, given any commuting square:



with e an epi in H, there exists a (unique) w rendering both triangles commutative. We also recall that a (strong) monic is called *regular* if it happens to be an equaliser.

In advance one does not know whether or not *all* strong monics are regular. Because H is complete and cocomplete this will follow from Kelly [18], provided the pushout in H of a strong monic is monic. This, in turn, will be the case if H has an injective cogenerator.

THEOREM 2.1. *H has $T = R/Z$ as an injective cogenerator.*

Proof. First observe that the canonical map $A \to [[A, T], T]$ is a strong monic in V for all $A \in H$ since, by definition of H, there is a strong monic $A \rightarrowtail \prod L_\lambda$ in V; thus we simply consider the diagram:

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$$\begin{array}{ccc} A & \longrightarrow & \prod L_\lambda \\ \downarrow & & \downarrow \wr\wr \\ [[A,\ T],\ T] & \longrightarrow & \prod\ [[L_\lambda,\ T],\ T]\ . \end{array}$$

Now consider the composite mapping

$$A \to [[A,\ T],\ T] \to [[A,\ T]^*,\ T] \cong T^{H(A,T)}\ ,$$

where $[A,\ T]^* \to [A,\ T]$ is the canonical bijection from the discrete space $[A,\ T]^*$ on the underlying set of $[A,\ T]$. This composite is thus a monic, so T is a cogenerator of H . The circle group T is injective in H by Kaplan [17], Theorem 1. //

PROPOSITION 2.2. *The pushout in* H *of a strong monic is monic.*

Proof. This is by a well-known argument. Let $i : A \to B$ be a strong monic in H and let $f : A \to C$ in H . Form the pushout

$$\begin{array}{ccc} A & \xrightarrow{\ i\ } & B \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ C & \xrightarrow{\ j\ } & P \end{array}$$

in H . Then

$$\begin{array}{ccc} H(P,\ T) & \longrightarrow & H(C,\ T) \\ \downarrow & & \downarrow \\ H(B,\ T) & \longrightarrow & H(A,\ T) \end{array}$$

is a pullback in Ens ; thus $H(j,\ T)$ is a surjection. The result now follows from considering the diagram

$$\begin{array}{ccc} C & \xrightarrow{\ j\ } & P \\ \downarrow & & \downarrow \\ T^{H(C,T)} & \rightarrowtail & T^{H(P,T)}\ . \end{array} \qquad //$$

COROLLARY 2.3. *In* H *all strong monics are regular.*

Proof. By Kelly [18], Proposition 5.10 and Proposition 5.14. //

We can now compute the epimorphisms in H . Let $e : A \to B$ be epic

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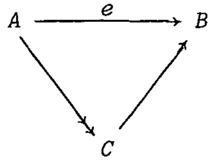
in H . Factor it into a continuous surjection followed by a subspace inclusion:



Since T is injective in H and $C \subset B$ is an epimorphism, we must have $H(B, T) \cong H(C, T)$. However, by Kaplan [17], Theorem 2, this is impossible unless $\overline{C} = B$. Thus the epimorphisms in H are precisely the epimorphisms in the category TAb_2 of Hausdorff topological abelian groups. This, in turn, means that the strong (equals regular) monics in H are precisely the closed subspace inclusions. Thus each closed subgroup of a product of L-groups is in fact a limit of L-groups.

THEOREM 2.4. *The limit closure* P *of* L *in* H *(as in* V *) is epireflexive in* H .

Proof. For each $A \in H$ we have a subspace inclusion $A \subset \int L^{V(A,L)}$. The reflexion of A into P is just the closure of A in $\int L^{V(A,L)}$. //

Much of the interest in this theorem centres around the fact that the limit closure P of L in V is cocomplete as well as being complete.

3. Duality in V

THEOREM 3.1. *Each* $A \in P$ *is Pontryagin reflexive in* V .

Proof. Each $A \in P$ admits an equaliser presentation $A \rightarrowtail \prod L_\lambda \rightrightarrows \prod L_\mu$, $L_\lambda, L_\mu \in L$. Thus we consider the following diagram:

$$\begin{array}{ccccc}
A & \rightarrowtail & \prod L_\lambda & \rightrightarrows & \prod L_\mu \\
\downarrow \eta & & \downarrow \eta & & \downarrow \eta \\
A^{**} & \overset{h}{\rightarrowtail} & \left(\prod L_\lambda\right)^{**} & \rightrightarrows & \left(\prod L_\mu\right)^{**} \\
\downarrow m & & \downarrow \eta^{-1} & & \downarrow \eta^{-1} \\
A & \rightarrowtail & \prod L_\lambda & \rightrightarrows & \prod L_\mu
\end{array} ,$$

where the A^{**} denotes the double dual of A. Each product $\prod L_\lambda$ is Pontryagin reflexive in V by Day [11], Corollary 5.1.5. By injectivity of T, h is a monic, so m is a monic. But $m\eta = 1$, so m and η are mutually inverse. //

COROLLARY 3.2. *Every closed subgroup of a product of* L*-groups is Pontryagin reflexive in* V.

A closely related subcategory of V is the category Q of all direct limits in V of locally compact Hausdorff abelian groups. Thus $A \in V$ is in Q if and only if there exists a coequaliser presentation

$\sum L_\mu \rightrightarrows \sum L_\lambda \to A$ of A in V, where $L_\mu, L_\lambda \in L$. Since Z is a projective generator of V, it is straightforward (by analogy with Section 2) to establish that Q is closed under colimits (equals direct limits) in V, and thus is coreflexive in V by the special adjoint-functor theorem. We denote the coreflexion by $R : V \to Q$.

The functor $[-, T] : Q^{op} \to P$ now has a left adjoint, namely the opposite of $R[-, T] : P^{op} \to Q$. In view of this adjunction we have

$$Q(B, R[A, T]) \cong P(A, [B, T]) .$$

Upon setting $B = Z$ we see that the canonical map $R[A, T] \to [A, T]$ is a continuous bijection.

Finally, we have the usual dual equivalence between $Fix\,[R[-, T], T] \subset P$ and $Fix\, R[[-, T], T]' \subset Q$. This equivalence extends the duality $L \simeq L^{op}$, as we shall see in the next section.

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4. Compactly generated limitspaces

We now introduce the cartesian closed category K of compactly generated limitspaces (K-spaces for brief). An object of K is a limitspace X for which there exists a strong epimorphism $\sum_{\lambda\in\Lambda} C_\lambda \to X$ in C where all the C_λ are compact Hausdorff spaces. Clearly K is closed under coproducts and strong epimorphisms in C, so $K \subset C$ has a right adjoint $W : C \to K$. Because $C(1, X) \cong C(1, WX)$ for all $X \in C$ we see that W does not alter underlying sets. Moreover, it is easily seen that K is cartesian closed, since the product *in* C of a finite number of K-spaces is again a K-space.

PROPOSITION 4.1. *Each locally compact Hausdorff space is, as a limitspace, a* K*-space.*

Proof. The embedding $T \subset C$ preserves all coproducts and those quotient maps $f : X \to Y$ which satisfy the following condition (see Day [10]): given any $y \in Y$ there exists a finite number of points $\{x_1, \ldots, x_n\} \subset f^{-1}y$ such that each neighbourhood of $\{x_1, \ldots, x_n\}$ maps to a neighbourhood of $y \in Y$. Now suppose Y is locally compact and Hausdorff. For each $y \in Y$ choose a compact Hausdorff neighbourhood C_y and give it the subspace topology in Y. Then $\sum_{y\in Y} C_y \to Y$ is clearly a quotient of the required form. //

A strong projective limit in T is a limit $\lim_{\lambda\in\Lambda} X_\lambda$ over a directed set Λ such that each projection $p_\lambda : \lim X_\lambda \to X_\lambda$ is an identification map in T. For example, a product $\prod_{\lambda\in\Lambda} X_\lambda$ may be regarded as a strong limit cofiltered over the set of finite subsets of Λ.

LEMMA 4.2. *Given a strong projective limit in* TAb *with projections* $p_\lambda : \lim A_\lambda \to A_\lambda$*, the collection* $\{\ker p_\lambda ; \lambda \in \Lambda\}$ *is a filter base on* $\lim A_\lambda$ *and it converges to zero.*

Proof. Since Λ is directed, the collection $\left\{p_\lambda^{-1}(V) ; V \text{ open in } A_\lambda\right\}$

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is a base for the topology on $\lim A_\lambda$ in TAb . Thus $\{\ker p_\lambda\} \to 0$. //

PROPOSITION 4.3. *Let* $\lim A_\lambda$ *be a strong projective limit in* TAb . *Then the continuous comparison map* $\operatorname{colim}[A_\lambda, T] \to \lim[A_\lambda, T]$ *is a homeomorphism* $\operatorname{colim} W[A_\lambda, T] \cong W[\lim A_\lambda, T]$.

Proof. Let $f : C \to [\lim A_\lambda, T]$ be a continuous (test) map from a compact Hausdorff space C . This transforms to a morphism $f' : \lim A_\lambda \to \{C, T\}$. But clearly $\{C, T\}$, which has the compact-open topology, has no small subgroups, so f' factors through some projection $p_\lambda : \lim A_\lambda \to A_\lambda$ (by the lemma). This then yields a morphism $C \to [A_\lambda, T]$ and the result follows since both $\operatorname{colim} W[A_\lambda, T]$ and $W[\lim A_\lambda, T]$ admit the same morphisms from compact Hausdorff spaces. //

For each strong limit $\lim A_\lambda$ in TAb of L-groups we have continuous bijections

$$\operatorname{colim}[A_\lambda, T] \to R[\lim A_\lambda, T] \to [\lim A_\lambda, T] .$$

PROPOSITION 4.4. *If* $\lim A_\lambda$ *is a strong projective limit in* TAb *of* L*-groups, then* $\lim A_\lambda \in Fix[R[-, T], T]$.

Proof. Firstly $R[\lim A_\lambda, T]$ is a K-space because this object is a quotient of a sum of locally compact Hausdorff spaces in V ; hence in C (simply filter each sum in V to obtain a quotient map in C , remembering that the forgetful functor $V \to C$ creates filtered colimits (see, for example [6])). Thus, by Proposition 4.3, we have

$$\operatorname{colim}[A_\lambda, T] \cong R[\lim A_\lambda, T] \cong W[\lim A_\lambda, T] ;$$

so

$$[\operatorname{colim}[A_\lambda, T], T] \cong [R[\lim A_\lambda, T], T] ,$$

whence $\lim A_\lambda \cong [R[\lim A_\lambda, T], T]$; so $\lim A_\lambda \in Fix[R[-, T], T]$, as required. //

From this fact we deduce that the dual equivalence $Fix[R[-, T], T] \simeq Fix\ R[[-, T], T]^{op}$ is larger than $L \simeq L^{op}$.

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