Canad. Math. Bull. Vol. **39** (3), 1996 pp. 294–307

CONJUGACY SEPARABILITY OF CERTAIN POLYGONAL PRODUCTS

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ABSTRACT. We show that polygonal products of polycyclic-by-finite groups amalgamating central cyclic subgroups, with trivial intersections, are conjugacy separable. Thus polygonal products of finitely generated abelian groups amalgamating cyclic subgroups, with trivial intersections, are conjugacy separable. As a corollary of this, we obtain that the group $A_1 *_{\langle a_1\rangle} A_2 *_{\langle a_2\rangle} \cdots *_{\langle a_{m-1}\rangle} A_m$ is conjugacy separable for the abelian groups A_i.

1. **Introduction.** A group G is called *conjugacy separable* (c.s.) iff to each pair $x, y \in G$ either x and y are conjugate in G ($x \sim_G y$) or their images are not conjugate in some finite quotient of G. For example, polycyclic-by-finite [5], free-by-finite [2], and Fuchsian [4] groups are c.s. In general, it is not known whether free products of those c.s. groups amalgamating a cyclic subgroup are c.s. However free products of free—or nilpotent—groups [3], certain finite extensions of free—or nilpotent—groups [16], and surface groups [15] amalgamating a cyclic subgroup are c.s. Also the conjugacy separability of certain free products of c.s. groups amalgamating a cyclic retract has been considered in [11, 8]. The purpose of this paper is to investigate the conjugacy separability of certain polygonal products of groups. We show that polygonal products of more than three polycyclic-by-finite groups amalgamating central cyclic subgroups with trivial intersections are c.s. (Theorem 4.1).

Polygonal products of groups were introduced by A. Karrass, A. Pietrowski and D. Solitar [7] in the study of the subgroup structure of the Picard group PSL$(2, Z[i])$, which is a polygonal product of four finite groups amalgamating cyclic subgroups, with trivial intersections. In [1], Allenby and Tang proved that polygonal products of four finitely generated (f.g.) free abelian groups, amalgamating cyclic subgroups with trivial intersections, are residually finite ($\mathcal{RF}$). And they gave an example of a polygonal product of f.g. nilpotent groups which is not $\mathcal{RF}$. However, in [12, 10], Tang and Kim showed that certain polygonal products of f.g. nilpotent groups are $\mathcal{RF}$ or π_c. In [9], Kim proved that polygonal products of more than three polycyclic-by-finite groups amalgamating central subgroups with trivial intersections are π_c; hence they are $\mathcal{RF}$. In [9], the subgroup separability of polygonal products is also considered. Kim and Tang [13] constructed a polygonal product of f.g. free abelian groups amalgamating cyclic

The author was supported by N.D.R.F., Korea Research Foundation, 1994.

Received by the editors Novmber 9, 1995.

AMS subject classification: Primary: 20E26, 20E06; secondary: 20F10.

Key words and phrases: Polygonal products, generalized free products, conjugacy separable, residually finite.

subgroups, with trivial intersections, which is not residually p-finite for any prime p. Thus we naturally ask whether polygonal products of f.g. abelian groups amalgamating cyclic subgroups, with trivial intersections, are c.s. In this paper, we obtain that those polygonal products are c.s. (Corollary 4.2).

2. **Preliminaries.** Briefly, polygonal products of groups can be described as follows [1]: Let P be a polygon. Assign a group G_v to each vertex v and a group G_e to each edge e of P. Let α_e and β_e be monomorphisms which embed G_e as a subgroup of the two vertex groups at the ends of the edge e. Then the *polygonal product* G is defined to be the group presented by the generators and relations of the vertex groups together with the extra relations obtained by identifying $g_e\alpha_e$ and $g_e\beta_e$ for each $g_e \in G_e$. By abuse of language, we say that G is the polygonal product of the (vertex) groups $G_0, G_1, \ldots, G_n$, amalgamating the (edge) subgroups $H_1, \ldots, H_n, H_0$ with *trivial intersections*, if $G_{i-1} \cap G_i = H_i$ and $H_{i-1} \cap H_i = 1$, where $0 \leq i \leq n$ and the subscripts i are taken modulo $n + 1$. We only consider the case $n \geq 3$ (see [1]).

We introduce some definitions and results that we shall use in this paper.

We write $x \sim_G y$ if there exists $g \in G$ such that $x = g^{-1}yg$ and we write $x \not\sim_G y$ otherwise. $\{x\}^G$ denotes the conjugacy class $\{y \in G : x \sim_G y\}$ of x in G. We use $\langle X\rangle^G$ to denote the normal closure of X in G. We also use $[x,y] = x^{-1}y^{-1}xy$ and $C_H(K) = \{h \in H : [h,k] = 1 \text{ for all } k \in K\}$.

We denote by $A *_H B$ the free product of A and B with their subgroup H amalgamated. If $G = A *_H B$ and $x \in G$ then $\|x\|$ denotes the amalgamated free product length of x in G. On the other hand we use $|x|$ to denote the order of x. We write $N \lhd_f G$ to denote that N is a normal subgroup of finite index in G. If $\bar{G}$ is a homomorphic image of G then we use $\bar{x}$ to denote the image of $x \in G$ in $\bar{G}$.

Let H be a subset of G. Then we say that G is *H-separable* if to each $x \in G \setminus H$ there exists $N \lhd_f G$ such that $x \notin NH$. A group G is said to be *residually finite* ($\mathcal{RF}$) if G is $\langle 1\rangle$-separable, and G is said to be π_c if G is $\langle x\rangle$-separable for any $x \in G$. We shall use the following results:

THEOREM 2.1 ([3]). *If A and B are* c.s. *and H is finite, then $A *_H B$ is* c.s.

THEOREM 2.2 ([9]). *Let G be the polygonal product of the polycyclic-by-finite groups $A_0, A_1, \ldots, A_n$ ($n \geq 3$), amalgamating any subgroups $H_1, \ldots, H_n, H_0$, with trivial intersections, where $H_i \subset Z(A_{i-1}) \cap Z(A_i)$ for all i, and where subscripts are taken modulo $n + 1$. Then G is π_c.*

LEMMA 2.3 ([9]). *Let A_i and H_i be as in Theorem 2.2, and let $E_m = A_0 *_{H_1} A_1 *_{H_2} \cdots *_{H_m} A_m$ ($m \geq 1$). Then E_m is ($H_0 * H_{m+1}$)-separable and H_0H_{m+1}-separable.*

For a graph Γ, with vertex set V and edge set E, assign a group G_v to each vertex $v \in V$. Then the group $\langle G_v; [G_v, G_w], \text{ for } \forall vw \in E\rangle$ is called the *graph product* of the groups G_v for the graph Γ. For example, the graph product of cyclic groups $\langle a_i\rangle$ ($i = 1, 2, \ldots, n$) for the n-gon is just the polygonal product of abelian groups $\langle a_1, a_2\rangle, \langle a_2, a_3\rangle, \ldots, \langle a_n, a_1\rangle$ amalgamating subgroups $\langle a_2\rangle, \langle a_3\rangle, \ldots, \langle a_1\rangle$ with trivial intersections. Hence such polygonal product is c.s. by the next result.

THEOREM 2.4 ([6, P104]). *Graph products of* c.s. *groups are* c.s.

As Dyer [3] observed, the main tool to prove the conjugacy separability of a free product with amalgamation is the following result, known as Solitar's theorem:

THEOREM 2.5 ([14]). *Let* $G = A *_H B$ *and* $x \in G$ *be of minimal length in its conjugacy class. Suppose* $y \in G$, y *is cyclically reduced, and* $x \sim_G y$.

(1) If $\|x\| = 0$, *then* $\|y\| \leq 1$ *and if* $y \in A$ *say, there is a sequence* $h_1, h_2, \ldots, h_r$ *of elements in* H *such that* $y \sim_A h_1 \sim_B h_2 \sim_A \cdots \sim_B h_r = x$.

(2) If $\|x\| = 1$, *then* $\|y\| = 1$ *and either* $x, y \in A$ *and* $x \sim_A y$, *or else* $x, y \in B$ *and* $x \sim_B y$.

(3) If $\|x\| \geq 2$, *then* $\|x\| = \|y\|$ *and* $y \sim_H x^*$ *where* x^* *is some cyclic permutation of* x.

3. **Some lemmas.** A group G is called *polycyclic-by-finite* if it has a normal subgroup N such that N is polycyclic and G/N is finite. Throughout this paper we consider that the A_i are polycyclic-by-finite groups and that $a_i, a_{i+1} \in Z(A_i)$, $\langle a_i \rangle \cap \langle a_{i+1} \rangle = 1$, and $A_i \cap A_{i+1} = \langle a_{i+1} \rangle$.

LEMMA 3.1. *Let* $E = A_1 *_{\langle a_2 \rangle} A_2 *_{\langle a_3 \rangle} \cdots *_{\langle a_{m-1} \rangle} A_{m-1}$ $(m \geq 3)$, *and* $H = \langle a_1 \rangle * \langle a_m \rangle$. *If* $x \sim_E y$ *for* $x, y \in H$, *then* $x \sim_H y$.

PROOF. We may assume that x and y are cyclically reduced in H and $x \neq 1 \neq y$.

First, suppose $x \in \langle a_1 \rangle$ (or $x \in \langle a_m \rangle$). Let $E = A_1 *_{\langle a_2 \rangle} E_1$, where $E_1 = A_2 *_{\langle a_3 \rangle} \cdots *_{\langle a_{m-1} \rangle} A_{m-1}$. Since $x \in Z(A_1)$ and $\langle a_1 \rangle \cap \langle a_2 \rangle = 1$, we have $\{x\}^{A_1} \cap \langle a_2 \rangle = \emptyset$. Thus x has the minimal length 1 in its conjugacy class in E. Thus by Theorem 2.5 $x \sim_E y$ implies $y \in A_1$ and $x \sim_{A_1} y$; hence $x = y$. Clearly $x \sim_H y$.

Second, suppose $\|x\| = 2n > 1$. Let $E = A_1 *_{\langle a_2 \rangle} E_1$ be as above. Since x has the minimal length $2n$ in its conjugacy class in E, by Theorem 2.5 we have $\|x\| = \|y\|$ and $x \sim_{\langle a_2 \rangle} y^*$ for some cyclic permutation y^* of y. If $m = 3$, *i.e.*, $E = A_1 *_{\langle a_2 \rangle} A_2$, then $x \sim_{\langle a_2 \rangle} y^*$ implies $x = y^*$; hence $x \sim_H y$. If $m > 3$, then $\langle a_2, a_m \rangle = \langle a_2 \rangle * \langle a_m \rangle$. Now suppose $x = a_1^{\epsilon_1} a_m^{\delta_1} \cdots a_1^{\epsilon_n} a_m^{\delta_n}$, $y^* = a_1^{\epsilon'_1} a_m^{\delta'_1} \cdots a_1^{\epsilon'_n} a_m^{\delta'_n}$, and $x = a_2^{-\lambda} y^* a_2^{\lambda}$. Then we have $a_1^{\epsilon_1} = a_2^{-\lambda} a_1^{\epsilon'_1} a_2^{\lambda_1}$, $a_m^{\delta_1} = a_2^{-\lambda_1} a_m^{\delta'_1} a_2^{t_1}$, $a_1^{\epsilon_2} = a_2^{-t_1} a_1^{\epsilon'_2} a_2^{\lambda_2}, \ldots$. Hence $a_2^{\lambda} = a_2^{\lambda_1}$ and $a_1^{\epsilon_1} = a_1^{\epsilon'_1}$, since $a_1 \in Z(A_1)$ and $\langle a_1 \rangle \cap \langle a_2 \rangle = 1$. Now, since $\langle a_2, a_m \rangle = \langle a_2 \rangle * \langle a_m \rangle$, we have $a_2^{-\lambda_1} = 1 = a_2^{t_1}$, and hence $a_2^{\lambda} = 1$. Thus $x = y^*$; hence $x \sim_H y$. ∎

LEMMA 3.2. *Let* P *be the polygonal product of the polycyclic-by-finite groups* $\langle a_0, a_1 \rangle, A_1, \ldots, A_m$ $(m \geq 3)$, *amalgamating the central subgroups* $\langle a_1 \rangle, \ldots, \langle a_m \rangle, \langle a_0 \rangle$ *with trivial intersections. Denote* $A = \langle a_0, a_1 \rangle$. *Then we have*

(a) If $x \sim_P y$ *for* $x, y \in A$, *then* $x = y$.

(b) If $x \sim_P y$ *for* $x \in A$ *and* $y \in A_1$, *then* $y \in A$ *and* $x = y$.

PROOF. (a) Let $E = \langle a_0, a_1 \rangle *_{\langle a_1 \rangle} A_1$, $F = A_m *_{\langle a_m \rangle} \cdots *_{\langle a_3 \rangle} A_2$, and $H = \langle a_0 \rangle * \langle a_2 \rangle$. Then $P = E *_H F$. We may assume $x \neq 1 \neq y$.

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CASE 1. Suppose $x \in \langle a_0 \rangle$.

Since $\|x\| = 0$ in P and $y \in \langle a_0, a_1 \rangle \subset E$, by Theorem 2.5 there exist $h_i \in H$ such that $y \sim_E h_1 \sim_F h_2 \sim_E \cdots \sim_E h_r = x$. It follows from Lemma 3.1 that $y \sim_E x$. Now $1 \neq x \in \langle a_0 \rangle$ and $a_1 \in Z(E)$; hence x has the minimal length 1 in its conjugacy class in $E = \langle a_0, a_1 \rangle *_{\langle a_1 \rangle} A_1$. Thus, by Theorem 2.5, $y \in A$ and $x \sim_A y$, where $A = \langle a_0, a_1 \rangle$ is abelian. Therefore $x = y$.

CASE 2. Suppose $x \notin \langle a_0 \rangle$. Clearly $x \notin H$.

First, we note that x has the minimal length 1 in its conjugacy class in $P = E *_H F$. For, if $x \sim_P h$ for some $h \in H$, then $x \sim_E h_1 \sim_F h_2 \sim_E \cdots \sim_E h_r = h$ for some $h_i \in H$. Then, by Lemma 3.1, we have $x \sim_E h$. Thus $x \sim_E h^*$ for a cyclically reduced cyclic permutation h^* of h. If $x \in \langle a_1 \rangle$, then $x = h^* \in \langle a_0, a_1 \rangle \cap H = \langle a_0 \rangle$. Hence, by assumption, $x \notin \langle a_1 \rangle$. Then x has the minimal length 1 in its conjugacy class in $E = \langle a_0, a_1 \rangle *_{\langle a_1 \rangle} A_1$. Thus by Theorem 2.5, $h^* \in A$ and $x \sim_A h^*$, where $A = \langle a_0, a_1 \rangle$. Hence $x = h^* \in H \cap A = \langle a_0 \rangle$, a contradiction. Therefore x has the minimal length 1 in its conjugacy class in $P = E *_H F$. Then, by Theorem 2.5, $x \sim_E y$. Now if $x \in \langle a_1 \rangle$ then $x = y$. If $x \notin \langle a_1 \rangle$ then x has the minimal length 1 in its conjugacy class in $E = A *_{\langle a_1 \rangle} A_1$, and then by Theorem 2.5 $x \sim_A y$. Thus $x = y$.

The proof of (b) is very similar to that of (a) above. ∎

LEMMA 3.3. *Let $F = A_0 *_{\langle a_1 \rangle} A_1 *_{\langle a_2 \rangle} \cdots *_{\langle a_m \rangle} A_m$ $(m \geq 1)$. If $[a_0^k, f] = 1$ for $a_0^k \neq 1$ and $f \in F$, then $f \in A_0$ and hence $[a_0, f] = 1$.*

PROOF. Let $F = A_0 *_{\langle a_1 \rangle} F_1$, where $F_1 = A_1 *_{\langle a_2 \rangle} \cdots *_{\langle a_m \rangle} A_m$. If $f \in F_1 \setminus \langle a_1 \rangle$, then clearly $f \neq a_0^{-k} f a_0^k$, since $\|a_0^{-k} f a_0^k\| = 3$. Thus suppose $f \notin A_0 \cup F_1$. Since $a_0^k \in Z(A_0)$, it suffices to consider $f = f_1 \alpha_1 \cdots \alpha_{n-1} f_n$, where $\alpha_i \in A_0 \setminus \langle a_1 \rangle$ and $f_i \in F_1 \setminus \langle a_1 \rangle$. Then $a_0^{-k} f a_0^k = a_0^{-k} f_1 \alpha_1 \cdots f_n a_0^k$ is reduced with length $\|f\| + 2$. Thus $f \neq a_0^{-k} f a_0^k$. Consequently, $f \in A_0$, and hence $[a_0, f] = 1$. ∎

LEMMA 3.4. *Let P be the polygonal product of the polycyclic-by-finite groups $A_0, A_1, \ldots, A_m$ $(m \geq 3)$, amalgamating the central subgroups $\langle a_1 \rangle, \ldots, \langle a_m \rangle, \langle a_0 \rangle$ with trivial intersections. Let $a_0^k \neq 1 \neq a_1^\ell$ and $p \in P$.*

(a) If $a_0^k \in C_{A_0}(p)$, then $a_0 \in C_{A_0}(p)$; hence $C_{A_0}(p) \cap \langle a_0 \rangle = \langle a_0 \rangle$.

(b) If $a_0^k a_1^\ell \in C_{A_0}(p)$, then $p \in A_0$; hence $C_{A_0}(p) \cap \langle a_0, a_1 \rangle = \langle a_0, a_1 \rangle$.

PROOF. (a) Let $E = A_0 *_{\langle a_1 \rangle} A_1$, $F = A_m *_{\langle a_m \rangle} \cdots *_{\langle a_3 \rangle} A_2$, and $H = \langle a_0 \rangle * \langle a_2 \rangle$. Then $P = E *_H F$.

First, if $p \in E$ (or $p \in F$), then by Lemma 3.3 we have $p \in A_0$ (or $p \in A_m$). Then $[a_0, p] = 1$, since $a_0 \in Z(A_0) \cap Z(A_m)$.

Second, if $p \notin E \cup F$, suppose $p = e_1 f_1 \cdots e_n f_n$, where $e_i \in E \setminus H$ and $f_i \in F \setminus H$ (the other cases are similar). Since $e_1 f_1 \cdots e_n f_n = a_0^{-k} e_1 f_1 \cdots e_n f_n a_0^k$, we have $e_1 = a_0^{-k} e_1 h_1$, $f_1 = h_1^{-1} f_1 k_1$, $e_2 = k_1^{-1} e_2 h_2, \ldots, e_n = k_{n-1}^{-1} e_n h_n$, and $f_n = h_n^{-1} f_n a_0^k$, for some $h_i, k_i \in H$. Then by Lemma 3.1, $a_0^k \sim_H h_1 \sim_H h_1^*$ for some cyclically reduced cyclic permutation h_1^* of h_1. Hence $a_0^k = h_1^*$, and it follows that $h_1 = w_1^{-1} a_0^k w_1$ for some $w_1 \in H$. Thus $e_1 = a_0^{-k} e_1 h_1 = a_0^{-k} e_1 w_1^{-1} a_0^k w_1$. By Lemma 3.3, $e_1 w_1^{-1} \in A_0$. Now $f_1 = h_1^{-1} f_1 k_1 =$

$w_1^{-1}a_0^{-k}w_1f_1k_1$; hence $w_1f_1 = a_0^{-k}\cdot w_1f_1\cdot k_1$. Then as before there exists $v_1 \in H$ such that $k_1 = v_1^{-1}a_0^kv_1$ and $w_1f_1v_1^{-1} \in A_m$. Inductively, suppose there exist $w_{n-1}, v_{n-1} \in H$ such that $k_{n-1} = v_{n-1}^{-1}a_0^kv_{n-1}$, and $w_{n-1}f_{n-1}v_{n-1}^{-1} \in A_m$. Then $e_n = k_{n-1}^{-1}e_nh_n = v_{n-1}^{-1}a_0^{-k}v_{n-1}e_nh_n$; hence as before there exists $w_n \in H$ such that $h_n = w_n^{-1}a_0^kw_n$, and $v_{n-1}e_nw_n^{-1} \in A_0$. Then $f_n = h_n^{-1}f_na_0^k = w_n^{-1}a_0^{-k}w_nf_na_0^k$. Hence, by Lemma 3.3, $w_nf_n \in A_m$. Therefore $p = e_1f_1\cdots e_nf_n = e_1w_1^{-1}\cdot w_1f_1v_1^{-1}\cdots v_{n-1}e_nw_n^{-1}\cdot w_nf_n$ is a product of elements in A_0 and A_m. Since $a_0 \in Z(A_0)\cap Z(A_m)$, we have $[a_0, p] = 1$.

(b) Let E, F, H be as above. If $p \in E$ then we have $a_0^kp = pa_0^k$. By Lemma 3.3, we have $p \in A_0$. Thus we shall show that if $p \notin E$ then $a_0^ka_1^\ell \notin C_{A_0}(p)$. If $p \in F\setminus H$ then clearly $a_0^ka_1^\ell p \neq pa_0^ka_1^\ell$, so suppose that $p \notin E\cup F$. If $p = f_1e_1\cdots$, or if $p = \cdots e_nf_n$, where $e_i \in E\setminus H$ and $f_i \in F\setminus H$, then clearly $a_0^ka_1^\ell p \neq pa_0^ka_1^\ell$. Thus we suppose $p = e_1f_1\cdots f_{n-1}e_n$, where $e_i \in E\setminus H$ and $f_i \in F\setminus H$. Now if $a_0^ka_1^\ell p = pa_0^ka_1^\ell$ then $a_0^ka_1^\ell e_1 \notin H$, $e_na_0^ka_1^\ell \notin H$, and we have $a_0^ka_1^\ell e_1 = e_1h_1$ for some $h_1 \in H$. Thus $a_0^ka_1^\ell \sim_E h_1 \sim_H h_1^*$ for some cyclically reduced cyclic permutation h_1^* of h_1. Since $a_0^ka_1^\ell$ has the minimal length 1 in its conjugate class in E, we have $h_1^* \in A_0$ and $a_0^ka_1^\ell \sim_{A_0} h_1^*$. Hence $a_0^ka_1^\ell = h_1^* \in A_0\cap H = \langle a_0\rangle$, a contradiction. ∎

Let P be as in Lemma 3.4. Then, for integers $s, t > 1$, we may construct a polygonal product $\bar{P}$ of $\bar{A}_0, \bar{A}_1, \ldots, \bar{A}_m$ $(m \geq 3)$, amalgamating subgroups $\langle\bar{a}_1\rangle, \ldots, \langle\bar{a}_m\rangle, \langle\bar{a}_0\rangle$, with trivial intersections, where $\bar{A}_0 = A_0/\langle a_0^s, a_1^t\rangle$, $\bar{A}_1 = A_1/\langle a_1^t\rangle$, $\bar{A}_m = A_m/\langle a_0^s\rangle$, and $\bar{A}_i = A_i$ for $i \neq 0, 1, m$. Then there exists a natural homomorphism $\phi_{s,t}: P \longrightarrow \bar{P}$ with $\ker\phi_{s,t} = \langle a_0^s, a_1^t\rangle^P$. Hence we may consider $\bar{P} = P/\langle a_0^s, a_1^t\rangle^P$.

LEMMA 3.5. *Let P be the polygonal product of the polycyclic-by-finite groups $A = \langle a_0, a_1\rangle, A_1, \ldots, A_m$ $(m \geq 3)$, amalgamating the central subgroups $\langle a_1\rangle, \ldots, \langle a_m\rangle, \langle a_0\rangle$ with trivial intersections. If $\{x\}^P\cap A = \emptyset$, where $x \in P$, then there exist s, t such that, in $\bar{P} = P/\langle a_0^s, a_1^t\rangle^P$, we have $\{\bar{x}\}^{\bar{P}}\cap\bar{A} = \emptyset$.*

PROOF. Let $E = A *_{\langle a_1\rangle} A_1$, $F = A_m *_{\langle a_m\rangle}\cdots *_{\langle a_3\rangle} A_2$, and $H = \langle a_0\rangle * \langle a_2\rangle$. Then $P = E *_H F$. Clearly $x \notin A$. We may assume that x has minimal length in its conjugacy class in $P = E *_H F$.

CASE 1. Suppose $x \in E$.

Thus $\{x\}^E\cap A = \emptyset$. Now $x \notin A$, and we may assume that x has minimal length in its conjugacy class in E. If $x \in A_1\setminus\langle a_1\rangle$, then $\bar{x} \in \bar{A}_1\setminus\langle\bar{a}_1\rangle$ for any s, t. Thus $\bar{x}$ has the minimal length 1 in its conjugacy class in $\bar{E} = \bar{A} *_{\langle\bar{a}_1\rangle}\bar{A}_1$. Thus we have $\{\bar{x}\}^{\bar{E}}\cap\bar{A} = \emptyset$. If $x = \alpha_1\beta_1\cdots\alpha_n\beta_n$, where $\alpha_i \in A\setminus\langle a_1\rangle$ and $\beta_i \in A_1\setminus\langle a_1\rangle$. Choose s, t so that $\alpha_i \notin \langle a_0^s, a_1^t\rangle\langle a_1\rangle$. Then, in $\bar{E}$, we have $\|x\| = \|\bar{x}\|$ and hence $\{\bar{x}\}^{\bar{E}}\cap\bar{A} = \emptyset$. Now, in $\bar{P} = P/\langle a_0^s, a_1^t\rangle^P$, we claim $\{\bar{x}\}^{\bar{P}}\cap\bar{A} = \emptyset$. For this, suppose $\bar{x} \sim_{\bar{P}} \bar{\alpha}$, for some $\alpha \in A$. If $\bar{\alpha} \in \langle\bar{a}_0\rangle$, then $\bar{x} \sim_{\bar{E}} \bar{h}_1 \sim_{\bar{F}} \bar{h}_2 \sim_{\bar{E}} \cdots \sim_{\bar{E}} \bar{h}_r = \bar{\alpha}$. Thus, by Lemma 3.1, $\bar{x} \sim_{\bar{E}} \bar{\alpha}$. If $\bar{\alpha} \notin \langle\bar{a}_0\rangle$ then, as in the proof of Lemma 3.2, $\bar{\alpha}$ has the minimal length 1 in its conjugacy class in $\bar{P} = \bar{E} *_{\bar{H}} \bar{F}$. Thus by Theorem 2.5 we have $\bar{\alpha} \sim_{\bar{E}} \bar{x}$, which contradicts the fact that $\{\bar{x}\}^{\bar{E}}\cap\bar{A} = \emptyset$. Therefore, we have $\{\bar{x}\}^{\bar{P}}\cap\bar{A} = \emptyset$.

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CASE 2. Suppose $x \in F \setminus H$.

Since F is H-separable (Lemma 2.3), there is s_1 such that $x \notin \langle a_0^{s_1}\rangle^F H$. Now $\{x\}^F \cap \langle a_0\rangle = \emptyset$, and we may assume that x is cyclically reduced in $F = A_m *_{\langle a_m\rangle} F_1$, where $F_1 = A_{m-1} *_{\langle a_{m-1}\rangle} \cdots *_{\langle a_3\rangle} A_2$. If $x = \alpha_1 f_1 \cdots$, where $\alpha_i \in A_m \setminus \langle a_m\rangle$ and $f_i \in F_1 \setminus \langle a_m\rangle$, then there exists s_2 such that $\alpha_i \notin \langle a_0^{s_2}\rangle\langle a_m\rangle$. Let $s = s_1 s_2$ and t be arbitrary. Then in $\bar{F} = A_m/\langle a_0^s\rangle *_{\langle a_m\rangle} F_1$, $\bar{x}$ is cyclically reduced with $\|\bar{x}\| = \|x\|$, and any $\bar{a}_0^\epsilon (\neq 1)$ has the minimal length 1 in its conjugacy class in $\bar{F}$. Hence, we have $\{\bar{x}\}^{\bar{F}} \cap \langle \bar{a}_0\rangle = \emptyset$. In $\bar{P} = P/\langle a_0^s, a_1^t\rangle^P$, if $\bar{x} \sim_{\bar{P}} \bar{\alpha}$ for some $\alpha \in A$ then, as in Case 1 above, $\bar{x} \sim_{\bar{F}} \bar{\alpha} \in \bar{A} \cap \bar{F} = \langle \bar{a}_0\rangle$. Therefore, since $\{\bar{x}\}^{\bar{F}} \cap \langle \bar{a}_0\rangle = \emptyset$, we have $\{\bar{x}\}^{\bar{P}} \cap \bar{A} = \emptyset$.

CASE 3. Suppose $x \notin E \cup F$.

Let $x = e_1 f_1 \cdots e_n f_n$, where $e_i \in E \setminus H$ and $f_i \in F \setminus H$. Since E, F are H-separable (Lemma 2.3), there exist s, t such that $e_i \notin \langle a_0^s, a_1^t\rangle^E H$ and $f_i \notin \langle a_0^s\rangle^F H$. Then, in $\bar{P} = P/\langle a_0^s, a_1^t\rangle^P$, $\|\bar{x}\| = \|x\| = 2n$. Thus $\{\bar{x}\}^{\bar{P}} \cap \bar{A} = \emptyset$. ∎

The following few lemmas are used to prove Lemma 3.9.

LEMMA 3.6. *Let $F = A_m *_{\langle a_m\rangle} \cdots *_{\langle a_3\rangle} A_2$, $H = \langle a_0\rangle * \langle a_2\rangle$, and let $f, f' \in F$.*
(a) If $f' \notin \langle a_0\rangle f H$, then there exists s such that $\bar{f}' \notin \langle \bar{a}_0\rangle \bar{f}\bar{H}$ in $\bar{F} = F/\langle a_0^s\rangle^F$.
(b) If $f' \notin \langle a_0\rangle f\langle a_2\rangle$, then there exists s such that $\bar{f}' \notin \langle \bar{a}_0\rangle \bar{f}\langle \bar{a}_2\rangle$ in $\bar{F} = F/\langle a_0^s\rangle^F$.
(c) If $f' \notin \langle a_0\rangle f\langle a_0\rangle$, then there exists s such that $\bar{f}' \notin \langle \bar{a}_0\rangle \bar{f}\langle \bar{a}_0\rangle$ in $\bar{F} = F/\langle a_0^s\rangle^F$.
(d) If $f' \notin \langle a_2\rangle f H$, then there exists s such that $\bar{f}' \notin \langle \bar{a}_2\rangle \bar{f}\bar{H}$ in $\bar{F} = F/\langle a_0^s\rangle^F$.
(e) If $f' \notin \langle a_2\rangle f\langle a_2\rangle$, then there exists s such that $\bar{f}' \notin \langle \bar{a}_2\rangle \bar{f}\langle \bar{a}_2\rangle$ in $\bar{F} = F/\langle a_0^s\rangle^F$.

PROOF. (a) We write $F = A_m *_{\langle a_m\rangle} F_1$, where $F_1 = A_{m-1} *_{\langle a_{m-1}\rangle} \cdots *_{\langle a_3\rangle} A_2$. For each $s > 1$, we have the natural homomorphism $\psi_s : A_m *_{\langle a_m\rangle} F_1 \longrightarrow A_m/\langle a_0^s\rangle *_{\langle a_m\rangle} F_1$ with $\operatorname{Ker}\psi_s = \langle a_0^s\rangle^F$. Since F is π_c and H-separable, there exists s such that $\|f\psi_s\| = \|f\|$, $\|f'\psi_s\| = \|f'\|$, and $(f^{-1}f')\psi_s \notin H\psi_s$.

CASE 1. Suppose $f \in A_m$ (or $f' \in A_m$).

Since $a_0 \in Z(A_m)$ and $(f^{-1}f')\psi_s \notin H\psi_s$, clearly $f'\psi_s \notin (\langle a_0\rangle f H)\psi_s$.

CASE 2. Suppose $f \in F_1 \setminus \langle a_m\rangle$ (or $f' \in F_1 \setminus \langle a_m\rangle$).

Considering Case 1, we may assume $f \notin \langle a_m\rangle\langle a_2\rangle$ and $f' \notin A_m$. Moreover, if $f' = f_1' f_2' \cdots$ is reduced with length ≥ 2, then we suppose $f_1' \notin \langle a_0\rangle\langle a_m\rangle$. Then in $\bar{F} = F\psi_s$, if $\bar{f}' = \bar{a}_0^\epsilon \bar{f}\bar{h}$ for $h \in H$, then we have $\bar{a}_0^\epsilon = 1$; hence $\bar{f}^{-1}\bar{f}' \in \bar{H}$. It contradicts the choice of s.

CASE 3. Suppose $\|f\|, \|f'\| \geq 2$.

Let $f = f_1 f_2 \cdots f_n$ and $f' = f_1' f_2' \cdots f_r'$ be reduced in $F = A_m *_{\langle a_m\rangle} F_1$. We may assume $f_n, f_r' \notin \langle a_m\rangle\langle a_0\rangle \cup \langle a_m\rangle\langle a_2\rangle$ and $f_1, f_1' \notin \langle a_0\rangle\langle a_m\rangle$. Moreover, if $f_1, f_1' \in A_m$, then we assume $f_1^{-1}f_1' \notin \langle a_0\rangle\langle a_m\rangle$. We shall show that $f'\psi_s \notin (\langle a_0\rangle f H)\psi_s$. For, supposing $\bar{f}' = \bar{a}_0^\epsilon \bar{f}\bar{h}$ for $h \in H$, where $\bar{F} = F\psi_s$, we derive a contradiction as follows:

(1) If $f_1, f_1' \in A_m$, then $\overline{f_1'} \in \bar{a}_0^\epsilon \bar{f}_1\langle \bar{a}_m\rangle$; hence $\overline{f_1^{-1}f_1'} \in \langle \bar{a}_0\rangle\langle \bar{a}_m\rangle$. Thus $f_1^{-1}f_1' \in \langle a_0\rangle\langle a_m\rangle$.

(2) If $f_1 \in A_m$ and $f_1' \in F_1$, then $\bar{a}_0^\epsilon \overline{f_1} \in \langle \bar{a}_m\rangle$, and hence $\overline{f_1} \in \langle \bar{a}_0\rangle\langle \bar{a}_m\rangle$. Therefore $f_1 \in \langle a_0\rangle\langle a_m\rangle$.

(3) If $f_1' \in A_m$ and $f_1 \in F_1$, then $\overline{f_1'} \in \langle \bar{a}_0\rangle\langle \bar{a}_m\rangle$. Therefore $f_1' \in \langle a_0\rangle\langle a_m\rangle$.

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(4) If $f_1, f_1' \in F_1$, then $\bar{a}_0^{\epsilon} = 1$; hence $\overline{f^{-1}f'} \in \bar{H}$.

We may prove (b) and (c) in a similar way. In particular, we may use for (b) the fact that F is also $\langle a_0\rangle\langle a_2\rangle$-separable (Lemma 2.3). The proofs of (d) and (e) are also similar to the proofs of (a) and (b), respectively, considering the homomorphism $\psi_s'\colon F_2 *_{\langle a_3\rangle} A_2 \longrightarrow F_2/\langle a_0^s\rangle^{F_2} *_{\langle a_3\rangle} A_2$, where $F_2 = A_m *_{\langle a_m\rangle} \cdots *_{\langle a_4\rangle} A_3$. ∎

LEMMA 3.7. *Let* $E = \langle a_0, a_1\rangle *_{\langle a_1\rangle} \langle a_1, a_2\rangle$, $F = A_m *_{\langle a_m\rangle} \cdots *_{\langle a_3\rangle} A_2$, *and* $H = \langle a_0\rangle * \langle a_2\rangle$. *Let* $P = E *_H F$. *Suppose* $p = f_1 e_1 \cdots e_{n-1} f_n$ *and* $q = f_1' e_1' \cdots e_{n-1}' f_n'$, *where* $e_i, e_i' \in E \setminus H$ *and* $f_i, f_i' \in F \setminus H$.

(a) *If* $q \notin \langle a_0\rangle p \langle a_2\rangle$, *then there exist* s, t *such that* $\bar{q} \notin \langle \bar{a}_0\rangle \bar{p} \langle \bar{a}_2\rangle$ *in* $\bar{P} = P/\langle a_0^s, a_1^t\rangle^P$.

(b) *If* $q \notin \langle a_0\rangle p \langle a_0\rangle$, *then there exist* s, t *such that* $\bar{q} \notin \langle \bar{a}_0\rangle \bar{p} \langle \bar{a}_0\rangle$ *in* $\bar{P} = P/\langle a_0^s, a_1^t\rangle^P$.

(c) *If* $q \notin \langle a_2\rangle p \langle a_2\rangle$, *then there exist* s, t *such that* $\bar{q} \notin \langle \bar{a}_2\rangle \bar{p} \langle \bar{a}_2\rangle$ *in* $\bar{P} = P/\langle a_0^s, a_1^t\rangle^P$.

PROOF. Since the proofs of (b) and (c) are very similar to (a), we only consider (a). Lemma 3.6 shows the result holds for $n = 1$. Note $e_i = a_1^{\epsilon_i} k_i$ and $e_i' = a_1^{\epsilon_i'} k_i'$, for some $k_i, k_i' \in H$. If $e_i' \notin He_iH$ for some i, that is $a_1^{\epsilon_i'} \neq a_1^{\epsilon_i}$, then one can easily find $\bar{P}$ such that $\overline{e_i'} \notin \overline{He_iH}$, $\|\bar{p}\| = \|p\|$, and $\|\bar{p}'\| = \|p'\|$. Then $\bar{q} \notin \langle \bar{a}_0\rangle \bar{p} \langle \bar{a}_2\rangle$. Hence it suffices to consider the case $q = f_1' a_1^{\epsilon_1} f_2' \cdots a_1^{\epsilon_{n-1}} f_n'$ and $p = f_1 a_1^{\epsilon_1} f_2 \cdots a_1^{\epsilon_{n-1}} f_n$. Now since F is H-separable, there exist s_1, t_1 such that $f_i, f_i' \notin \langle a_0^{s_1}\rangle^F H$ and $a_1^{\epsilon_i} \notin \langle a_1^{t_1}\rangle$. Then $\|p\phi_{s_1,t_1}\| = \|p\|$ and $\|q\phi_{s_1,t_1}\| = \|q\|$, where $\phi_{s,t}$ is as on p.298.

If $f_1' \notin \langle a_0\rangle f_1 H$, then by Lemma 3.6, there exists s_2 such that $f_1' \notin \langle a_0^{s_2}\rangle^F \langle a_0\rangle f_1 H$. Let $s = s_1 s_2$ and $t = t_1$. Then, in $\bar{P} = P\phi_{s,t}$, we have $\bar{q} \notin \langle \bar{a}_0\rangle \bar{p} \langle \bar{a}_2\rangle$, since $\|\bar{p}\| = \|p\|$, $\|\bar{q}\| = \|q\|$, and $\overline{f_1'} \notin \langle \bar{a}_0\rangle \overline{f_1} \bar{H}$.

So, suppose $f_1' = a_0^{\mu} f_1 k_1$, for some $k_1 \in H$.

CASE 1. Suppose $f_1^{-1} a_0 f_1 \notin H$.

Then, by Lemmas 3.1 and 3.3, we have $f_1^{-1} a_0^i f_1 \notin H$ for any $a_0^i \neq 1$. Then $q \notin \langle a_0\rangle p \langle a_2\rangle$ iff $(f_2 a_1^{\epsilon_2} \cdots f_n)^{-1} k_1 (f_2' a_1^{\epsilon_2} \cdots f_n') \notin \langle a_2\rangle$. Since P is π_c (Theorem 2.2) and H-separable, there exist s_2, t_2 such that $(f_2 a_1^{\epsilon_2} \cdots f_n)^{-1} k_1 (f_2' a_1^{\epsilon_2} \cdots f_n') \notin \langle a_0^{s_2}, a_1^{t_2}\rangle^P \langle a_2\rangle$, and $f_1^{-1} a_0 f_1 \notin \langle a_0^{s_2}, a_1^{t_2}\rangle^P H$. Let $s = s_1 s_2$ and $t = t_1 t_2$. Then, in $\bar{P} = P\phi_{s,t}$, we have $\|\bar{p}\| = \|p\|$, $\|\bar{q}\| = \|q\|$, $\overline{f_1^{-1} a_0 f_1} \notin \bar{H}$, and $\overline{(f_2 a_1^{\epsilon_2} \cdots f_n)^{-1} k_1 (f_2' a_1^{\epsilon_2} \cdots f_n')} \notin \overline{\langle a_2\rangle}$. These imply that $\bar{q} \notin \langle \bar{a}_0\rangle \bar{p} \langle \bar{a}_2\rangle$, as required.

CASE 2. Suppose $f_1^{-1} a_0 f_1 = h_1 \in H$.

Then $a_0 \sim_F h_1^*$ for a cyclically reduced cyclic permutation h_1^* of h_1. Thus $a_0 \sim_{A_m} h_1^*$; hence $a_0 = h_1^*$. Thus there exists $w_1 \in H$ such that $h_1 = w_1^{-1} a_0 w_1$. Now we have $[w_1 f_1^{-1}, a_0] = 1$. Then we note that $q \notin \langle a_0\rangle p \langle a_2\rangle$ iff $a_0^{\mu} f_1 k_1 a_1^{\epsilon_1} f_2' \cdots f_n' \notin \langle a_0\rangle f_1 w_1^{-1} \cdot w_1 \cdot a_1^{\epsilon_1} f_2 \cdots f_n \langle a_2\rangle$ iff $f_1 w_1^{-1} \cdot a_1^{\epsilon_1} \cdot w_1 k_1 f_2' \cdots f_n' \notin \langle a_0\rangle f_1 w_1^{-1} \cdot a_1^{\epsilon_1} \cdot w_1 f_2 \cdots f_n \langle a_2\rangle$ iff $w_1 k_1 f_2' \cdots f_n' \notin \langle a_0\rangle w_1 f_2 \cdots f_n \langle a_2\rangle$. By induction, there exist s_2, t_2 such that $(w_1 k_1 f_2' \cdots f_n')\phi_{s_2,t_2} \notin (\langle a_0\rangle w_1 f_2 \cdots f_n \langle a_2\rangle)\phi_{s_2,t_2}$. Let $s = s_1 s_2$ and $t = t_1 t_2$. Then, in $\bar{P} = P\phi_{s,t}$, we have $\bar{q} \notin \langle \bar{a}_0\rangle \bar{p} \langle \bar{a}_2\rangle$, since $\|\bar{p}\| = \|p\|$, $\|\bar{q}\| = \|q\|$, and $\overline{w_1 k_1 f_2' \cdots f_n'} \notin \overline{\langle a_0\rangle w_1 f_2 \cdots f_n \langle a_2\rangle}$. ∎

LEMMA 3.8. *Let* $E = \langle a_0, a_1\rangle *_{\langle a_1\rangle} \langle a_1, a_2\rangle$, $F = A_m *_{\langle a_m\rangle} \cdots *_{\langle a_3\rangle} A_2$, *and* $H = \langle a_0\rangle * \langle a_2\rangle$. *Let* $P = E *_H F$, $A = \langle a_0, a_1\rangle$, *and* $B = \langle a_1, a_2\rangle$. *Then we have the following, for* $p, q \in P$:

(a) *If* $q \notin ApB$, *then there exist* s, t *such that* $\bar{q} \notin \bar{A}\bar{p}\bar{B}$ *in* $\bar{P} = P/\langle a_0^s, a_1^t\rangle^P$.

(b) *If* $q \notin BpA$, *then there exist* s, t *such that* $\bar{q} \notin \bar{B}\bar{p}\bar{A}$ *in* $\bar{P} = P/\langle a_0^s, a_1^t\rangle^P$.

(c) *If* $q \notin ApA$, *then there exist* s, t *such that* $\bar{q} \notin \bar{A}\bar{p}\bar{A}$ *in* $\bar{P} = P/\langle a_0^s, a_1^t\rangle^P$.

(d) *If* $q \notin BpB$, *then there exist* s, t *such that* $\bar{q} \notin \bar{B}\bar{p}\bar{B}$ *in* $\bar{P} = P/\langle a_0^s, a_1^t\rangle^P$.

PROOF. (a) and (b) are equivalent, since $q \in ApB$ iff $q^{-1} \in Bp^{-1}A$. And the proofs of (c) and (d) are similar to that of (a); hence we only consider (a).

CASE 1. Suppose $p \in E$ (or $q \in E$).

If $q \notin E$ then, considering the length of q in P, one can easily choose $\bar{P}$ such that $\bar{q} \notin \bar{E}$. Then clearly $\bar{q} \notin \bar{A}\bar{p}\bar{B}$. If $q \in E$, then $p = a_1^{\epsilon_1}h_1$ and $q = a_1^{\epsilon_2}h_2$ for $h_1, h_2 \in H$. Note that $q \in ApB$ iff $h_2 \in \langle a_0\rangle h_1\langle a_2\rangle$. By Lemma 3.6, there exists s such that $h_2\psi_s \notin (\langle a_0\rangle h_1\langle a_2\rangle)\psi_s$. For any t, we have $\bar{q} \notin \bar{A}\bar{p}\bar{B}$ in $\bar{P} = P/\langle a_0^s, a_1^t\rangle^P$.

CASE 2. Suppose $p, q \in F \setminus H$.

There exists s_1 such that $p, q \notin \langle a_0^{s_1}\rangle^F H$. By Lemma 3.6, there exists s_2 such that $q\psi_{s_2} \notin (\langle a_0\rangle p\langle a_2\rangle)\psi_{s_2}$. Let $s = s_1s_2$ and let t be arbitrary. Then, in $\bar{P} = P\phi_{s,t}$, we have $\bar{q} \notin \bar{A}\bar{p}\bar{B}$, since $\bar{q} \notin \langle\bar{a}_0\rangle\bar{p}\langle\bar{a}_2\rangle$.

CASE 3. Suppose $p \notin E \cup F$ (or $q \notin E \cup F$).

Since $AH = E = HB$, we may assume $q = f_1' a_1^{\epsilon_1'} f_2' \cdots a_1^{\epsilon_{r-1}'} f_r'$ and $p = f_1 a_1^{\epsilon_1} f_2 \cdots a_1^{\epsilon_{n-1}} f_n$, where $f_i, f_i' \in F \setminus H$ and $a_1^{\epsilon_i} \neq 1 \neq a_1^{\epsilon_i'}$. Then $q \in ApB$ iff $r = n$ and $q \in \langle a_0\rangle p\langle a_2\rangle$. If $r \neq n$ then we can easily find $\bar{P}$ such that $\bar{q} \notin \bar{A}\bar{p}\bar{B}$, by a length preserving homomorphism. Hence we let $r = n$ and $q \notin \langle a_0\rangle p\langle a_2\rangle$. Then, by Lemma 3.7, there exist s, t such that $\bar{q} \notin \langle\bar{a}_0\rangle\bar{p}\langle\bar{a}_2\rangle$, $\|\bar{q}\| = \|q\|$, and $\|\bar{p}\| = \|p\|$, where $\bar{P} = P/\langle a_0^s, a_1^t\rangle^P$. Then we have $\bar{q} \notin \bar{A}\bar{p}\bar{B}$. ∎

LEMMA 3.9. *Let* $E = \langle a_0, a_1\rangle *_{\langle a_1\rangle} \langle a_1, a_2\rangle$, $F = A_m *_{\langle a_m\rangle} \cdots *_{\langle a_3\rangle} A_2$, *and* $H = \langle a_0\rangle * \langle a_2\rangle$. *Let* $P = E *_H F$, *and* $P_1 = P *_B A_1$, *where* $B = \langle a_1, a_2\rangle$. *Suppose* $x, y \in P_1$ *are such that* $x \notin AyA$, *where* $A = \langle a_0, a_1\rangle$. *Then there exist* s, t *such that* $\bar{x} \notin \bar{A}\bar{y}\bar{A}$ *in* $\overline{P_1} = P_1/\langle a_0^s, a_1^t\rangle^{P_1}$.

PROOF.

CASE 1. Suppose $x, y \in P$.

Then by Lemma 3.8 there exist s, t such that $x \notin \langle a_0^s, a_1^t\rangle^P AyA$. Then clearly, $\bar{x} \notin \overline{AyA}$ in $\overline{P_1} = P_1/\langle a_0^s, a_1^t\rangle^{P_1} = \bar{P} *_{\bar{B}} \bar{A}_1$, where $\bar{P} = P/\langle a_0^s, a_1^t\rangle^P$, $\bar{A}_1 = A_1/\langle a_1^t\rangle$.

CASE 2. Suppose $x \notin P$ and $y \in P$.

If $x \in A_1 \setminus B$ then, for any s, t, we have $\bar{x} \in \bar{A}_1 \setminus \bar{B}$; hence $\bar{x} \notin \overline{AyA}$, where $\overline{P_1} = \bar{P} *_{\bar{B}} \bar{A}_1$. Suppose $\|x\| \geq 2$, say $x = p_1\alpha_1 \cdots$, where $p_i \in P \setminus B$ and $\alpha_i \in A_1 \setminus B$. Then there exist s, t such that $p_i \notin \langle a_0^s, a_1^t\rangle^P B$, by Lemma 3.8. Then $\alpha_i \notin \langle a_0^s, a_1^t\rangle^P B$, thus $\|\bar{x}\| = \|x\| \geq 2$, and hence $\bar{x} \notin \overline{AyA}$, where $\overline{P_1} = \bar{P} *_{\bar{B}} \bar{A}_1$.

CASE 3. Suppose $x, y \in A_1 \setminus B$.

Since $x \notin AyA$, we have $x \notin \langle a_1\rangle y\langle a_1\rangle = y\langle a_1\rangle$. Thus we can choose t such that $y^{-1}x \notin \langle a_1^t\rangle$. Then for any s, in $\overline{P_1} = P_1/\langle a_0^s, a_1^t\rangle^{P_1}$, we have $\bar{x} \notin \overline{AyA}$.

CASE 4. Suppose $x \in A_1 \setminus B$ and $\|y\| \geq 2$ (or $y \in A_1 \setminus B$ and $\|x\| \geq 2$).

Considering the above cases, we may assume that $y = y_1y_2\cdots y_n$ is reduced and $y_1 \notin AB$, $y_n \notin BA$. As in Case 2, we can find $\overline{P_1} = \bar{P} *_{\bar{B}} \bar{A}_1$ such that $\|\bar{y}\| = \|y\| \geq 2$, $\bar{y}_1 \notin \overline{AB}$, and $\bar{y}_n \notin \overline{BA}$ (using Lemma 3.8). Then clearly $\bar{x} \notin \overline{AyA}$, if $\|\bar{y}\| = \|y\| \geq 4$. If $n = 2, 3$ and $\bar{x} \in \overline{AyA}$, then we have $\bar{y}_1 \in \overline{AB}$, or $\bar{y}_n \in \overline{BA}$, a contradiction.

CASE 5. Suppose $\|x\| \geq 2$ and $\|y\| \geq 2$.

Suppose that $x = x_1x_2\cdots x_n$ and $y = y_1y_2\cdots y_r$ are reduced in P_1 and $x_1, y_1 \notin AB$, $x_n, y_r \notin BA$. Then as above, there exist s_1, t_1 such that, in $\overline{\overline{P_1}} = P_1/\langle a_0^{s_1}, a_1^{t_1}\rangle^{P_1}$, $\|\bar{\bar{x}}\| = \|x\|$, $\|\bar{\bar{y}}\| = \|y\|$, $\overline{\overline{x_1}}, \overline{\overline{y_1}} \notin \overline{\overline{AB}}$, and $\overline{\overline{x_n}}, \overline{\overline{y_r}} \notin \overline{\overline{BA}}$.

If x_1 and y_1 are in different factors, say $x_1 \in A_1$ and $y_1 \in P$, then $\bar{x} \notin \overline{AyA}$, where $\overline{P_1} = \overline{\overline{P_1}}$, since $\bar{y}_1 \in \bar{P} \setminus \overline{AB}$ and $\bar{x}_1 \in \bar{A}_1 \setminus \bar{B}$. Hence we may assume that x_1 and y_1 are in the same factor of P_1. Similarly, considering $x^{-1} \notin Ay^{-1}A$, we may assume that x_n and y_r are in the same factor of P_1. In this case, if $n \neq r$, then clearly $\bar{x} \notin \overline{AyA}$, where $\overline{P_1} = \overline{\overline{P_1}}$. Thus we only consider the case $n = r$.

SUBCASE 1. Suppose $x = p_1'\alpha_1'\cdots\alpha_{n-1}'p_n'$ and $y = p_1\alpha_1\cdots\alpha_{n-1}p_n$, where $p_i, p_i' \in P \setminus B$ and $\alpha_i, \alpha_i' \in A_1 \setminus B$.

If $\alpha_i^{-1}\alpha_i' \notin B$ for some i, then $\overline{\alpha_i^{-1}\alpha_i'} \notin \bar{B}$; hence $\bar{x} \notin \overline{AyA}$, where $\overline{P_1} = \overline{\overline{P_1}}$. So it suffices to consider $x = p_1'\alpha_1\cdots\alpha_{n-1}p_n'$ and $y = p_1\alpha_1\cdots\alpha_{n-1}p_n$. Note that $x \in AyA$ iff $p_1' = d_1p_1b_1$, $\alpha_1 = b_1^{-1}\alpha_1b_1$, $p_2' = b_1^{-1}p_2b_2, \ldots, p_n' = b_{n-1}^{-1}p_nd_2$, where $b_i \in B$ and $d_1, d_2 \in A$. Now if $p_1' \notin Ap_1B$, or $p_i' \notin Bp_iB$ $(1 < i < n)$, or $p_n' \notin Bp_nA$, then by Lemma 3.8, we can find s, t ($s_1|s$ and $t_1|t$) such that $\overline{p_1'} \notin \overline{Ap_1B}$, or $\overline{p_i'} \notin \overline{Bp_iB}$, or $\overline{p_n'} \notin \overline{Bp_nA}$. Then, since $\|\bar{x}\| = \|\bar{\bar{x}}\| = \|x\|$, $\|\bar{y}\| = \|\bar{\bar{y}}\| = \|y\|$, we have $\bar{x} \notin \overline{AyA}$.

Thus we assume $p_1' \in Ap_1B$, $p_i' \in Bp_iB$ $(1 < i < n)$, and $p_n' \in Bp_nA$. Then one of the following holds:

$$(*)\qquad p_1'\alpha_1\cdots\alpha_{i-1}p_i' \in Ap_1\alpha_1\cdots\alpha_{i-1}p_iB, \quad \text{but}$$
$$p_1'\alpha_1\cdots\alpha_ip_{i+1}' \notin Ap_1\alpha_1\cdots\alpha_ip_{i+1}B \quad \text{for } i < n-1, \quad \text{or}$$
$$(**)\qquad p_1'\alpha_1\cdots\alpha_{n-2}p_{n-1}' \in Ap_1\alpha_1\cdots\alpha_{n-2}p_{n-1}B, \quad \text{but}$$
$$p_1'\alpha_1\cdots\alpha_{n-1}p_n' \notin Ap_1\alpha_1\cdots\alpha_{n-1}p_nA.$$

If $(*)$ holds, then let $p_1'\alpha_1\cdots\alpha_{i-1}p_i' = d_1p_1\alpha_1\cdots\alpha_{i-1}p_iu$ and $p_{i+1}' = vp_{i+1}w$ for $u, v, w \in B$, $d_1 \in A$. Since $p_1'\alpha_1\cdots\alpha_ip_{i+1}' \notin Ap_1\alpha_1\cdots\alpha_ip_{i+1}B$, we have $uv \notin C_{\langle a_1\rangle}(p_1\alpha_1\cdots p_i\alpha_i)C_B(p_{i+1}) = S$. Then $S = 1$, or $\langle a_1\rangle$, or $\langle a_2\rangle$ by Lemma 3.4. Now since P_1 is π_c by Lemma 2.2, there exist s, t ($s_1|s$, $t_1|t$) such that $\overline{uv} \notin \bar{S}$, and such that $C_{\langle \bar{a}_1\rangle}(\overline{p_1\alpha_1\cdots p_i\alpha_i})C_{\bar{B}}(\overline{p_{i+1}}) = \bar{S}$ by Lemma 3.4. Then we note that $\overline{p_1'\cdots p_i'\alpha_ip_{i+1}'} \notin \overline{Ap_1\alpha_1\cdots\alpha_ip_{i+1}B}$. For, if $\overline{p_1'\cdots p_i'\alpha_ip_{i+1}'} \in \overline{Ap_1\alpha_1\cdots\alpha_ip_{i+1}B}$, then we have $\overline{p_1\cdots p_iuv\alpha_ip_{i+1}w} \in \overline{Ap_1\alpha_1\cdots\alpha_ip_{i+1}B}$. Hence, by Lemma 3.2, for some $u_1 \in B$, and $d_2 \in A$, we have $\bar{p}_1 = \overline{d_2p_1u_1}$, $\bar{\alpha}_1 = \overline{u_1^{-1}\alpha_1u_1}$, $\bar{p}_2 = \overline{u_1^{-1}p_2u_1}, \ldots, \overline{p_iuv} = \bar{u}_1^{-1}\bar{p}_i\bar{u}_1\overline{uv}$, $\bar{\alpha}_i = (\overline{u_1uv})^{-1}\bar{\alpha}_i\overline{u_1uv}$, and $\overline{p_{i+1}} = (\overline{u_1uv})^{-1}\overline{p_{i+1}u_1uv}$. By Lemma 3.2, $\overline{d_2} \in \bar{A} \cap \bar{B} = \langle \bar{a}_1\rangle$, and $\bar{d}_2^{-1} = \bar{u}_1$. Now $\bar{u}_1 \in C_{\bar{B}}(\overline{p_1\alpha_1\cdots p_i\alpha_i}) \cap \langle \bar{a}_1\rangle$ and $\overline{u_1uv} \in C_{\bar{B}}(\overline{p_{i+1}})$. Thus $\overline{uv} \in \bar{S}$, a contradiction.

The case $(**)$ can be similarly handled.

SUBCASE 2. Suppose $x = \alpha'_1 p'_1 \cdots \alpha'_n p'_n$ and $y = \alpha_1 p_1 \cdots \alpha_n p_n$, where $p_i, p'_i \in P \setminus B$ and $\alpha_i, \alpha'_i \in A_1 \setminus B$.

If $\alpha_i^{-1}\alpha'_i \notin B$ for some i, then $\overline{\alpha_i^{-1}\alpha'_i} \notin \bar{B}$; hence $\bar{x} \notin \overline{AyA}$, where $\overline{P_1} = \overline{\overline{P_1}}$. So it suffices to consider $x = \alpha_1 p'_1 \cdots \alpha_n p'_n$ and $y = \alpha_1 p_1 \cdots \alpha_n p_n$. Note that $x \in AyA$ iff $x' \in \langle a_1 \rangle y' A$, where $x' = p'_1 \alpha_2 \cdots p'_n$ and $y' = p_1 \alpha_2 \cdots p_n$. Thus if $x' \notin Ay'A$ then we can find $\overline{P_1}$, by Subcase 1, such that $\bar{x'} \notin \overline{Ay'A}$. Then $\bar{x} \notin \overline{AyA}$. Now if $x' \in Ay'A \setminus \langle a_1 \rangle y' A$, let $x' = d_1 p_1 \alpha_2 \cdots p_n d_2$, where $d_1, d_2 \in A$ and $d_1 \notin \langle a_1 \rangle$. Choose s, t ($s_1 | s$ and $t_1 | t$) such that $\bar{d}_1 \notin \langle \bar{a}_1 \rangle$. Now if $\bar{x} \in \overline{AyA}$, then $\bar{x'} \in \langle \bar{a}_1 \rangle \bar{y'} \bar{A}$; hence $\overline{d_1 p_1} = \bar{a}_1^{\epsilon} \bar{p}_1 \bar{u}_1$ for some $u_1 \in B$. Thus by Lemma 3.2 we have $\bar{d}_1 = \bar{a}_1^{\epsilon} \bar{u}_1 \in \bar{A} \cap \bar{B} = \langle \bar{a}_1 \rangle$, a contradiction. Therefore $\bar{x} \notin \overline{AyA}$.

SUBCASE 3. Suppose $x = p'_1 \alpha'_1 \cdots p'_n \alpha'_n$ and $y = p_1 \alpha_1 \cdots p_n \alpha_n$, where $p_i, p'_i \in P \setminus B$ and $\alpha_i, \alpha'_i \in A_1 \setminus B$.

This case is similar to Subcase 2, since $x^{-1} \notin Ay^{-1}A$.

SUBCASE 4. Suppose $x = \alpha'_1 p'_1 \cdots p'_n \alpha'_{n+1}$ and $y = \alpha_1 p_1 \cdots p_n \alpha_{n+1}$, where $p_i, p'_i \in P \setminus B$ and $\alpha_i, \alpha'_i \in A_1 \setminus B$.

If $\alpha_i^{-1}\alpha'_i \notin B$ for some i, then $\overline{\alpha_i^{-1}\alpha'_i} \notin \bar{B}$; hence $\bar{x} \notin \overline{AyA}$, where $\overline{P_1} = \overline{\overline{P_1}}$. So it suffices to consider $x = \alpha_1 p'_1 \cdots p'_n \alpha_{n+1}$ and $y = \alpha_1 p_1 \cdots p_n \alpha_{n+1}$. Note that $x \in AyA$ iff $x \in \langle a_1 \rangle y \langle a_1 \rangle$ iff $x' \in \langle a_1 \rangle y' \langle a_1 \rangle$ iff $x' \in Ay'A$, where $x' = p'_1 \alpha_2 \cdots p'_n$ and $y' = p_1 \alpha_2 \cdots p_n$. Thus if $x' \notin Ay'A$ then we can find $\overline{P_1}$ such that $\bar{x'} \notin \overline{Ay'A}$ by Subcase 1. Then $\bar{x} \notin \overline{AyA}$. Now if $x' \in Ay'A \setminus \langle a_1 \rangle y' \langle a_1 \rangle$, let $x' = d_1 y' d_2 = d_1 p_1 \alpha_2 \cdots p_n d_2$, where $d_1, d_2 \in A$ and $d_1 \notin \langle a_1 \rangle$ or $d_2 \notin \langle a_1 \rangle$. Choose s, t ($s_1 | s$ and $t_1 | t$) such that $\bar{d}_1, \bar{d}_2 \notin \langle \bar{a}_1 \rangle$. Now if $\bar{x} \in \overline{AyA}$, then $\overline{\alpha_1 p'_1 \cdots p'_n \alpha_{n+1}} = \overline{d_3 \alpha_1 p_1 \cdots p_n \alpha_{n+1} d_4}$, for some $d_3, d_4 \in A$. Then $\bar{d}_3 \in \bar{A} \cap \bar{B} = \langle \bar{a}_1 \rangle$ and $\bar{d}_4 \in \langle \bar{a}_1 \rangle$. It follows that $\overline{d_3 p_1 \alpha_2 \cdots p_n d_4} = \overline{d_1 p_1 \alpha_2 \cdots p_n d_2}$; hence $\bar{d}_3 \bar{p}_1 = \bar{d}_1 \bar{p}_1 \bar{u}_1$ for some $u_1 \in B$. Thus, by Lemma 3.2, $\bar{d}_1^{-1} \bar{d}_3 = \bar{u}_1 \in \bar{A} \cap \bar{B} = \langle \bar{a}_1 \rangle$. Hence $\bar{d}_1 \in \langle \bar{a}_1 \rangle$, and similarly, $\bar{d}_2 \in \langle \bar{a}_1 \rangle$, which contradicts the choice of s, t. ■

4. Main result.

THEOREM 4.1. *Let P be the polygonal product of the polycyclic-by-finite groups $A_0, A_1, \ldots, A_m$ $(m \geq 3)$, amalgamating the central subgroups $\langle a_1 \rangle, \ldots, \langle a_m \rangle, \langle a_0 \rangle$ with trivial intersections. Then P is* c.s.

PROOF. First, we note that the reduced polygonal product P_0, which is a polygonal product of abelian groups $\langle a_0, a_1 \rangle, \langle a_1, a_2 \rangle, \ldots, \langle a_m, a_0 \rangle$ amalgamating cyclic subgroups $\langle a_1 \rangle, \ldots, \langle a_m \rangle, \langle a_0 \rangle$, with trivial intersections, is a graph product of the cyclic groups $\langle a_1 \rangle, \ldots, \langle a_m \rangle, \langle a_0 \rangle$. Hence P_0 is c.s. by Theorem 2.4. Let $P_i = \Big(\cdots \big((P_0 *_{B_m} A_m) *_{B_{m-1}} A_{m-1} \big) \cdots \Big) *_{B_{m-i+1}} A_{m-i+1}$, where $B_j = \langle a_j, a_{j+1} \rangle$ with subscripts taken modulo $m+1$. Then P_i is the polygonal product of $\langle a_0, a_1 \rangle, \ldots, \langle a_{m-i}, a_{m-i+1} \rangle$, $A_{m-i+1}, \ldots, A_m$ amalgamating the central subgroups $\langle a_1 \rangle, \ldots, \langle a_{m-i+1} \rangle, \ldots, \langle a_0 \rangle$, with trivial intersections, and $P_{i+1} = P_i *_{B_{m-i}} A_{m-i}$. Since P_0 is c.s., for an induction we assume that P_m is c.s. and we show that $P = P_{m+1} = P_m *_{B_0} A_0$ is c.s. By the assumption, every polygonal product of polycyclic-by-finite groups $\langle c_0, c_1 \rangle, C_1, \ldots, C_m$, amalgamating the central subgroups $\langle c_1 \rangle, \ldots, \langle c_m \rangle, \langle c_0 \rangle$ with trivial intersections, is c.s. Hence

https://doi.org/10.4153/CMB-1996-037-3 Published online by Cambridge University Press

$\overline{P_m} = P_m/\langle a_0^s, a_1^t\rangle^{P_m}$ is c.s. for any $s, t > 1$, since $\overline{P_m}$ is the polygonal product of $\langle\overline{a_0}, \overline{a_1}\rangle$, $A_1/\langle a_1^t\rangle, A_2, \ldots, A_{m-1}, A_m/\langle a_0^s\rangle$, amalgamating the subgroups $\langle\bar{a}_1\rangle, \langle a_2\rangle, \ldots, \langle a_m\rangle, \langle\bar{a}_0\rangle$. Thus $\bar{P} = P/\langle a_0^s, a_1^t\rangle^P = \overline{P_m} *_{\bar{A}} \bar{A}_0$ is c.s. for any $s, t > 1$, where $\bar{A}_0 = A_0/\langle a_0^s, a_1^t\rangle$, since $\bar{A} = \langle\bar{a}_0, \bar{a}_1\rangle$ is finite. Hence, for each pair $x, y \in P$ with $x \not\sim_P y$, we shall find s, t such that $\bar{x} \not\sim_{\bar{P}} \bar{y}$.

Let $x, y \in P = P_m *_{B_0} A_0$ such that $x \not\sim_P y$, each of minimal length in its conjugacy class in $P_m *_{B_0} A_0$. Throughout the proof, we denote $A = B_0 = \langle a_0, a_1\rangle$ and

$$\phi_{s,t}: P_m *_A A_0 \longrightarrow \overline{P_m} *_{\bar{A}} \bar{A}_0,$$

where $\overline{P_m} = P_m/\langle a_0^s, a_1^t\rangle^{P_m}$, $\bar{A}_0 = A_0/\langle a_0^s, a_1^t\rangle$, and $\bar{A} = \langle\bar{a}_0, \bar{a}_1\rangle$. By Lemma 3.9, P_m is A-separable. Hence there exist s_0, t_0 such that $\|x\| = \|x\phi_{s_0,t_0}\|$ and $\|y\| = \|y\phi_{s_0,t_0}\|$.

Since P is $\mathcal{RF}$ by Theorem 2.2, we may assume $x \neq 1 \neq y$.

CASE 1. $\|x\| = 0$ and $\|y\| = 1$ (or, similarly, $\|y\| = 0$ and $\|x\| = 1$).

Firstly, we suppose $y \in A_0 \setminus A$. Let $s = s_0$, $t = t_0$. Then $\bar{y} \notin \bar{A}$; hence $\{\bar{y}\}^{\bar{A}_0} \cap \bar{A} = \emptyset$, thus $\bar{x} \not\sim_{\bar{P}} \bar{y}$.

Secondly, suppose $y \in P_m \setminus A$. By Lemma 3.5, there exist s_1, t_1 such that $\{y\phi_{s_1,t_1}\}^{P_m\phi_{s_1,t_1}} \cap A\phi_{s_1,t_1} = \emptyset$. Let $s = s_0 s_1$ and $t = t_0 t_1$. Then $\{\bar{y}\}^{\overline{P_m}} \cap \bar{A} = \emptyset$ and $\bar{y} \notin \bar{A}$. Hence $\bar{x} \not\sim_{\bar{P}} \bar{y}$ by Theorem 2.5.

CASE 2. $\|x\| \neq \|y\|$ and $\|x\| \geq 2$ (or, similarly, $\|x\| \neq \|y\|$ and $\|y\| \geq 2$).

Since x has minimal length in its conjugacy class in P, x is cyclically reduced. Let $s = s_0$ and $t = t_0$. Then $\|\bar{x}\| = \|x\| \neq \|y\| = \|\bar{y}\|$. Thus $\bar{x} \not\sim_{\bar{P}} \bar{y}$ by Theorem 2.5.

CASE 3. $\|x\| = \|y\| = 0$.

Since P is $\mathcal{RF}$, there exist s_1, t_1 such that $y^{-1}x \notin \langle a_0^{s_1}, a_1^{t_1}\rangle^P$. Let $s = s_1$ and $t = t_1$. If $\bar{x} \sim_{\bar{P}} \bar{y}$, then $\bar{x} \sim_{\overline{P_m}} \bar{\alpha}_1 \sim_{\bar{A}_0} \cdots \sim_{\bar{A}_0} \bar{\alpha}_r = \bar{y}$ for $\bar{\alpha}_i \in \bar{A}$. It follows by Lemma 3.2 that $\bar{x} = \bar{\alpha}_i = \bar{y}$, since $\bar{A} \in Z(\bar{A}_0)$. Hence $\bar{x} \not\sim_{\bar{P}} \bar{y}$ by Theorem 2.5.

CASE 4. $\|x\| = \|y\| = 1$.

Firstly, suppose both x and y are in $P_m \setminus A$. Now $\{x\}^{P_m} \cap A = \emptyset$ and $x \not\sim_{P_m} y$. There exist s_1, t_1 such that $\{x\phi_{s_1,t_1}\}^{P_m\phi_{s_1,t_1}} \cap A\phi_{s_1,t_1} = \emptyset$ and there exist s_2, t_2 such that $x\phi_{s_2,t_2} \not\sim_{P_m\phi_{s_2,t_2}} y\phi_{s_2,t_2}$, since P_m is c.s. by the induction hypothesis. Let $s = s_0 s_1 s_2$ and $t = t_0 t_1 t_2$. Then $\{\bar{x}\}^{\overline{P_m}} \cap \bar{A} = \emptyset$, $\bar{x} \not\sim_{\overline{P_m}} \bar{y}$, and $\|\bar{x}\| = \|\bar{y}\| = 1$. Hence $\bar{x} \not\sim_{\bar{P}} \bar{y}$ by Theorem 2.5.

Secondly, suppose $x, y \in A_0 \setminus A$. Since $x \not\sim_{A_0} y$, and since A_0 is c.s., there exist s_1, t_1 such that $x\phi_{s_1,t_1} \not\sim_{A_0\phi_{s_1,t_1}} y\phi_{s_1,t_1}$. Let $s = s_0 s_1$ and $t = t_0 t_1$. Then $\bar{x} \not\sim_{\bar{A}_0} \bar{y}$ and $\{\bar{x}\}^{\bar{A}_0} \cap \bar{A} = \emptyset$; hence, by Lemma 3.2 and Theorem 2.5, we have $\{\bar{x}\}^{\bar{P}} \cap \bar{A} = \emptyset$. It follows that $\bar{x} \not\sim_{\bar{P}} \bar{y}$ by Theorem 2.5.

Finally, suppose $x \in A_0 \setminus A$ and $y \in P_m \setminus A$. Let $s = s_0$ and $t = t_0$. Then as before $\{\bar{x}\}^{\bar{P}} \cap \bar{A} = \emptyset$. Hence $\bar{x} \not\sim_{\bar{P}} \bar{y}$ by Theorem 2.5, since $\bar{y} \in \overline{P_m} \setminus \bar{A}$.

CASE 5. $\|x\| = \|y\| = 2n$.

Let $x = p_1\alpha_1 \cdots p_n\alpha_n$ and $y = p_1'\alpha_1' \cdots p_n'\alpha_n'$, where $p_j, p_j' \in P_m \setminus A$ and $\alpha_j, \alpha_j' \in A_0 \setminus A$ for all j. Since $x \not\sim_P y$, we have $x \not\sim_A y^*$ for all cyclic permutation y^* of y. Thus each of the equations

$$(j)\ p_j'\alpha_j' \cdots p_n'\alpha_n' p_1'\alpha_1' \cdots p_{j-1}'\alpha_{j-1}' = a^{-1} p_1\alpha_1 \cdots p_n\alpha_n a$$

has no solution $a \in A$. We shall find s_j, t_j such that $(j)\phi_{s_j,t_j}$ has no solution $a\phi_{s_j,t_j} \in A\phi_{s_j,t_j}$ for each j. Then, for $s = s_0 s_1 \cdots s_n$ and $t = t_0 t_1 \cdots t_n$, we have $\|\bar{x}\| = \|x\| = \|y\| = \|\bar{y}\|$ and $\bar{x} \not\sim_{\bar{A}} \bar{y}^*$ for any cyclic permutation $\bar{y}^*$ of $\bar{y}$. Hence we have $\bar{x} \not\sim_{\bar{P}} \bar{y}$ as required.

Here we only consider the case $j = 1$, since the others are similar.

CLAIM. *If $p_1'\alpha_1' \cdots p_n'\alpha_n' \not\sim_A p_1\alpha_1 \cdots p_n\alpha_n$ then there exist s, t such that $\overline{p_1'\alpha_1' \cdots p_n'\alpha_n'} \not\sim_{\bar{A}} \overline{p_1\alpha_1 \cdots p_n\alpha_n}$.*

If $\alpha_i^{-1}\alpha_i' \notin A$ for some i then, taking $s = s_0$ and $t = t_0$, we have $\overline{\alpha_i^{-1}\alpha_i'} \notin \bar{A}$; hence clearly $\bar{x} \not\sim_{\bar{A}} \bar{y}$. Thus it suffices to consider the case $\alpha_i = \alpha_i'$ for all i. Now if $p_i' \notin Ap_iA$ for some i then, by Lemma 3.9, there exist s_1, t_1 such that $p_i'\phi_{s_1,t_1} \notin (Ap_iA)\phi_{s_1,t_1}$. Let $s = s_0 s_1$ and $t = t_0 t_1$. Then $\overline{p_i'} \notin \overline{Ap_iA}$; hence $\bar{x} \not\sim_{\bar{A}} \bar{y}$. Therefore, we suppose $\alpha_i' = \alpha_i$ and $p_i' \in Ap_iA$ for all i. Then one of the following is true:

$$(*)\qquad p_1'\alpha_1 \cdots \alpha_{i-1}p_i' \in Ap_1\alpha_1 \cdots \alpha_{i-1}p_iA, \quad \text{but}$$
$$p_1'\alpha_1 \cdots \alpha_i p_{i+1}' \notin Ap_1\alpha_1 \cdots \alpha_i p_{i+1}A, \quad \text{for some } i, \text{ or}$$
$$(**)\qquad p_1'\alpha_1 \cdots p_n'\alpha_n \in Ap_1\alpha_1 \cdots p_n\alpha_nA, \quad \text{but}$$
$$p_1'\alpha_1 \cdots p_n'\alpha_n \not\sim_A p_1\alpha_1 \cdots p_n\alpha_n.$$

If $(*)$ is true, then let $p_1'\alpha_1 \cdots \alpha_{i-1}p_i' = d_1p_1\alpha_1 \cdots \alpha_{i-1}p_id_2$ and $p_{i+1}' = d_3p_{i+1}d_4$ for $d_k \in A$. Since $p_1'\alpha_1 \cdots \alpha_ip_{i+1}' \notin Ap_1\alpha_1 \cdots \alpha_ip_{i+1}A$, we have $d_2d_3 \notin C_A(p_1\alpha_1 \cdots p_i\alpha_i)C_A(p_{i+1})$. Hence, by Lemma 3.4, $C_A(p_1\alpha_1 \cdots p_i\alpha_i)C_A(p_{i+1})$ is a cyclic subgroup of A. Now since P is π_c by Lemma 2.2, there exist s_1, t_1 such that, in $\bar{\bar{P}} = P/\langle a_0^{s_1}, a_1^{t_1}\rangle^P$, $\overline{\overline{d_2d_3}} \notin \overline{\overline{C_A(p_1\alpha_1 \cdots p_i\alpha_i)C_A(p_{i+1})}}$, $C_{\bar{\bar{A}}}(\overline{\overline{p_1 \cdots p_i\alpha_i}}) = \overline{\overline{C_A(p_1 \cdots p_i\alpha_i)}}$, and $C_{\bar{\bar{A}}}(\overline{\overline{p_{i+1}}}) = \overline{\overline{C_A(p_{i+1})}}$. Let $s = s_0s_1$ and $t = t_0t_1$. Then, in $\bar{P} = P/\langle a_0^s, a_1^t\rangle^P$, we have $\overline{d_2d_3} \notin \overline{C_A(p_1\alpha_1 \cdots p_i\alpha_i)C_A(p_{i+1})}$, $C_{\bar{A}}(\overline{p_1 \cdots \alpha_ip_i}) = \overline{C_A(p_1 \cdots \alpha_ip_i)}$, and $C_{\bar{A}}(\overline{p_{i+1}}) = \overline{C_A(p_{i+1})}$. Now we note that $\bar{x} \not\sim_{\bar{A}} \bar{y}$. For, if $\bar{x} \sim_{\bar{A}} \bar{y}$, then $\overline{p_1' \cdots p_i'\alpha_ip_{i+1}'} \in \overline{Ap_1\alpha_1 \cdots \alpha_ip_{i+1}A}$, and hence $\overline{p_1 \cdots p_id_2\alpha_id_3p_{i+1}} = \overline{d_5p_1\alpha_1 \cdots \alpha_ip_{i+1}d_6}$ for some $d_5, d_6 \in A$. Then, by Lemma 3.2, and since $\bar{A} \subset Z(\bar{A}_0)$, we have $\bar{p}_1 = \overline{d_5p_1d_5^{-1}}$, $\bar{\alpha}_1 = \overline{d_5\alpha_1d_5^{-1}}, \ldots,$ $\bar{p}_i\bar{d}_2\bar{d}_3 = \bar{d}_5\bar{p}_i\bar{d}_5^{-1}(\overline{d_2d_3})$, $\bar{\alpha}_i = (\overline{d_2d_3})^{-1}\bar{d}_5\bar{\alpha}_i\bar{d}_5^{-1}\overline{d_2d_3}$, and $\overline{p_{i+1}} = (\overline{d_2d_3})^{-1}\bar{d}_5\overline{p_{i+1}}\bar{d}_6$; hence $\bar{d}_6 = \bar{d}_5^{-1}\overline{d_2d_3}$ by Lemma 3.2. It follows that $\bar{d}_5^{-1}\overline{d_2d_3} \in C_{\bar{A}}(\overline{p_{i+1}})$, and $\bar{d}_5 \in C_{\bar{A}}(\overline{p_1 \cdots \alpha_ip_i})$. Thus $\overline{d_2d_3} \in C_{\bar{A}}(\overline{p_1 \cdots \alpha_ip_i})C_{\bar{A}}(\overline{p_{i+1}}) = \overline{C_A(p_1\alpha_1 \cdots p_i\alpha_i)C_A(p_{i+1})}$, a contradiction.

If $(**)$ is true, then let $p_1'\alpha_1 \cdots p_n'\alpha_n = d_1p_1\alpha_1 \cdots p_n\alpha_nd_2$, where $d_1, d_2 \in A$ and $d_1d_2 \neq 1$. Choose s_1, t_1 such that $d_1d_2 \notin \langle a_0^{s_1}, a_1^{t_1}\rangle^P$. Let $s = s_0s_1$ and $t = t_0t_1$.

We note that $\bar{x} \not\sim_{\bar{A}} \bar{y}$ in $\bar{P} = P/\langle a_0^s, a_1^t\rangle^P$. For, if $\bar{x} \sim_{\bar{A}} \bar{y}$, then $\overline{d_1 p_1 \cdots p_n \alpha_n d_2} = \overline{d_3^{-1} p_1 \alpha_1 \cdots p_n \alpha_n d_3}$, for some $d_3 \in A$. Hence, by Lemma 3.2, we have $\overline{d_1 p_1} = \bar{d}_3^{-1}\bar{p}_1\overline{d_3 d_1}$, $\bar{\alpha}_1 = (\overline{d_3 d_1})^{-1}\bar{\alpha}_1\overline{d_3 d_1}, \ldots, \bar{p}_n = (\overline{d_3 d_1})^{-1}\bar{p}_n\overline{d_3 d_1}$, $\overline{\alpha_n d_2} = (\overline{d_3 d_1})^{-1}\bar{\alpha}_n\bar{d}_3$. Thus we have $\bar{d}_2 = \bar{d}_1^{-1}$, which contradicts the choice of s_1, t_1. This completes the proof. ∎

COROLLARY 4.2. *Let P be the polygonal product of the* f.g. *abelian groups $A_0, A_1, \ldots, A_m$ ($m \geq 3$), amalgamating subgroups $\langle a_1\rangle, \ldots, \langle a_m\rangle, \langle a_0\rangle$, with trivial intersections. Then P is* c.s.

Corollary 4.2 generalizes Theorem 3.4 in [1]. We also have the following.

COROLLARY 4.3. *Let $E_m = A_1 *_{\langle a_2\rangle} A_2 *_{\langle a_3\rangle} \cdots *_{\langle a_{m-1}\rangle} A_{m-1}$ ($m \geq 3$), where the A_i are polycyclic-by-finite and $a_i \in Z(A_{i-1}) \cap Z(A_i)$ with $\langle a_i\rangle \cap \langle a_{i+1}\rangle = 1$. Then E_m is* c.s.

PROOF. Let $E = \langle a_0, a_1\rangle *_{\langle a_1\rangle} (\langle a_1\rangle \times A_1) *_{\langle a_2\rangle} \cdots *_{\langle a_{m-1}\rangle} (A_{m-1} \times \langle a_m\rangle) *_{\langle a_m\rangle} \langle a_m, a_{m+1}\rangle$, and $F = \langle a_0, a_{m+2}\rangle *_{\langle a_{m+2}\rangle} \langle a_{m+2}, a_{m+1}\rangle$, where $\langle a_0, a_1\rangle$, $\langle a_m, a_{m+1}\rangle$, $\langle a_0, a_{m+2}\rangle$, and $\langle a_{m+2}, a_{m+1}\rangle$ are free abelian groups of rank 2. We write $H = \langle a_0, a_{m+1}\rangle = \langle a_0\rangle * \langle a_{m+1}\rangle$, and $P = E *_H F$. Then P is a polygonal product of polycyclic-by-finite groups, amalgamating cyclic central subgroups with trivial intersections. Hence P is c.s. by our main result. Note that there is a natural homomorphism $\pi: P \longrightarrow E_m$ such that $a_i\pi = 1$ for $i = 0, 1, m+1, m+2$ and $\pi|_{E_m}$ is the identity map on E_m. Simply, E_m is a retract of P. It follows immediately that E_m is c.s., since P is c.s. ∎

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