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OSCILLATION AND GLOBAL ATTRACTIVITY IN A PERIODIC DELAY EQUATION

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ABSTRACT. Consider the delay differential equation

$$\dot{x}(t) = -\alpha(t)x(t) + \beta(t)e^{-x(t-m\omega)}, \quad t \geq 0,$$

where $\alpha(t)$ and $\beta(t)$ are positive, periodic, and continuous functions with period $\omega > 0$, and m is a nonnegative integer. We show that this equation has a positive periodic solution $x^*(t)$ with period ω. We also establish a necessary and sufficient condition for every solution of the equation to oscillate about $x^*(t)$ and a sufficient condition for $x^*(t)$ to be a global attractor of all solutions of the equation.

1. **Introduction.** Our aim in this paper is to investigate the asymptotic behavior of solutions of the nonlinear delay differential equation

$$\dot{x}(t) = -\alpha(t)x(t) + \beta(t)e^{-x(t-m\omega)}, \quad t \geq 0, \tag{1.1}$$

where $\alpha(t)$ and $\beta(t)$ are positive, periodic, and continuous functions with period $\omega > 0$, and m is a nonnegative integer. We will first show that equation (1.1) has a positive periodic solution $x^*(t)$ with period ω. Then we will establish a necessary and sufficient condition for every solution of equation (1.1) to oscillate about $x^*(t)$ and a sufficient condition for $x^*(t)$ to be a global attractor of all solutions of equation (1.1).

When $m\omega = \tau$, $\alpha(t) \equiv \alpha$, and $\beta(t) \equiv \beta\gamma$ with α, β, and γ positive constants, equation (1.1) reduces to the autonomous equation

$$\dot{y}(t) = -\alpha y(t) + \beta e^{-\gamma y(t-\tau)}, \quad t \geq 0 \tag{1.2}$$

where $y(t) = x(t)/\gamma$. Equation (1.2) was used by Wazewska-Czyzewska and Lasota [10] as a model for the survival of red blood cells in an animal; see also Arino and Kimmel [1]. Here $y(t)$ denotes the number of red blood cells at time t, α is the probability of death of a red blood cell, β and γ are positive constants related to the production of red blood cells per unit time, and τ is the time required to produce a red blood cell. The oscillation and the global attractivity of equation (1.2) has been studied by Kulenovic and Ladas [5], and by Kulenovic, Ladas and Sficas [6], respectively; see also Györi and Ladas [3].

Recently, the asymptotic behavior of solutions of some periodic population models has been studied, for example, in Gopalsamy, Kulenovic and Ladas [2], Zhang and

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Gopalsamy [11], and Lalli and Zhang [7]. As stated in [7], the effects of a periodically varying environment are important for evolutionary theory as the selective forces on systems in a fluctuating environment differ from those in a stable environment. Thus, the assumptions of periodicity of the parameters are a way of incorporating the periodicity of the environment. It has been suggested by Nicholson [8] that any periodic change of climate tends to impose its period upon oscillations of internal origin or to cause such oscillations to have a harmonic relation to periodic climactic changes. In view of this, it is realistic to assume that α and β are periodic functions of period ω and that the delay is an integral multiple of the periodicity of the environment. Hence, we are motivated to investigate the asymptotic behavior of solutions of the periodic delay differential equation (1.1).

2. **Existence of positive periodic solutions.** In this section, we study the existence of periodic solutions of equation (1.1). The result is the following.

THEOREM 1. *Equation (1.1) has a positive periodic solution $x^*(t)$ with period ω.*

PROOF. First, consider equation (1.1) without delay, that is,

$$\dot{x}(t) = -\alpha(t)x(t) + \beta(t)e^{-x(t)}, \quad t \geq 0. \tag{2.1}$$

Observe that there is a unique $r(t) > 0$ such that

$$-\alpha(t)r(t) + \beta(t)e^{-r(t)} = 0 \quad \text{for } t > 0.$$

Set

$$A = \min_{0 \leq t \leq \omega} \{r(t)\} \quad \text{and} \quad B = \max_{0 \leq t \leq \omega} \{r(t)\},$$

and let $x(t) = x(t, 0, x_0)$ denote the unique solution of equation (2.1) through $(0, x_0)$. We claim that

$$x_0 \in [A, B] \quad \text{implies that } x(t) = x(t, 0, x_0) \in [A, B].$$

We first show that $x(t) \leq B$. Otherwise,

$$t^* = \inf\{t > 0 : x(t) > B\} < \infty.$$

Then, it is easy to see that there exists a $t_1 > t^*$ such that

$$x(t_1) > B, \quad \text{and} \quad x'(t_1) > 0.$$

Hence, it follows from (2.1) that

$$\begin{aligned} 0 < x'(t_1) &= -\alpha(t_1)x(t_1) + \beta(t_1)e^{-x(t_1)} \\ &\leq -\alpha(t_1)B + \beta(t_1)e^{-B} \\ &\leq -\alpha(t_1)r(t_1) + \beta(t_1)e^{-r(t_1)} = 0, \end{aligned}$$

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which is a contradiction. Hence, $x(t) \leq B$. By a similar argument, we can show that $x(t) \geq A$ also. Thus, in particular,

$$x_\omega = x(\omega, 0, x_0) \in [A, B].$$

Now define a mapping F: $[A, B] \longrightarrow [A, B]$ as follows: for each $x_0 \in [A, B]$

$$F(x_0) = x_\omega.$$

Since the solution $x(t, 0, x_0)$ depends continuously on the initial value x_0, the mapping F is continuous and maps the interval $[A, B]$ into itself. Therefore, F has a fixed point x_0^* by Brouwer's fixed point theorem. Thus, the unique positive solution $x^*(t) = x(t, 0, x_0^*)$ is periodic with period ω.

Finally, by noting that $x^*(t) = x^*(t - m\omega)$, we see that $x^*(t)$ is also a periodic solution of equation (1.1). This completes the proof.

REMARK 1. From the proof of the theorem, we see that $x^*(t)$ satisfies

$$A \leq x^*(t) \leq B$$

which gives an estimate for the location of the periodic solution. Clearly, when $\alpha(t)$ and $\beta(t)$ are both constants, then $A = B$ and so $x^*(t) \equiv A$ becomes a positive equilibrium of equation (1.1).

3. **Oscillation of equation (1.1).** In this section, we study the oscillatory behavior of solutions of equation (1.1). The following theorem is our main result.

THEOREM 2. *Every solution of equation (1.1) oscillates about* $x^*(t)$ *if and only if*

$$\exp\Big(\int_0^{m\omega} \alpha(t)\, dt\Big) \int_0^{m\omega} \beta(t) e^{-x^*(t)}\, dt > \frac{1}{e}. \tag{3.1}$$

PROOF. Let $x(t)$ be a solution of equation (1.1) and let $x(t) - x^*(t) = y(t)$. Then equation (1.1) reduces to

$$\dot{y}(t) + \alpha(t)y(t) + \beta(t)e^{-x^*(t)}[1 - e^{-y(t-m\omega)}] = 0. \tag{3.2}$$

Clearly, $x(t)$ oscillates about $x^*(t)$ if and only if $y(t)$ oscillates about zero.

Set $g(u) = 1 - e^{-u}$. By noting that

$$g'(u) = e^{-u} > 0 \quad \text{and} \quad e^{-u} \geq 1 - u$$

for any u, we see that

$$g(u) \leq g'(0)u \quad \text{for } u > 0.$$

Hence, by the linearized oscillation theorem established in [4], it follows that every solution of equation (3.2) oscillates if and only if every solution of the equation

$$\dot{z}(t) + \alpha(t)z(t) + \beta(t)e^{-x^*(t)}z(t - m\omega) = 0 \tag{3.3}$$

oscillates. Next, observe that the oscillation invariant transformation

$$w(t) = z(t)\exp\Big(\int_0^t \alpha(s)\,ds\Big)$$

reduces equation (3.3) to

$$\dot{w}(t) + Q(t)w(t - m\omega) = 0 \tag{3.4}$$

where

$$Q(t) = \exp\Big(\int_0^{m\omega} \alpha(t)\,dt\Big)\beta(t)e^{-x^*(t)}.$$

It is well-known, for example, see [3, p. 42], that every solution of equation (3.4) oscillates provided

$$\lim_{t\to\infty} \int_{t-m\omega}^{t} Q(s)\,ds > \frac{1}{e},$$

and that equation (3.4) has a positive solution if

$$\sup_{t\geq m\omega} \int_{t-m\omega}^{t} Q(s)\,ds \leq \frac{1}{e}.$$

Hence, by noting that $Q(t)$ is periodic with period ω, we see that every solution of equation (3.4) oscillates if and only if

$$\int_0^{m\omega} Q(s)\,ds > \frac{1}{e},$$

that is, (3.1) holds. This completes the proof.

4. **Global attractivity of equation (1.1).** In this section, we study the global attractivity of equation (1.1). The following lemma extracted from [9] with a slight modification is needed in the proof of our main result.

LEMMA 1. *Consider the difference equation*

$$A_{n+1} = h(A_n) \tag{4.1}$$

where

$$h \in C^1[R, R].$$

Assume that h is a nonincreasing function and has a unique fixed point A^*. Suppose also that

$$h(\infty) = \lim_{u\to\infty} h(u)$$

exists and that

$$h'(u)h'\big(h(u)\big) < 1 \quad \textit{for } u > A^*.$$

Then the solution $\{A_n\}$ of equation (4.1) with $A_0 = h(\infty)$ tends to A^* as n tends to ∞.

Our main result in this section is the following.

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THEOREM 3. *Assume that*

$$\int_0^{m\omega} \beta(t)e^{-x^*(t)}\,dt \le 1. \tag{4.2}$$

Then every solution of equation (1.1) tends to $x^*(t)$ as t tends to ∞, that is,

$$\lim_{t\to\infty}[x(t) - x^*(t)] = 0. \tag{4.3}$$

PROOF. Since the transformation $y(t) = x(t) - x^*(t)$ reduces equation (1.1) to

$$\dot y(t) = -\alpha(t)y(t) + \beta(t)e^{-x^*(t)}(e^{-y(t-m\omega)} - 1), \tag{4.4}$$

it suffices to show that every solution of equation (4.4) tends to zero as t tends to ∞.

First, we show that every nonoscillatory solution of equation (4.4) tends to zero. We assume that $y(t)$ is eventually positive; the proof for the case $y(t)$ eventually negative is similar and will be omitted. Since $y(t) > 0$ eventually, from (4.4) we see that there is a $T > 0$ such that

$$\dot y(t) < 0 \quad \text{for } t > T,$$

and so there exists a constant $l \ge 0$ such that $\lim_{t\to\infty} y(t) = l$ and

$$\dot y(t) \le -l\alpha(t) + \beta(t)e^{-x^*(t)}(e^{-l} - 1) \quad \text{for } t \ge T + m\omega.$$

Hence, it follows that

$$l - y(T + m\omega) \le -l \int_{T+m\omega}^{\infty} \alpha(t)\,dt,$$

which, clearly, implies that $l = 0$.

Next, assume that $y(t)$ oscillates. We claim that for any $n \ge 0$, there exists a constant $T(n) \ge 0$ such that

$$y_{2n} \le y(t) \le y_{2n+1} \quad \text{for } t \ge T(n) \tag{4.5}$$

where $\{y_n\}$ is defined by

$$\begin{cases} y_{n+1} = \left(\int_0^{m\omega} \beta(t)e^{-x^*(t)}\,dt\right)(e^{-y_n} - 1) \\ y_0 = -\int_0^{m\omega} \beta(t)e^{-x^*(t)}\,dt. \end{cases}$$

Since $y(t)$ oscillates, there is an increasing sequence $\{t_n\}$ such that

$$t_{n+1} - t_n \ge 2m\omega + 1 \quad \text{and} \quad y(t_n) = 0 \quad \text{for } n = 0, 1, \ldots. \tag{4.6}$$

Let $s_n \in (t_n, t_{n+1})$ be a point where $y(t)$ obtains its local maximum or local minimum in (t_n, t_{n+1}). Hence, $\dot y(s_n) = 0$ and it follows from equation (4.4) that

$$-\alpha(s_n)y(s_n) + \beta(s_n)e^{-x^*(s_n)}(e^{-y(s_n-m\omega)} - 1) = 0,$$

which implies that

$$y(s_n)y(s_n - m\omega) < 0, \quad n = 0, 1, \ldots.$$

Hence, there is a $\xi_n \in (s_n - m\omega, s_n)$ such that

$$y(\xi_n) = 0, \quad n = 0, 1, \dots. \tag{4.7}$$

From equation (4.4) we find that

$$\frac{d}{dt}[y(t)e^{\int_0^t \alpha(s)\,ds}] = \beta(t)e^{\int_0^t \alpha(s)\,ds}e^{-x^*(t)}(e^{-y(t-m\omega)} - 1).$$

Then, by integrating both sides of this equation from ξ_n to s_n and by noting (4.7), we see that

$$\begin{aligned} y(s_n)e^{\int_0^{s_n} \alpha(s)\,ds} &= \int_{\xi_n}^{s_n} \beta(t)e^{\int_0^t \alpha(s)\,ds}e^{-x^*(t)}(e^{-y(t-m\omega)} - 1)\,dt \\ &\geq -e^{\int_0^{s_n} \alpha(s)\,ds}\int_{\xi_n}^{s_n} \beta(t)e^{-x^*(t)}\,dt \\ &\geq -e^{\int_0^{s_n} \alpha(s)\,ds}\int_0^{m\omega} \beta(t)e^{-x^*(t)}\,dt, \end{aligned} \tag{4.8}$$

and so

$$y(s_n) \geq -\int_0^{m\omega} \beta(t)e^{-x^*(t)}\,dt = y_0 \quad \text{for } n = 0, 1, \dots.$$

Hence, it follows that

$$y(t) \geq y_0 \quad \text{for } t \geq t_0. \tag{4.9}$$

By noting the decreasing nature of e^{-u} and by using (4.9) in (4.8), we find that

$$y(s_n)e^{\int_0^{s_n} \alpha(s)\,ds} \leq e^{\int_0^{s_n} \alpha(s)\,ds}\Big(\int_0^{m\omega} \beta(t)e^{-x^*(t)}\,dt\Big)(e^{-y_0} - 1),$$

and so

$$y(s_n) \leq \Big(\int_0^{m\omega} \beta(t)e^{-x^*(t)}\,dt\Big)(e^{-y_0} - 1) = y_1 \quad \text{for } n = 1, 2, \dots.$$

Hence,

$$y(t) \leq y_1 \quad \text{for } t \geq t_1$$

and so

$$y_0 \leq y(t) \leq y_1 \quad \text{for } t \geq t_1.$$

Now assume that

$$y_{2k} \leq y(t) \leq y_{2k+1} \quad \text{for } t \geq t_{2k+1}. \tag{4.10}$$

Then, by using (4.10) in (4.8), we find that

$$y(s_n)e^{\int_0^{s_n} \alpha(s)\,ds} \geq e^{\int_0^{s_n} \alpha(s)\,ds}\Big(\int_0^{m\omega} \beta(t)e^{-x^*(t)}\,dt\Big)(e^{-y_{2k+1}} - 1),$$

and so

$$\begin{aligned} y(s_n) &\geq \Big(\int_0^{m\omega} \beta(t)e^{-x^*(t)}\,dt\Big)(e^{-y_{2k+1}} - 1) \\ &= y_{2(k+1)}, \quad n = 2(k+1), \quad 2(k+1)+1, \dots. \end{aligned}$$

Hence,

(4.11)
$$y(t) \geq y_{2(k+1)} \quad \text{for } t \geq t_{2(k+1)}.$$

By using (4.11) in (4.8) we find that

$$y(s_n)e^{\int_0^{s_n} \alpha(s)\,ds} \leq e^{\int_0^{s_n} \alpha(s)\,ds}\Big(\int_0^{m\omega} \beta(t)e^{-x^*(t)}\,dt\Big)(e^{-y_{2(k+1)}} - 1),$$

and so

$$\begin{aligned} y(s_n) &\leq \Big(\int_0^{m\omega} \beta(t)e^{-x^*(t)}\,dt\Big)(e^{-y_{2(k+1)}} - 1) \\ &= y_{2(k+1)+1}, \quad n = 2(k+1)+1, \quad 2(k+2), \ldots . \end{aligned}$$

Hence,

$$y(t) \leq y_{2(k+1)+1} \quad \text{for } t \geq t_{2(k+1)+1},$$

and so

$$y_{2(k+1)} \leq y(t) \leq y_{2(k+1)+1} \quad \text{for } t \geq t_{2k+1}.$$

Therefore, by induction, we see that (4.5) holds.

Now, we claim that

$$\lim_{n\to\infty} y_n = 0.$$

To this end, set

$$h(u) = \Big(\int_0^{m\omega} \beta(t)e^{-x^*(t)}\,dt\Big)(e^{-u} - 1).$$

Then,

$$h'(u) = -\Big(\int_0^{m\omega} \beta(t)e^{-x^*(t)}\,dt\Big)e^{-u},$$

and so

(4.12)
$$h'(u)h'\big(h(u)\big) = \Big(\int_0^{m\omega} \beta(t)e^{-x^*(t)}\,dt\Big)^2 e^{-(u+h(u))}.$$

Observe that

$$u + h(u) = u + \Big(\int_0^{m\omega} \beta(t)e^{-x^*(t)}\,dt\Big)(e^{-u} - 1),$$

which, in view of (4.2) and the fact that $e^{-u} \geq 1 - u$, implies that

$$u + h(u) \geq u + (e^{-u} - 1) > 0 \quad \text{for } u > 0.$$

Hence, it follows from (4.12) that

$$h'(u)h'\big(h(u)\big) < \Big(\int_0^{m\omega} \beta(t)e^{-x^*(t)}\,dt\Big)^2 \leq 1 \quad \text{for } u > 0,$$

and so by Lemma 1,

$$\lim_{n\to\infty} y_n = 0.$$

Then, in view of (4.5), we see that

$$\lim_{n\to\infty} y(t) = 0.$$

This completes the proof.

REMARK 2. Set $x(t) = \gamma y(t)$. Then equation (1.2) reduces to

$$\dot{x}(t) = -\alpha x(t) + \beta\gamma e^{-x(t-\tau)}, \quad t \geq 0. \tag{4.13}$$

equation (1.2) has a unique positive equilibrium y^* and so $x^* = \gamma y^*$ is the unique positive equilibrium of equation (4.13). Clearly, every solution of equation (1.2) oscillates about y^* if and only if every solution of equation (4.13) oscillates about x^*; every solution of equation (1.2) tends to y^* if and only if every solution of equation (4.13) tends to x^*. Hence, by employing Theorem 2, we see that every solution of equation (1.2) oscillates about y^* if and only if

$$e^{\alpha\tau}\beta\gamma\tau e^{-\gamma y^*} > \frac{1}{e}. \tag{4.14}$$

Since $\alpha y^* = \beta e^{-\gamma y^*}$, (4.14) is equivalent to

$$\alpha\gamma\tau y^* e^{\alpha\tau} > \frac{1}{e}. \tag{4.15}$$

It has been shown in [5] that (4.15) is a necessary and sufficient condition for every positive solution of equation (1.2) to oscillate about y^*.

By employing Theorem 3, we see that every solution of equation (1.2) tends to y^* as t tends to ∞ provided

$$\beta\gamma\tau e^{-\gamma y^*} \leq 1,$$

that is,

$$\alpha\gamma\tau y^* \leq 1. \tag{4.16}$$

It has been shown in [6] that if

$$\exp\big(\gamma y^*(1 - e^{-\alpha\tau})\big) < 2, \tag{4.17}$$

then every positive solution of equation (1.2) tends to y^* as t tends to ∞. Clearly, (4.16) is a different condition from (4.17).

EXAMPLE 1. Consider the delay differential equation

$$\dot{x}(t) = -\frac{x(t)}{2\pi \ln(e + 1 + \sin t)} + \frac{e + 1 + \sin t + 2\pi \cos t}{2\pi} e^{-x(t-2\pi)}. \tag{4.18}$$

It is easy to check that $x^*(t) = \ln(e + 1 + \sin t)$ is a periodic solution of equation (4.18) with period 2π. Observe that

$$\begin{aligned}
&\exp\left(\int_0^{2\pi} \frac{1}{2\pi \ln(e + 1 + \sin t)}\, dt\right)\left(\int_0^{2\pi} \frac{e + 1 + \sin t + 2\pi\cos t}{2\pi} e^{-x^*(t)}\, dt\right) \\
&\quad = \exp\left(\int_0^{2\pi} \frac{1}{2\pi \ln(e + 1 + \sin t)}\, dt\right)\left(\int_0^{2\pi} \frac{1}{2\pi}\left(1 + \frac{2\pi\cos t}{e + 1 + \sin t}\right) dt\right) \\
&\quad \geq \int_0^{2\pi} \frac{1}{2\pi}\left(1 + \frac{2\pi\cos t}{e + 1 + \sin t}\right) dt \\
&\quad = 1 \geq \frac{1}{e}.
\end{aligned}$$

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Hence, by Theorem 2, every solution of equation (4.18) oscillates about $x^*(t)$. Also, observe that

$$\int_0^{2\pi} \frac{e+1+\sin t+2\pi\cos t}{2\pi} e^{-x^*(t)}\,dt = \int_0^{2\pi} \frac{1}{2\pi}\left(1+\frac{2\pi\cos t}{e+1+\sin t}\right)dt = 1.$$

Hence, by Theorem 3, every solution of equation (4.18) tends to $x^*(t)$ as t tends to ∞.

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