Canad. J. Math. Vol. **54** (3), 2002 pp. 634–647

Frames and Single Wavelets for Unitary Groups

Eric Weber

Abstract. We consider a unitary representation of a discrete countable abelian group on a separable Hilbert space which is associated to a cyclic generalized frame multiresolution analysis. We extend Robertson's theorem to apply to frames generated by the action of the group. Within this setup we use Stone's theorem and the theory of projection valued measures to analyze wandering frame collections. This yields a functional analytic method of constructing a wavelet from a generalized frame multiresolution analysis in terms of the frame scaling vectors. We then explicitly apply our results to the action of the integers given by translations on $L^2(\mathbb{R})$.

1 Introduction

The theory of wavelets and multiresolution analyses are inseparably interconnected. Indeed, the idea of generating a wavelet from a multiresolution analysis (MRA) is the foundation of the theory [7], [13]. However, Journé introduced a wavelet that did not arise from a multiresolution analysis, so the standard techniques did not apply. Later, the idea of a generalized multiresolution analysis (GMRA) was introduced by Baggett, Medina and Merrill [3]. A GMRA is a way of associating a wavelet that does not arise from a MRA to a multiresolution structure; however, no analogous technique for the construction of a wavelet was presented.

The procedure utilized by Daubechies and Mallat is to start with a potential low pass filter function which generates a scaling function, from which a wavelet can be constructed. Recently, the work of Baggett, Courter and Merrill [2] and independently Papadakis [14] introduced techniques for generating a wavelet from non-MRA structures, though their approaches are different. In [2] the technique is similar to Daubechies' in which they start with a potential low pass filter, from which a GMRA is constructed and then subsequently a wavelet. Whereas in [14] the technique is to start with a generalized frame multiresolution analysis (GFMRA), presented in terms of the frame scaling vectors, from which a low pass filter function is generated, and then finally the wavelet(s).

We present an alternate technique for constructing wavelets. We will start with a cyclic GFMRA and construct an explicit formula for a wavelet associated with that GFMRA. Our methods are more functional analytic in nature: we use Stone's theorem for representations of abelian groups and the theory of spectral multiplicity [3], [8]. In so doing, we bypass the step of producing a low pass filter. Furthermore, the constructions will be valid for a setting more general than $L^2(\mathbb{R})$, though since that is the motivating example, we shall consider such a special case.

Received by the editors August 28, 2001; revised January 30, 2002.
AMS subject classification: Primary: 42C40, 43A25; secondary: 42C15, 46N99.
Keywords: wavelet, multiresolution analysis, unitary group representation, frame.

There is another important idea in the theory, that of frames. Section 2 is devoted to analyzing frames that arise from the action of a discrete countable abelian group. Out of this analysis comes our construction technique, which is presented in Section 3. Finally, in Section 4, we consider several special cases, including the traditional case of translations by integers on $\mathbb{R}$. Additionally, we show by example that our technique is not, in a certain sense, exhaustive.

Let G be a discrete, countable abelian group, and let $\pi\colon G \to B(H)$ be a unitary representation of G on a separable Hilbert space H. Denote $\pi(g)$ by π_g; let $\widehat{G}$ denote the dual group of G; let λ denote normalized Haar measure on $\widehat{G}$. By Stone's theorem there exists a projection valued measure p on $\widehat{G}$ such that

$$\pi_g = \int_{\widehat{G}} g(\xi)\, dp(\xi).$$

Then, by the theory of projection valued measures, there exists a probability measure μ on $\widehat{G}$, a multiplicity function $m\colon \widehat{G} \to \{0, 1, \dots, \infty\}$ and a unitary operator

$$U\colon H \to \bigoplus_{j=1}^{k} L^2(F_j, \mathbb{C}^j, \mu),$$

where $F_j = m^{-1}(j)$. Note that k could be infinite, in which case we adjoin the direct summand $L^2\big(F_\infty, l^2(\mathbb{Z}), \mu\big)$ above. The operator U intertwines the projection valued measure on H and the canonical projection valued measure on $\widehat{G}$, and can be thought of as the Fourier transform on H. Indeed, if $x \in H$, we will denote Ux by $\hat{x}$. Furthermore, we will write $H \simeq \bigoplus_{j=1}^{k} L^2(F_j, \mu)$ to denote that such an intertwining unitary U exists.

Suppose there exists another unitary operator D on H, such that $G_1 = D^*\pi(G)D$ is a subgroup of finite index in $\pi(G)$. As noted in [1], not all groups admit this type of affine condition. Denote by Γ the quotient group of $\pi(G)/G_1$. The unitary D is called the dilation operator, and the group $\pi(G)$ is called the translation group.

A (orthonormal) wavelet is a vector $\psi \in H$ such that the collection $\{D^n \pi_g \psi : n \in \mathbb{Z}, g \in G\}$ is an orthonormal basis of H. A frame wavelet is such that the same collection is a frame for H. A generalized multiresolution analysis of H is a sequence of closed subspaces $\{V_j\}_{j\in\mathbb{Z}}$ such that the following conditions hold:

1. $V_j \subset V_{j+1}$,
2. $DV_j = V_{j+1}$,
3. $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$ and $\bigcup_{j\in\mathbb{Z}} V_j$ has dense span in H,
4. the *core subspace* V_0 is invariant under the action of π_g.

By number 1 above, we can define a second sequence of subspaces $\{W_j\}$ given by $V_{j+1} = V_j \oplus W_j$. For our purposes, we will assume that the subspace W_0 is cyclic under the action of $\pi(G)$; we shall call such a GMRA a *cyclic GMRA*.

If $\psi \in H$ is an orthonormal wavelet, then the subspaces defined by $V_j = \overline{\operatorname{span}}\{D^n \pi_g \psi : n < j,\ g \in G\}$ satisfy the 4 conditions above [1]. Conversely, a routine calculation shows that given a GMRA $\{V_j\}$, if ψ is such that $\{\pi_g\psi : g \in G\}$ is an orthonormal basis (or frame) of W_0, then ψ is a wavelet (or frame wavelet).

We shall assume a stronger condition than 4 above: suppose there exist vectors $\{\phi_j\}_{j\in J} \subset V_0$ such that $\{\pi_g\phi_j : g \in G, j \in J\}$ is a frame for V_0. We call such a structure a generalized frame multiresolution analysis [14]. For $\gamma \in \Gamma$, let $\pi_{g_\gamma} \in \pi(G)$ be a representative of the γ coset of $G_1 \subset \pi(G)$. For $\gamma \in \Gamma$ and $j \in J$, define the vectors

$$\varphi_{\gamma,j} = P_{W_0} D\pi_{g_\gamma}\phi_j.$$

Our main result is the following.

Theorem 1 *Suppose the vectors $\{\phi_j\}_{j\in J}$ are the frame scaling vectors for a GFMRA $\{V_k\}$. Define the $\varphi_{\gamma,j}$'s as above. For appropriately chosen sets $E_{\gamma,j} \subset \widehat{G}$, the vector ψ defined by*

$$\hat{\psi}(\xi) = \sum_{\gamma\in\Gamma}\sum_{j\in J} \frac{\chi_{E_{\gamma,j}}(\xi)}{|\hat{\varphi}_{\gamma,j}(\xi)|}\hat{\varphi}_{\gamma,j}(\xi)$$

is a normalized tight frame wavelet. If $\bigcup_{\gamma\in\Gamma}\bigcup_{j\in J} E_{\gamma,j} = \widehat{G}$, then ψ is an orthonormal wavelet.

We note several things regarding our presentation here. First, we will be decomposing the representation of G on V_0 as outlined above. It turns out, that the assumption that there exist frame scaling vectors ϕ_j is equivalent to the condition that the measure μ is the restriction of Haar measure to F_1 (see Lemma 2, Theorem 2, and Proposition 2). It is still open as to whether a wavelet can be associated to a GMRA that is not a GFMRA. Additionally, one motivation is to construct a wavelet from a GFMRA, which requires knowing the scaling functions. If one knows the core space V_0, one can decompose the representation and find the frame vectors. However, in practice the GFMRA is presented in terms of the scaling vectors, which is the approach we use here.

2 Wandering Frame Collections

A sequence $\{f_i\}_{i\in I}$ is a frame for a separable Hilbert space H if there exist positive constants A, B such that, for any $x \in H$,

$$A\|x\|^2 \le \sum_{i\in I} |\langle x, f_i\rangle|^2 \le B\|x\|^2.$$

The frame $\{f_i\}_{i\in I}$ is called *tight* if $A = B$ and *normalized tight* if $A = B = 1$. We make the following definition:

Definition 1 A collection of vectors $W = \{w_1, \dots, w_n\}$ will be called a *wandering frame collection* (or *complete wandering frame collection*) for π_g if the collection $S = \{\pi_g w_i : g \in G, i = 1, \dots, n\}$ is a frame for its closed linear span (or for H). If $X = \{x_1, \dots, x_n\}$ and $Y = \{y_1, \dots, y_m\}$ are wandering frame collections for π_g on H, we will say that X and Y are *complementary* if $X \cup Y$ is a complete wandering frame collection for π_g on H.

Our purpose here is to describe the relationship between a wandering frame collection and a complete wandering frame collection. Our results here can be considered as generalizations of Robertson's theorem [16], which describes the relationship between a wandering subspace and a complete wandering subspace. The proof extends an orthonormal basis for a wandering subspace to an orthonormal basis for a complete wandering subspace. If the dimension of the complete wandering subspace is finite, then this procedure is exact in the sense that the number of wandering vectors that is required is precisely the difference of the two dimensions. Robertson's theorem is valid if we replace orthonormal bases with Riesz bases [12]; again the procedure is exact. Frames, however, lack this exactness property, so we present two versions.

Note We do not require that the subspaces generated by X and Y to be orthogonal. We shall see that they may be chosen that way, however.

We begin with a technical lemma. The proof appears in [9], but we include it here for completeness.

Lemma 1 *Let $\{x_i\}$ be a frame for the Hilbert space H, let K be a closed subspace, and let P be the projection of H onto K. Then $\{Px_i\}$ is a frame for K. In particular, if $\{x_i\}$ is a normalized tight frame for H, then $\{Px_i\}$ is a normalized tight frame for K.*

Proof For $z \in K$, we have

$$A\|z\|^2 \leq \sum_{i\in I} |\langle z, x_i\rangle|^2 = \sum_{i\in I} |\langle Pz, x_i\rangle|^2 = \sum_{i\in I} |\langle z, Px_i\rangle|^2 \leq B\|z\|^2,$$

from which the lemma follows. ■

Proposition 1 *Suppose that the representation π_g admits a complete wandering frame collection W. Suppose that Y is a wandering frame collection for a subspace. Then there exists a wandering frame collection that is complementary to Y. This complementary collection may be chosen so that the resulting subspaces are orthogonal.*

Proof Let K be the subspace generated by Y, let P be the projection onto $K^{\perp}$, and define $x_i = Pw_i$. It follows from the lemma and the fact that P commutes with the representation that $X = \{x_1, \dots x_n\}$ satisfies the statement. ■

This result is not optimal in the sense that our complementary collection is in general bigger than necessary. Our second result, which can be considered as an extension of Theorem 4 in [10], improves on the first in that our complementary collection Y is smaller. The idea relies on Stone's theorem for unitary representations of abelian groups, and the decomposition of projection valued measures, see [3]. However, we first require this technical result.

Lemma 2 *A representation admits a finite complete wandering frame collection if and only if the representation is unitarily equivalent to a sub-representation of a finite multiple of the regular representation.*

Proof This statement is shown in [9, Theorem 3.11] for any countable group but for vectors which generate a normalized tight frame. We shall show, then, that any representation that admits a complete wandering frame collection also admits a complete wandering frame collection that generates a normalized tight frame.

Given a frame $\{f_n\}$ of a Hilbert space H, define an operator $S\colon H \to H$ given by $Sf = \sum_n \langle f, f_n\rangle f_n$; this is called the frame operator. It is a positive, self-adjoint invertible operator, and the collection $\{S^{-1/2} f_n\}$ is a normalized tight frame for H (see [5]).

Suppose that $\{w_1, \dots, w_n\}$ is a complete wandering frame collection for π_g. We first show that the frame operator S commutes with the representation. Let $f \in H$, $h \in G$ and compute:

$$S\pi_h f = \sum_{i=1}^{n} \sum_{g\in G} \langle \pi_h f, \pi_g w_i\rangle \pi_g w_i = \pi_h \sum_{i=1}^{n} \sum_{g\in G} \langle f, \pi_g w_i\rangle \pi_g w_i = \pi_h S f.$$

Since the commutant of $\pi(G)$ is a von Neumann algebra, it follows that $S^{-1/2}$ commutes with $\pi(G)$. Hence, the collection $\{S^{-1/2}w_1, \dots, S^{-1/2}w_n\}$ is a complete wandering normalized tight frame collection for π. ■

Theorem 2 *Suppose that the representation π_g admits a finite complete wandering frame collection W. Suppose that Y is a wandering frame collection for a subspace. Then there exists an integer k, independent of the cardinality of W and Y, such that:*

1. *there exists a finite wandering frame collection X that is complementary to Y and the cardinality of X is k, and;*
2. *if Z is complementary to Y, its cardinality is at least k.*

Moreover, the frame generated by X can be taken to be normalized tight, regardless of whether the frames generated by W and Y are normalized tight.

Proof A special case of this statement was proven in [6]. Let K be the closed subspace spanned by $\{\pi(G)Y\}$. By combining Proposition 1 and Lemma 2, the representation of G on $K^{\perp}$ is equivalent to a multiple of the regular representation. Then by Stone's theorem and the theory of spectral multiplicity it follows that there exists a unitary operator

$$U\colon K^{\perp} \to \bigoplus_{j=1}^{k} L^2(E_j, \mu),$$

where $\widehat{G} \supset E_1 \supset E_2 \supset \cdots \supset E_k$ and μ is the restriction of Haar measure to E_1, such that U intertwines the projection valued measure on H and the canonical projection valued measure on $\widehat{G}$. Since $\{G\}$ forms an orthonormal basis of $L^2(\widehat{G}, \lambda)$, where λ is Haar measure, the functions $g(\xi)\chi_{E_j}(\xi)$ form a normalized tight frame for $L^2(E_j, \mu)$. It follows that the functions

$${}_g x_j(\xi) = \big(0, \dots, \underbrace{g(\xi)\chi_{E_j}(\xi)}_{j\text{-th position}}, \dots, 0\big)$$

form a normalized tight frame for $UK^{\perp}$. Hence, if we let $X = \{U^{-1}x_j\}$, then X satisfies condition 1 of the theorem.

To establish condition 2, note that the decomposition is unique (up to unitary equivalence), and that the summands are maximal cyclic subspaces, whence it follows that any complete wandering frame collection must have at least k elements in it. ∎

As we stated earlier, the technique in Proposition 1 yields (possibly) more vectors than we would want in practice. On the other hand, the technique in Theorem 2 is not reasonably constructive. So we wish to now demonstrate a technique for reducing the size of the collection from Proposition 1 by combining several vectors into one.

We first need to go back to the decomposition theorem as presented in the proof of Theorem 2. In the theory of the decomposition of projection valued measures, there exists a multiplicity function; in the case of the decomposition given in the proof above, the multiplicity function $m\colon \widehat{G} \to \mathbb{N}$ is given by $m(\xi) = \sum_{j=1}^{n} \chi_{E_j}(\xi)$ a.e. ξ. For our purposes, the multiplicity function will provide us a way of "counting" what parts of the regular representation we have. Lemma 3 follows from the theory.

Lemma 3 *Suppose $K = \overline{\operatorname{span}}\{\pi_g x\}$, whence the representation on K is cyclic. Then the associated multiplicity function is given by χ_E where $E = \{\xi \in \widehat{G} : |\hat{x}(\xi)| > 0\}$. If $X = \{x_1, \dots, x_k\} \in K$, then the multiplicity function associated to the subspace generated by X is χ_F where $F = \big\{ \xi \in \widehat{G} : \max_{i=1,\dots,k}\big(|\hat{x}_i(\xi)|\big) > 0 \big\}$ up to a set of measure* 0.

Our next proposition is a generalization of Theorem 3.4 in [4].

Proposition 2 *Suppose that the representation π_g is cyclic on K. Then the collection $\{w_1, \dots, w_k\}$, where k could be infinite, is a complete wandering frame collection if and only if the following two conditions hold:*

1. *$K \simeq L^2(E, \lambda|_E)$,*
2. *there exists $A, B > 0$ such that for almost every $\xi \in E$,*

$$A \leq \sum_{i=1}^{k} |\hat{w}_i(\xi)|^2 \leq B.$$

Moreover, the frame bounds are given by

$$A = \operatorname{ess\,inf} \sum_{i=1}^{k} |\hat{w}_i(\xi)|^2, \quad B = \operatorname{ess\,sup} \sum_{i=1}^{k} |\hat{w}_i(\xi)|^2.$$

Proof We have established the equivalence of condition 1 to the existence of a complete wandering frame collection (for a cyclic representation). Hence, we shall only consider condition 2. We first show the necessity of the upper bound in condition 2 by contrapositive. Suppose $B > 0$ is given and suppose that $\sum_{i=1}^{k} |\hat{w}_i(\xi)|^2 > B$ for some set $F \subset E$ of positive measure. Then $\hat{w}_i \chi_F \in L^2(E, \lambda)$, and consider the following calculation:

$$\sum_{i=1}^{k} \sum_{g \in G} |\langle \chi_F, \widehat{\pi}_g \hat{w}_i \rangle|^2 = \sum_{i=1}^{k} \left| \int_E \chi_F(\xi) \overline{g(\xi) \hat{w}_i(\xi)} \, d\lambda \right|^2$$

$$= \sum_{i=1}^{k} \|\chi_F \hat{w}_i\|^2,$$

since G forms an orthonormal basis of $L^2(\widehat{G}, \lambda)$. We have:

$$\sum_{i=1}^{k} \|\chi_F \hat{w}_i\|^2 = \sum_{i=1}^{k} \int_{\widehat{G}} |\chi_F(\xi) \hat{w}_i(\xi)|^2 \, d\lambda$$

$$= \int_{\widehat{G}} |\chi_F(\xi)|^2 \sum_{i=1}^{k} |\hat{w}_i(\xi)|^2 \, d\lambda$$

$$> B \int_{\widehat{G}} |\chi_F(\xi)|^2 \, d\lambda = B \|\chi_F\|^2,$$

whence B cannot be an upper frame bound. The necessity of the lower bound in 2 can be shown by an analogous calculation.

Likewise, to establish the sufficiency, let $x \in K$ and consider:

$$\sum_{i=1}^{k} \sum_{g \in G} |\langle x, \pi_g w_i \rangle|^2 = \sum_{i=1}^{k} \sum_{g \in G} |\langle \hat{x}, \widehat{\pi}_g \hat{w}_i \rangle|^2$$

$$= \sum_{i=1}^{k} \int_{\widehat{G}} \hat{x}(\xi) \overline{g(\xi) \hat{w}_i(\xi)} \, d\lambda$$

$$= \sum_{i=1}^{k} \|\hat{x} \hat{w}_i\|^2$$

since, by condition 2, $\hat{x}\hat{w}_i \in L^2(E)$. Moreover, by a calculation similar to above,

$$A \|\hat{x}\|^2 \leq \sum_{i=1}^{k} \|\hat{x} \hat{w}_i\|^2 \leq B \|\hat{x}\|^2.$$

Finally, the frame bounds follow from a calculation analogous to the first calculation above. ■

The idea we present here is to "fuse" two vectors from the wandering frame collection into one. This cannot always be done. When doing so we have two requirements: the first is that the resulting collection is cyclic for the entire space, and the second is that the collection retains frame bounds.

Definition 2 A collection $X = \{x_1, \dots, x_m\} \subset H$ is called a *cyclic collection* if the collection $\{\pi_g x_i\}$ has dense span in H. A cyclic collection X will be called *reducible* if, after an appropriate reordering of X, there exists a $y_1 \in H$ such that $\{x_1, \dots, x_{n-2}, y_1\}$ is also a cyclic collection. We shall say that the vectors x_{n-1} and x_n are *fusable.*

Each element of a cyclic collection generates a cyclic subspace, whence we have the following lemma.

Lemma 4 *A cyclic collection is reducible if and only if two of its vectors are elements of a common cyclic subspace. Equivalently, two vectors of a cyclic collection are fusable if and only if they are elements of a common cyclic subspace.*

Lemma 5 *Let $x, y \in H$, and assume that $|\hat{x}|, |\hat{y}| \in L^\infty(\widehat{G}, \lambda)$. Let $F_1 = \{\xi \in \widehat{G} : \hat{x}(\xi) \neq 0\}$ and $F_2 = \{\xi \in \widehat{G} : \hat{y}(\xi) \neq 0\}$. The two vectors x and y are elements of a common cyclic subspace in H if and only if there exists a scalar valued function $h(\xi) \in L^2(\widehat{G}, \lambda, \mathbb{C})$ such that for a.e. $\xi \in F_1 \cap F_2$, $\hat{y}(\xi) = h(\xi)\hat{x}(\xi)$.*

Proof Let K be the closed subspace generated by x and y; then we have that U restricted to K has the form $UK = L^2(E, \lambda, \mathbb{C}^2)$ for some set E. Consider w defined by $\hat{w} = \hat{x}\chi_{F_1} + \hat{y}\chi_{F_2 \setminus F_1}$. It follows that $w \in K$; let K_0 denote the closed subspace generated by w. A standard argument shows $UK_0 = \{f(\xi)\hat{w}(\xi) : f(\xi) \in L^2(\widehat{G}, \lambda, \mathbb{C})\}$. Therefore, $\chi_{F_1}\hat{w} = \hat{x} \in K_0$; likewise, $[\chi_{F_2 \cap F_1}(\xi)h(\xi) + \chi_{F_2 \setminus F_1}(\xi)]\hat{w}(\xi) \in K_0$. Thus, $K \subset K_0$, which establishes the if part.

The only if part follows immediately. ■

The condition that $|\hat{x}|, |\hat{y}| \in L^\infty(\widehat{G}, \lambda)$ may seem artificial; without it however, we would not have as nice of a characterization of UK_0 above. In light of Proposition 2, however, this is a prerequisite to fusing two frame vectors into one anyway.

Theorem 3 *Suppose the complete wandering frame collection $W = \{w_1, \dots, w_n\}$ for π_g is reducible (as a cyclic collection), with the vectors w_{n-1} and w_n fusable. Then x_1 can be chosen such that the cyclic collection $\widetilde{W} = \{w_1, \dots, w_{n-2}, x_1\}$ is a complete wandering frame collection.*

Proof By Lemma 4, w_{n-1} and w_n are in a common cyclic subspace K. Since the projection of W onto K yields a wandering frame collection for K and K is cyclic, we have that $K \simeq L^2(E, \lambda)$ for some $E \subset \widehat{G}$.

We shall construct x_1 in the following manner: let $F_1 = \{\xi : |\hat{w}_{n-1}(\xi)| \geq |\hat{w}_n(\xi)|\}$ and let $F_2 = E \setminus F_1$. Then define $\hat{x}_1 = \hat{w}_{n-1}\chi_{F_1} + \hat{w}_n\chi_{F_2}$. It follows from Lemmas 3 and 5 that the cyclic subspace generated by x_1 is the same as the subspace generated by w_{n-1} and w_n.

To show that $\widetilde{W}$ generates a frame, we need show that there exist frame bounds $\tilde{A}, \tilde{B}$. Let A, B be the frame bounds for W; we shall show that $\tilde{A} = \frac{A}{2}$ and $\tilde{B} = 2B$

suffice. First suppose that $y \in K$. By Proposition 2, $|\hat{x}_1|^2 \leq B$, whence by the calculation above,

$$\sum_{g \in G} |\langle y, \pi_g x_1 \rangle|^2 = \|\hat{y}\hat{x}_1\|^2$$
$$= \int_{\widehat{G}} |\hat{y}|^2 \, |\hat{x}_1|^2 \, d\lambda$$

and

$$\sum_{g \in G} |\langle y, \pi_g w_{n-1} \rangle|^2 + \sum_{g \in G} |\langle y, \pi_g w_n \rangle|^2 = \|\hat{y}\hat{w}_{n-1}\|^2 + \|\hat{y}\hat{w}_n\|^2$$
$$= \int_{\widehat{G}} |\hat{y}|^2 (|\hat{w}_{n-1}|^2 + |\hat{w}_n|^2) \, d\lambda.$$

By our construction of x_1,

$$\frac{1}{2}(|\hat{w}_{n-1}|^2 + |\hat{w}_n|^2) \leq |\hat{x}_1|^2 \leq (|\hat{w}_{n-1}|^2 + |\hat{w}_n|^2),$$

whence,

$$\frac{1}{2}\Big(\sum_{g \in G} |\langle y, \pi_g w_{n-1} \rangle|^2 + \sum_{g \in G} |\langle y, \pi_g w_n \rangle|^2\Big)$$
$$\leq \sum_{g \in G} |\langle y, \pi_g x_1 \rangle|^2 \leq \sum_{g \in G} |\langle y, \pi_g w_{n-1} \rangle|^2 + \sum_{g \in G} |\langle y, \pi_g w_n \rangle|^2.$$

Now let $y \in H$, P_K be the projection onto K and compute

$$\sum_{i=1}^{n-2} \sum_{g \in G} |\langle y, \pi_g w_i \rangle|^2 + \sum_{g \in G} |\langle y, \pi_g x_1 \rangle|^2$$
$$= \sum_{i=1}^{n-2} \sum_{g \in G} |\langle y, \pi_g w_i \rangle|^2 + \sum_{g \in G} |\langle P_K y, \pi_g x_1 \rangle|^2$$
$$\geq \frac{1}{2}\Big(\sum_{i=1}^{n-2} \sum_{g \in G} |\langle y, \pi_g w_i \rangle|^2 + \sum_{g \in G} |\langle P_K y, \pi_g w_{n-1} \rangle|^2 + \sum_{g \in G} |\langle P_K y, \pi_g w_n \rangle|^2\Big)$$
$$\geq \frac{1}{2} A \|y\|^2.$$

A similar calculation shows that $\widetilde{W}$ also has an upper frame bound of $2B$. ∎

https://doi.org/10.4153/CJM-2002-023-3 Published online by Cambridge University Press

3 Proof of Theorem 1

We shall now prove Theorem 1. Let $\{V_j\}$ be a GFMRA, where W_0 is cyclic. Suppose the vectors $\{\phi_j : j \in J\}$ generate a frame, under the action of $\pi(G)$, in V_0 with lower and upper frame bounds of A and B, respectively. Define the vectors $\{\varphi_{\gamma,j} : \gamma \in \Gamma; j \in J\}$ as in the introduction.

Claim The vectors $\{\varphi_{\gamma,j} : \gamma \in \Gamma; j \in J\}$ generate a frame for W_0 with frame bounds A' and B', where $A \leq A'$ and $B' \leq B$.

Proof Note that by our definition of GFMRA, the subspace W_0 is invariant under the action of $\pi(G)$, whence the projection P_{W_0} commutes with the representation. Let g_γ be as in the introduction. In light of the proof of Proposition 1, it suffices to show that the vectors $\{D\pi_{g_\gamma}\phi_j : \gamma \in \Gamma; j \in J\}$ forms a complete wandering frame collection for V_1. Since D is a unitary operator, we have that $\{D\pi_g\phi_j : g \in G; j \in J\}$ forms a frame for V_1 with the same frame bounds.

Define a homomorphism $\sigma\colon \pi(G) \to \pi(G)$ by $\sigma(\pi_g) = D^*\pi_g D$; recall that the image of σ is the subgroup G_1. We have that

$$\begin{aligned}\bigcup_{\gamma\in\Gamma}\bigcup_{g\in G}\bigcup_{j\in J}\pi_g\{D\pi_{g_\gamma}\phi_j\} &= \bigcup_{\gamma\in\Gamma}\bigcup_{g\in G}\bigcup_{j\in J}\{D\sigma(\pi_g)\pi_{g_\gamma}\phi_j\}\\ &= \bigcup_{j\in J}\{D\pi_g\phi_j : g \in G\}.\end{aligned}$$ ■

By Proposition 2 and Lemma 3, the multiplicity function for the subrepresentation of π on W_0 is χ_E for some $E \subset \widehat{G}$. We wish to fuse all of the $\varphi_{\gamma,j}$ into one vector, a wavelet. Define the sets

$$F_{\gamma,j} = \left\{\xi \in \widehat{G} : |\hat{\varphi}_{\gamma,j}(\xi)|^2 \geq \frac{A}{|\Gamma|2^{j+1}}\right\}$$

where $|\Gamma|$ is the cardinality of Γ, for $\gamma \in \Gamma$, and $j \in \mathbb{N}$.

Claim The measure of $E \cap (\bigcup_{\gamma,j} F_{\gamma,j})$ is 0.

Proof Suppose, by contrapositive, that there exists a set $F \subset E$ of non-zero measure such that $F \cap (\bigcup_{\gamma,j} F_{\gamma,j}) = \varnothing$. For $\xi \in F$ we have that

$$\sum_{\gamma,j}|\hat{\varphi}_{\gamma,j}(\xi)|^2 < |\Gamma|\sum_{j=1}^{\infty}\frac{A}{|\Gamma|2^{j+1}} = A,$$

a violation of Proposition 2. ■

We now wish to "orthogonalize" the sets $F_{\gamma,j}$ in some way. Enumerate the elements of Γ (it doesn't matter how). Define $E_{\gamma_0,1} = F_{\gamma_0,1}$ and recursively define $E_{\gamma_0,j} =$

$F_{\gamma_0,j} \setminus \bigcup_{k<j} E_{\gamma_0,k}$. Let $E_0 = \bigcup E_{\gamma_0,j}$, and likewise define $E_{\gamma_1,j} = F_{\gamma_1,j} \setminus (\bigcup_{k<j} F_{\gamma_1,k} \cup E_0)$, with $E_{\gamma_1,1} = F_{\gamma_1,1} \setminus E_0$. Continue this procedure through all elements of Γ.

We now can construct a wavelet associated with the given GFMRA. Define the functions $f_{\gamma,j}$ by

$$f_{\gamma,j}(\xi) = \frac{\chi_{E_{\gamma,j}}(\xi)}{|\hat{\varphi}_{\gamma,j}(\xi)|}.$$

Note that each $f_{\gamma,j} \in L^2(\widehat{G}) \cap L^\infty(\widehat{G})$ since $|\hat{\varphi}_{\gamma,j}(\xi)|$ is bounded away from 0 on $E_{\gamma,j}$. It follows that $f_{\gamma,j}\varphi_{\gamma,j} \in \widehat{W}_0$. Now define the vector ψ by

$$\hat{\psi}(\xi) = \sum_{\gamma,j} f_{\gamma,j}(\xi)\hat{\varphi}_{\gamma,j}(\xi).$$

Claim The sum above converges in $L^2(\widehat{G})$.

Proof We compute:

$$\begin{aligned}
\|f_{\gamma,j}\hat{\varphi}_{\gamma,j}\|^2 &= \int_{\widehat{G}} |f_{\gamma,j}(\xi)\hat{\varphi}_{\gamma,j}(\xi)|^2 \, d\lambda \\
&= \int_{\widehat{G}} \left| \frac{\chi_{E_{\gamma,j}}(\xi)}{|\hat{\varphi}_{\gamma,j}(\xi)|} \hat{\varphi}_{\gamma,j}(\xi) \right|^2 d\lambda \\
&= \int_{\widehat{G}} \chi_{E_{\gamma,j}}(\xi) \, d\lambda \\
&= \lambda(E_{\gamma,j}).
\end{aligned}$$

Since the $E_{\gamma,j}$'s are disjoint and $\lambda(E) \leq 1$, it follows that the sequence of partial sums is Cauchy, whence the sum converges. ■

We have that $\psi \in W_0$ since each $U^{-1}(f_{\gamma,j}\hat{\varphi}_{\gamma,j}) \in W_0$. Furthermore, by definition, $|\hat{\psi}(\xi)|^2 = 1$ a.e. $\xi \in E$, therefore by Proposition 2, $\{\pi_g\psi : g \in G\}$ is a normalized tight frame for W_0. Whence, from the stucture of a GFMRA, the collection $\{D^n\pi_g\psi : n \in \mathbb{Z};\ g \in G\}$ is a normalized tight frame for H.

Moreover, if the set E above is all of $\widehat{G}$, except for possibly a set of measure 0, then $\{\pi_g\psi : g \in G\}$ is an orthonormal basis for W_0 and thus ψ is an orthonormal wavelet.

4 Special Cases

We now apply our results to classical wavelet theory on $L^2(\mathbb{R})$. The dilation operator D is given by $Df(x) = \sqrt{2}f(2x)$. The group in question is the integers; the representation on $L^2(\mathbb{R})$ given by $\pi_l = T^l$, where $T^l f(x) = f(x - l)$. We shall normalize the Fourier transform on $L^2(\mathbb{R})$ as follows; for $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$

$$\hat{f}(\xi) = \int_{\mathbb{R}} f(x)e^{-2\pi i x\xi} \, dx.$$

For $\phi \in L^2(\mathbb{R})$, define
$$|\vec{\hat{\phi}}(\xi)|^2 = \sum_{l\in\mathbb{Z}} |\hat{\phi}(\xi + l)|^2.$$

In the case of the representation of the integers on $L^2(\mathbb{R})$ given by translations, the multiplicity theory gives that
$$L^2(\mathbb{R}) \simeq L^2\big(\widehat{G}, l^2(\mathbb{Z})\big),$$
where the unitary operator yielding the equivalence is precisely the Fourier transform. Hence, the representation on W_0 is equivalent to $L^2(\widehat{G}, \mathbb{C})$ (via a unitary operator U), and also equivalent to a subspace of $L^2\big(\widehat{G}, l^2(\mathbb{Z})\big)$ (via the Fourier transform), where each "fiber" is one dimensional. For $f \in W_0$ then, we have that
$$|Uf(\xi)|^2 = \sum_{l\in\mathbb{Z}} |\hat{f}(\xi + l)|^2.$$

Moreover, we shall associate $\widehat{G} = S^1$ to $[0, 1)$ in the standard way.

Consider a finite-height GFMRA of $L^2(\mathbb{R})$ $\{V_k\}$ [17], *i.e.*, there exists a wandering frame collection $\{\phi_j : j = 1, \dots, n\} \subset V_0$. This may seem like a considerable restriction, but actually all but the most pathological of wavelets on $L^2(\mathbb{R})$ are associated to a finite-height GFMRA [6], [15].

Our construction from section 3 would give that
$$\hat{\psi}(\xi) = \sum_{i=1,2} \sum_{j=1}^{n} \frac{\chi_{E_{i,j}}(\xi)}{|\vec{\hat{\varphi}}_{i,j}(\xi)|} \hat{\varphi}_{i,j}(\xi).$$

However, here we can be more flexible. For example, for any $0 < \epsilon \leq \frac{1}{2n}$, define the sets $F^{\epsilon}_{i,j} = \{\xi \in \mathbb{R} : |\vec{\hat{\varphi}}_{i,j}(\xi)|^2 \geq \epsilon\}$. Then define the sets $E^{\epsilon}_{i,j}$ as above, and consider the vector ψ_ϵ given by
$$\hat{\psi}(\xi) = \sum_{i=1,2} \sum_{j=1}^{n} \chi_{E^{\epsilon}_{i,j}(\xi)} \hat{\varphi}_{i,j}(\xi).$$

The ψ_ϵ above is a frame wavelet (but not a normalized tight frame wavelet). Since the number of scaling vectors is finite, we do not need to normalize as we go along; we have that $|\vec{\hat{\psi}}(\xi)|^2 \geq \epsilon$. Note that if we did normalize, we would get a normalized tight frame wavelet, or possibly an orthonormal wavelet.

We actually have three degrees of freedom in our construction. The first is by altering the threshold in computing the sets $F_{i,j}$ as illustrated above. The second is that we could permute the $F_{i,j}$'s before orthogonalizing them. The third is to alter our choice of g_0 and g_1 above, the representatives of the cosets of G_1 in G. One may ask then, can all wavelets in W_0, given the ϕ_j's fixed, be generated by appropriately altering these parameters? The answer is no, as we will show in the following example.

It can be shown that given a GFMRA $\{V_k\}$, all wavelets $\eta \in W_0$ can be written as $\hat{\eta} = g\hat{\psi}$ for a fixed wavelet $\psi \in W_0$ and a 1-periodic unimodular function g. Recall that a MRA is just as a GFMRA, but that there exists a single scaling function $\phi \in V_0$.

Example 1 Consider the MRA of $L^2(\mathbb{R})$ given by $\widehat{V}_k = L^2\big(2^k[-\frac{1}{2},\frac{1}{2})\big)$. This is the MRA associated to the Shannon wavelet set $W = [-1,-\frac{1}{2}) \cup [\frac{1}{2},1)$. An easy computation shows that $\hat{\phi} = \chi_{[-\frac{1}{2},\frac{1}{2})}$ is a scaling function for this MRA, and that $\widehat{W}_0 = L^2\big([-1,-\frac{1}{2}) \cup [\frac{1}{2},1)\big)$. Additionally, we have

$$\hat{\varphi}_0(\xi) = \frac{1}{\sqrt{2}}\chi_{[-1,-\frac{1}{2})\cup[\frac{1}{2},1)}(\xi); \quad \hat{\varphi}_1 = \frac{1}{\sqrt{2}}e^{-i\frac{\xi}{2}}\chi_{[-1,-\frac{1}{2})\cup[\frac{1}{2},1)}(\xi),$$

where we have chosen $g_0 = 0$ and $g_1 = 1$. For any $0 < \epsilon \leq \frac{1}{2}$, we have $F_0 = F_1 = \widehat{G}$. Thus, depending upon the ordering of the F's, we get that $\hat{\psi}(\xi) = \chi_{[-1,-\frac{1}{2})\cup[\frac{1}{2},1)}(\xi)$ or $\hat{\psi}(\xi) = e^{-i\frac{\xi}{2}}\chi_{[-1,-\frac{1}{2})\cup[\frac{1}{2},1)}(\xi)$. Clearly not all wavelets in W_0 can be generated in this fashion.

We wish to consider one other special case, which results in a remarkably simple construction. Notice that in Example 1 above, $\hat{\psi} = \sqrt{2}\hat{\varphi}_0$ is a wavelet; the reason is because the MRA comes from a MSF wavelet [11]. However, this is true more generally than for just MSF wavelets. Define the operator $T_{1/2}$ on $L^2(\mathbb{R})$ by $T_{1/2}f(x) = f(x-1/2)$. We have a much stronger result if the subspace V_0 reduces $T_{1/2}$. (See [18] for a discussion of when this occurs.)

Proposition 3 *Let ϕ be the scaling vector for a MRA, and suppose that the subspace V_0 reduces $T_{1/2}$. Then the function given by $\psi = \sqrt{2}\varphi_0 = \sqrt{2}P_{W_0}D\phi$ is an orthonormal wavelet.*

Proof The subspace V_0 reduces $T_{1/2}$ if and only if W_0 reduces $T_{1/2}$. If so, then we have

$$T_{1/2}\varphi_0 = T_{1/2}P_{W_0}D\phi = P_{W_0}DT\phi = \varphi_1.$$

Since $\hat{T}_{1/2}$ is a unitary multiplication operator, it follows that

$$|\vec{\hat{\varphi}}_0(\xi)|^2 = |\vec{\hat{\varphi}}_1(\xi)|^2 = \frac{1}{2}.$$

Therefore, $\{T^l\phi_0 : l \in \mathbb{Z}\}$ forms a tight frame for W_0, with frame bound $\frac{1}{2}$. Since the representation on W_0 is equivalent to the regular representation, $\{T^l\varphi_0\}$ forms an orthogonal set. Whence, by normalizing φ_0 by the factor of $\sqrt{2}$, we get an orthonormal basis of W_0, as required. ∎

Acknowledgements We thank Larry Baggett, David Larson, and Manos Papadakis for helpful conversations.

References

[1] L. Baggett, A. Carey, W. Moran and P. Ohring, *General Existence Theorems for Orthonormal Wavelets, An Abstract Approach.* Publ. Res. Inst. Math. Sci. (1) **31**(1995), 95–111.

[2] L. Baggett, J. Courter and K. Merrill, *Construction of Wavelets from Generalized Conjugate Mirror Filters.* Preprint, 2000.

[3] L. Baggett, H. Medina and K. Merrill, *Generalized Multiresolution Analyses, and a Construction Procedure for All Wavelet Sets in* $\mathbb{R}^n$. J. Fourier Anal. Appl. (6) **5**(1999), 563–573.
[4] J. Benedetto and S. Li, *The Theory of Multiresolution Analysis Frames and Applications to Filter Banks*. Appl. Comput. Harmon. Anal. (4) **5**(1998), 389–427.
[5] P. Casazza, *The Art of Frame Theory.* Taiwanese J. Math. (2) **4**(2000), 129–201.
[6] J. Courter, *Construction of Dilation-d Orthonormal Wavelets.* Ph.D. thesis, University of Colorado, Boulder, 1999.
[7] I. Daubechies, *Orthonormal bases of compactly supported wavelets.* Comm. Pure Appl. Math. (7) **41**(1988), 909–996.
[8] P. Halmos, *Introduction to Hilbert Space and the Theory of Spectral Multiplicity.* Chelsea, New York, 1957.
[9] D. Han and D. Larson, *Frames, Bases and Group Representations.* Mem. Amer. Math. Soc. (697) **147**(2000).
[10] D. Han, D. Larson, M. Papadakis and T. Stavropoulos, *Multiresolution Analyses of Abstract Hilbert Spaces and Wandering Subspaces.* In: The Functional and Harmonic Analysis of Wavelets and Frames, Contemp. Math. **247**(1999), 259–284.
[11] E. Hernandez and G. Weiss, *An Introduction to Wavelets.* CRC Press, Boca Raton, 1998.
[12] S. Lee, H. Tan and W. Tang, *Wavelet Bases for a Unitary Operator.* Proc. Edinburgh Math. Soc. **38**(1995), 233–260.
[13] S. Mallat, *Multiresolution Approximations and Wavelet Orthonormal Bases of* $L^2(\mathbb{R})$. Trans. Amer. Math. Soc. (1) **315**(1989), 69–87.
[14] M. Papadakis, *Generalized Frame MRAs of Abstract Hilbert Spaces.* Preprint, 2000.
[15] ______, *On the dimension function of a wavelet.* Proc. Amer. Math. Soc. (7) **128**(2000), 2043–2049.
[16] J. Robertson, *On Wandering Subspaces for Unitary Operators.* Proc. Amer. Math. Soc. **16**(1965), 233–236.
[17] S. Schaffer, *Generalized Multiresolution Analyses and Applications of their Multiplicity Functions to Wavelets.* Ph.D. thesis, University of Colorado, Boulder, 2000.
[18] E. Weber, *On the translation invariance of wavelet subspaces.* J. Fourier Anal. Appl. (5) **6**(2000), 551–558.

Department of Mathematics
Texas A&M University
College Station, TX 77843-3368
USA
email: weber@math.tamu.edu