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DIMENSIONAL CHARACTERISTICS OF THE NONWANDERING SETS OF OPEN BILLIARDS

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An open billiard is a dynamical system in which a pointlike particle moves at constant speed in an unbounded domain, reflecting off a boundary according to the classical laws of optics [4]. This thesis is an investigation of dimensional characteristics of the nonwandering set of an open billiard in the exterior of three or more strictly convex bodies satisfying Ikawa's no-eclipse condition [5, 11]. The billiard map for these systems is an axiom A diffeomorphism with a finite Markov partition. The nonwandering set is a hyperbolic set with stable and unstable manifolds satisfying a certain reflection property. The characteristics we investigate include the topological and measure-theoretic entropy, topological pressure, Lyapunov exponents, lower and upper box dimensions and the Hausdorff dimension of the nonwandering set. In particular, we investigate the dependence of the Hausdorff dimension on deformations to the boundary of the billiard obstacles. While the dependence of dimensional characteristics on perturbations of a system has been studied before [6, 9], this is the first time this question has been answered for dynamical billiards.

We find upper and lower bounds for the Hausdorff dimension using two different methods: one involving bounding the size of curves on convex fronts and the other using Bowen's equation and the variational principle for topological pressure. Both methods lead to the same upper and lower bounds. In the first method, we use a well-known recurrence relation for the successive curvatures of convex fronts to find bounds on the size of the fronts. This allows us to construct Lipschitz (but not bi-Lipschitz) homeomorphisms between the nonwandering set and the one-sided symbol space. From there we obtain estimates of the dimension. Kenny [7] used this method for open billiards in the plane. We extend it to higher dimensions and make improvements to the results in the plane. This work has been published in [13].

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The second method is a more general approach from the dimension theory of dynamical systems. In the plane, the billiard map is conformal, meaning that its derivative is a multiple of an isometry. For conformal maps, the Hausdorff dimension of nonwandering sets is well understood and satisfies Bowen's equation [2, 8]. In higher dimensions, the billiard map is nonconformal and the dimension only satisfies some estimates [3].

We consider what happens to the nonwandering set when one or more obstacles in a billiard are deformed or shifted. Consider a billiard with C^r-smooth boundary, deformed $C^{r'}$-smoothly with respect to some parameter α. In two and higher dimensions, we show that all points in the nonwandering set depend $C^{\min\{r-1,r'\}}$-smoothly on α. We use a well-known lemma [10] about the position of periodic points in a noneclipsing billiard, and differentiate these points to get a cyclic tridiagonal system of equations. In more than two dimensions, the system is block cyclic tridiagonal. We extend a theorem from [12] to bound the solutions to these equations.

For billiards in the plane, we use Bowen's equation to further show that the Hausdorff dimension depends smoothly on the deformations. Specifically, if the boundary of the billiard obstacles are C^r and they are deformed $C^{r'}$-smoothly with respect to a parameter α, then the Hausdorff dimension is a $C^{\min\{r-3,r'-1\}}$-smooth function of α. We find an upper bound for the derivative of the Hausdorff dimension with respect to α. Finally, we show that when the billiard deformation is real analytic, the Hausdorff dimension is also real analytic with respect to α. This work has been submitted for publication (see [14]). For higher dimensions, there is no exact equation for the Hausdorff dimension to differentiate because the billiard ball map is nonconformal.

We obtain some new results for nonconformal hyperbolic dynamics. The concept of an average conformal repeller was developed as a generalisation of a conformal repeller [1]. We extend this idea by generalising conformal hyperbolic sets to average conformal hyperbolic sets. A hyperbolic set is average conformal if it has only two distinct Lyapunov exponents, one positive and one negative. We obtain an equation for the Hausdorff dimension of an average conformal hyperbolic set. While we know that a billiard in three or more dimensions is never conformal, it is unknown whether there exist billiards that are average conformal. This work has been submitted for publication (see [15]).

Finally, we consider several examples of billiards with deformations and apply the techniques developed in this thesis to obtain numerical upper and lower bounds for the Hausdorff dimension and its derivative.

References

[1] J. Ban, Y. Cao and H. Hu, 'The dimensions of a non-conformal repeller and an average conformal repeller', *Trans. Amer. Math. Soc.* **362**(2) (2010), 727–751.

[2] L. Barreira, 'A non-additive thermodynamic formalism and applications to dimension theory of hyperbolic dynamical systems', *Ergod. Th. & Dynam. Sys.* **16**(5) (1996), 871–927.

[3] L. Barreira, 'Dimension estimates in nonconformal hyperbolic dynamics', *Nonlinearity* **16**(5) (2003), 1657–1672.

https://doi.org/10.1017/S0004972715000817 Published online by Cambridge University Press

[4] N. Chernov and R. Markarian, *Chaotic Billiards*, Mathematical Surveys and Monographs, 127 (American Mathematical Society, Providence, RI, 2006).
[5] M. Ikawa, 'Decay of solutions of the wave equation in the exterior of several convex bodies', *Ann. Inst. Fourier* **38**(2) (1988), 113–146.
[6] A. Katok, G. Knieper, M. Pollicott and H. Weiss, 'Differentiability and analyticity of topological entropy for Anosov and geodesic flows', *Invent. Math.* **98**(3) (1989), 581–597.
[7] R. Kenny, 'Estimates of Hausdorff dimension for the non-wandering set of an open planar billiard', *Canad. J. Math.* **56**(1) (2004), 115–133.
[8] Y. Pesin, *Dimension Theory in Dynamical Systems: Contemporary Views and Applications*, Chicago Lectures in Mathematics (University of Chicago Press, 1997).
[9] D. Ruelle, 'Differentiation of SRB states', *Comm. Math. Phys.* **187**(1) (1997), 227–241.
[10] L. Stoyanov, 'An estimate from above of the number of periodic orbits for semi-dispersed billiards', *Comm. Math. Phys.* **124**(2) (1989), 217–227.
[11] L. Stoyanov, 'Non-integrability of open billiard flows and Dolgopyat-type estimates', *Ergod. Th. & Dynam. Sys.* **32**(1) (2011), 295–313.
[12] J. Varah, 'A lower bound for the smallest singular value of a matrix', *Linear Algebra Appl.* **11**(1) (1975), 3–5.
[13] P. Wright, 'Estimates of Hausdorff dimension for non-wandering sets of higher dimensional open billiards', *Canad. J. Math.* **65** (2013), 1384–1400.
[14] P. Wright, 'Differentiability of Hausdorff dimension of the non-wandering set in a planar open billiard', Preprint, 2014, arXiv:1401.1002v3.
[15] P. Wright, 'Hausdorff dimension of non-wandering sets for average conformal hyperbolic maps', Preprint, 2014, arXiv:1401.1005v2.

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