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On Zindler Curves in Normed Planes

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Abstract. We extend the notion of Zindler curve from the Euclidean plane to normed planes. A characterization of Zindler curves for general normed planes is given, and the relation between Zindler curves and curves of constant area-halving distances in such planes is discussed.

1 Introduction

Let C be a rectifiable simple closed curve in the Euclidean plane. A pair of points $p, q \in C$ is said to be a *halving pair* of C if the length of each part of C connecting p and q is one half of the perimeter of C, and the distance between a halving pair is called the corresponding *halving distance*. In particular, such a curve C is said to be a *Zindler curve* (see [14]) if it is of constant halving distance. In the Euclidean case, Zindler curves are not necessarily circles, have many interesting characterizations, and are strongly related to other concepts, such as *curves of constant area-halving distance* (defined in our final section) and *curves of constant width* (see Section 2 of the survey [7]). Also, Zindler curves are related to the construction of graphs of low geometric dilation; see [2–4] for investigations of the geometric dilation problem in the Euclidean plane, and [11] for the extension of this problem to normed planes. The aim of this paper is to extend the notion of Zindler curve from the Euclidean plane to normed planes and to give a characterization of Zindler curves for general normed planes. In this framework, we will also discuss the relation between Zindler curves and closed curves of constant area-halving distances.

By X we denote a (*normed* or) *Minkowski plane* with *origin* o, *norm* $\|\cdot\|$, *unit disc* B_X (which is a compact, convex set with non-empty interior centered at its interior point o) and its boundary, the *unit circle* S_X of X. We refer to [7–9, 13] for more information about the geometry of Minkowski planes and spaces. Any homothet of S_X is said to be a *circle* in X. By $[p, q]$ we denote the *segment* (possibly degenerate) between two points $p, q \in X$, for $p \neq q$, by $[p, q\rangle$ the *ray* with starting point p passing through q, and by $\langle p, q\rangle$ the *line* passing through p and q. The *convex hull* of a set S is denoted by conv(S). Let $x, y \in X$. We say that x is *Birkhoff orthogonal* to y if $\|x + ty\| \geq \|x\|$ holds for any real number t, and this situation is denoted by $x \perp_B y$

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(see [1, 6]); x is said to be (*James* or) *isosceles orthogonal* to y if $\|x+y\| = \|x-y\|$ holds, and for this case we write $x \perp_I y$ (see [5]).

By a *curve* in X we mean the range of a continuous function ϕ that maps a closed bounded interval $[\alpha, \beta]$ into X. Furthermore, a curve defined by $\phi\colon [\alpha, \beta] \mapsto X$ is called *closed* if $[\alpha, \beta]$ is replaced by a Euclidean circle, say, and it is *simple* if it has no self-intersections. Moreover, such a curve C is said to be *rectifiable* if the set of all Riemann sums

$$\Big\{ \sum_{i=1}^{n} \|\phi(\gamma_i) - \phi(\gamma_{i-1})\| : (\gamma_0, \gamma_1, \dots, t_n) \text{ is a partition of } [\alpha, \beta] \Big\}$$

with respect to the norm $\|\cdot\|$ of X is bounded from above. If C is rectifiable, then we denote by $|C|$ its length, *i.e.*,

$$|C| := \sup\Big\{ \sum_{i=1}^{n} \|\phi(\gamma_i) - \phi(\gamma_{i-1})\| : (\gamma_0, \gamma_1, \dots, \gamma_n) \text{ is a partition of } [\alpha, \beta] \Big\}.$$

A parametrization c of C is said to be *regular* if it has non-vanishing one-sided derivatives everywhere, and the curve C is said to be *regular* if it admits a regular parametrization. We denote by $\mathcal{C}$ the set of simple, regular, rectifiable, closed curves that are piecewise continuously differentiable and have one-sided derivatives. Throughout this paper each curve C that we will consider is a curve in an arbitrary Minkowski plane X, and it is either from the family $\mathcal{C}$ or a closed convex curve, *i.e.,* the boundary of a compact, convex set with non-empty interior. Let C be such a curve. Two points $p, q \in C$ are said to form a *(Minkowskian) halving pair* of C if they split C regarding its Minkowskian length into two equal parts, and the distance $\|p - q\|$ between them is called the corresponding *(Minkowskian) halving distance*. A closed curve is said to be a *(Minkowskian) Zindler curve* if it is of constant halving distance. A parametrization $c\colon [0, |C|) \to C$ of C is said to be a *halving pair parametrization* if every pair of points $c(\gamma)$ and $c(\gamma + \frac{1}{2}|C|))$ is a halving pair of the curve C.

Let $c\colon [0, |C|) \to C$ be a parametrization of a closed curve C. The curve M_c (which is called the *midpoint curve*) and the curve C^*_c corresponding to C with respect to c are defined by the parametrizations

$$m_c(\gamma) := \frac{1}{2}\Big(c(\gamma) + c(\gamma + \frac{1}{2}|C|)\Big) \quad \text{and} \quad c^*_c(\gamma) := \frac{1}{2}\Big(c(\gamma) - c(\gamma + \frac{1}{2}|C|)\Big),$$

respectively. It is clear that the curve C^*_c is centrally symmetric and that if c is piecewise continuously differentiable, then both parametrizations are piecewise continuously differentiable.

We shall frequently use the *arc-length parametrization* $\bar{c}\colon [0, |C|) \to C$ of a rectifiable closed curve C, which is continuous, bijective, and has the property that $\|\dot{c}(\gamma)\| = 1$ whenever the derivative exists. As in the Euclidean case, one can prove that every curve $C \in \mathcal{C}$ in an arbitrary Minkowski plane admits an arc-length parametrization which is piecewise continuously differentiable.

2 A Characterization of Zindler Curves

First we prove a lemma showing the relation between isosceles orthogonality and halving pair parametrizations.

Lemma 2.1 *Let $c\colon [0,|C|) \to C$ be a piecewise continuously differentiable parametrization of a closed curve $C \in \mathcal{C}$. The parametrization c is a halving pair parametrization of C if and only if $\dot{c}_c^*(\gamma) \perp_I \dot{m}_c(\gamma)$ whenever the derivatives exist.*

Proof Clearly, c is a halving pair parametrization of C if and only if

$$f(\gamma) := \int_{\gamma}^{\gamma+\frac{1}{2}|C|} \|\dot{c}(\tau)\| \, d\tau \equiv \frac{1}{2}|C|,$$

which is, whenever the derivatives exist, equivalent to the situation that the condition

$$\begin{aligned} f'(\gamma) &= \Big\| \dot{c}\Big(\gamma + \frac{1}{2}|C|\Big) \Big\| - \|\dot{c}(\gamma)\| \\ &= \|\dot{m}_c(\gamma) - \dot{c}_c^*(\gamma)\| - \|\dot{m}_c(\gamma) + \dot{c}_c^*(\gamma)\| \\ &= 0 \end{aligned}$$

holds. Hence c is a halving pair parametrization of C if and only if $\dot{c}_c^*(\gamma) \perp_I \dot{m}_c(\gamma)$ whenever the derivatives exist. ■

We note that Lemma 2.1 is also true when C is a closed convex curve.

Let K be a *convex body* (*i.e.*, a compact, convex set with non-empty interior) in X containing o as interior point. For $x, y \in X$, x is said to be *K-normal* to y ($x \perp_K y$) if $\|x\| \, \|y\| = 0$ or there exists a line which is parallel to the line $\langle -y, y \rangle$ and supports K at the point of intersection of the ray $[o, x\rangle$ and ∂K (the *boundary* of K). It is clear that the implication $x \perp_K y \Rightarrow \beta x \perp_K \alpha y$ holds for any $\beta \geq 0$ and $\alpha \in \mathbb{R}$, and that $x \not\perp_K x$ unless $x = o$. Let $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ be two sequences in X. If $x_n \perp_K y_n$ holds for each n, then, since K is convex, $\lim_{n\to+\infty} x_n \perp_K \lim_{n\to+\infty} y_n$ whenever the limits exist. Also, it follows from the convexity of K that for every $x \in S_X$ with a countable set of exceptions, there exists a unique $y \in S_X$ (up to the sign) such that $x \perp_K y$.

A curve $C \in \mathcal{C}$ is said to be a *K-curve* if

- the bounded region enclosed by C contains o as interior point,
- there exists an arc-length parametrization $\bar{c}\colon [0,|C|) \to C$ of C with $\bar{c}(\gamma) \perp_K \dot{\bar{c}}(\gamma)$ holding for any $\gamma \in [0,|C|)$ whenever the derivative exists.

Note that if such an arc-length parametrization exists, then the above holds for all such parametrizations.

Theorem 2.2 *For any convex body K containing o as interior point and any K-curve C there exists a real number $\alpha > 0$ such that $\alpha C = \partial K$.*

To prove Theorem 2.2, we need the following lemma.

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Lemma 2.3 *Let K be a convex body containing o as interior point, and C be a K-curve with an arc-length parametrization $\bar{c}\colon [0,|C|) \to C$ which is piecewise continuously differentiable. Then the ray $[o,\bar{c}(\gamma)\rangle$ intersects C only once for any $\gamma \in [0,|C|)$.*

Proof First we show that for any $\gamma \in [0,|C|)$, the ray $[o,\bar{c}(\gamma)\rangle$ intersects C at most finitely many times. Suppose the contrary, namely, that there exists a number $\mu_0 \in [0,|C|)$ such that the ray $[o,\bar{c}(\mu_0)\rangle$ intersects C infinitely many times. Then we can obtain an infinite series $\{\gamma_n\}_{n=1}^{\infty} \subset [0,|C|)$ such that $[o,\bar{c}(\mu_0)\rangle = [o,\bar{c}(\gamma_n)\rangle$ holds for each n and that $\bar{c}(\gamma_0') = \lim_{n\to\infty} \bar{c}(\gamma_n)$ exists. Assume that, without loss of generality, there exists an infinite convergent subsequence $\{\gamma_{n_k}\}_{k=1}^{\infty} \subset \{\gamma_n\}_{n=1}^{\infty} \cap (\gamma_0',|C|)$. Since the one-sided derivatives of $\bar{c}$ both exist at γ_0', we have

$$\dot{\bar{c}}(\gamma_0'+) := \lim_{\gamma\to\gamma_0'^{+}} \frac{\bar{c}(\gamma)-\bar{c}(\gamma_0')}{\gamma-\gamma_0'} = \lim_{k\to\infty} \frac{\bar{c}(\gamma_{n_k})-\bar{c}(\gamma_0')}{\gamma_{n_k}-\gamma_0'}.$$

We note that $\|\dot{\bar{c}}(\gamma_0'+)\| \neq 0$ since $\bar{c}$ is an arc-length parametrization of C which is piecewise continuously differentiable. Since $\dot{\bar{c}}(\gamma_0'+)$ is a non-zero multiple of $\bar{c}(\gamma_0')$, we have that

$$\bar{c}(\gamma_0') = \lim_{\gamma\to\gamma_0'^{+}} \bar{c}(\gamma) \not\perp_K \dot{\bar{c}}(\gamma_0'+) = \lim_{\gamma\to\gamma_0'^{+}} \dot{\bar{c}}(\gamma),$$

which is a contradiction.

Now suppose the contrary, namely that there exists a number $\gamma_0 \in [0,|C|)$ such that the ray $[o,\bar{c}(\gamma_0)\rangle$ intersects C more than once. Let γ_1 and γ_2 be two numbers with $0 \le \gamma_1 < \gamma_2 < |C|$ such that

$$[o,\bar{c}(\gamma_0)\rangle \cap \bar{c}([\gamma_1,|C|)) = \{\bar{c}(\gamma_1),\bar{c}(\gamma_2)\},$$

where

$$\bar{c}([\gamma_1,|C|)) := \{\bar{c}(\gamma) : \gamma_1 \le \gamma < |C|\}.$$

These two numbers exist, since for any $\gamma \in [0,|C|)$ the ray $[o,\bar{c}(\gamma)\rangle$ intersects C at most finitely many times.

Let $\bar{k}\colon [0,|\partial K|) \to \partial K$ be an arc-length parametrization of ∂K such that $\bar{k}(0)$ is a positive multiple of $\bar{c}(\gamma_0)$. Let $f\colon [\gamma_1,\gamma_2] \to [0,|\partial K|)$ be a function mapping each $\gamma \in [\gamma_1,\gamma_2]$ to the number $f(\gamma) \in [0,|\partial K|)$ such that $\bar{k}(f(\gamma))$ is a positive multiple of $\bar{c}(\gamma)$. Since $[o,\bar{c}(\gamma_0)\rangle \cap \bar{c}((\gamma_1,\gamma_2)) = \varnothing$, the function f is continuous and attains its maximum value at a number $\eta_0 \in [\gamma_1,\gamma_2]$. Without loss of generality, we may assume that $f(\eta_0) > 0$, since otherwise the ray $[o,\bar{c}(\gamma_0)\rangle$ has to intersect $\bar{c}([\gamma_1,\gamma_2])$ infinitely many times, which is impossible.

Also we can find a number $\delta_0 > 0$ such that $[\eta_0-\delta_0,\eta_0+\delta_0] \subset [\gamma_1,\gamma_2]$ and that $f(\gamma) < f(\eta_0)$ holds for any number $\gamma \in [\eta_0-\delta_0,\eta_0+\delta_0]\backslash\{\eta_0\}$. Otherwise, for any number $\delta > 0$ there exists a number $\gamma \in [\eta_0-\delta,\eta_0+\delta]\backslash\{\eta_0\}$ such that $f(\gamma) = f(\eta_0)$. Then the ray $[o,\bar{c}(\eta_0)\rangle$ intersects $\bar{c}([\gamma_1,\gamma_2])$ infinitely many times, which is a contradiction.

Note that $\dot{\bar{c}}(\eta_0)$ does not exist. If this were not true, then $\langle -\bar{c}(\eta_0),\bar{c}(\eta_0)\rangle$ would be the unique line tangent to C at $\bar{c}(\eta_0)$, which would imply that $\bar{c}(\eta_0) \not\perp_K \dot{\bar{c}}(\eta_0)$, which is a contradiction. Furthermore, there exist two numbers $0 < \delta_1,\delta_2 \le \delta_0$

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such that for each $\gamma \in [\eta_0 - \delta_1, \eta_0] \subset [\eta_0 - \delta_0, \eta_0 + \delta_0]$, the ray $[o, \bar{c}(\gamma)\rangle$ intersects $\bar{c}([\eta_0 - \delta_1, \eta_0])$ precisely once, and for each $\gamma' \in [\eta_0, \eta_0 + \delta_2] \subset [\eta_0 - \delta_0, \eta_0 + \delta_0]$ the ray $[o, \bar{c}(\gamma')\rangle$ intersects $\bar{c}([\eta_0, \eta_0 + \delta_2])$ only once. If the number δ_1 does not exist, then, for any number $\delta > 0$ with $[\eta_0 - \delta, \eta_0] \subset [\gamma_1, \gamma_2]$, there exist two numbers η_1', $\eta_1'' \in [\eta_0 - \delta, \eta_0]$ with $\eta_1' > \eta_1''$ such that $f(\eta_1') = f(\eta_1'')$. Then, replacing γ_1 and γ_2 in the previous arguments by η_1' and η_1'', respectively, we can find a number $\eta_1 \in [\eta_1', \eta_1'']$ such that $\bar{c}(\eta_1)$ is not a smooth point of C. In the next step we consider the interval $[\eta_1', \eta_0]$ instead of $[\eta_0 - \delta, \eta_0]$, and we can find another number η_2 such that $\bar{c}(\eta_2)$ is not a smooth point of C. As this process goes on, we can find a sequence of non-smooth points of C tending to $\bar{c}(\eta_0)$. This contradicts the fact that C is piecewise continuously differentiable. The existence of the number δ_2 can be proved in a similar way. We may assume that, without loss of generality, $f(\eta_0 - \delta_1) = f(\eta_0 + \delta_2)$. Otherwise, suppose that, without loss of generality, $f(\eta_0 - \delta_1) < f(\eta_0 + \delta_2)$. Since the function f is continuous on $[\gamma_1, \gamma_2]$ and $f(\eta_0) > f(\eta_0 + \delta_2)$, we can choose δ_1 to be the number such that $f(\eta_0 - \delta_1) = f(\eta_0 + \delta_2)$ instead.

Let $C_1 := \bar{c}([\eta_0 - \delta_1, \eta_0])$ and $C_2 := \bar{c}([\eta_0, \eta_0 + \delta_2])$. Now assume that the underlying plane is endowed with Euclidean structure and a system of polar coordinates. Let $\rho = g_i(\theta)$ be the polar equation of C_i, $i = 1, 2$. We may also assume that $g_i(0) = \bar{c}(\eta_0)$, $i = 1, 2$, and that there exists a positive number θ_0 such that $g_1(\theta_0) = \bar{c}(\eta_0 - \delta_1)$ and $g_2(\theta_0) = \bar{c}(\eta_0 + \delta_2)$. Then, with a countable set of exceptions, for every $\theta \in [0, \theta_0]$ the vector $\dot{g}_i(\theta)$, $i = 1, 2$, exists and the tangent directions of C_1 and C_2 corresponding to a given θ are parallel to each other. Consequently,

$$\frac{\dot{g}_1(\theta)}{g_1(\theta)} = \frac{\dot{g}_2(\theta)}{g_2(\theta)}$$

holds for all $\theta \in [0, \theta_0]$ with the same set of exceptions. This implies that

$$\frac{g_1(\theta)}{g_1(0)} = \frac{g_2(\theta)}{g_2(0)}$$

for all $\theta \in [0, \theta_0]$. Since $g_1(0) = g_2(0)$, this yields that $g_1(\theta) = g_2(\theta)$ holds for all $\theta \in [0, \theta_0]$, which contradicts the fact that C is a simple curve. ■

Proof of Theorem 2.2 By Lemma 2.3, the polar equation $\rho = f_1(\theta)$ of C exists. Let $\rho = f_2(\theta)$ be the polar equation of ∂K. Then $\dot{f}_i(\theta)$, $i = 1, 2$, exists for every $\theta \in [0, 2\pi]$ with a countable set of exceptions, and the tangent directions of C and ∂K corresponding to a given θ are parallel to each other. Thus we have

$$\frac{\dot{f}_1(\theta)}{f_1(\theta)} = \frac{\dot{f}_2(\theta)}{f_2(\theta)}$$

for all θ with the same set of exceptions. This implies that

$$\frac{f_1(\theta)}{f_1(0)} = \frac{f_2(\theta)}{f_2(0)}$$

for all θ. Then $\alpha = \frac{f_2(0)}{f_1(0)}$ is the number such that $\alpha C = \partial K$. ■

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Theorem 2.4 *Let $\bar{c}\colon [0,|C|) \to C$ be an arc-length parametrization of C that is piecewise continuously differentiable. Then C is a Zindler curve if and only if $c^*_{\bar{c}}(\gamma) \perp_B \dot{c}^*_{\bar{c}}(\gamma)$ whenever the derivative exists.*

Proof Let $C^* := C^*_{\bar{c}}$, and c^* be the corresponding parametrization of C^*. Then C is a Zindler curve if and only if C^* is a circle centered at o. If C^* is a circle, then the region bounded by C^* is a disc centered at o. From the definition of Birkhoff orthogonality it follows that $c^*(\gamma) \perp_B \dot{c}^*(\gamma)$ holds whenever the derivative exists. Now suppose that $c^*(\gamma) \perp_B \dot{c}^*(\gamma)$ holds whenever the derivative exists. Then it follows from Theorem 2.2 that C^* is a circle. ■

Theorem 2.4 also holds when the curve C is closed and convex. The corresponding proof can be done by applying [12, 4A] instead of Theorem 2.2.

3 Constant Area-Halving Distance

For a Minkowski plane X, there exists a unique (up to a scalar factor) Haar measure on X; see [13, §1.4]. Thus we may assume that the underlying Minkowski plane is endowed with Euclidean structure, and therefore we can use the corresponding Lebesgue measure to calculate Minkowskian areas. In this section we will consider only closed convex curves. A chord of a closed convex curve C bisecting the area of $\operatorname{conv}(C)$ is called an *area-halving chord*, and the length of such a chord is said to be the corresponding *area-halving distance*. Let C be a closed convex curve. A parametrization $c\colon [0,|C|) \to C$ of C is an *area-bisecting parametrization* if every chord $[c(\gamma), c(\gamma+\frac{1}{2}|C|)]$ is an area-halving chord. Clearly, a chord $[c(\gamma), c(\gamma+\frac{1}{2}|C|)]$ bisects the Euclidean area of $\operatorname{conv}(C)$ if and only if it bisects the Minkowskian area of $\operatorname{conv}(C)$. Thus Lemma 3.5 from [4], referring to the Euclidean case, can be carried over to Minkowski planes, and we have the following lemma.

Lemma 3.1 (based on [4, Lemma 3.5]) *Let $c\colon [0,|C|) \to C$ be a piecewise continuously differentiable parametrization of a closed convex curve C in a Minkowski plane. It is an area-bisecting parametrization if and only if $\dot{m}_c(\gamma)$ is parallel to $c^*_c(\gamma)$ whenever the derivatives $\dot{c}(\gamma)$ and $\dot{c}(\gamma+\frac{1}{2}|C|)$ exist.*

The following lemma can be proved in a way analogous to that in the proof of Theorem 2.4.

Lemma 3.2 *Let $c\colon [0,|C|) \to C$ be a piecewise continuously differentiable area-bisecting parametrization of a closed convex curve C. Then C is a curve of constant area-halving distance if and only if $c^*_c(\gamma) \perp_B \dot{c}^*_c(\gamma)$ whenever the derivative exists.*

Lemma 3.1 and Lemma 3.2 imply the following.

Theorem 3.3 *Let $c\colon [0,|C|) \to C$ be a piecewise continuously differentiable area-bisecting parametrization of a closed convex curve C. Then C is a curve of constant area-halving distance if and only if we have $\dot{m}_c(\gamma) \perp_B \dot{c}^*_c(\gamma)$ whenever the derivatives $\dot{c}(\gamma)$ and $\dot{c}(\gamma+\frac{1}{2}|C|)$ exist.*

Zindler [14, Section 7] (see also [2, Theorem 4]) proved that for a closed, convex curve C in the Euclidean plane the following statements are equivalent:

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- All halving chords of C have the same length.
- All chords of C bisecting the area have the same length.
- The halving chords and the area-halving chords of C coincide.

But it is known that Birkhoff orthogonality and isosceles orthogonality are different orthogonality types in non-Euclidean Minkowski planes. Moreover, there exist a Minkowski plane X and two points $x, y \in S_X$ with $x \perp_B y$ such that y is the unique point (up to the sign) in S_X to which x is Birkhoff orthogonal, and that $x \not\perp_I ty$ holds for any number $t \neq 0$ (see [10, Remark 2.10]). Thus it follows from Lemma 2.1 and Lemma 3.1 that a halving pair parametrization of a closed convex curve C of constant halving distance is not necessarily an area-bisecting parametrization of C. Similarly, an area-bisecting parametrization of a closed convex curve of constant area-halving distance is not necessarily a halving pair parametrization of C.

References

[1] G. Birkhoff, *Orthogonality in linear metric spaces*. Duke Math. J. **1**(1935), no. 2, 169–172. http://dx.doi.org/10.1215/S0012-7094-35-00115-6

[2] A. Dumitrescu, A. Ebbers-Baumann, A. Grüne, R. Klein, and G. Rote, *On the geometric dilation of closed curves, graphs, and point sets*. Comput. Geom. **36**(2007), no. 1, 16–38. http://dx.doi.org/10.1016/j.comgeo.2005.07.004

[3] A. Ebbers-Baumann, A. Grüne, and R. Klein, *Geometric dilation of closed planar curves: new lower bounds*. Comput. Geom. **37**(2007), no. 3, 188–208. http://dx.doi.org/10.1016/j.comgeo.2004.12.009

[4] A. Grüne, *Geometric dilation and halving distance*. Ph.D. Dissertation, Universität Bonn. 2006.

[5] R. C. James, *Orthogonality in normed linear spaces*. Duke Math. J. **12**(1945), 291–301. http://dx.doi.org/10.1215/S0012-7094-45-01223-3

[6] ———, *Orthogonality and linear functionals in normed linear spaces*. Trans. Amer. Math. Soc. **61**(1947), 265–292. http://dx.doi.org/10.1090/S0002-9947-1947-0021241-4

[7] H. Martini and K. J. Swanepoel, *The geometry of Minkowski spaces — a survey. II*. Expo. Math. **22**(2004), no. 2, 93–144.

[8] ———, *Antinorms and Radon curves*. Aequationes Math. **72**(2006), no. 1-2, 110–138. http://dx.doi.org/10.1007/s00010-006-2825-y

[9] H. Martini, K. J. Swanepoel, and G. Weiß, *The geometry of Minkowski spaces — a survey. I*. Expo. Math. **19**(2001), no. 2, 97–142.

[10] H. Martini and S. Wu, *Radial projections of bisectors in Minkowski spaces*. Extracta Math. **23**(2008), no. 1, 7–28.

[11] ———, *Geometric dilation of closed curves in normed planes*. Comput. Geom. **42**(2009), no. 4, 315–321. http://dx.doi.org/10.1016/j.comgeo.2008.06.001

[12] J. J. Schäffer, *Geometry of Spheres in Normed Spaces*. Lecture Notes in Pure and Applied Mathematics 20. Marcel Dekker, New York-Basel, 1976.

[13] A. C. Thompson, *Minkowski Geometry*. Encyclopedia of Mathematics and Its Applications 63. Cambridge University Press, Cambridge, 1996.

[14] K. Zindler, *Über konvexe Gebilde*. Monatsh. Math. Phys. **31**(1921), no. 1, 25–56. http://dx.doi.org/10.1007/BF01702711

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