SEQUENCES OF EULER GRAPHS

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(received October 26, 1965)

A graph is called Euler if it has no isolated vertices and every vertex has even or infinite degree. The graphs we consider may have multiple edges but no loops. Loosely speaking, we will be concerned with the conditional compactness of the set of all countable Euler subgraphs of a given graph. For a precise enunciation of what we mean by "conditional compactness" see Theorem 2 below.

Let $\alpha = (X_n)_{n \in N}$ (N the natural numbers) be a sequence of graphs. By α_* we shall denote the smallest graph whose edge set, $E(\alpha_*)$, is $\liminf E(X_n)$ (in the usual set-theoretic sense). Similarly, we define α^* to be the smallest graph with $E(\alpha^*) = \limsup E(X_n)$. The graphs α_* and α^* will be called limit inferior and limit superior of α, respectively. If $\alpha_* = \alpha^*$ we say that α converges to $\lim \alpha = \alpha_* = \alpha^*$. We denote $\bigcup_{n \in N} X_n$ by U_α.

For a vertex $x \in X$ we shall denote by $E(x;X)$ the set of all edges incident with x. The degree of x in X is $d(x;X) = |E(x;X)|$.

In the proof of Theorem 1 we shall make use of the obvious fact that if E is a finite subset of $E(\alpha_*)$ then $E \subset E(X_n)$ for all $n > n_o$.

THEOREM 1. *Let* $\alpha = (X_n)_{n \in N}$ *be a sequence of graphs,* $A = \{a_1, a_2, \ldots,\}$ *a finite or countable set of vertices of* α_*

Canad. Math. Bull. vol. 9, no. 2, 1966

such that (i) $d(a_i;X_n)$ is even whenever $a_i \in X_n$, and (ii) $d(a_i; U_\alpha) < \infty$, $i = 1, 2, \ldots$. Then there exists a subsequence β of α such that $d(a_i;\beta_*)$ is even, $i = 1, 2, \ldots$, and $E(a_i;\beta_*) = E(a_i;\gamma_*)$ for any subsequence γ of β.

Proof. Assume first that $A = \{a\}$. $a \in \alpha_*$ implies $a \in X_n$ for all $n > n_o$, so that $d(a;X_n) < d(a; U_\alpha) < \infty$ for all $n > n_o$. Hence there exists a sequence $n_o \leqq n_1 < n_2 < \ldots$ such that

$$d(a;X_{n_1}) = d(a;X_{n_2}) = \ldots = 2d < \infty.$$

Put $X_{n_k} = Y_k$, $k = 1, 2, \ldots$. $E(a;\alpha_*) \subset E(a; U_\alpha)$, hence $E(a;\alpha_*)$ is finite. By the remark preceding Theorem 1 this implies $E(a;\alpha_*) \subset E(a;Y_k)$, and hence

$$d(a;\alpha_*) \leq d(a;Y_k) = 2d, \ k = 1, 2, \ldots .$$

If $d(a;\alpha_*) = 2d$, then $E(a;\alpha_*) = E(a;Y_k)$, $k = 1, 2, \ldots$; hence if we put $\beta = (Y_k)_{k \in N}$ the proof is complete. If $d(a;\alpha_*) < 2d$, then $E(a;\alpha_*)$ is a proper subset of $E(a;Y_k)$ for all k. Let $e_k \in E(a;Y_k) - E(a;\alpha_*)$, $k = 1, 2, \ldots$. Clearly $e_k \in E(a; U_\alpha)$. Since $d(a; U_\alpha) < \infty$ it follows that only finitely many of the edges e_k are distinct, i.e., there is an infinite sequence $k_1 < k_2 < \ldots$ such that $e_{k_1} = e_{k_2} = \ldots$. Thus

$$e_{k_1} \in E(a;Y_{k_m}), \ m = 1, 2, \ldots .$$

Put $\alpha^{(1)} = (Y_{k_m})_{m \in N}$. Then $e_{k_1} \in E(a;\alpha_*^{(1)})$. Since $\alpha^{(1)}$ is a subsequence of α, $E(a;\alpha_*) \subset E(a;\alpha_*^{(1)})$, and since $e_{k_1} \notin E(a;\alpha_*)$ this inclusion is proper. Thus $d(a;\alpha_*) < d(a;\alpha_*^{(1)}) \leqq 2d$. If $d(a;\alpha_*^{(1)}) < 2d$ the above procedure can be repeated to yield a subsequence $\alpha^{(2)}$ of $\alpha^{(1)}$ such that $d(a;\alpha_*^{(1)})$

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$< d(a;\alpha_*^{(2)}) \leq 2d$. In view of the finiteness of $2d$ it follows that there is a subsequence $\alpha^{(s)}$ of α such that $d(a;\alpha^{(s)}) = 2d$, and $E(a;\alpha_*^{(s)}) = E(a;\gamma_*)$ for every subsequence γ of $\alpha^{(s)}$.

Now let $A = \{a_1, a_2, \ldots\}$. By the first part of the proof there is a subsequence $\beta^{(1)}$ of α such that $d(a_1;\beta_*^{(1)})$ is even and $E(a_1;\beta_*^{(1)}) = E(a_1;\gamma_*^{(1)})$ for every subsequence $\gamma^{(1)}$ of $\beta^{(1)}$. By induction there is a subsequence $\beta^{(i)}$ of $\beta^{(i-1)}$ having the same properties with respect to a_i. Then β, the diagonal sequence of the sequences $\beta^{(i)}$, has the required properties.

REMARK 1. For $A = \{a\}$ we have actually proved a slightly stronger result, viz., the following: Given any d such that $d(a;X_n) = 2d$ for infinitely many n, there exists a subsequence β of α with $d(a;\beta_*) = 2d$, and $E(a;\beta_*) = E(a;\gamma_*)$ for any subsequence γ of β.

Theorem 1 has the following

COROLLARY. Let α be a convergent sequence of Euler graphs such that $d(x; U_\alpha) < \infty$ for any $x \in \lim \alpha$ with $d(x; \lim \alpha) < \infty$. Then $\lim \alpha$ is Euler. In particular, $\lim \alpha$ is Euler if U_α is locally finite.

Proof. Take a vertex $x \in \lim \alpha$ with $d(x;\lim \alpha) < \infty$ and apply Theorem 1 to $A = \{x\}$. This yields the existence of a subsequence β of α such that $d(x;\beta_*)$ is even. But $\beta_* = \lim \alpha$.

REMARK 2. Both Theorem 1 and its corollary are false if the boundedness condition (ii) is removed. In the graph of Figure 1 let X_n be the triangle (a_1, a_2, x_n), $n = 1, 2, \ldots$. This sequence converges to (e), the graph consisting of e and a_1, a_2, but $d(a_i;(e)) = 1$, $i = 1, 2$.

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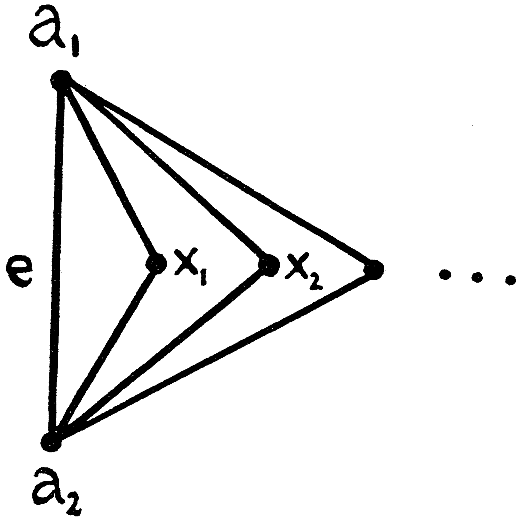

Figure 1

This example also shows that (ii) can not even be replaced by the weaker condition that $d(a_i;\alpha^*) < \infty$ for every $a_i \in A$.

THEOREM 2. Let α be a sequence of countable Euler such that U_α is locally finite. Then there exists a subsequence β of α such that β_* is Euler.

Proof. Suppose no such subsequence exists. Put $\beta^{(o)} = \alpha$ and for each countable ordinal τ define $\beta^{(\tau)}$ inductively as follows. If τ is not a limit ordinal then application of Theorem 1 to the sequence $\beta^{(\tau-1)}$ and the set $A = V(\beta_*^{(\tau-1)})$ yields the existence of a subsequence $\beta^{(\tau)}$ such that $d(a;\beta_*^{(\tau)})$ is even for every $a \in \beta_*^{(\tau-1)}$. $\beta_*^{(\tau-1)} \subset \beta_*^{(\tau)}$ and this inclusion is proper, otherwise $\beta_*^{(\tau)}$ would be Euler. If τ is a limit ordinal, let $(\sigma_i)_{i \in N}$ be a sequence of ordinals which is cofinal in τ, and define $\beta^{(\tau)}$ to be the diagonal sequence of the sequences $\beta^{(\sigma_i)}$. Note that $\beta_*^{(\tau)} \subset \alpha^*$ for all τ. Hence α^* has at least as many edges as there are countable non-limit ordinals, i.e., α^* is uncountable, a contradiction.

REMARK 3. In the hypothesis of Theorem 2, the word "countable" may be replaced by "connected". For if U_α is locally finite, then so is every X_n, and every connected locally finite graph is countable ([C], p.28, Theorem 2.4.1).

REMARK 4. By a similar argument one can prove the following: Let α be a sequence of countable Euler graphs. Then there exists a subsequence β of α such that $d(x;\beta_*)$ is even for any $x \in \beta_*$ with $d(x; U_\alpha) < \infty$.

REMARK 5. Starting with $\beta^{(o)} = \alpha$ define the sequences $\beta^{(\tau)}$, $\tau < \omega_1$, as in the proof of Theorem 2. Then

$X_\tau = \bigcup_{\sigma < \tau} \beta_*^{(\sigma)}$ is Euler for every countable limit ordinal τ.

For, given $x \in X_\tau$, let σ be the smallest ordinal such that $x \in \beta_*^{(\sigma)}$. Then $E(x;\beta_*^{(\sigma)}) \subset E(x;\beta_*^{(\rho)})$ for all $\rho > \sigma$. Hence $d(x;X_\tau) = d(x;\beta_*^{(\sigma+1)})$, and thus is even. Hence X_τ is Euler.

We conclude by showing that the closely related class of finitely cyclicly coverable (f.c.c.) graphs does not have the property stated in Theorem 2. We recall ([2], p.822) that a graph is f.c.c. if it is the union of a set of mutually edge-disjoint finite circuits. Every f.c.c. graph is Euler, but not conversely (not even in the locally finite case).

In the graph of Figure 2 let X_n be the 2n-circuit $(x_1, \ldots, x_n, y_n, \ldots, y_1)$, $n = 1, 2, \ldots$. Trivially, every X_n is f.c.c., and U_α is locally finite. But the sequence $\alpha = (X_n)_{n \in N}$ converges to the infinite circuit $(\ldots, y_2, y_1, x_1, x_2, \ldots)$ which, of course, is not f.c.c.

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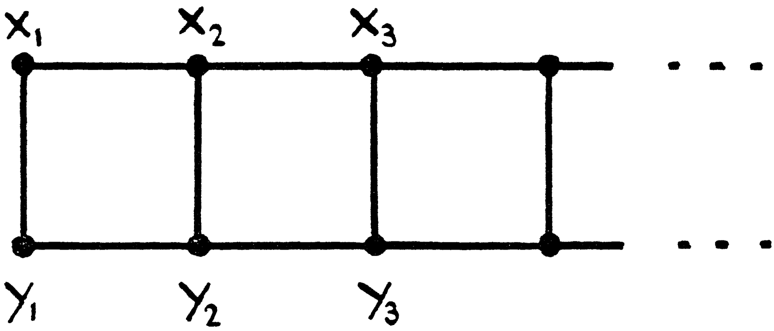

Figure 2

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