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ON THE HÖLDER SEMI-NORM OF THE REMAINDER IN POLYNOMIAL APPROXIMATION

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Suppose the qth derivative of a function f is Hölder continuous of index α, where $0 < \alpha \leqslant 1$, on the interval $[-1,1]$. Suppose further that p_n is any polynomial of degree at most n such that $|r_n(x)| = |f(x) - p_n(x)| \leqslant c\left\{\max\left(\left(1-x^2\right)^{1/2}/n, 1/n^2\right)\right\}^{q+\alpha}$ on $[-1,1]$. If

$$\|r_n\|_\beta = \sup_{\substack{x,y\in[-1,1]\\ x\neq y}} |r_n(x) - r_n(y)| \,/\, |x-y|^\beta,$$

then it is shown that

$$\|r_n\|_\beta \leqslant cn^{-q-\alpha+\beta}, \quad 0 < \beta \leqslant 1.$$

1. INTRODUCTION

Suppose that a function f is Hölder continuous of order α, where $0 < \alpha \leqslant 1$, on the compact interval $[-1,1]$. That is, there exists $\alpha \in (0,1]$ such that for every pair of points $x, y \in [-1,1]$ we have

$$(1) \qquad |f(x) - f(y)| \leqslant L\,|x-y|^\alpha,$$

where L is independent of x and y. We write $f \in H_\alpha[-1,1]$. The Hölder semi-norm $\|f\|_\alpha$ is the smallest L for which (1) is satisfied so that we define

$$(2) \qquad \|f\|_\alpha := \sup_{\substack{x,y\in[-1,1]\\ x\neq y}} \frac{|f(x)-f(y)|}{|x-y|^\alpha}.$$

It also follows that if $f \in H_\alpha[-1,1]$ then $f \in H_\beta[-1,1]$ for any β such that $0 < \beta < \alpha$.

Suppose now that for every $n \in \mathbb{N}$, the set of all natural numbers, f is approximated on $[-1,1]$ by some polynomial p_n say, of degree at most n, and let the remainder be denoted and defined by

$$(3) \qquad r_n := f - p_n.$$

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Obviously if f is in $H_\alpha[-1,1]$ then so is r_n.

In this paper we shall discuss estimates for $\|r_n\|_\beta$, where β lies in some interval dependent upon both α and the characteristics of the polynomial approximations p_n to f. Such estimates have proved to be extremely useful, for example, in the approximate evaluation of Cauchy principal value integrals and the solution of singular integral equations with Cauchy kernel (see, for example, Elliott and Paget [2] and Elliott [3]).

The first result for $\|r_n\|_\beta$ was given by Kalandiya [6].

THEOREM 1.1. *Suppose $f \in H_\alpha[-1,1]$. Then for every polynomial p_n such that*

$$\|r_n\|_\infty \leqslant A_1 n^{-\alpha}, \quad n \in \mathbb{N}, \tag{4}$$

it follows that

$$\|r_n\|_\beta \leqslant A_2 n^{-\alpha+2\beta}, \tag{5}$$

where $0 < \beta < \alpha/2$.

(Note that we shall be introducing a sequence of constants A_1, A_2, et cetera throughout the paper. These will always be positive and independent of n. Note also that $\|\cdot\|_\infty$ denotes the uniform norm whereas $\|\cdot\|_\alpha$, for $0 < \alpha \leqslant 1$, will always denote the Hölder semi-norm of (2). There should be no confusion.)

The presence of the factor 2β instead of β in (5) comes about because Kalandiya's proof uses the well known fact (see, for example, Lorentz [7, Chapter 3, Theorem 5]) that for any polynomial p_n of degree n, $\|p_n'\|_\infty \leqslant n^2\|p_n\|_\infty$, with equality occurring when $p_n = T_n$, the Chebyshev polynomial of the first kind of degree n. Ioakimidis [5], however, later gave the following result.

THEOREM 1.2. *Suppose $f \in H_\alpha[-1,1]$. For each $n \in \mathbb{N}$, there exists a polynomial p_n such that*

$$\|r_n\|_\infty \leqslant A_1 n^{-\alpha} \tag{6}$$

and for which

$$\|r_n\|_\beta \leqslant A_3 n^{-\alpha+\beta}, \tag{7}$$

where $0 < \beta < \alpha$.

This result intrigued O.V. Davydov who wondered what condition on p_n should replace (6) so that (7) was true for every such polynomial. Before stating Davydov's theorem we shall, following Lorentz [7], introduce the function $\Delta_n(x)$ defined by

$$\Delta_n(x) := \max\left\{\frac{\sqrt{1-x^2}}{n}, \frac{1}{n^2}\right\}, \tag{8}$$

where $-1 \leqslant x \leqslant 1$ and $n \in \mathbb{N}$. A slightly modified statement of Davydov's theorem [**1**] is given as follows.

THEOREM 1.3. *Suppose $f \in H_\alpha[-1,1]$, $0 < \alpha \leqslant 1$. For every $n \in \mathbb{N}$, and for every polynomial p_n for which*

$$|f(x) - p_n(x)| \leqslant A_4(\Delta_n(x))^\alpha, \quad -1 \leqslant x \leqslant 1, \tag{9}$$

then

$$\|r_n\|_\beta \leqslant A_5 n^{-\alpha+\beta}, \tag{10}$$

where

$$0 < \beta \leqslant \alpha < 1, \qquad \text{or} \qquad 0 < \beta < \alpha \leqslant 1. \tag{11}$$

Condition (9) ensures that the polynomials p_n are in general a better approximation to f near the end points ± 1 than are the polynomials which satisfy only (6). The proof of Davydov's theorem makes use of an interpolation theorem due to Riesz [**11**] and follows an argument similar to that given by Nikolskii [**10**]. Polynomials satisfying (9) have been discussed by Nikolskii [**9**] when $\alpha = 1$ and by Timan [**12**] for $0 < \alpha < 1$. An explicit construction of such polynomials was given by Grünwald [**4**], see Mills and Varma [**8**].

So much for the background. The purpose of this note is to give an extension of Davydov's theorem. We shall now state our principal result.

THEOREM 1.4. *Suppose $q \in \mathbb{N}$ and $f^{(q)} \in H_\alpha[-1,1]$, where $0 < \alpha \leqslant 1$. For every $n \in \mathbb{N}$, if p_n denotes any polynomial of degree at most n such that*

$$|f(x) - p_n(x)| \leqslant A_6(\Delta_n(x))^{q+\alpha}, \quad -1 \leqslant x \leqslant 1, \tag{12}$$

then

$$\|r_n\|_\beta \leqslant A_7 n^{-q-\alpha+\beta}, \tag{13}$$

where $0 < \beta \leqslant 1$.

2. PROOF OF THEOREM 1.4

Before giving the proof of theorem 1.4 we need to quote some further results from Lorentz [**7**].

LEMMA 2.1. *With $\Delta_n(x)$ defined as in (8),*

$$\Delta_n(x)/4 \leqslant \Delta_{2n}(x) \leqslant \Delta_n(x)/2 \tag{14}$$

for $-1 \leqslant x \leqslant 1$, $n \in \mathbb{N}$.

PROOF: This is straightforward. □

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Lemma 2.2. *Let $r \in \mathbb{N}_0$ and $0 < \alpha \leqslant 1$, where $\mathbb{N}_0 = \{0,1,2,\ldots\}$. If a polynomial p_n of degree n satisfies*

$$|p_n(x)| \leqslant (\Delta_n(x))^{r+\alpha}, \quad -1 \leqslant x \leqslant 1, \tag{15}$$

then

$$|p_n'(x)| \leqslant A_8(\Delta_n(x))^{r-1+\alpha}, \quad -1 \leqslant x \leqslant 1. \tag{16}$$

Proof: See Lorentz [7, Chapter 5, Theorem 3]. □

Lemma 2.3. *Suppose $q \in \mathbb{N}_0$ and $f^{(q)} \in H_\alpha[-1,1]$, $0 < \alpha \leqslant 1$. Then there exists a sequence $\{p_n\}$ of polynomials such that*

$$|f(x) - p_n(x)| \leqslant A_9(\Delta_n(x))^{q+\alpha}, \tag{17}$$

for $-1 \leqslant x \leqslant 1$ and $n \geqslant q$.

Proof: See Lorentz [7, Chapter 5, Theorems 1 and 2]. □

We are now in a position to supply the proof of theorem 1.4.

Proof of theorem 1.4. We first note that lemma 2.3 tells us that a sequence of polynomials satisfying (12) exists. Since $q \geqslant 1$, f' and consequently r_n' exist and are certainly continuous on $[-1,1]$. If x and y are any two distinct points of $[-1,1]$ then the mean value theorem gives

$$\frac{|r_n(x) - r_n(y)|}{|x-y|^\beta} = |x-y|^{1-\beta}\,|r_n'(\xi)|, \tag{18}$$

where ξ is some point between x and y. Now (17) shows that the sequence of polynomials $\{p_n\}$ converges uniformly to f on $[-1,1]$. In particular, for a given value of $n \in \mathbb{N}$, we can write

$$f(x) - p_n(x) = \sum_{k=0}^{\infty}(p_{2^{k+1}n}(x) - p_{2^k n}(x)). \tag{19}$$

Now

$$\begin{aligned} |p_{2^{k+1}n}(x) - p_{2^k n}(x)| &= |(p_{2^{k+1}n}(x) - f(x)) + (f(x) - p_{2^k n}(x))| \\ &\leqslant A_6\left[(\Delta_{2^{k+1}n}(x))^{q+\alpha} + (\Delta_{2^k n}(x))^{q+\alpha}\right], \quad \text{by (12)}, \\ &\leqslant A_{10}(\Delta_{2^{k+1}n}(x))^{q+\alpha}, \quad \text{by Lemma 2.1.} \end{aligned}$$

From Lemma 2.2 it follows at once that

$$\begin{aligned} \left|p_{2^{k+1}n}'(x) - p_{2^k n}'(x)\right| &\leqslant A_{11}(\Delta_{2^{k+1}n}(x))^{q+\alpha-1} \\ &\leqslant A_{12}\left(1/2^{k+1}n\right)^{q+\alpha-1}, \quad \text{see (8).} \end{aligned} \tag{20}$$

Since $q \geqslant 1$ and $\sum_{k=0}^{\infty} (2^{k+1})^{-\alpha}$ is convergent it follows by Weierstrass' M-test that

$$r_n'(x) = f'(x) - p_n'(x) = \sum_{k=0}^{\infty} \left(p'_{2^{k+1}n}(x) - p'_{2^k n}(x)\right), \tag{21}$$

the convergence of the series being uniform on $[-1,1]$. Again, using (20) we have that

$$|r_n'(x)| \leqslant A_{12} \sum_{k=0}^{\infty} \left(\frac{1}{2^{k+1}n}\right)^{q+\alpha-1} = \frac{A_{13}}{n^{q+\alpha-1}}. \tag{22}$$

We are now in a position to prove inequality (13). Firstly, let us suppose that $|x-y| < 1/n$. Then from (18) and (22) we have

$$\frac{|r_n(x) - r_n(y)|}{|x-y|^{\beta}} \leqslant A_{14} n^{-q-\alpha+\beta}, \tag{23}$$

for $0 < \beta \leqslant 1$. Next, suppose that $|x-y| \geqslant 1/n$. Then

$$\frac{|r_n(x) - r_n(y)|}{|x-y|^{\beta}} \leqslant n^{\beta} \left\{|r_n(x)| + |r_n(y)|\right\}. \tag{24}$$

But from (8) and (12) we have

$$|r_n(x)| \leqslant A_{15} n^{-q-\alpha}$$

for every $x \in [-1,1]$. Consequently,

$$\frac{|r_n(x) - r_n(y)|}{|x-y|^{\beta}} \leqslant A_{16} n^{-q-\alpha+\beta}. \tag{25}$$

From (23) and (25) we have

$$\|r_n\|_{\beta} = \sup_{\substack{x \neq y \\ x,y \in [-1,1]}} \frac{|r_n(x) - r_n(y)|}{|x-y|^{\beta}} \leqslant A_{17} n^{-q-\alpha+\beta}, \tag{26}$$

which proves the theorem. □

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