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SOME PROPERTIES OF EQUATIONS IN INTEGERS

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1. **Introduction.** In certain boundary value problems associated with two-parameter ordinary differential equations defined and having $2p$, $(p > 1)$, turning points in a given interval, there arises certain equations in integers whose solutions determine the coefficients in the asymptotic expansions for the eigenvalues [1, 2, 3, pp. 134–139].

As an example consider the system discussed in [1]; we have here the differential equation in the two parameters λ and μ, $y''(x)+(\lambda+\mu a(x)+q(x))y(x)=0$, $0\leq x\leq 1$, $'=d/dx$, together with a pair of linear, homogeneous boundary conditions, and where in [0, 1] $a(x)$ and $q(x)$ are real-valued continuous functions, $a(x)\in C_4$ and attains its absolute maximum in [0, 1] at the points $\{h_i\}^p_{=1}$, $0<h_1<\cdots<h_p<1$, $p>1$, with $a''(h_i)<0$, $i=1,\ldots,p$. For fixed integer $m\geq 0$, let $\lambda_m(\mu)$ denote the m^{th} eigenvalue of our system; then we have shown in [1] that as $\mu\to\infty$, $\lambda_m(\mu)=\mu[B_0(r,n)+B_1(r,n)\mu^{-1/2}+B_2(r,n)\mu^{-1}+o(\mu^{-1})]$, for some integer tuple (r,n), and where $B_i(r,n)=B_i(a^{(0)}(h_r),\ldots,a^{(4)}(h_r),n)$, $i=0,1,2$, and $a^{(j)}(h_r)=d^ja(h_r)/dx^j$, $j=0,\ldots,4$. Hence in order to deduce the coefficients in the asymptotic formula for $\lambda_m(\mu)$, it remains to determine the tuple (r,n). To this end we put $A=\sup a(x)$ in [0, 1], and for $i=1,\ldots,p$, $a_i=-a^{(2)}(h_i)/2$, $\nu_i(\mu)=[(4\mu a_i)^{-1/2}(\lambda_m(\mu)+\mu A)-\frac{1}{2}]$, $\mu>0$, and for μ sufficiently large we approximate an eigenfunction corresponding to $\lambda_m(\mu)$ in the neighbourhood of h_i by means of the parabolic cylinder function $D_{\nu_i(\mu)}(s_i)$, $s_i=(4\mu h_i)^{1/4}(x-h_i)$. It can then be shown that $\nu_i(\mu)$ tends to a finite limit, say ν_i, as $\mu\to\infty$, $-\frac{1}{2}<\nu_i$, $i=1,\ldots,p$, and at least one such limit is an integer. If precisely one such limit is an integer then we must have $\nu_r=n$; and if $g(\nu)$ denotes the number of real zeros of $D_\nu(s)$, [4, p. 126], then $g(n)+\sum_{i=1}^{\prime p} g(\nu_i)=m$, $'$ implies $i\neq r$. Since $(a_i)^{1/2}(\nu_i+\frac{1}{2})=(a_r)^{1/2}(n+\frac{1}{2})$, $i=1,\ldots,p$, we see that the tuple (r,n) must be chosen as to render soluble the equation in integers $f_r(n)=m$, where $f_r(n)=g(n)+\sum_{i=1}^{\prime p} g((a_r/a_i)^{1/2}(n+\frac{1}{2})-\frac{1}{2})$. But then one may ask whether there is a tuple (r,n) such that $f_r(n)=m$, or if there is, is it unique? It is precisely these questions which are discussed in the sequel; and for further discussion and application of these and similar results to our two-parameter eigenvalue problem we again refer to [1].

2. **Equations in integers.** Let $\{a_i\}_{i=1}^p$, $p\geq 2$, be a set of p positive numbers. For $r,s=1,\ldots,p$, and $x\geq 0$, let $A_{r,s}(x)$ denote the greatest positive integer less than

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$(x(a_r/a_s)^{1/2}+\frac{1}{2})$ or zero if such a positive integer does not exist; (here and in the sequel the positive square root is always assumed). Let us denote by R the subset of the rationals consisting of all numbers of the form $((2k+1)/(2q+1))^2$, k, q integers; and for nonnegative integer n put:

$$(1) \qquad f_r(n) = n+\sum_{\substack{s=1\\ s\neq r}}^{p} A_{r,s}(n+\tfrac{1}{2}), \qquad r = 1,\ldots,p;$$

then we shall prove the following theorem.

THEOREM 1. *If* $(a_i/a_j)\notin R$, $i,j=1,\ldots,p$, $i\neq j$, *and* m *is any nonnegative integer, then there is an* r_0 *and an* n_0 *such that* $f_{r_0}(n_0)=m$. *The tuple* (r_0, n_0) *is unique.*

First for simplicity of notation, let us put $b(i)=(a_i)^{1/2}$, $i=1,\ldots,p$, and $b(i,j)=(a_i/a_j)^{1/2}$, $i,j=1,\ldots,p$. Then before proving Theorem 1, let us observe a case where the hypothesis of this theorem is violated. Put $p=2$, $a_1=1$, $a_2=9$; then $f_1(0)=0$, $f_1(1)=1$, $f_1(n)\geq 3$ for $n\geq 2$, $f_2(0)=1$, $f_2(1)=5$, $f_2(n)\geq 9$ for $n\geq 2$. Hence $f_r(n)=m$ is (I) uniquely soluble if $m=0$, (II) soluble, but not uniquely if $m=1$, (III) not soluble if $m=2$.

Now with the assumption that $(a_i/a_j)\notin R$, $i,j=1,\ldots,p$, $i\neq j$, it is clear that without loss of generality we may assume $a_1<a_2<\cdots<a_p$. Under both these assumptions then, let us first prove the following lemma and then Theorem 1.

LEMMA 1. *If* $f_i(n_1)=f_j(n_2)$, *then* $i=j$ *and* $n_1=n_2$.

Proof. Under our hypotheses we see that for any integer $n\geq 0$, $n\leq f_1(n)\leq np$, $f_r(n+1)\geq f_r(n)+1$, $r=1,\ldots,p$, and $f_{r+1}(n)\geq f_r(n)+1$, $r=1,\ldots,(p-1)$; and clearly our lemma is true if $i=j$. Now let us assume $f_i(n_1)=f_j(n_2)$ for $i<j$, say; then $n_2<n_1$, and from equation (1) we have

$$(2) \qquad \sum_{\substack{s=1\\ s\neq i,j}}^{p} [A_{j,s}(n_2+\tfrac{1}{2})-A_{i,s}(n_1+\tfrac{1}{2})]+A_{j,i}(n_2+\tfrac{1}{2})-A_{i,j}(n_1+\tfrac{1}{2}) = n_1-n_2$$

If $b(j)(n_2+\frac{1}{2})>b(i)(n_1+\frac{1}{2})$, then $A_{j,i}(n_2+\frac{1}{2})\geq n_1+1$, $A_{i,j}(n_1+\frac{1}{2})\leq n_2$, and $A_{j,s}(n_2+\frac{1}{2})\geq A_{i,s}(n_1+\frac{1}{2})$, $s=1,\ldots,p$, $s\neq i,j$; and hence the left hand side of (2) is not less than n_1-n_2+1, which is a contradiction. Similarly if $b(j)(n_2+\frac{1}{2})<b(i)(n_1+\frac{1}{2})$, then the left hand side of (2) is not greater than n_1-n_2-1, which again is a contradiction; and this completes the proof of our lemma.

Proof of Theorem 1. First we note that Lemma 1 proves the uniqueness part of our theorem; then since $f_1(0)=0$, our theorem is true for $m=0$. Let us now assume $m\geq 1$; and since $f_1(m)\geq m$, we see that our theorem is proved once we show that the set of integers $\{r\}_{r=0}^{f_1(m)}$ is contained in the set $\{f_r(n) \mid n=0,\ldots,m, r=1,\ldots,p\}$.

Now we observe that if $b(1,2)(m+\frac{1}{2})<\frac{1}{2}$, then $f_1(n)=n$, $n=0,\ldots,m$, and hence our theorem is true. So let us then suppose that for some ν, $2\leq\nu\leq p$

$b(1, \nu)(m+\frac{1}{2})>\frac{1}{2}$, and if $\nu<p$, $b(1, \nu+1)(m+\frac{1}{2})<\frac{1}{2}$. Then for $i=2, \ldots, \nu$, let us introduce the positive integers $n(i,j)$, $j=1, 2, \ldots$, with the property that $b(1, i)(n(i,j)+\frac{1}{2})>(j-\frac{1}{2})$, $b(1, i)(n(i,j)-\frac{1}{2})<(j-\frac{1}{2})$, and where $n(i, 1)<n(i, 2)< \ldots, n(i, m_i)\leq m$, $n(i, m_i+1)>m$, $m_i\geq 1$. We observe that $1\leq n(2, 1)\leq n(3, 1)\leq \cdots \leq n(\nu, 1)$, and $m\geq m_2\geq m_3\geq \cdots \geq m_\nu$; and for $i=2, \ldots, \nu$ and $j\geq 1$, we have

$$(3) \qquad (n(i,j)-\tfrac{1}{2})/(j-\tfrac{1}{2}) < b(i, 1) < (n(i,j)+\tfrac{1}{2})/(j-\tfrac{1}{2}).$$

Let us remark now that if $\nu<p$, then,

(I) $A_{1,s}(n+\frac{1}{2})=0$, $n=0, \ldots, m$, $s=\nu+1, \ldots, p$, and,
(II) for $i=2, \ldots, \nu$, $A_{i,s}(n+\frac{1}{2})=0$, $n=0, \ldots, (m_i-1)$, $s=\nu+1, \ldots, p$.

Statement (I) follows from our definition of ν, and statement (II) from the fact that for $i=2, \ldots, \nu$,

$$b(i, \nu+1)(m_i-\tfrac{1}{2}) = b(1, \nu+1)b(i, 1)(m_i-\tfrac{1}{2}) < b(1, \nu+1)(n(i, m_i)+\tfrac{1}{2})$$
$$\leq b(1, \nu+1)(m+\tfrac{1}{2}) < \tfrac{1}{2},$$

using equation (3).

We now conclude that, (I) $f_1(m)=m+\sum_{=2}^{\nu} m_i$ and, (II) for $i=2, \ldots, \nu$, $A_{i,1}(m_i-\frac{1}{2})=n(i, m_i)$. Statement (I) follows from above and statement (II) from equation (3) if j there is replaced by m_i, $i=2, \ldots, \nu$. Hence if $\nu=2$, $f_2(m_2-1)= m_2-1+n(2, m_2)\leq m+m_2-1<f_1(m)$; and since the $(f_1(m)+1)$ elements of the set $\{f_r(n) \mid n=0, \ldots, (m_r-1), r=1, 2, m_1=m+1\}$ are all distinct and each does not exceed $f_1(m)$, then it is clear that these elements are precisely the integers $\{r\}_{r=0}^{f_1(m)}$ and so our theorem follows for this case.

Therefore let us assume $\nu>2$; we shall now show that if $2\leq i, j\leq \nu$ and $i\neq j$ then $A_{i,j}(m_i-\frac{1}{2})\leq m_j$ and $A_{j,i}(m_j-\frac{1}{2})\leq m_i$. To this end select the integers k, q so that $2\leq k, q\leq \nu$, $k\neq q$; and fix the integer $s\geq 1$ so that $n(q, s)\geq n(k, 1)$ and denote by $n(k, r)$ the largest number from the set $\{n(k,j)\}_{j=1}^{\infty}$ not exceeding $n(q, s)$. Then $b(1, q)<(s+\frac{1}{2})/(n(k, r)+\frac{1}{2})$ and $b(1, k)<(r+\frac{1}{2})/(n(q, s)+\frac{1}{2})$. Also $b(k, 1)< (n(k, r)+\frac{1}{2})/(r-\frac{1}{2})$ and $b(q, 1)<(n(q, s)+\frac{1}{2})/(s-\frac{1}{2})$, as seen from equation (3); thus we conclude

$$(4) \qquad b(k, q)(r-\tfrac{1}{2}) < (s+\tfrac{1}{2}), \quad \text{and} \quad b(q, k)(s-\tfrac{1}{2}) < (r+\tfrac{1}{2}).$$

Hence going back to the first statement of this paragraph we see that if $n(i, m_i)\leq n(j, m_j)$ then our result follows from equation (4) if we put $k=i$, $r=m_i$, $q=j$, and $s=m_j$; while if $n(j, m_j)<n(i, m_i)$ then our result again follows if we put $k=j$, $r=m_j$, $q=i$, and $s=m_i$.

Thus we see that if $\nu>2$, then for $i=2, \ldots, \nu$,

$$f_i(m_i-1) \leq m_i-1+n(i, m_i)+\sum_{\substack{j=2 \\ j\neq i}}^{\nu} m_j \leq m-1+\sum_{j=2}^{\nu} m_j < f_1(m);$$

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and since the $(f_1(m)+1)$ elements of the set $\{f_r(n) \mid n=0, \ldots, (m_r-1), r=1, \ldots, \nu, m_1=m+1\}$ are all distinct and each does not exceed $f_1(m)$, then it is clear that they are precisely the integers $\{r\}_{r=0}^{f_1(m)}$. This completes the proof of our theorem.

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