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Abstract

We study some basic properties of schematic homotopy types and the schematization functor. We describe two different algebraic models for schematic homotopy types, namely cosimplicial Hopf alegbras and equivariant cosimplicial algebras, and provide explicit constructions of the schematization functor for each of these models. We also investigate some standard properties of the schematization functor that are helpful for describing the schematization of smooth projective complex varieties. In a companion paper, these results are used in the construction of a non-abelian Hodge structure on the schematic homotopy type of a smooth projective variety.

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1. Introduction

The schematization functor is a device which converts a pair (X, k), consisting of a topological space X and a field k, into an algebraic–geometric object $(X \otimes k)^{\mathrm{sch}}$. The characteristic property

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of $(X \otimes k)^{\mathrm{sch}}$ is that it encodes the k-linear part of the homotopy type of X; that is, $(X \otimes k)^{\mathrm{sch}}$ captures all the information about local systems of k-vector spaces on X and their cohomology. For a simply connected X and $k = \mathbb{Q}$ (respectively, $k = \mathbb{F}_p$), the object $(X \otimes k)^{\mathrm{sch}}$ is a model for the rational (respectively, p-adic) homotopy type of X. An important advantage of $(X \otimes k)^{\mathrm{sch}}$ is that it makes sense for non-simply connected X and detects non-nilpotent information.

The object $(X \otimes k)^{\mathrm{sch}}$ belongs to a special class of algebraic ∞-stacks over k, called schematic homotopy types [Toe06]. The existence and functoriality of the schematization $(X \otimes k)^{\mathrm{sch}}$ have been proven in [Toe06], but the construction is somewhat abstract and unwieldy. In this paper we supplement [Toe06] by describing explicit algebraic models for $(X \otimes k)^{\mathrm{sch}}$. We also study in detail some basic properties of the schematization, the Van Kampen theorem, the schematization of homotopy fibers, de Rham models, etc. These results are used in an essential way in [KPT08], where we construct mixed Hodge structures on schematizations of smooth complex projective varieties.

The paper is organized in three parts. In § 2 we briefly review the definition of schematic homotopy types and existence results for the schematization functor from [Toe06]. In § 3 we present two different algebraic models for the schematization of a space, namely equivariant cosimplicial k-algebras and cosimplicial Hopf k-algebras. These generalize two well-known ways of modelling rational homotopy types: via dg algebras and via nilpotent dg Hopf algebras [Tan83]. Each of the two models utilizes a different facet of the homotopy theory of a space X. The equivariant cosimplicial k-algebras codify the cohomology of X with k-local system coefficients together with their cup-product structure, whereas the cosimplicial Hopf k-algebra is the algebra of representative functions on the simplicial loop group associated with X via Kan's construction [GJ99, § V.5]. The two models have different ranges of applicability; for instance, the cosimplicial Hopf algebra model is needed in constructing the weight tower for the mixed Hodge structure (MHS) on the schematization of a smooth projective variety [KPT08]. On the other hand, the Hodge decomposition on the schematic homotopy type of a smooth projective variety is defined in terms of the equivariant cosimplicial model.

Section 4 gathers some useful facts about the behavior of the schematization functor. As an application of the equivariant cosimplicial algebra model, we describe the schematization of differentiable manifolds in terms of de Rham complexes of flat connections. This description generalizes a theorem of Sullivan [Sul77] which expresses the real homotopy theory of a manifold in terms of its de Rham complex. To facilitate computations, we prove a schematic analogue of the van Kampen theorem, which allows us to build schematizations by gluing schematizations of local pieces. Since in the algebraic–geometric setting we cannot use contractible neighborhoods as building blocks, we are forced to study the schematizations of Artin neighborhoods and, more generally, of $K(\pi, 1)$. This leads to the notion of k-algebraically good groups, which are precisely groups Γ with the property that the schematization of $K(\Gamma, 1)$ has no higher homotopy groups. We give various examples of such groups and prove that the fundamental groups of Artin neighborhoods are algebraically good. Note that an analogous statement in rational homotopy theory is unknown and probably false. Finally, we prove two exactness properties of the schematization functor: we establish a Lefschetz-type right exactness property of schematizations, which will be useful for understanding homotopy types of hyperplane sections; we also give sufficient conditions under which the schematization commutes with taking homotopy fibers. This criterion is used in the construction of new examples [KPT08] of non-Kähler homotopy types.

Recently, a similar approach to some constructions and questions raised in this paper has been proposed by J. P. Pridham. In a sequence of interesting works [Pri08, Pri07, Pri06], Pridham introduced pro-algebraic homotopy theory and studied its structural, Galois and Hodge-theoretic

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properties in a systematic way. Even though Pridham's formalism is, in certain circumstances, equivalent to ours and leads to similar results, it differs in the details and the implementation, and is of independent interest. One of the main differences between the two approaches lies in Pridham's treatment [Pri08] of the weight filtration on the homotopy type, which is much more straightforward and tractable than ours [KPT08]. It would be interesting to continue studying the relationship between the two approaches and probe the boundaries of their applicability.

Conventions: We will fix two universes $\mathbb{U}$ and $\mathbb{V}$, with $\mathbb{U} \in \mathbb{V}$, and assume that $\mathbb{N} \in \mathbb{U}$.

We denote by k a base field in $\mathbb{U}$. We shall consider Aff, the category of affine schemes over $\operatorname{Spec} k$ belonging to $\mathbb{U}$. The category Aff is a $\mathbb{V}$-small category. We endow it with the faithfully flat and quasi-compact topology, and consider the model category $\mathsf{SPr}(k)$ of presheaves of $\mathbb{V}$-simplicial sets on the site $(\mathsf{Aff}, \mathrm{fpqc})$. We will use the local projective model structure on simplicial presheaves described in [Bla01, Toe06] (note that as the site Aff is $\mathbb{V}$-small, the model category $\mathsf{SPr}(k)$ exists). We denote by $\mathsf{SPr}(k)_*$ the model category of pointed objects in $\mathsf{SPr}(k)$. The term *stacks* will always refer to objects in $\mathrm{Ho}(\mathsf{SPr}(k))$. Similarly, *morphisms of stacks* refers to morphisms in $\mathrm{Ho}(\mathsf{SPr}(k))$.

2. Review of the schematization functor

In this section, we review the theory of affine stacks and schematic homotopy types introduced in [Toe06]. The main goal is to recall the theory and fix the notation and terminology.

We will denote by Alg^{Δ} the category of cosimplicial (commutative) k-algebras that belong to the universe $\mathbb{V}$. The category Alg^{Δ} is endowed with a simplicial closed model category structure for which the fibrations are the epimorphisms and the equivalences are the quasi-isomorphisms. This model category is known to be cofibrantly generated, and even finitely generated [Toe06, Theorem 2.1.2].

There is a natural spectrum functor

$$\operatorname{Spec} : (\mathsf{Alg}^{\Delta})^{\mathrm{op}} \longrightarrow \mathsf{SPr}(k)$$

defined by the formula

$$\begin{aligned} \operatorname{Spec} A : \quad & \mathsf{Aff}^{\mathrm{op}} \longrightarrow \mathsf{SSet} \\ & \operatorname{Spec} B \mapsto \underline{\mathrm{Hom}}(A, B). \end{aligned}$$

As usual, $\underline{\mathrm{Hom}}(A, B)$ denotes the simplicial set of morphisms from the cosimplicial algebra A to the algebra B. Explicitly, if A is given by a cosimplicial object $[n] \mapsto A_n$, then the presheaf of n-simplices of $\operatorname{Spec} A$ is given by $(\operatorname{Spec} B) \mapsto \mathrm{Hom}(A_n, B)$.

The functor Spec is a right Quillen functor, and its right derived functor is denoted by

$$\mathbb{R}\operatorname{Spec} : \mathrm{Ho}(\mathsf{Alg}^{\Delta})^{\mathrm{op}} \longrightarrow \mathrm{Ho}(\mathsf{SPr}(k)).$$

The restriction of $\mathbb{R}\operatorname{Spec}$ to the full sub-category of $\mathrm{Ho}(\mathsf{Alg}^{\Delta})$ consisting of objects isomorphic to a cosimplicial algebra in $\mathbb{U}$ is fully faithful (see [Toe06, Corollary 2.2.3]). By definition, an affine stack is an object $F \in \mathrm{Ho}(\mathsf{SPr}(k))$ that is isomorphic to an object of the form $\mathbb{R}\operatorname{Spec} A$, for some cosimplicial algebra A in $\mathbb{U}$. Moreover, by [Toe06, Theorems 2.4.1 and 2.4.5], the following conditions are equivalent for a given pointed stack F.

(i) The pointed stack F is affine and connected.

(ii) The pointed stack F is connected and, for all $i > 0$, the sheaf $\pi_i(F, *)$ is represented by an affine unipotent group scheme.

(iii) There exists a cohomologically connected cosimplicial algebra A (i.e. $H^0(A) \simeq k$) which belongs to $\mathbb{U}$ and is such that $F \simeq \mathbb{R}\,\mathrm{Spec}\,A$.

Next, recall that for a pointed simplicial presheaf F, one can define its simplicial presheaf of loops $\Omega_* F$. The functor $\Omega_* : \mathsf{SPr}_*(k) \longrightarrow \mathsf{SPr}(k)$ is right Quillen, and can be derived to a functor defined on the level of homotopy categories,

$$\mathbb{R}\Omega_* F : \mathrm{Ho}(\mathsf{SPr}_*(k)) \longrightarrow \mathrm{Ho}(\mathsf{SPr}(k)).$$

A pointed and connected stack $F \in \mathrm{Ho}(\mathsf{SPr}_*(k))$ is called a *pointed affine ∞-gerbe* if the loop stack $\mathbb{R}\Omega_* F \in \mathrm{Ho}(\mathsf{SPr}(k))$ is affine. A pointed schematic homotopy type is a pointed affine ∞-gerbe which, in addition, satisfies a cohomological condition (see [Toe06, Definition 3.1.2] for details).

The main result on affine stacks that we need is the existence theorem of [Toe06]. Embed the category SSet into the category $\mathsf{SPr}(k)$ by viewing a simplicial set X as a constant simplicial presheaf on $(\mathsf{Aff}, \mathrm{ffqc})$. With this convention, we have the following important definition.

Definition 2.1 [Toe06, Definition 3.3.1]. Let X be a pointed and connected simplicial set in $\mathbb{U}$. The schematization of X over k is a pointed schematic homotopy type $(X \otimes k)^{\mathrm{sch}}$ together with a morphism

$$u : X \longrightarrow (X \otimes k)^{\mathrm{sch}}$$

in $\mathrm{Ho}(\mathsf{SPr}_*(k))$ which is a universal for morphisms from X to pointed schematic homotopy types (in the category $\mathrm{Ho}(\mathsf{SPr}_*(k))$).

We have stated the above definition only for simplicial sets so as to simplify the exposition. However, by using the *singular functor* Sing, which attaches to each topological space T the simplicial set of singular chains in T (see, e.g., [Hov98] for details), one can define the schematization of a pointed connected topological space. In what follows, we will always assume implicitly that the functor Sing has been applied when necessary, and we will generally not distinguish between topological spaces and simplicial sets when considering the schematization functor.

Finally, let us recall the main existence theorem.

Theorem 2.2 [Toe06, Theorem 3.3.4]. *Any pointed and connected simplicial set (X, x) in $\mathbb{U}$ possesses a schematization over k. Furthermore, for any $i > 0$, the sheaf $\pi_i((X \otimes k)^{\mathrm{sch}}, x)$ is represented by an affine group scheme which is commutative and unipotent for $i > 1$.*

Let (X, x) be a pointed connected simplicial set in $\mathbb{U}$, and let $(X \otimes k)^{\mathrm{sch}}$ be its schematization. Then, one has the following properties.

(i) The affine group scheme $\pi_1((X \otimes k)^{\mathrm{sch}}, x)$ is naturally isomorphic to the pro-algebraic completion of the discrete group $\pi_1(X, x)$ over k.

(ii) Let V be a local system of finite-dimensional k-vector spaces on X; in particular, V corresponds to a linear representation of $\pi_1((X \otimes k)^{\mathrm{sch}}, x)$ and gives rise to a local system $\mathcal{V}$ on $(X \otimes k)^{\mathrm{sch}}$. Then, there is a natural isomorphism

$$H^\bullet(X, V) \simeq H^\bullet((X \otimes k)^{\mathrm{sch}}, \mathcal{V}).$$

(iii) If X is simply connected and of finite type (i.e. the homotopy type of a simply connected finite CW complex), then for any $i > 1$ the group scheme $\pi_i((X \otimes k)^{\mathrm{sch}}, x)$ is naturally isomorphic to the pro-unipotent completion of the discrete groups $\pi_i(X,x)$. In other words, for any $i > 1$,

$$\pi_i((X \otimes k)^{\mathrm{sch}}, x) \simeq \pi_i(X,x) \otimes_{\mathbb{Z}} \mathbb{G}_a \qquad \text{if } \operatorname{char}(k) = 0,$$
$$\pi_i((X \otimes k)^{\mathrm{sch}}, x) \simeq \pi_i(X,x) \otimes_{\mathbb{Z}} \mathbb{Z}_p \qquad \text{if } \operatorname{char}(k) = p > 0.$$

Here, the groups $\pi_i(X,x)$ appearing on the right-hand sides are thought of as constant group schemes over k.

3. Algebraic models

In this section, we discuss two different algebraic models for pointed schematic homotopy types: cosimplicial commutative Hopf algebras and equivariant cosimplicial commutative algebras. These models give rise to two different explicit formulas for the schematization of a space, each having its advantages and disadvantages depending on the situation. Our first model will allow us to complete the proof of [Toe06, Theorem 3.2.9].

We start by introducing an intermediate model category structure on the category of simplicial affine k-group schemes. This model category structure will then be localized in order to get the right homotopy category of cosimplicial Hopf algebras suited to the setting of schematic homotopy types. We will present this intermediate model category in a separate initial subsection, as we believe it may be of independent interest.

3.1 Simplicial affine group schemes

By a *Hopf algebra* we mean a *unital and co-unital commutative Hopf k-algebra*. The category of Hopf algebras will be denoted by Hopf, which is equivalent to the opposite of the category GAff of affine k-group schemes. Recall that every Hopf algebra is equal to the colimit of its Hopf subalgebras of finite type (see [DG70a, III, § 3, No. 7]). In particular, the category Hopf is the category of ind-objects in the category of Hopf algebras of finite type and is therefore complete and co-complete [AGV72, Exposé 1, Proposition 8.9.1(b)]. We consider the category of cosimplicial Hopf algebras Hopf^{Δ}, which is dual to the category of simplicial affine group schemes sGAff. When there is a need to specify universes, we shall write $\mathsf{Hopf}_{\mathbb{U}}$, $\mathsf{Hopf}^{\Delta}_{\mathbb{U}}$, $\mathsf{GAff}_{\mathbb{U}}$, $\mathsf{sGAff}_{\mathbb{U}}$ and so on.

The category $\mathsf{GAff}_{\mathbb{U}}$ has all $\mathbb{U}$-small limits and colimits. In particular, the category of simplicial objects $\mathsf{sGAff}_{\mathbb{U}}$ is naturally endowed with tensor and co-tensor structures over the category $\mathsf{SSet}_{\mathbb{U}}$ of $\mathbb{U}$-small simplicial sets. By duality, the category $\mathsf{Hopf}^{\Delta}_{\mathbb{U}}$ also has a natural tensor and co-tensor structure over $\mathsf{SSet}_{\mathbb{U}}$. For $X \in \mathsf{SSet}_{\mathbb{U}}$ and $B_* \in \mathsf{Hopf}^{\Delta}$, $G_* = \operatorname{Spec} B_*$, we will use the standard notation (see [GJ99, Theorem 2.5])

$$X \otimes B_*, \quad B_*^X, \quad X \otimes G_*, \quad G_*^X.$$

Explicitly, for a simplicial affine group scheme G_* and a simplicial set X, we have $(X \otimes G)_n = \coprod_{X_n} G_n$, where the coproduct is taken in the category of affine group schemes. To describe G^X, we first define $(G^X)_0$ as the equalizer

$$(G^X)_0 \longrightarrow \prod_n G_n^{X_n} \underset{b}{\overset{a}{\rightrightarrows}} \prod_{p \xrightarrow{u} q} G_q^{X_p}.$$

Here a is the composition $\prod_n G_n^{X_n} \twoheadrightarrow G_p^{X_p} \to G_q^{X_p}$, where the second map is induced from u; similarly, b is the composition $\prod_n G_n^{X_n} \twoheadrightarrow G_q^{X_q} \to G_q^{X_p}$, where the second map is induced from u.

With this definition, we can now set $(G^X)_n = (G^{X\times\Delta_n})_0$. The definitions of $X \otimes B_*$ and B_*^X are analogous. Note that

$$\mathrm{Spec}(X \otimes B_*) \simeq G_*^X \qquad \text{and} \qquad \mathrm{Spec}(B_*^X) \simeq X \otimes B_*.$$

For any simplicial set K and any simplicial affine group scheme G_*, we will use the notation

$$\mathrm{Map}(K, G_*) := (G_*^K)_0.$$

Note that $\mathrm{Map}(\Delta^n, G_*) \simeq G_n$.

Let $n \geq 0$. We consider the simplicial sphere $S^n := \partial\Delta^{n+1}$, pointed by the vertex $0 \in \Delta^{n+1}$. There is a natural morphism of affine group schemes

$$\mathrm{Map}(S^n, G_*) \longrightarrow \mathrm{Map}(*, G_*) \simeq G_0.$$

The kernel of this morphism will be denoted by $\mathrm{Map}_*(S^n, G_*)$. In the same way, the kernel of the morphism

$$\mathrm{Map}(\Delta^{n+1}, G_*) \longrightarrow \mathrm{Map}(*, G_*) = G_0$$

will be denoted by $\mathrm{Map}_*(\Delta^{n+1}, G_*)$. The inclusion $S^n \subset \Delta^{n+1}$ induces a morphism

$$\mathrm{Map}_*(\Delta^{n+1}, G_*) \longrightarrow \mathrm{Map}_*(S^n, G_*).$$

The cokernel of this morphism, taken in sGAff, will be denoted by $\Pi_n(G_*)$.

For any n and any $0 \leq k \leq n$, we denote by $\Lambda^{n,k}$ the kth horn of Δ^n, which by definition is obtained from $\partial\Delta^n$ by removing its kth face. For a morphism $G_* \longrightarrow H_*$ in sGAff, we then have a morphism of affine group schemes

$$\mathrm{Map}(\Delta^n, G_*) \longrightarrow \mathrm{Map}(\Lambda^{n,k}, G_*) \times_{\mathrm{Map}(\Lambda^{n,k}, G_*)} \mathrm{Map}(\Delta^n, H_*)$$

or, equivalently,

$$G_n \longrightarrow \mathrm{Map}(\Lambda^{n,k}, G_*) \times_{\mathrm{Map}(\Lambda^{n,k}, G_*)} H_n.$$

Definition 3.1. Let $f : G_* \longrightarrow H_*$ be a morphism in sGAff.

(i) The morphism f is an *equivalence* if for all n, the induced morphism

$$\Pi_n(G_*) \longrightarrow \Pi_n(H_*)$$

is an isomorphism.

(ii) The morphism f is a *fibration* if for all n and all $0 \leq k \leq n$, the induced morphism

$$G_n \longrightarrow \mathrm{Map}(\Lambda^{n,k}, G_*) \times_{\mathrm{Map}(\Lambda^{n,k}, G_*)} H_n$$

is a faithfully flat morphism of schemes.

(iii) The morphism f is a *cofibration* if it has the left lifting property with respect to all morphisms which are fibrations and equivalences.

It is useful to note here that a morphism of affine group schemes $G \longrightarrow H$ is faithfully flat if and only if the induced morphism of Hopf algebras $\mathcal{O}(H) \longrightarrow \mathcal{O}(G)$ is injective (see [DG70a, III, § 3, No. 7]).

Theorem 3.2. *The above definition makes* $\mathsf{sGAff}_{\mathbb{U}}$ *a model category.*

Proof. We are going to apply [Hov98, Theorem 2.1.19] to the opposite category $\mathsf{sGAff}_{\mathbb{U}}^{\mathrm{op}} = \mathsf{Hopf}_{\mathbb{U}}^{\Delta}$. To do this, we let I be a set of representatives of all morphisms in $\mathsf{sGAff}_{\mathbb{U}}$ which are fibrations $G_* \longrightarrow H_*$ such that the group schemes G_n and H_n are of finite type (as schemes over $\mathrm{Spec}\, k$) for any $n \geq 0$. In the same way, we let J be the subset corresponding to morphisms in I which are also equivalences. We need to prove that the category $\mathsf{sGAff}_{\mathbb{U}}^{\mathrm{op}}$ is equal to $\mathsf{Hopf}_{\mathbb{U}}^{\Delta}$, the class W of equivalences, and that the sets I and J satisfy the following six conditions.

(i) The sub-category W has the two-out-of-three property and is closed under retracts.

(ii) The domains and codomains of I are small (relative to I-cell).

(iii) The domains and codomains of J are small (relative to J-cell).

(iv) $J\text{-cell} \subset W \cap I\text{-cof}$.

(v) $I\text{-inj} \subset W \cap J\text{-inj}$.

(vi) $W \cap I\text{-cell} \subset J\text{-cof}$.

Establishing these properties will prove the existence of a model category structure on $\mathsf{Hopf}_{\mathbb{U}}^{\Delta}$ whose equivalences are as in Definition 3.1 and whose cofibrations are generated by the set I. Before proceeding further, we check that the cofibrations generated by the set I are precisely as given in Definition 3.1 (note that, by duality, fibrations in $\mathsf{Hopf}_{\mathbb{U}}^{\Delta}$ correspond to cofibrations in $\mathsf{sGAff}_{\mathbb{U}}$, and conversely).

Lemma 3.3.

(1) *A morphism $G_* \longrightarrow H_*$ in $\mathsf{sGAff}_{\mathbb{U}}$ such that, for each n, the morphism $G_n \longrightarrow H_n$ is faithfully flat is a fibration.*

(2) *A morphism in sGAff has the left lifting property with respect to I if and only if it has the left lifting property with respect to every fibration.*

Proof. (1) Let K_* be the kernel of the morphism $G_* \longrightarrow H_*$. Take $0 \leq k \leq n$ and set

$$L_n := H_n \times_{\mathrm{Map}(\Lambda^{n,k}, H_*)} \mathrm{Map}(\Lambda^{n,k}, G_*).$$

We have the following commutative diagram of affine group schemes.

$$\begin{array}{ccccc} K_n & \longrightarrow & G_n & \longrightarrow & H_n \\ \downarrow & & \downarrow & & \downarrow \mathrm{id} \\ \mathrm{Map}(\Lambda^{n,k}, K_*) & \longrightarrow & L_n & \longrightarrow & H_n \end{array}$$

Using a version of the five lemma, we see that it is enough to prove that $H_n \longrightarrow \mathrm{Map}(\Lambda^{n,k}, K_*)$ is faithfully flat. In other words, we can assume that $H_* = \{e\}$.

The simplicial presheaf h_{G_*} represented by G_* is a presheaf in simplicial groups on the site of all affine schemes. It is a globally fibrant simplicial presheaf (see [May67, Theorem 17.1]). As the functor $G_* \mapsto h_{G_*}$ commutes with exponentiation by simplicial sets (because it commutes with arbitrary limits), for $0 \leq n \leq n$ the morphism

$$G_n \longrightarrow \mathrm{Map}(\Lambda^{n,k}, G_*)$$

induces a surjective morphism on the associated presheaves. As $\mathrm{Map}(\Lambda^{n,k}, G_*)$ is an affine scheme, this morphism has, in fact, a section and thus, by [DG70a, III. § 3, No. 7], is faithfully flat.

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(2) Let $G_* \longrightarrow H_*$ be a morphism having the left lifting property with respect to I, and let $K_* \longrightarrow L_*$ be a fibration together with a commutative diagram as follows.

$$\begin{array}{ccc} G_* & \longrightarrow & K_* \\ \downarrow & & \downarrow \\ H_* & \longrightarrow & L_* \end{array}$$

We consider the factorization $K_* \longrightarrow K'_* \longrightarrow L_*$ into an faithfully flat morphism followed by an injective morphism. The morphism $K'_* \longrightarrow L_*$ stays a fibration, as shown by the following commutative square with faithfully flat rows.

$$\begin{array}{ccc} K_n & \longrightarrow & K'_n \\ \downarrow & & \downarrow \\ \mathrm{Map}(\Lambda^{n,k}, K_*) \times_{\mathrm{Map}(\Lambda^{n,k}, L_*)} L_n & \longrightarrow & \mathrm{Map}(\Lambda^{n,k}, K'_*) \times_{\mathrm{Map}(\Lambda^{n,k}, L_*)} L_n \end{array}$$

Moreover, owing to part (1), the morphism $K_* \longrightarrow K'_*$ is also a fibration. Thus, the problem is reduced to treating two cases: either $K_* \longrightarrow L_*$ is faithfully flat, or it is injective.

We begin with the case where $K_* \longrightarrow L_*$ is injective. Here, the induced morphism of simplicial sheaves

$$h_{K_*} \longrightarrow h_{L_*}$$

is a monomorphism and a local fibration. The local lifting property with respect to the inclusion $* \hookrightarrow \Delta^1$ and the fact that the morphism is mono imply that the induced morphism $\pi_0(h_{K_*}) \longrightarrow \pi_0(h_{L_*})$ is a monomorphism of sheaves. As monomorphisms and local fibrations are stable with respect to exponentiation by a finite simplicial set, we see that the induced morphism $\pi_i(h_{K_*}) \longrightarrow \pi_i(h_{L_*})$ is a monomorphism of sheaves for all $i \geq 0$. Finally, the local lifting property for the inclusion $* \hookrightarrow \Delta^n$ and the fact the morphism is mono also imply that the induced morphism $\pi_n(h_{K_*}) \longrightarrow \pi_n(h_{L_*})$ is surjective for all $n > 0$. In other words, the square

$$\begin{array}{ccc} h_{K_*} & \longrightarrow & h_{L_*} \\ \downarrow & & \downarrow \\ \pi_0(h_{K_*}) & \longrightarrow & \pi_0(h_{L_*}) \end{array}$$

is cartesian; equivalently, the square

$$\begin{array}{ccc} K_* & \longrightarrow & L_* \\ \downarrow & & \downarrow \\ \Pi_0(K_*) & \longrightarrow & \Pi_0(L_*) \end{array}$$

is cartesian. Hence, to prove the existence of a lifting $H_* \longrightarrow K_*$, we can replace K_* by $\Pi_0(K_*)$ and L_* by $\Pi_0(L_*)$. It therefore remains to prove the following fact about affine group schemes: if $p : G \longrightarrow H$ is a morphism of affine group schemes having the left lifting property with respect to every morphism between affine group schemes of finite type, then p is an isomorphism. This last assertion follows easily from the fact that the category of affine group schemes is the category of pro-objects in the category of affine group schemes of finite type.

We now turn to the case where $K_* \longrightarrow L_*$ is faithfully flat. Consider a certain set X, whose elements are cosimplicial Hopf subalgebras $A_* \subset \mathcal{O}(K_*)$ such that:

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- the morphism $\mathcal{O}(L_*) \longrightarrow \mathcal{O}(K_*)$ factors through A_*;
- there exists a morphism $H_* \longrightarrow \operatorname{Spec} A_*$ that makes the diagram

$$\begin{array}{ccc} G_* & \longrightarrow & \operatorname{Spec} A_* \\ \downarrow & \nearrow & \downarrow \\ H_* & \longrightarrow & L_* \end{array}$$

  commutative.

This set is non-empty, since the image of $\mathcal{O}(L_*) \hookrightarrow \mathcal{O}(K_*)$ is an element of X. Next, we order X by the order induced by inclusion of cosimplicial Hopf subalgebras. The ordered set X is inductive, and we let $A_* \subset \mathcal{O}(K_*)$ be a maximal element. Assume that $A_* \neq \mathcal{O}(K_*)$. We choose a lift $H_* \longrightarrow \operatorname{Spec} A_*$ and consider the following commutative diagram.

$$\begin{array}{ccc} G_* & \longrightarrow & K_* \\ \downarrow & & \downarrow \\ H_* & \longrightarrow & \operatorname{Spec} A_* \end{array}$$

As $A_* \neq \mathcal{O}(K_*)$, there exists a cosimplicial Hopf subalgebra $D_* \subset \mathcal{O}(K_*)$ such that for any n the algebra D_n is of finite type, and $A_* \nsubseteq D_*$. We let $D'_* := A_* \cap D_*$ and let B_* be the cosimplicial Hopf subalgebra of $\mathcal{O}(K_*)$ which is generated by D_* and A_*. There exists a commutative diagram

$$\begin{array}{ccccccc} G_* & \longrightarrow & K_* & \longrightarrow & \operatorname{Spec} B_* & \longrightarrow & \operatorname{Spec} D_* \\ \downarrow & & \downarrow & \swarrow & & & \downarrow \\ H_* & \longrightarrow & \operatorname{Spec} A_* & & \longrightarrow & & \operatorname{Spec} D'_* \end{array}$$

where, furthermore, the square on the right-hand side is cartesian. Finally, as $D'_* \longrightarrow D_*$ is injective, $\operatorname{Spec} D_* \longrightarrow \operatorname{Spec} D'_*$ is a fibration (owing to part (1)) between simplicial affine group schemes of finite type. By assumption, a lift $H_* \longrightarrow \operatorname{Spec} B_*$ exists; however, this contradicts the maximality of A_*. Therefore, $A_* = \mathcal{O}(K_*)$ and a lift $H_* \longrightarrow K_*$ exists. □

Let us now check that the conditions (i) to (v) are satisfied. Firstly, (i) is true because equivalences are the morphisms inducing isomorphisms on Π_n (and isomorphisms of group schemes do have the two-out-of-three property and are closed under retracts). Properties (ii) and (iii) hold since any object in the category of Hopf algebras is small (with respect to some cardinal depending on this object) relative to the whole category, and therefore the domains and codomains of I and J are also small with respect to the whole category $\mathsf{Hopf}^{\Delta}_{\mathbb{U}}$.

Let us now prove that (iv) holds. We have $J \subset I \subset I$-cof, and hence J-cell $\subset I$-cof because I-cof is stable by pushouts and transfinite compositions. To show that J-cell $\subset W$, it is enough to establish the following two properties.

(a) The trivial fibrations in $\mathsf{sGAff}_{\mathbb{U}}$ are stable by base change.

(b) The trivial fibrations in $\mathsf{sGAff}_{\mathbb{U}}$ are stable by ($\mathbb{U}$-small) filtered limits.

Let

$$\begin{array}{ccc} G'_* & \longrightarrow & G_* \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ H'_* & \longrightarrow & H_* \end{array}$$

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be a cartesian diagram in $\mathsf{sGAff}_{\mathbb{U}}$. For any simplicial set K, the diagram

$$\begin{array}{ccc} \mathrm{Map}(K, G'_*) & \longrightarrow & \mathrm{Map}(K, G_*) \\ \downarrow & & \downarrow f \\ \mathrm{Map}(K, H'_*) & \longrightarrow & \mathrm{Map}(K, H_*) \end{array}$$

is a cartesian diagram of affine group schemes. Therefore, for any n and any $0 \leq k \leq n$, we have a cartesian diagram of affine group schemes as follows.

$$\begin{array}{ccc} G'_n & \longrightarrow & G_n \\ \downarrow & & \downarrow \\ \mathrm{Map}(\Lambda^{n,k}, G'_*) \times_{\mathrm{Map}(\Lambda^{n,k}, G'_*)} H'_n & \longrightarrow & \mathrm{Map}(\Lambda^{n,k}, G_*) \times_{\mathrm{Map}(\Lambda^{n,k}, G_*)} H_n \end{array}$$

As the faithfully flat morphisms are stable by base change, this implies that if f is a fibration, then so is f'.

Consider the functor $h : \mathsf{sGAff}_{\mathbb{U}} \longrightarrow \mathsf{SPr}_*(k)$ sending a simplicial affine group scheme G_* to the simplicial presheaf $X \mapsto \mathrm{Hom}(X, G_*)$, pointed at the unit of G_0. Because the functor sending an affine group scheme to the sheaf of groups it represents commutes with limits and quotients (see [DG70a, III, § 3, No. 7]), we see that the homotopy sheaves of $\pi_n(h_{G_*})$ are representable by the group schemes $\Pi_n(G_*)$. Moreover, a morphism $f : G_* \longrightarrow H_*$ is a fibration in our sense if and only if the induced morphism $h_{G_*} \longrightarrow h_{H_*}$ is a local fibration in the sense of [Jar87, § 1] (i.e. it satisfies the local right lifting property with respect to $\Lambda^{n,k} \subset \Delta^n$). In particular, this morphism of simplicial presheaves provides a long exact sequence on homotopy sheaves when f is a fibration (see [Jar87, Lemma 1.15]). We deduce from this that if K_* denotes the kernel of the morphism f and f is a fibration, then there exists a long exact sequence of affine group schemes

$$\ldots \Pi_{n+1}(H_*) \longrightarrow \Pi_n(K_*) \longrightarrow \Pi_n(G_*) \longrightarrow \Pi_n(H_*) \longrightarrow \Pi_{n-1}(K_*) \longrightarrow \cdots.$$

Using this fact, we deduce that a fibration in $\mathsf{sGAff}_{\mathbb{U}}$ is an equivalence if and only if its kernel K_* is acyclic (i.e. $\Pi_n(K_*) = 0$). As kernels are stable by base change, trivial fibrations must also be stable by base change. This proves property (a).

In order to prove property (b), let $G_* = \lim_\alpha G_*^{(\alpha)}$ be a filtered limit of objects in $\mathsf{sGAff}_{\mathbb{U}}$.

Lemma 3.4. *For any $n \geq 0$, the natural morphism*

$$\Pi_n(G_*) \longrightarrow \lim_\alpha \Pi_n(G_*^\alpha)$$

is an isomorphism.

Proof. For any n, we have the following cocartesian square of affine group schemes.

$$\begin{array}{ccc} \mathrm{Map}_*(\partial\Delta^{n+1}, G_*) & \longrightarrow & \mathrm{Map}_*(\Delta^{n+1}, G_*) \\ \downarrow & & \downarrow \\ \{e\} & \longrightarrow & \Pi_n(G_*) \end{array}$$

As the functor $\mathrm{Map}(K, G_*)$ commutes with limits, we need only show that filtered limits of affine group schemes preserve cocartesian squares. Let

$$\begin{array}{ccc} E^{(\alpha)} & \longrightarrow & F^{(\alpha)} \\ \downarrow & & \downarrow \\ G^{(\alpha)} & \longrightarrow & H^{(\alpha)} \end{array}$$

be a filtered diagram of cocartesian squares in GAff, and let

$$\begin{array}{ccc} E & \longrightarrow & F \\ \downarrow & & \downarrow \\ G & \longrightarrow & H \end{array}$$

be the limit diagram. We need to prove that for any $G_0 \in \mathsf{sGAff}$, the induced diagram of sets

$$\begin{array}{ccc} \mathrm{Hom}(H, G_0) & \longrightarrow & \mathrm{Hom}(F, G_0) \\ \downarrow & & \downarrow \\ \mathrm{Hom}(G, G_0) & \longrightarrow & \mathrm{Hom}(E, G_0) \end{array}$$

is cartesian. Since any affine group scheme G_0 is the projective limit of its quotients of finite type, we can assume that G_0 is of finite type. But then $\mathrm{Hom}(-, G_0)$ sends filtered limits to filtered colimits, as the Hopf algebra corresponding to G_0 is of finite type as an algebra and thus is a compact object in the category of Hopf algebras (i.e. $\mathrm{Hom}(\mathcal{O}(G_0), -)$ commutes with filtered colimits). Therefore, the above diagram of sets is isomorphic to the following one.

$$\begin{array}{ccc} \mathrm{colim}_\alpha \mathrm{Hom}(H^{(\alpha)}, G_0) & \longrightarrow & \mathrm{colim}_\alpha \mathrm{Hom}(F^{(\alpha)}, G_0) \\ \downarrow & & \downarrow \\ \mathrm{colim}_\alpha \mathrm{Hom}(G^{(\alpha)}, G_0) & \longrightarrow & \mathrm{colim}_\alpha \mathrm{Hom}(E^{(\alpha)}, G_0) \end{array}$$

This last diagram is cartesian because filtered colimits in sets preserve finite limits. □

The previous lemma implies that equivalences in $\mathsf{sGAff}_{\mathbb{U}}$ are stable by filtered limits and, in particular, by transfinite composition on the left. Moreover, faithfully flat morphisms of affine group schemes are also stable by filtered limits, because injective morphisms of Hopf algebras are stable by filtered colimits. By the definition of fibrations, this implies that the fibrations are also stable by filtered limits. This finishes the proof of (b) above. Property (iv) is thus proven.

Before proving the last two properties, (v) and (vi), we need some additional lemmas.

Lemma 3.5. *A morphism $G_* \longrightarrow H_*$ in $\mathsf{sGAff}_{\mathbb{U}}$ is a fibration if and only if for any trivial cofibration $A \subset B$ of $\mathbb{U}$-small simplicial sets, the natural morphism*

$$\mathrm{Map}(B, G_*) \longrightarrow \mathrm{Map}(A, G_*) \times_{\mathrm{Map}(A, H_*)} \mathrm{Map}(B, H_*)$$

is faithfully flat.

Proof. This follows from the definition, the fact that the morphisms $\Lambda^{n,k} \subset \Delta^n$ generate the trivial cofibrations of simplicial sets (see [Hov98, Definition 3.2.1]), and the fact that faithfully flat morphisms of affine group schemes are stable by pull-backs and filtered compositions (because injective morphisms of Hopf algebras are stable by filtered colimits). □

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LEMMA 3.6. *Let $G_* \in \mathsf{sGAff}_{\mathbb{U}}$ and let $A \subset B$ be a cofibration of $\mathbb{U}$-small simplicial sets. Then the morphism $G_*^B \longrightarrow G_*^A$ is a fibration.*

Proof. This follows from the formula

$$\mathrm{Map}(K, G_*^A) \simeq \mathrm{Map}(A \times K, G_*)$$

together with Lemma 3.5 and the fact that for any n and any k, the natural morphism

$$(A \times \Delta^n) \coprod_{A \times \Lambda^{n,k}} B \times \Lambda^{n,k} \longrightarrow B \times \Delta^n$$

is a trivial cofibration of simplicial sets. □

We have $J \subset I$ and hence I-inj $\subset J$-inj. In order to prove (v), we need to show that I-inj $\subset W$. Let $i : G_* \longrightarrow H_*$ be a morphism in $\mathsf{sGAff}_{\mathbb{U}}$ that has the left lifting property with respect to I. From Lemma 3.3, we know that it also has the left lifting property with respect to all fibrations. Moreover, Lemma 3.3(i) implies that G_* itself is fibrant, and the lifting property for the diagram

$$\begin{array}{ccc} G_* & \longrightarrow & G_* \\ \downarrow & & \downarrow \\ H_* & \longrightarrow & \{e\} \end{array}$$

implies the existence of $r : H_* \longrightarrow G_*$ such that $r \circ i = \mathrm{id}$. Lemma 3.6 implies that $H_*^{\Delta^1} \longrightarrow H_*^{\partial\Delta^1} = H^{\Delta^1} \times H^{\Delta^1}$ is a fibration. Therefore, the lifting property for the diagram

$$\begin{array}{ccc} G_* & \xrightarrow{i} & H_*^{\Delta^1} \\ \downarrow & & \downarrow \\ H_* & \xrightarrow[\mathrm{id}, i \circ r]{} & H_* \times H_* \end{array}$$

implies the existence of a morphism

$$h : H_* \longrightarrow H_*^{\Delta^1}$$

which is a homotopy between the identity and $i \circ r$. This implies that $i : h_{G_*} \longrightarrow h_{H_*}$ is a homotopy equivalence of simplicial presheaves and thus induces isomorphisms on homotopy sheaves. As we have already seen, the homotopy sheaves of h_{G_*} and h_{H_*} are representable by the homotopy group schemes of G_* and H_*. Therefore, i itself is an equivalence in $\mathsf{sGAff}_{\mathbb{U}}$.

It only remains to verify that property (v) is satisfied. However, as we have already seen in the course of proving (iv), the morphisms in I-cell are fibrations in sGAff (because fibrations are stable by base change and filtered limits). In particular, any morphism in $I\text{-cell} \cap W$ is a trivial fibration. Lemma 3.3 then implies that $I\text{-cell} \cap W \subset J$-cof. This completes the proof of Theorem 3.2. □

Before continuing further, we make an important remark concerning dependence on the universe $\mathbb{U}$ of the model category $\mathsf{Hopf}^{\Delta}_{\mathbb{U}}$. If $\mathbb{U} \in \mathbb{V}$ are two universes, one gets a natural inclusion functor $\mathsf{Hopf}^{\Delta}_{\mathbb{U}} \longrightarrow \mathsf{Hopf}^{\Delta}_{\mathbb{V}}$. An important fact about this inclusion is that the sets I and J defined in the proof of Theorem 3.2 are independent of the choice of universe. Therefore, the sets I and J are generating sets of cofibrations and trivial cofibrations for model categories $\mathsf{Hopf}^{\Delta}_{\mathbb{U}}$ and $\mathsf{Hopf}^{\Delta}_{\mathbb{V}}$. Since we know that a morphism is a cofibration if and only if it is a retract of a relative I-cell complex (see [Hov98, Definition 2.1.9] and [Hov98, Proposition 2.1.18(b)]), a morphism in

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$\mathsf{Hopf}^{\Delta}_{\mathbb{U}}$ is a cofibration if and only if it is a cofibration in $\mathsf{Hopf}^{\Delta}_{\mathbb{V}}$. In the same way, fibrations are morphisms that have the right lifting property with respect to J, and thus a morphism in $\mathsf{Hopf}^{\Delta}_{\mathbb{U}}$ is a fibration if and only if it is a fibration in $\mathsf{Hopf}^{\Delta}_{\mathbb{V}}$. Moreover, using the functorial factorization of cosimplicial algebras via the small-object argument with respect to I and J (as described in the proof above), one sees that the full sub-category $\mathsf{Hopf}^{\Delta}_{\mathbb{U}}$ is stable by the functorial factorizations in $\mathsf{Hopf}^{\Delta}_{\mathbb{V}}$. In particular, a path object (see [Hov98, Definition 1.2.4(2)]) in $\mathsf{Hopf}^{\Delta}_{\mathbb{U}}$ is also a path object in $\mathsf{Hopf}^{\Delta}_{\mathbb{V}}$. Consequently, two morphisms in $\mathsf{Hopf}^{\Delta}_{\mathbb{U}}$ between fibrant and cofibrant objects are homotopic (see [Hov98, Definition 1.2.4(5)]) if and only if they are homotopic in $\mathsf{Hopf}^{\Delta}_{\mathbb{V}}$.

A consequence of these remarks and the fact that the homotopy category of a model category is equivalent to the category of fibrant and cofibrant objects and homotopy classes of objects between them (see [Hov98, Theorem 1.2.10]) is that the induced functor

$$\mathrm{Ho}(\mathsf{Hopf}^{\Delta}_{\mathbb{U}}) \longrightarrow \mathrm{Ho}(\mathsf{Hopf}^{\Delta}_{\mathbb{V}})$$

is fully faithful. The essential image of this functor consists of all cosimplicial Hopf algebras in $\mathbb{V}$ which are equivalent to a cosimplicial Hopf algebra in $\mathbb{U}$.

The conclusion of this short discussion is that increasing the size of the ambient universe does not destroy the homotopy theory of cosimplicial Hopf algebras, and its only effect is to add more objects to the corresponding homotopy category.

An important additional property of the model category $\mathsf{sGAff}_{\mathbb{U}}$ is its right properness.

COROLLARY 3.7. *The model category* $\mathsf{sGAff}_{\mathbb{U}}$ *is right proper (i.e. equivalences are stable by pull-backs along fibrations).*

Proof. This result follows from Lemma 3.3(i), which implies, in particular, that any object in $\mathsf{sGAff}_{\mathbb{U}}$ is fibrant (see [Hir03, Corollary 13.1.3]). This could also have been checked directly using the long exact sequence in homotopy for fibrations (see the proof of property (iv) for Theorem 3.2). □

Another interesting property is the compatibility between the model structure and the simplicial enrichment.

COROLLARY 3.8. *Together with its natural simplicial enrichment, the model category* $\mathsf{sGAff}_{\mathbb{U}}$ *is a simplicial model category.*

Proof. This is a consequence of Lemma 3.5. Indeed, we need to verify axiom M7 of [Hir03, Definition 9.1.6]. To do this, let $i : A \subset B$ be a cofibration of $\mathbb{U}$-small simplicial sets and $f : G_* \longrightarrow H_*$ a fibration in $\mathsf{sGAff}_{\mathbb{U}}$. The fact that

$$G_*^B \longrightarrow G_*^A \times_{H_*^A} H_*^B$$

is again a fibration follows from the definition, Lemma 3.5, and the fact that the morphism

$$B \times \Lambda^{n,k} \coprod_{A \times \Lambda^{n,k}} B \times \Delta^n \longrightarrow B \times \Delta^n$$

is a trivial cofibration.

If, moreover, i or f is an equivalence, then, because of [Jar87, Corollary 1.5] and [Jar87, Lemma 1.1.5], the morphism

$$h_{G_*} \longrightarrow h_{G_*^A \times_{H_*^A} H_*^B} \simeq h_{G_*}^A \times_{h_{H_*}^A} h_{H_*}^B$$

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is a local equivalence. This implies that

$$G_*^B \longrightarrow G_*^A \times_{H_*^A} H_*^B$$

is an equivalence. □

The simplicial Homs of the simplicial category $\mathsf{SGAff}_{\mathbb{U}}$ will be denoted by $\underline{\mathrm{Hom}}$. By Proposition 3.8, these simplicial Homs possess a derived version (see [Hov98, § 4.3])

$$\mathbb{R}\,\underline{\mathrm{Hom}}(-,-) : \mathrm{Ho}(\mathsf{sGAff}_{\mathbb{U}})^{\mathrm{op}} \times \mathrm{Ho}(\mathsf{sGAff}_{\mathbb{U}}) \longrightarrow \mathrm{Ho}(\mathsf{SSet}_{\mathbb{U}}).$$

3.2 The P-localization

Let K be an affine group scheme of finite type over k, and let V be a finite-dimensional linear representation of K. We view V as an affine group scheme (its group law being the addition), and also consider the simplicial affine group scheme $K(V,n)$ for $n \geq 0$. By definition, $K(V,n)$ is the classifying space of $K(V,n-1)$, defined, for instance, as in [Toe06, § 1.3]; we set $K(V,0)=V$. The group scheme K acts on V and therefore acts on $K(V,n)$. We will denote by $K(K,V,n)$ the simplicial group scheme which is the semi-direct product of K (considered as a constant simplicial object) and $K(V,n)$. We therefore have a split exact sequence of simplicial affine group schemes

$$1 \longrightarrow K(V,n) \longrightarrow K(K,V,n) \longrightarrow K \longrightarrow 1.$$

Ideally, we would like to construct the left Bousfield localization of $\mathsf{sGAff}_{\mathbb{U}}$ with respect to the set of objects $K(K,V,n)$, for all K, V and n. From a dual point of view, this would correspond to performing a right Bousfield localization of $\mathsf{Hopf}_{\mathbb{U}}^{\Delta}$ with respect to the corresponding set of objects. The only general result we are aware of that ensures the existence of a right Bousfield localization is [Hir03, Theorem 5.1.1], which requires the model category to be cellular. Unfortunately, the model category $\mathsf{Hopf}_{\mathbb{U}}^{\Delta}$ is not cellular, as cofibrations are simply not monomorphisms. It is therefore unclear whether the localized model structure exists (although we think it does). In this section, we will show the existence of a localization functor

$$(-)^P : \mathrm{Ho}(\mathsf{sGAff}_{\mathbb{U}}) \longrightarrow \mathrm{Ho}^P(\mathsf{sGAff}_{\mathbb{U}}),$$

which will be enough to prove the equivalence between pointed schematic homotopy types and cosimplicial Hopf algebras up to P-equivalences.

We start by defining our new equivalences in $\mathsf{sGAff}_{\mathbb{U}}$. We will see later (in Corollary 3.17) that they are precisely the quasi-isomorphisms of cosimplicial Hopf algebras.

Definition 3.9. A morphism $f : G_* \longrightarrow H_*$ is a *P-equivalence* if for any affine group scheme of finite type K, any finite-dimensional linear representation V of K and any $n \geq 1$, the induced morphism

$$f^* : \mathbb{R}\,\underline{\mathrm{Hom}}(H_*, K(K,V,n)) \longrightarrow \mathbb{R}\,\underline{\mathrm{Hom}}(G_*, K(K,V,n))$$

is an isomorphism in $\mathrm{Ho}(\mathsf{SSet}_{\mathbb{U}})$.

The preceding definition gives a new class of equivalences on $\mathsf{sGAff}_{\mathbb{U}}$. The localization of the category $\mathsf{sGAff}_{\mathbb{U}}$ with respect to P-equivalences will be denoted by

$$\mathrm{Ho}^P(\mathsf{sGAff}_{\mathbb{U}}).$$

As an equivalence is a P-equivalence, we have a natural functor

$$l : \mathrm{Ho}(\mathsf{sGAff}_{\mathbb{U}}) \longrightarrow \mathrm{Ho}^P(\mathsf{sGAff}_{\mathbb{U}}).$$

Proposition 3.10. *The functor*

$$l : \mathrm{Ho}(\mathsf{sGAff}_{\mathbb{U}}) \longrightarrow \mathrm{Ho}^P(\mathsf{sGAff}_{\mathbb{U}})$$

possesses a right adjoint

$$j : \mathrm{Ho}^P(\mathsf{sGAff}_{\mathbb{U}}) \longrightarrow \mathrm{Ho}(\mathsf{sGAff}_{\mathbb{U}})$$

which is fully faithful. The essential image of j is of the smallest full sub-category of $\mathrm{Ho}(\mathsf{sGAff}_{\mathbb{U}})$ *that contains the objects $K(K, V, n)$ and which is stable by homotopy limits.*

Proof. This involves an application of the existence of a cellularization functor to the model category $\mathsf{Hopf}_{\mathbb{U}}^{\Delta}$ and to the set of objects $\{K(K, V, n)\}$ (see [Hir03, Propositions 5.2.3 and 5.2.4]). For the reader's convenience, we shall reproduce the argument here.

Let X be a set of representatives for the morphisms

$$K(K, V, n)^{\Delta^m} \longrightarrow K(K, V, n)^{\partial\Delta^m}$$

for all affine group schemes of finite type K, all finite-dimensional linear representations V of K, and all integers $n \geq 1$ and $m \geq 0$. By Corollary 3.8, all the morphisms in X are fibrations. For a given cofibrant object $G_* \in \mathsf{sGAff}_{\mathbb{U}}$, we construct a tower of cofibrant objects in $G_*/\mathsf{sGAff}_{\mathbb{U}}$,

$$G_* \longrightarrow \ldots G_*^{(i)} \longrightarrow G_*^{(i-1)} \longrightarrow \cdots \longrightarrow G_*^{(0)} = *,$$

defined inductively in the following way. Let I_i be the set of all commutative squares in $\mathsf{sGAff}_{\mathbb{U}}$ of the form

$$\begin{array}{ccc} G_* & \longrightarrow & H_* \\ \downarrow & & \downarrow u \\ G_*^{(i-1)} & \longrightarrow & H'_* \end{array}$$

with $u \in X$. The set I_i is then $\mathbb{U}$-small.

We now define an object $G_*(i)$ via the following pull-back square.

$$\begin{array}{ccc} G_*(i) & \longrightarrow & \prod_{j \in I_i} H_* \\ \downarrow & & \downarrow \\ G_*^{(i-1)} & \longrightarrow & \prod_{j \in I_i} H'_* \end{array}$$

Let $G_* \to G_*^{(i)} = Q(G_*(i))$ be the cofibrant replacement of $F \to G_*(i)$ in $G_*/\mathsf{sGAff}_{\mathbb{U}}$; this defines the tower inductively on i. Finally, we consider the morphism

$$\alpha : G_* \longrightarrow \widetilde{G_*} := \mathrm{Lim}_i\, G_*^{(i)}.$$

First, we claim that the object $\widetilde{G_*}$ is P-local in the sense that for any P-equivalence $H_* \longrightarrow H'_*$, the induced morphism

$$\mathbb{R}\,\underline{\mathrm{Hom}}(H'_*, G_*) \longrightarrow \mathbb{R}\,\underline{\mathrm{Hom}}(H_*, G_*)$$

is an isomorphism in $\mathrm{Ho}(\mathsf{SSet})$. Indeed, by construction it is a $\mathbb{U}$-small homotopy limit of P-local objects.

Thus, it only remains to show that the morphism α is a P-equivalence, as this would imply formally that $G_* \longrightarrow \widetilde{G_*}$ is a P-localization (i.e. a universal P-equivalence with a P-local object). This, in turn, would yield the result, since the functor $G_* \mapsto \widetilde{G_*}$ would then identify the localization $\mathrm{Ho}^P(\mathsf{sGAff}_{\mathbb{U}})$ with the full sub-category of P-local objects in $\mathrm{Ho}(\mathsf{sGAff}_{\mathbb{U}})$. Moreover, the construction of $\widetilde{G_*}$ shows that the P-local objects are obtained via successive homotopy limits of objects of the form $K(K, V, n)$.

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Let us, then, consider $K(K,V,n)$ for a given affine group scheme of finite type K, a finite-dimensional linear representation V of K, and an integer $n \geq 1$. Using the fact that the simplicial affine group scheme $K(K,V,n)$ is levelwise of finite type and is n-truncated, we see that it is a ω-cosmall object in $\mathsf{sGAff}_{\mathbb{U}}$. Moreover, since $\widetilde{G_*}$, $G_*^{(i)}$ and G_* are all cofibrant, the morphism

$$\alpha^* : \mathbb{R}\,\underline{\mathrm{Hom}}(\widetilde{G_*}, K(K,V,n)) \longrightarrow \mathbb{R}\,\underline{\mathrm{Hom}}(\widetilde{G_*}, K(K,V,n))$$

is isomorphic in $\mathrm{Ho}(\mathsf{SSet})$ to the natural morphism

$$\alpha^* : \mathrm{Colim}_i\,\underline{\mathrm{Hom}}(G_*^{(i)}, K(K,V,n)) \longrightarrow \underline{\mathrm{Hom}}(G_*, K(K,V,n)).$$

Thus, by the inductive construction of the tower, we deduce that for any m, the morphism

$$\mathbb{R}\,\underline{\mathrm{Hom}}(\widetilde{G_*}, K(K,V,n))^{\partial\Delta^m} \longrightarrow \mathbb{R}\,\underline{\mathrm{Hom}}(\widetilde{G_*}, K(K,V,n))^{\Delta^m} \times^h_{\mathbb{R}\underline{\mathrm{Hom}}(G_*,K(K,V,n))^{\Delta^m}} \mathbb{R}\,\underline{\mathrm{Hom}}(G_*, K(K,V,n))^{\partial\Delta^m}$$

is surjective on connected components. This implies that α^* is an isomorphism and, therefore, that $G_* \longrightarrow \widetilde{G_*}$ is a P-equivalence as required. $\square$

Using the previous proposition, we can (and will) always implicitly identify the category $\mathrm{Ho}^P(\mathsf{sGAff}_{\mathbb{U}})$ with the full sub-category of $\mathrm{Ho}(\mathsf{sGAff}_{\mathbb{U}})$ consisting of P-local objects, and also with the smallest full sub-category of $\mathrm{Ho}(\mathsf{sGAff}_{\mathbb{U}})$ that contains the $K(K,V,n)$ and which is stable by homotopy limits. The left adjoint of the inclusion functor

$$\mathrm{Ho}^P(\mathsf{sGAff}_{\mathbb{U}}) \hookrightarrow \mathrm{Ho}(\mathsf{sGAff}_{\mathbb{U}})$$

is isomorphic to the localization functor and can be identified with the construction $G_* \mapsto \widetilde{G_*}$ given in the proof of the theorem. We will often denote this functor by

$$G_* \mapsto G_*^P.$$

We now give a more explicit description of the P-equivalences related to the Hochschild cohomology of affine group schemes with coefficients in linear representations. For a Hopf algebra B and a B-comodule V (which most of the time will be assumed to be of finite dimension, though this is not strictly needed), one can consider the cosimplicial k-vector space

$$\begin{array}{rccc} C^*(B,V): & \Delta & \longrightarrow & k\text{-}\mathsf{Vect} \\ & [n] & \mapsto & C^n(B,V) := V \otimes B^{\otimes n}, \end{array}$$

where the transition morphisms $V \otimes B^{\otimes n} \longrightarrow V \otimes B^{\otimes m}$ are given by the co-action and co-unit morphisms. From a dual point of view, V corresponds to a linear representation of the affine group scheme $G = \mathrm{Spec}\, B$, and can be considered as a quasi-coherent sheaf $\mathbb{V}$ on the simplicial affine scheme BG. The cosimplicial space $C^*(B,V)$ is, by definition, the cosimplicial space of sections $\Gamma(BG,\mathbb{V})$ of this sheaf on BG.

The cosimplicial vector space $C^*(B,V)$ has an associated total complex, whose cohomology groups will be denoted by

$$H^i(B,V) := H^i(\mathrm{Tot}(C^*(B,V))).$$

These are the Hochschild cohomology groups of B with coefficients in V. From a dual point of view, the complex $\mathrm{Tot}(C^*(B,V))$ also computes the cohomology of the affine group scheme G with coefficients in the linear representation V. We will use the notation

$$C^*(G,V) := C^*(B,V), \quad H^*(G,V) := H^*(B,V).$$

Now, let $B_* \in \mathsf{Hopf}_{\mathbb{U}}^{\Delta}$ be a cosimplicial Hopf algebra, and consider the Hopf algebra $H^0(B_*)$ of zeroth cohomology of B_*. We have $\mathrm{Spec}\, H^0(B_*) \simeq \Pi_0(G_*)$, where $G_* := \mathrm{Spec}\, B_*$.

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By construction, $H^0(B_*)$ is the limit (in the category of Hopf algebras) of the cosimplicial diagram $[n] \mapsto B_n$, and therefore comes equipped with a natural co-augmentation $H^0(B_*) \to B_*$. In particular, if V is any $H^0(B_*)$-comodule, V can also be considered naturally as a comodule over each B_n. In this way, we obtain a cosimplicial object in the category of cosimplicial vector spaces (i.e. a bi-cosimplicial vector space):

$$[n] \mapsto C^*(B_n, V).$$

The diagonal associated to this bi-cosimplicial space will be denoted by

$$C^*(B_*, V) := \mathrm{Diag}\left([n] \mapsto C^*(B_n, V)\right) := \left([n] \mapsto C^n(B_n, V)\right).$$

We let $G_* = \mathrm{Spec}\, B_*$ be the associated simplicial affine group scheme, and also write

$$C^*(G_*, V) := C^*(B_*, V).$$

The cohomology of the total complex associated to $C^*(B_*, V)$ is called the *Hochschild cohomology* of the cosimplicial algebra B_* (or, equivalently, of the simplicial affine group scheme $G_* = \mathrm{Spec}\, B_*$) with coefficients in the comodule V; it is written as

$$H^n(B_*, V) := H^n(\mathrm{Tot}(C^*(B_*, V))), \quad H^n(G_*, V) := H^n(\mathrm{Tot}(C^*(G_*, V))).$$

We see that $C^*(G_*, V)$ is the cosimplicial space of sections of V considered as a quasi-coherent sheaf on h_{BG_*}, represented by the simplicial affine scheme BG_*. As the sheaf V is quasi-coherent, we have natural isomorphisms

$$H^i(\mathrm{Tot}(C^*(G_*, V))) \simeq H^i(h_{BG_*}, V) := \pi_0(\mathrm{Map}_{\mathsf{SPr}(k)/h_{BG_*}}(h_{BG_*}, F(V, i))),$$

where $F(V, i) \longrightarrow h_{BG_*}$ is the relative Eleinberg–MacLane construction on the sheaf of abelian groups V. This can be deduced easily from the special case of a non-simplicial affine group scheme treated in [Toe06, §§ 1.3 and 1.5], simply by noticing that h_{BG_*} is naturally equivalent to the homotopy colimit of the h_{BG_n} as n varies in Δ^{op}.

Proposition 3.11. *A morphism $f : G_* \longrightarrow H_*$ is a P-equivalence if and only if it satisfies the following two properties.*

(i) *For any finite-dimensional linear representation V of $\Pi_0(H_*)$, the induced morphism*

$$f^* : H^*(H_*, V) \longrightarrow H^*(G_*, V)$$

is an isomorphism.

(ii) *The induced morphism $\Pi_0(G_*) \longrightarrow \Pi_0(H_*)$ is an isomorphism.*

Proof. Let $G_* \in \mathsf{sGAff}_{\mathbb{U}}$, let K be an affine group scheme of finite type, and let V be a finite-dimensional linear representation of K. We assume that G_* is a cofibrant object. Then there exists a natural morphism of simplicial sets,

$$\underline{\mathrm{Hom}}(G_*, K(K, V, n)) \longrightarrow \underline{\mathrm{Hom}}_*(Bh_{G_*}, Bh_{K(K,V,n)}),$$

where $\underline{\mathrm{Hom}}_*$ denotes the simplicial set of morphisms of the category $\mathsf{SPr}_*(k)$. Upon composing this with the fibrant and cofibrant replacement functors in $\mathsf{SPr}_*(k)$, we get a natural morphism in $\mathrm{Ho}(\mathsf{SSet}_*)$:

$$\mathbb{R}\,\underline{\mathrm{Hom}}(G_*, K(K, V, n)) \longrightarrow \mathbb{R}\,\underline{\mathrm{Hom}}_*(Bh_{G_*}, Bh_{K(K,V,n)}).$$

This morphism comes equipped with a natural projection to the set $\mathrm{Hom}(\Pi_0(G_*), K)$, as follows.

$$\begin{array}{ccc} \mathbb{R}\,\underline{\mathrm{Hom}}(G_*, K(K,V,n)) & \longrightarrow & \mathbb{R}\,\underline{\mathrm{Hom}}_*(Bh_{G_*}, Bh_{K(K,V,n)}) \\ & \searrow \quad \swarrow & \\ & \mathrm{Hom}(\Pi_0(G_*), K) & \end{array}$$

Moreover, the homotopy fiber F of the right-hand side morphism is such that

$$\pi_i(F) \simeq H^{n-i}(G_*, V).$$

This shows that in order to prove the lemma, it is enough to prove that the horizontal morphism

$$\underline{\mathrm{Hom}}(G_*, K(K,V,n)) \simeq \mathbb{R}\,\underline{\mathrm{Hom}}(G_*, K(K,V,n)) \longrightarrow \mathbb{R}\,\underline{\mathrm{Hom}}_*(Bh_{G_*}, Bh_{K(K,V,n)})$$

is an isomorphism (recall that G_* is cofibrant and that $K(K,V,n)$ is always fibrant).

To establish this, we use the fact that $\underline{\mathrm{Hom}}(G_*, K(K,V,n))$ is naturally isomorphic to the total space (see [Hir03, Definition 18.6.3]) of the cosimplicial space $m \mapsto \underline{\mathrm{Hom}}(G_m, K(K,V,n))$, that is,

$$\mathbb{R}\,\underline{\mathrm{Hom}}(G_*, K(K,V,n)) \simeq \underline{\mathrm{Hom}}(G_*, K(K,V,n)) \simeq \mathrm{Tot}(m \mapsto \underline{\mathrm{Hom}}(G_m, K(K,V,n))).$$

In the same way, the simplicial presheaf Bh_{G_*} is equivalent to the homotopy colimit of the diagram $n \mapsto Bh_{G_n}$, and thus we have

$$\mathbb{R}\,\underline{\mathrm{Hom}}_*(Bh_{G_*}, Bh_{K(K,V,n)}) \simeq \mathrm{Holim}_n\, \mathbb{R}\,\underline{\mathrm{Hom}}_*(Bh_{G_m}, Bh_{K(K,V,n)}).$$

Therefore, proving the lemma involves verifying the following two statements.

(a) The natural morphism

$$\mathrm{Tot}\,(m \mapsto \underline{\mathrm{Hom}}(G_m, K(K,V,n))) \longrightarrow \mathrm{Holim}\,(m \mapsto \underline{\mathrm{Hom}}(G_m, K(K,V,n)))$$

(see [Hir03, Definition 18.7.3]) is an isomorphism in $\mathrm{Ho}(\mathsf{SSet})$.

(b) For any m, the natural morphism

$$\underline{\mathrm{Hom}}(G_m, K(K,V,n)) \longrightarrow \mathbb{R}\,\underline{\mathrm{Hom}}_*(Bh_{G_m}, Bh_{K(K,V,n)})$$

is an isomorphism in $\mathrm{Ho}(\mathsf{SSet})$.

For property (a), we make use of the fact that $K(K,V,n)$ are abelian group objects in the category of simplicial affine group schemes over K. This implies that the morphism

$$\mathrm{Tot}\,(m \mapsto \underline{\mathrm{Hom}}(G_m, K(K,V,n))) \longrightarrow \mathrm{Holim}\,(m \mapsto \underline{\mathrm{Hom}}(G_m, K(K,V,n)))$$

is a morphism of abelian group objects in the category of simplicial sets over the set $\mathrm{Hom}(G_m; K)$ or, in other words, is a morphism between disjoint unions of simplicial abelian groups. Property (a) then follows from the fact that for any cosimplicial object X_* in the category of simplicial abelian groups, the natural morphism

$$\mathrm{Tot}(X_*) \longrightarrow \mathrm{Holim}(X_*)$$

is a weak equivalence (see, e.g., [BK72, X § 4.9, XI § 4.4]).

Property (b) clearly holds for $n = 0$, because we can simply find a bijection of sets

$$\mathrm{Hom}(G_m, K \rtimes V) \simeq \mathrm{Hom}_*(Bh_{G_m}, Bh_{K \rtimes V}).$$

Let us assume that $n > 0$. Since we have

$$\pi_i(\underline{\mathrm{Hom}}(G_m, K(K,V,n))) \simeq \pi_{i-1}(\underline{\mathrm{Hom}}(G_m, K(K,V,n-1))),$$
$$\pi_i(\mathbb{R}\,\underline{\mathrm{Hom}}_*(Bh_{G_m}, Bh_{K(K,V,n)})) \simeq \pi_{i-1}(\mathbb{R}\,\underline{\mathrm{Hom}}_*(Bh_{G_m}, Bh_{K(K,V,n-1)})),$$

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by induction it is enough to show that the natural morphism

$$\pi_0(\underline{\mathrm{Hom}}(G_m, K(K,V,n))) \longrightarrow \pi_0(\mathbb{R}\,\underline{\mathrm{Hom}}_*(Bh_{G_m}, Bh_{K(K,V,n)}))$$

is bijective. For this it is, in turn, sufficient to prove that both of the morphisms

$$\pi_0(\underline{\mathrm{Hom}}(G_m, K(K,V,n))) \longrightarrow \mathrm{Hom}(G_m, K)$$
$$\pi_0(\mathbb{R}\,\underline{\mathrm{Hom}}_*(Bh_{G_m}, Bh_{K(K,V,n)})) \longrightarrow \mathrm{Hom}(G_m, K)$$

are isomorphisms. As the natural projection $K(K,V,n) \longrightarrow K$ possesses a section, these two morphisms are surjective; so it remains to show that these morphisms are injective. This will follow from the next lemma.

Lemma 3.12. *Let G_* be a cofibrant simplicial affine group scheme. Then, for any $m \geq 0$ and any faithfully flat morphism of affine group schemes $H \longrightarrow H'$, the induced morphism*

$$\mathrm{Hom}(G_m, H') \longrightarrow \mathrm{Hom}(G_m, H)$$

is surjective.

Proof. We consider the evaluation functor $H_* \mapsto H_m$ and its right adjoint

$$i_m^* : \mathsf{GAff}_{\mathbb{U}} \longrightarrow \mathsf{sGAff}_{\mathbb{U}}.$$

We have

$$i_m^*(H)_p \simeq H^{\mathrm{Hom}([m],[p])},$$

which shows that i_m^* preserves faithfully flat morphisms. Moreover, for any simplicial set A we have that

$$\mathrm{Map}(A, i_m^*(H)) \simeq H^{A_m},$$

which easily implies that $\Pi_i(i_m^*(H)) = 0$ for any i. In particular, if $H \longrightarrow H'$ is a faithfully flat morphism, then we see from Lemma 3.3 that the induced morphism

$$i_m^*(H) \longrightarrow i_m^*(H')$$

is a trivial fibration. The right lifting property for G_* with respect to this last morphism then says precisely that $\mathrm{Hom}(G_m, H') \longrightarrow \mathrm{Hom}(G_m, H)$ is surjective. □

The above lemma tells us that for any linear representation V of G_m, we have $H^2(G_m, V) = 0$, since this group classifies extensions

$$1 \longrightarrow V \longrightarrow J \longrightarrow G_m \longrightarrow 1.$$

This, in turn, implies that $H^i(G_m, V)$ for any $i > 1$, and thus that the natural morphism

$$\pi_0(\mathbb{R}\,\underline{\mathrm{Hom}}_*(Bh_{G_m}, Bh_{K(K,V,n)})) \longrightarrow \mathrm{Hom}(G_m, K)$$

is injective, since its fibers are in bijection with $H^n(G_m, V)$. For the other morphism, we consider the morphism of affine group schemes

$$\mathrm{Map}(\Delta^1, K(K,V,n)) \longrightarrow \mathrm{Map}(\Delta^1, K) \times_{\mathrm{Map}(\partial\Delta^1, K)} \mathrm{Map}(\partial\Delta^1, K(K,V,n)).$$

Since the morphism $K(K,V,n) \longrightarrow K$ is a fibration (by Lemma 3.3(i)) and is relatively (1)-connected (i.e. its fibers are 1-connected), the above morphism is faithfully flat. The right lifting property of G_m with respect to this morphism (which is guaranteed by Lemma 3.12) implies that the morphism

$$\pi_0(\underline{\mathrm{Hom}}(G_m, K(K,V,n))) \longrightarrow \mathrm{Hom}(G_m, K)$$

is injective. This finishes the proof of Proposition 3.11. □

From the proof of Proposition 3.11 we also extract the following important corollary that will be used later.

Corollary 3.13. *Let $G_* \in \mathsf{sGAff}_{\mathbb{U}}$, let K be a affine group scheme of finite type, let V be a finite-dimensional linear representation of K, and let $n \geq 1$. Then the natural morphism*

$$\mathbb{R}\,\underline{\mathrm{Hom}}(G_*, K(K,V,n)) \longrightarrow \mathbb{R}\,\underline{\mathrm{Hom}}_{\mathsf{SGp(k)}}(h_{G_*}, h_{K(K,V,n)}) \simeq \mathbb{R}\,\underline{\mathrm{Hom}}_{\mathsf{SPr}_*(\mathsf{k})}(Bh_{G_*}, Bh_{K(K,V,n)})$$

is an isomorphism in $\mathrm{Ho}(\mathsf{SSet})$.

3.3 Cosimplicial Hopf algebras and schematic homotopy types

We are now ready to explain how cosimplicial Hopf algebras are models for schematic homotopy types.

We consider $\mathsf{SGp}(k)$, the category of presheaves of $\mathbb{V}$-simplicial groups on $(\mathsf{Aff}, \mathrm{fpqc})$. It will be endowed with the model category structure for which equivalences and fibrations are defined via the forgetful functor $\mathsf{SGp}(k) \longrightarrow \mathsf{SPr}(k)$: a morphism in $\mathsf{SGp}(k)$ is a fibration (respectively, an equivalence) if and only if it is a fibration (respectively, an equivalence) as a morphism in $\mathsf{SPr}(k)$ (by forgetting the group structure). We consider the Yoneda functor

$$h_- : \mathsf{sGAff}_{\mathbb{U}} \longrightarrow \mathsf{SGp}(k).$$

The functor h sends equivalences to local equivalences of simplicial presheaves and thus induces a functor

$$h : \mathrm{Ho}(\mathsf{sGAff}_{\mathbb{U}}) \longrightarrow \mathrm{Ho}(\mathsf{SGp}(k)).$$

Consider the classifying space functor

$$\boldsymbol{B} : \mathsf{SGp}(k) \longrightarrow \mathsf{SPr}(k)_*$$

from the category of presheaves of simplicial groups to the category of pointed simplicial presheaves. It is well-known (see, for example, [Toe06, Theorem 1.4.3] and [Toe02, Proposition 1.5]) that this functor preserves equivalences and induces a fully faithful functor on the homotopy categories,

$$\boldsymbol{B} : \mathrm{Ho}(\mathsf{SGp}(k)) \longrightarrow \mathrm{Ho}(\mathsf{SPr}_*(k)).$$

Upon composing this with the functor

$$h : \mathrm{Ho}(\mathsf{sGAff}_{\mathbb{U}}) \longrightarrow \mathrm{Ho}(\mathsf{SGp}(k)),$$

one obtains a functor

$$\boldsymbol{B}h : \mathrm{Ho}(\mathsf{sGAff}_{\mathbb{U}}) \longrightarrow \mathrm{Ho}(\mathsf{SPr}_*(k)).$$

With this notation we have the following theorem.

Theorem 3.14. *The functor*

$$\boldsymbol{B}h\ : \mathrm{Ho}(\mathsf{sGAff}_{\mathbb{U}}) \longrightarrow \mathrm{Ho}(\mathsf{SPr}_*(k))$$

is fully faithful when restricted to the full sub-category $\mathrm{Ho}^P(\mathsf{sGAff}_{\mathbb{U}}) \subset \mathrm{Ho}(\mathsf{sGAff}_{\mathbb{U}})$ *consisting of P-local objects. Its essential image consists precisely of all psht.*

Proof. First, we analyze the full faithfulness properties of $\boldsymbol{B}h$. Let G_* and G'_* be two P-local and cofibrant simplicial affine group schemes. We need to prove that the natural morphism of simplicial sets

$$\underline{\mathrm{Hom}}(G_*, G'_*) \to \underline{\mathrm{Hom}}_{\mathsf{SPr}_*(k)}(\boldsymbol{B}h_{G'}, \boldsymbol{B}h_{G'_*}) \to \mathbb{R}\,\underline{\mathrm{Hom}}_{\mathsf{SPr}_*(k)}(\boldsymbol{B}h_{G'}, \boldsymbol{B}h_{G'_*})$$

is a weak equivalence. As the functor B is fully faithful, it is enough to show that the induced morphism

$$\underline{\mathrm{Hom}}(G_*, G'_*) \to \mathbb{R}\,\underline{\mathrm{Hom}}_{\mathsf{SGp}(k)}(h_{G'}, h_{G'_*})$$

is an equivalence. Because G'_* is P-local, it can be written as a transfinite composition of homotopy pull-backs of objects of the form $K(K, V, n)$ (see [Hir03, Theorem 5.1.5] and also the proof of Proposition 3.10), for some affine group scheme K of finite type, some finite-dimensional linear representation V of K, and some integer $n \geq 1$.

LEMMA 3.15. *The functor*

$$h : \mathsf{sGAff}_{\mathbb{U}} \longrightarrow \mathsf{SGp}(k)$$

commutes with homotopy limits of P-local objects.

Proof. It suffices to show that h preserves homotopy pull-backs and (possibly infinite) homotopy products of P-local objects. The case of homotopy pull-backs follows from [Jar87, Lemma 1.15], as we have already seen that the functor h sends fibrations to local fibrations of simplicial presheaves. The case of infinite homotopy products of P-local objects would follow from the fact that h commutes with products and sends P-local objects to fibrant objects in $\mathsf{SGp}(k)$. As the P-local objects are obtained by transfinite composition of homotopy pull-backs of objects of the form $K(K, V, n)$, it is enough to check that $h_{K(K,V,n)}$ is a fibrant simplicial presheaf; but this is clear because, as a simplicial presheaf, it is isomorphic to $h_K \times K(\mathbb{G}_a, n)^d$, where d is the dimension of V, which is known to be fibrant since both h_K and $K(\mathbb{G}_a, n)$ are fibrant (see, e.g., [Toe06, Lemma 1.1.2]). □

By virtue of Lemma 3.15, the problem is reduced to proving that the natural morphism

$$\underline{\mathrm{Hom}}(G_*, K(K, V, n)) \longrightarrow \mathbb{R}\,\underline{\mathrm{Hom}}_{\mathsf{SPr}_*(k)}(\boldsymbol{B}h_{G'}, \boldsymbol{B}h_{K(K,V,n)})$$

is an equivalence; but this is something we have already established while proving Proposition 3.11 (see Corollary 3.13).

To complete the proof of Theorem 3.14, it remains to show that the image of the functor $\boldsymbol{B}h$ consists precisely of all psht. That the image of $\boldsymbol{B}h$ is contained in the category of psht follows from [Toe06, Corollary 3.2.7] and the fact that the P-local affine group schemes are generated via homotopy limits by the objects $K(K, V, n)$ (note that $\boldsymbol{B}h_{K(K,V,n)}$ is a psht with $\pi_1 = K$, $\pi_n = V$ and $\pi_i = 0$ for $i \neq n$). Now suppose that F is a psht. We will show that F is P-equivalent to an object of the form $Bh\,G_*$, for some P-local object $G_* \in \mathsf{sGAff}_{\mathbb{U}}$ (see [Toe06, Definition 3.1.1] for the notion of P-local equivalences). Since, by definition, a psht is a P-local object, this would imply that F is isomorphic to $Bh\,G_*$ as required. To prove our claim, let us assume that F is cofibrant as an object in $\mathsf{SPr}_*(k)$; we shall construct a P-local model of F by the small-object argument. Let $\mathscr{K}$ be a $\mathbb{U}$-small set of representatives of all psht of the form $\boldsymbol{B}h_{K(K,V,n)}$, for some affine group scheme K which is of finite type as a k-algebra, some finite-dimensional linear representation V of K, and some $n \geq 1$. We consider the $\mathbb{U}$-small set K of all morphisms of the form

$$G^{\Delta^m} \longrightarrow G^{\partial\Delta^m},$$

for $G \in \mathscr{K}$ and $m \geq 0$. Finally, we let $\Lambda(K)$ be the $\mathbb{U}$-small set of morphisms in $\mathsf{SPr}_*(k)$ which are fibrant approximations of the morphisms of K. We construct a tower of cofibrant objects in $F/\mathsf{SPr}_*(k)$,

$$F \longrightarrow \ldots F_i \longrightarrow F_{i-1} \longrightarrow \cdots \longrightarrow F_0 = *,$$

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defined inductively in the following way. Let I_i be the set of all commutative squares in $\mathsf{SPr}_*(k)$:

$$\begin{array}{ccc} F & \longrightarrow & G_1 \\ \downarrow & & \downarrow u \\ F_{i-1} & \longrightarrow & G_2 \end{array}$$

where $u \in \Lambda(K)$. Let J_i be a subset in I_i of representatives for the isomorphism classes of objects in the homotopy category of commutative squares in $\mathsf{SPr}_*(k)$ (i.e. J_i is a subset of representatives for the equivalence classes of squares in I_i). Note that the set J_i is $\mathbb{U}$-small, since all stacks in the previous diagrams are affine ∞-gerbes.

We now define an object F_i' as the pull-back square

$$\begin{array}{ccc} F_i' & \longrightarrow & \prod_{j\in J_i} G_1 \\ \downarrow & & \downarrow \\ F_{i-1} & \longrightarrow & \prod_{j\in J_i} G_2 \end{array}$$

and define $F \to F_i = Q(F_i')$ as a cofibrant replacement of $F \to F_i$. This gives the ith step of the tower from its $(i-1)$th step. Finally, we consider the morphism

$$\alpha : F \longrightarrow \widetilde{F} := \mathrm{Lim}_i\, F_i.$$

We first claim that the object $\widetilde{F} \in \mathrm{Ho}(\mathsf{SPr}_*(k))$ lies in the essential image of the functor $\boldsymbol{B}\,\mathrm{Spec}$. Indeed, it is a $\mathbb{U}$-small homotopy limit of objects belonging to this essential image, and as the functor $\boldsymbol{B}h$ is fully faithful and commutes with homotopy limits of P-local objects (see Lemma 3.15), we see that $\widetilde{F}$ stays in the essential image of $\boldsymbol{B}h$. It remains to show that the morphism α is a P-equivalence. For this, let F_0 be a psht of the form $\boldsymbol{B}h_{K(K,V,n)}$. Using the fact that $\widetilde{F}$ and F_0 are both in the image of $\boldsymbol{B}h$, we see that the morphism

$$\alpha^* : \mathbb{R}\,\underline{\mathrm{Hom}}(\widetilde{F}, F_0) \longrightarrow \mathbb{R}\,\underline{\mathrm{Hom}}(F, F_0)$$

is isomorphic in $\mathrm{Ho}(\mathsf{SSet})$ to the natural morphism

$$\alpha^* : \mathrm{Colim}_i\, \mathbb{R}\,\underline{\mathrm{Hom}}(F_i, F_0) \longrightarrow \mathbb{R}\,\underline{\mathrm{Hom}}(F, F_0).$$

Thus, by inductive construction of the tower, it is clear that for any m, the morphism

$$\mathbb{R}\,\underline{\mathrm{Hom}}(\widetilde{F}, F_0)^{\partial\Delta^m} \longrightarrow \mathbb{R}\,\underline{\mathrm{Hom}}(\widetilde{F}, F_0)^{\Delta^m} \times^h_{\mathbb{R}\,\underline{\mathrm{Hom}}(F,F_0)^{\Delta^m}} \mathbb{R}\,\underline{\mathrm{Hom}}(F, F_0)^{\partial\Delta^m}$$

is surjective on connected components. This implies that α^* is an isomorphism and, therefore, that $F \longrightarrow \widetilde{F}$ is a P-equivalence. □

The following corollary completes the characterization of psht given in [Toe06, Theorem 3.2.4 and Proposition 3.2.9].

Corollary 3.16. *An object $F \in \mathrm{Ho}(\mathsf{SPr}_*(k))$ is a psht if and only if $\pi_1(F, *)$ is an affine group scheme and $\pi_i(F, *)$ is an affine unipotent group scheme.*

Proof. This follows from [Toe06, Corollary 3.2.7] and the essential surjectivity part of Theorem 3.14. □

We are now ready to explain the relations between cosimplicial Hopf algebras and schematic homotopy types. The first step is to identify the P-equivalences in $\mathsf{Hopf}^{\Delta}_{\mathbb{U}}$.

Corollary 3.17. *A morphism of cosimplicial Hopf algebras*

$$f : B_* \longrightarrow B'_*$$

is a P-equivalence if and only if the induced morphism on the total complexes

$$\mathrm{Tot}(B_*) \longrightarrow \mathrm{Tot}(B'_*)$$

is a quasi-isomorphism.

Proof. We take $G_* \in \mathsf{sGAff}_{\mathbb{U}}$ and let $G_* \longrightarrow G_*^P$ be a P-localization. Using Corollary 3.13, we see that the induced morphism

$$\boldsymbol{B}h_{G_*} \longrightarrow \boldsymbol{B}h_{G_*^P}$$

is a P-equivalence of simplicial presheaves. In the same way, it has been proved in [Toe06, Corollary 3.2.7] that the natural morphism

$$\boldsymbol{B}h_{G_*} \longrightarrow \boldsymbol{B}\mathbb{R}\mathrm{Spec}\,\mathcal{O}(G_*)$$

is a P-equivalence of simplicial presheaves. As both $\boldsymbol{B}h_{G_*^P}$ and $\boldsymbol{B}\mathbb{R}\mathrm{Spec}\,\mathcal{O}(G_*)$ are P-local simplicial presheaves, we deduce that they are naturally equivalent as pointed simplicial presheaves.

From this and Theorem 3.14, we deduce that a morphism $G_* \longrightarrow G'_*$ of simplicial affine group schemes is a P-equivalence if and only if the induced morphism of simplicial presheaves

$$\boldsymbol{B}\mathbb{R}\mathrm{Spec}\,\mathcal{O}(G_*) \longrightarrow \boldsymbol{B}\mathbb{R}\mathrm{Spec}\,\mathcal{O}(G'_*)$$

is an equivalence. This is also equivalent to saying that the induced morphism

$$\mathbb{R}\mathrm{Spec}\,\mathcal{O}(G_*) \longrightarrow \mathbb{R}\mathrm{Spec}\,\mathcal{O}(G'_*)$$

is an equivalence. By [Toe06, Corollary 2.2.3], this is the same as saying that

$$\mathcal{O}(G'_*) \longrightarrow \mathcal{O}(G_*)$$

is an equivalence of cosimplicial algebras or, in other words, that

$$\mathrm{Tot}(\mathcal{O}(G'_*)) \longrightarrow \mathrm{Tot}(\mathcal{O}(G_*))$$

is a quasi-isomorphism. □

Putting Corollary 3.17 and Theorem 3.14 together, we find the following result, which explains in what sense cosimplicial Hopf algebras are algebraic models for pointed schematic homotopy types.

Corollary 3.18. *Let $B_* \mapsto B_*^P$ be a P-localization in $\mathsf{Hopf}^{\Delta}_{\mathbb{U}}$. Then the composite functor*

$$B_* \mapsto B_*^P \mapsto \boldsymbol{B}h_{\mathrm{Spec}\,B_*^P}$$

induces an equivalence between $\mathrm{Ho}^P(\mathsf{Hopf}^{\Delta}_{\mathbb{U}})^{\mathrm{op}}$, the localized category of cosimplicial Hopf algebras along quasi-isomorphisms, and the homotopy category of pointed schematic homotopy types.

By using cosimplicial Hopf algebras as models for psht, we can describe the schematization functor $X \mapsto (X \otimes k)^{\mathrm{sch}}$ of [Toe06] in the following explicit way.

Let X be a pointed and connected simplicial set in $\mathbb{U}$. As X is connected, one can choose an equivalent X which is a reduced simplicial set (i.e. a simplicial set with $X_0 = *$). We then apply Kan's loop group construction which associates to X a simplicial group GX. Kan's loop group functor is left adjoint to the classifying space functor that associates to each simplicial group its classical classifying space. Explicitly (see [GJ99, § V]), GX is the simplicial group whose group of n-simplices is the free group generated by the set $X_{n+1} - s_0(X_n)$. Upon applying the pro-algebraic completion functor levelwise, we turn GX into a simplicial affine group scheme GX^{alg} in $\mathbb{U}$. The object GX^{alg} can be thought of as an *algebraic loop space for* X. The corresponding cosimplicial algebra $B_* := \mathcal{O}(GX^{\mathrm{alg}})$ of k-valued regular functions on GX^{alg} is such that the level-n component B_n of B_* is the sub-Hopf algebra of $k^{GX_n} = \mathrm{Hom}(GX_n, k)$ of functions spanning a finite-dimensional sub-GX_n-module.

Corollary 3.19. *With the above notation, there exists a natural isomorphism in* $\mathrm{Ho}(\mathsf{SPr}_*(k))$,

$$(X \otimes k)^{\mathrm{sch}} \simeq \boldsymbol{B}\,\mathrm{Spec}\, B_*^P,$$

where B_*^P *is a P-local model for* B_*.

Proof. As shown in the proof of [Toe06, Theorem 3.3.4], the schematization $(X \otimes k)^{\mathrm{sch}}$ is given by $\boldsymbol{B}\mathbb{R}\mathrm{Spec}\, B_*$. Also, we have already seen in the proof of Corollary 3.17 that there exists a natural equivalence

$$\boldsymbol{B}\mathbb{R}\mathrm{Spec}\, B_* \simeq \boldsymbol{B}\,\mathrm{Spec}\, B_*^P. \qquad \square$$

3.4 Equivariant cosimplicial algebras

In this section we provide another algebraic model for psht. This model is based on equivariant cosimplicial algebras and is quite similar in approach to [BS93]. The main difference between these two approaches is that we work equivariantly over a affine group scheme, whereas [BS93] works over a discrete group.

Equivariant stacks. We start with some reminders on the notion of equivariant stacks.

Throughout this section we fix a presheaf of groups G on $(\mathsf{Aff})_{\mathrm{ffqc}}$, which will be considered as a group object in $\mathsf{SPr}(k)$. We will make the assumption that G is cofibrant as an object of $\mathsf{SPr}(k)$. For example, G could be representable (i.e. an affine group scheme) or a constant presheaf associated to a group in $\mathbb{U}$.

Because the direct product makes the category $\mathsf{SPr}(k)$ a cofibrantly generated symmetric monoidal closed model category for which the monoid axiom of [SS00, § 3] is satisfied, the category G-$\mathsf{SPr}(k)$ of simplicial presheaves equipped with a left action of G is again a closed model category [SS00]. Recall that the fibrations (respectively, equivalences) in G-$\mathsf{SPr}(k)$ are defined to be the morphisms that induce fibrations (respectively, equivalences) between the underlying simplicial presheaves. The model category G-$\mathsf{SPr}(k)$ will be called the *model category of G-equivariant simplicial presheaves*, and the objects in $\mathrm{Ho}(G\text{-}\mathsf{SPr}(k))$ will be called *G-equivariant stacks*. For any two G-equivariant stacks F and F', we will denote by $\underline{\mathrm{Hom}}_G(F, F')$ the simplicial set of morphisms in G-$\mathrm{SPr}(k)$, and by $\mathbb{R}\,\underline{\mathrm{Hom}}_G(F, F')$ its derived version.

On the other hand, to any group G one can associate its classifying simplicial presheaf $BG \in \mathsf{SPr}_*(k)$ (see [Toe06, § 1.3]). The object $BG \in \mathrm{Ho}(\mathsf{SPr}_*(k))$ is uniquely characterized, up to a unique isomorphism, by the properties

$$\pi_0(BG) \simeq *, \qquad \pi_1(BG, *) \simeq G, \qquad \pi_i(BG, *) = 0 \quad \text{for } i > 1.$$

Consider the comma category $\mathsf{SPr}(k)/BG$ of objects over the classifying simplicial presheaf BG, endowed with its natural simplicial closed model structure (see [Hov98]). Note that since the model structure on $\mathsf{SPr}(k)$ is proper, the model category $\mathsf{SPr}(k)/BG$ is an invariant, up to a Quillen equivalence, of the isomorphism class of BG in $\mathrm{Ho}(\mathsf{SPr}_*(k))$. This implies, in particular, that one is free to choose any model for $BG \in \mathrm{Ho}(\mathsf{SPr}(k))$ when dealing with the homotopy category $\mathrm{Ho}(\mathsf{SPr}(k)/BG)$. We will set $BG := EG/G$, where EG is a cofibrant model of $*$ in G-$\mathsf{SPr}(k)$ which is fixed once and for all.

We now define a pair of adjoint functors

$$G\text{-}\mathsf{SPr}(k) \underset{\mathrm{Mo}}{\overset{\mathrm{De}}{\rightleftarrows}} \mathsf{SPr}(k)/BG,$$

where De stands for *descent* and Mo for *monodromy*. If F is a G-equivariant simplicial presheaf, then $\mathrm{De}(F)$ is defined to be $(EG \times F)/G$, where G acts diagonally on $EG \times F$. Note that there is a natural projection $\mathrm{De}(F) \longrightarrow EG/G = BG$, so $\mathrm{De}(F)$ is naturally an object in $\mathsf{SPr}(k)/BG$. This adjunction is easily seen to be a Quillen adjunction. Furthermore, using the same reasoning as in [Toe02], one can also show that this Quillen adjunction is actually a Quillen equivalence. For future reference, we state this as a lemma.

LEMMA 3.20. *The Quillen adjunction* (De, Mo) *is a Quillen equivalence.*

Proof. This is essentially the same as the proof of [Toe02, 2.22] and is left to the reader. □

The previous lemma implies that the derived Quillen adjunction induces an equivalence of categories:

$$\mathrm{Ho}(G\text{-}\mathsf{SPr}(k)) \simeq \mathrm{Ho}(\mathsf{SPr}(k)/BG).$$

For any G-equivariant stack $F \in \mathrm{Ho}(G\text{-}\mathsf{SPr}(\mathbb{C}))$, we define the *quotient stack* $[F/G]$ of F by G as the object $\mathbb{L}\,\mathrm{De}(F) \in \mathrm{Ho}(\mathsf{SPr}(k)/BG)$ corresponding to $F \in \mathrm{Ho}(G\text{-}\mathsf{SPr}(k))$. By construction, the homotopy fiber (taken at the distinguished point of BG) of the natural projection

$$p : [F/G] \longrightarrow BG$$

is canonically isomorphic to the underlying stack of the G-equivariant stack F.

Equivariant cosimplicial algebras and equivariant affine stacks. Suppose that G is an affine group scheme and consider the category of k-linear representations of G. This category will be denoted by $\mathsf{Rep}(G)$. Note that it is an abelian k-linear tensor category which admits all $\mathbb{V}$-limits and $\mathbb{V}$-colimits. The category of cosimplicial G-modules is defined to be the category $\mathsf{Rep}(G)^{\Delta}$ of cosimplicial objects in $\mathsf{Rep}(G)$. For any object $E \in \mathsf{Rep}(G)^{\Delta}$, one can construct the normalized cochain complex $\boldsymbol{N}(E)$ associated to E, which is a cochain complex in $\mathsf{Rep}(G)$. Its cohomology representations $H^i(\boldsymbol{N}(E)) \in \mathsf{Rep}(G)$ will simply be denoted by $H^i(E)$. This construction is obviously functorial and gives rise to various cohomology functors $H^i : \mathsf{Rep}(G)^{\Delta} \longrightarrow \mathsf{Rep}(G)$. As the category $\mathsf{Rep}(G)$ is $\mathbb{V}$-complete and $\mathbb{V}$-co-complete, its category of cosimplicial objects $\mathsf{Rep}(G)^{\Delta}$ has the natural structure of a simplicial category [GJ99, II, Example 2.8].

Following the argument of [Qui67, II.4], it can be seen that there exists a simplicial finitely generated closed model structure on the category $\mathsf{Rep}(G)^{\Delta}$ with the following properties.

- A morphism $f : E \longrightarrow E'$ is an equivalence if and only if for any i, the induced morphism $H^i(f) : H^i(E) \longrightarrow H^i(E')$ is an isomorphism.
- A morphism $f : E \longrightarrow E'$ is a cofibration if and only if for any $n > 0$, the induced morphism $f_n : E_n \longrightarrow E'_n$ is a monomorphism.
- A morphism $f : E \longrightarrow E'$ is a fibration if and only if it is an epimorphism whose kernel K is such that for any $n \geq 0$, K_n is an injective object in $\mathsf{Rep}(G)$.

The category $\mathsf{Rep}(G)^{\Delta}$ is endowed with a symmetric monoidal structure, given by the tensor product of cosimplicial G-modules (defined levelwise). In particular, we can consider the category G-Alg^{Δ} of commutative unital monoids in $\mathsf{Rep}(G)^{\Delta}$. It is reasonable to view the objects in G-Alg^{Δ}

as cosimplicial algebras equipped with an action of the group scheme G. Motivated by this remark, we shall refer to the category $G\text{-}\mathsf{Alg}^{\Delta}$ as the *category of G-equivariant cosimplicial algebras*. From another point of view, the category $G\text{-}\mathsf{Alg}^{\Delta}$ is also the category of simplicial affine schemes in $\mathbb{V}$ equipped with an action of G.

Every G-equivariant cosimplicial algebra A has an underlying cosimplicial G-module, again denoted by $A \in \mathsf{Rep}(G)^{\Delta}$. This defines a forgetful functor

$$G\text{-}\mathsf{Alg}^{\Delta} \longrightarrow \mathsf{Rep}(G)^{\Delta},$$

which has a left adjoint L given by the free commutative monoid construction.

Proposition 3.21. *There exists a simplicial cofibrantly generated closed model structure on the category $G\text{-}\mathsf{Alg}^{\Delta}$ satisfying the following conditions.*

- *A morphism $f : A \longrightarrow A'$ is an equivalence if and only if the induced morphism in $\mathsf{Rep}(G)^{\Delta}$ is an equivalence.*
- *A morphism $f : A \longrightarrow A'$ is a fibration if and only if the induced morphism in $\mathsf{Rep}(G)^{\Delta}$ is a fibration.*

Proof. This is again an application of the small-object argument and, more precisely, of [Hov98, Theorem 2.1.19].

Let I and J be sets of generating cofibrations and trivial cofibrations in $\mathsf{Rep}(G)^{\Delta}$; that is, I is the set of monomorphisms between finite-dimensional cosimplicial G-modules, and J is the set of trivial cofibrations between finite-dimensional G-modules. Consider the forgetful functor $G\text{-}\mathsf{Alg}^{\Delta} \longrightarrow \mathsf{Rep}(G)^{\Delta}$ and its left adjoint $L : \mathsf{Rep}(G)^{\Delta} \longrightarrow G\text{-}\mathsf{Alg}^{\Delta}$. The functor L sends a cosimplicial representation V of G to the free commutative G-equivariant cosimplicial algebra generated by V. We will apply the small-object argument to the sets $L(I)$ and $L(J)$.

By construction, the morphisms in $L(I)$-inj are precisely the morphisms inducing surjective quasi-isomorphisms on the associated normalized cochain complexes; similarly, the morphisms in $L(J)$-inj are precisely the morphisms inducing surjections on the associated normalized cochain complexes. From this, it is easy to see that among the conditions (i) to (vi), only (vi), as well as the inclusion J-cell $\subset W$, requires a proof. However, note that (vi) can be replaced by $W \cap I$-inj $\subset I$-inj, which is easily seen to hold from the previous descriptions. Therefore, it only remains to check that for any morphism $A \longrightarrow B$ in J and any morphism $L(A) \longrightarrow C$, the induced morphism $C \longrightarrow C \coprod_{L(A)} L(B)$ is an equivalence; but since the forgetful functor $G\text{-}\mathsf{Alg}^{\Delta} \longrightarrow \mathsf{Alg}^{\Delta}$ is a left adjoint, it commutes with colimits and so $C \longrightarrow C \coprod_{L(A)} L(B)$ will be an equivalence owing to the fact that Alg^{Δ} is endowed with a model category structure for which $L(A) \longrightarrow L(B)$ is a trivial cofibration (see [Toe06, Theorem 2.1.2]). □

We will also need the following result.

Lemma 3.22. *The forgetful functor $G\text{-}\mathsf{Alg}^{\Delta} \longrightarrow \mathsf{Alg}^{\Delta}$ is a left Quillen functor.*

Proof. The forgetful functor has a right adjoint

$$F : \mathsf{Alg}^{\Delta} \longrightarrow G\text{-}\mathsf{Alg}^{\Delta},$$

which assigns to a cosimplicial algebra $A \in \mathsf{Alg}^{\Delta}$ the G-equivariant algebra $F(A) := \mathcal{O}(G) \otimes A$, where G acts on $\mathcal{O}(G)$ by left translation.

By adjunction, it is enough to show that F is right Quillen. By definition, the functor F preserves equivalences. Moreover, if $A \longrightarrow B$ is a fibration of cosimplicial algebras (i.e. an epimorphism), then the map $\mathcal{O}(G) \otimes A \longrightarrow \mathcal{O}(G) \otimes B$ is again an epimorphism. The kernel of

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the latter morphism is then isomorphic to $\mathcal{O}(G) \otimes K$, where K is the kernel of $A \longrightarrow B$. But, for any vector space V, $\mathcal{O}(G) \otimes V$ is always an injective object in $\mathsf{Rep}(G)$. This implies that $\mathcal{O}(G) \otimes K_n$ is an injective object in $\mathsf{Rep}(G)^{\Delta}$, therefore $F(A) \longrightarrow F(B)$ is a fibration in $G\text{-}\mathsf{Alg}^{\Delta}$ (see Proposition 3.21). □

For any G-equivariant cosimplicial algebra A, one can define its (geometric) spectrum $\operatorname{Spec}_G A \in G\text{-}\mathsf{SPr}(k)$ by taking the usual spectrum of its underlying cosimplicial algebra and keeping track of the G-action. Explicitly, if A is given by a morphism of cosimplicial algebras $A \longrightarrow A \otimes \mathcal{O}(G)$, one finds a morphism of simplicial schemes

$$G \times \operatorname{Spec} A \simeq \operatorname{Spec}(A \otimes \mathcal{O}(G)) \longrightarrow \operatorname{Spec} A,$$

which induces a well-defined G-action on the simplicial scheme $\operatorname{Spec} A$. Hence, by passing to the simplicial presheaves represented by G and $\operatorname{Spec} A$, one gets the G-equivariant simplicial presheaf $\operatorname{Spec}_G(A)$. This procedure defines a functor

$$\operatorname{Spec}_G : (G\text{-}\mathsf{Alg}^{\Delta})^{\mathrm{op}} \longrightarrow G\text{-}\mathsf{SPr}(k),$$

and we have the following corollary.

Corollary 3.23. *The functor* Spec_G *is right Quillen.*

Proof. Clearly, Spec_G commutes with $\mathbb{V}$-limits. Furthermore, the category $G\text{-}\mathsf{Alg}^{\Delta}$ possesses a small set of small generators in $\mathbb{V}$. For example, one can take a set of representatives of G-equivariant cosimplicial algebras A with A_n of finite type for any $n \geq 0$. This implies that $(G\text{-}\mathsf{Alg}^{\Delta})^{\mathrm{op}}$ possesses a small set of small co-generators in $\mathbb{V}$. The existence of the left adjoint to Spec_G then follows from the special adjoint theorem of [Mac71, § V.8].

To prove that Spec_G is right Quillen, it remains to prove that it preserves fibrations and trivial fibrations. In other words, one needs to show that if $A \longrightarrow A'$ is a (trivial) cofibration of G-equivariant cosimplicial algebras, then $\operatorname{Spec}_G(A') \longrightarrow \operatorname{Spec}_G(A)$ is a (trivial) fibration in $G\text{-}\mathsf{SPr}(k)$. As fibrations and equivalences in $G\text{-}\mathsf{SPr}(k)$ are defined on the underlying object, the desired result follows immediately from Lemma 3.22 together with the fact that the non-equivariant Spec is right Quillen. □

The left adjoint of Spec_G will be denoted by $\mathcal{O}_G : G\text{-}\mathsf{SPr}(\mathbb{C}) \longrightarrow G\text{-}\mathsf{Alg}^{\Delta}$.

The previous corollary allows one to form the right derived functor of Spec_G,

$$\mathbb{R}\operatorname{Spec}_G : \operatorname{Ho}(G\text{-}\mathsf{Alg}^{\Delta})^{\mathrm{op}} \longrightarrow \operatorname{Ho}(G\text{-}\mathsf{SPr}(k)),$$

which possesses a left adjoint $\mathbb{L}\mathcal{O}_G$. One can then compose this functor with the quotient stack functor $[-/G]$ and obtain a functor

$$[\mathbb{R}\operatorname{Spec}_G(-)/G] : \operatorname{Ho}(G\text{-}\mathsf{Alg}^{\Delta})^{\mathrm{op}} \longrightarrow \operatorname{Ho}(\mathsf{SPr}(k)/BG),$$

which still possesses a left adjoint owing to the fact that $[-/G]$ is an equivalence of categories. We will denote this left adjoint again by

$$\mathbb{L}\mathcal{O}_G : \operatorname{Ho}(\mathsf{SPr}(k)/BG) \longrightarrow \operatorname{Ho}(G\text{-}\mathsf{Alg}^{\Delta})^{\mathrm{op}}.$$

Proposition 3.24. *If* $A \in \operatorname{Ho}(G\text{-}\mathsf{Alg}^{\Delta})^{\mathrm{op}}$ *is isomorphic to some* G*-equivariant cosimplicial algebra in* $\mathbb{U}$*, then the adjunction morphism*

$$A \longrightarrow \mathbb{L}\mathcal{O}_G(\mathbb{R}\operatorname{Spec}_G A)$$

is an isomorphism.

In particular, the functors $\mathbb{R}\,\mathrm{Spec}_G$ and $[\mathbb{R}\,\mathrm{Spec}_G(-)/G]$ become fully faithful when restricted to the full sub-category of $\mathrm{Ho}(G\text{-}\mathsf{Alg}^{\Delta})$ consisting of G-equivariant cosimplicial algebras isomorphic to some object in $\mathbb{U}$.

Proof. Let A be a cofibrant G-equivariant cosimplicial algebra in $\mathbb{U}$, and let $F := \mathrm{Spec}_G\, A$. Since F and $\mathrm{hocolim}_{n\in\Delta^{\mathrm{op}}}\, \mathrm{Spec}_G\, A_n$ are weakly equivalent in $G\text{-}\mathsf{SPr}(k)$ and $\mathcal{O}_G$ is a left Quillen functor, we get that

$$\mathbb{L}\mathcal{O}_G(F) \simeq \mathrm{holim}_{n\in\Delta}\, \mathbb{L}\mathcal{O}_G(\mathrm{Spec}_G\, A_n) \simeq \mathrm{holim}_{n\in\Delta}\, A_n \simeq A.$$

So the proposition is proven. □

Definition 3.25. An equivariant stack $F \in \mathrm{Ho}(G\text{-}\mathsf{SPr}(k))$ is a G-equivariant affine stack if it is isomorphic to some $\mathbb{R}\,\mathrm{Spec}_G(A)$, with A a G-equivariant cosimplicial algebra in $\mathbb{U}$.

We conclude this section with a proposition showing that stacks of the form $[\mathbb{R}\,\mathrm{Spec}_G(A)/G]$ are often pointed schematic homotopy types.

Proposition 3.26. *Let $A \in G\text{-}\mathsf{Alg}^{\Delta}$ be a G-equivariant cosimplicial algebra in $\mathbb{U}$ such that the underlying algebra of A has an augmentation $x : A \longrightarrow k$. Assume also that A is cohomologically connected, i.e. that $H^0(A) \simeq k$. Then, the quotient stack $[\mathbb{R}\,\mathrm{Spec}_G(A)/G]$ is a pointed schematic homotopy type.*

Proof. The augmentation map $x : A \longrightarrow k$ induces a morphism $* \longrightarrow \mathbb{R}\,\mathrm{Spec}\, A$ and hence gives rise to a well-defined point

$$* \longrightarrow \mathbb{R}\,\mathrm{Spec}\, A \longrightarrow [\mathbb{R}\,\mathrm{Spec}_G(A)/G].$$

Consider the natural morphism in $\mathrm{Ho}(\mathsf{SPr}_*(k))$

$$[\mathbb{R}\,\mathrm{Spec}_G(A)/G] \longrightarrow BG.$$

Its homotopy fiber is isomorphic to $\mathbb{R}\,\mathrm{Spec}\, A$. Using the general results [Toe06, Corollary 2.4.10 and Theorem 3.2.4] and the long exact sequence on homotopy sheaves, it is then enough to show that $\mathbb{R}\,\mathrm{Spec}\, A$ is a connected affine stack. Equivalently, we need to show that A is cohomologically connected, which is true by hypothesis. Therefore the proposition is proven. □

We now define a category $\mathrm{Ho}(\mathsf{EqAlg}^{\Delta,*}_{\mathbb{U}})$ of *equivariant augmented and 1-connected cosimplicial algebras* in the following way. The objects of $\mathrm{Ho}(\mathsf{EqAlg}^{\Delta,*}_{\mathbb{U}})$ are triplets (G, A, u) consisting of

- an affine group scheme G in $\mathbb{U}$;
- a G-equivariant cosimplicial algebra A in $\mathbb{U}$ such that $H^0(A) \simeq k$ and $H^1(A) = 0$;
- a morphism $u : A \to \mathcal{O}(G)$ of G-equivariant cosimplicial algebras (where $\mathcal{O}(G)$ is the algebra of functions on G considered as a G-equivariant cosimplicial algebra via the regular representation of G).

A morphism $f : (G, A, u) \longrightarrow (H, B, v)$ in $\mathrm{Ho}(\mathsf{EqAlg}^{\Delta,*}_{\mathbb{U}})$ is a pair (ϕ, a) consisting of:

- a morphism $\phi : G \longrightarrow H$ of affine group schemes;
- a morphism $a : B \longrightarrow A$ in $\mathrm{Ho}(G\text{-}\mathsf{Alg}^{\Delta}/\mathcal{O}(G))$ (the homotopy category of the model category of objects over $\mathcal{O}(G)$), where B is considered as an object in $G\text{-}\mathsf{Alg}^{\Delta}/\mathcal{O}(G)$ by composing the action and the augmentation with the morphisms $G \to H$ and $\mathcal{O}(H) \to \mathcal{O}(G)$.

This defines the category $\mathrm{Ho}(\mathsf{EqAlg}^{\Delta,*}_{\mathbb{U}})$. Note, however, that this category is not (a priori) the homotopy category of a model category $\mathsf{EqAlg}^{\Delta,*}_{\mathbb{U}}$ (at least we did not define any such model category) and that it is defined in an ad hoc manner.

Let $(G, A, u) \in \mathrm{Ho}(\mathsf{EqAlg}_{\mathbb{U}}^{\Delta,*})$. We consider the associated G-equivariant stack $\mathbb{R}\,\mathrm{Spec}_G\, A$. The morphism $u: A \longrightarrow \mathcal{O}(G)$ induces a natural morphism

$$\mathbb{R}\,\mathrm{Spec}_G\,\mathcal{O}(G) \simeq G \longrightarrow \mathbb{R}\,\mathrm{Spec}_G\, A,$$

where G acts on itself by left translations. Therefore $\mathbb{R}\,\mathrm{Spec}_G\, A$ can be considered, in a natural way, as an object in $\mathrm{Ho}(G/G\text{-}\mathsf{SPr}(k))$, the homotopy category of G-equivariant stacks under G. Applying the quotient stack functor of [KPT08, Definition 1.2.2], we get a well-defined object

$$([G/G] \simeq * \longrightarrow [\mathbb{R}\,\mathrm{Spec}_G\, A/G]) \in \mathrm{Ho}([G/G]/\mathsf{SPr}(k)) \simeq \mathrm{Ho}(\mathsf{SPr}_*(k)).$$

Tracing carefully through the definitions, it is straightforward to check that the previous construction defines a functor

$$\begin{aligned} \mathrm{Ho}(\mathsf{EqAlg}_{\mathbb{U}}^{\Delta,*})^{\mathrm{op}} &\longrightarrow \mathrm{Ho}(\mathsf{SPr}_*(k)) \\ (G, A, u) &\longmapsto [\mathbb{R}\,\mathrm{Spec}_G\, A/G]. \end{aligned} \tag{1}$$

Proposition 3.27. *The functor (1) is fully faithful and its essential image consists exactly of psht.*

Proof. The full faithfulness follows easily from Proposition 3.24. Also, Proposition 3.26 implies that this functor takes values in the sub-category of psht. It remains to prove that any psht is in the essential image of (1).

Let F be a psht. By Theorem 3.14 we can write F as $\boldsymbol{B}G_*$, where G_* is a fibrant object in $\mathsf{sGAff}_{\mathbb{U}}$. We consider the projection $F \longrightarrow BG$, where $G := \pi_0(G_*) \simeq \pi_1(F)$, as well as the cartesian square

$$\begin{array}{ccc} F & \longrightarrow & BG \\ \uparrow & & \uparrow \\ F^0 & \longrightarrow & EG \end{array}$$

where, as usual, EG is the simplicial presheaf with $EG_m := G^{m+1}$ and the face and degeneracy maps are given by the projections and diagonal embeddings (see [Toe06, § 1.3] for details). The simplicial presheaf F^0 is a pointed affine scheme in $\mathbb{U}$ equipped with a natural action of G. Taking its cosimplicial algebra of functions, one gets a G-equivariant cosimplicial algebra $A := \mathcal{O}(F^0)$. As the fiber of $F^0 \longrightarrow F$ is naturally isomorphic to G, one further obtains a G-equivariant morphism $G \longrightarrow F^0$, giving rise to an G-equivariant morphism $u: A \longrightarrow \mathcal{O}(G)$. One can then check that F^0 is naturally isomorphic in $\mathrm{Ho}([G/G] - \mathsf{SPr}(k))$ to the image of the object $(G, A, u) \in \mathrm{Ho}(G\text{-}\mathsf{Alg}^{\Delta}/\mathcal{O}(G))^{\mathrm{op}}$ by the functor $\mathbb{R}\,\mathrm{Spec}_G$ (this follows from the fact that F^0 is an affine stack). Finally, one has the following isomorphisms in $\mathrm{Ho}(\mathsf{SPr}_*(k))$:

$$(* \to F) \simeq \big([G/G] \to [F^0/G]\big) \simeq ([\mathbb{R}\,\mathrm{Spec}_G\,\mathcal{O}(G)/G] \to [\mathbb{R}\,\mathrm{Spec}_G\, A/G]),$$

which show that F belongs to the essential image of the required functor. □

Corollary 3.28. *The categories* $\mathrm{Ho}(\mathsf{Hopf}_{\mathbb{U}}^{\Delta})$ *and* $\mathrm{Ho}(\mathsf{EqAlg}_{\mathbb{U}}^{\Delta,*})$ *are equivalent.*

Proof. Indeed, by Theorem 3.14 and Proposition 3.27, both of these categories are equivalent to the full sub-category of $\mathrm{Ho}(\mathsf{SPr}_*(k))$ consisting of psht. □

Remark 3.29. The previous corollary is a generalization of the equivalence between reduced nilpotent Hopf dg-algebras and 1-connected reduced dga given by the bar and cobar constructions (see [Tan83, § 0]). It would be interesting to produce explicit functors between $\mathrm{Ho}(\mathsf{Hopf}_{\mathbb{U}}^{\Delta})$ and $\mathrm{Ho}(\mathsf{EqAlg}_{\mathbb{U}}^{\Delta,*})$ without passing through the category of psht.

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An explicit model for $(X \otimes k)^{\mathrm{sch}}$. Let X be a pointed and connected simplicial set in $\mathbb{U}$. Here we shall give an explicit model for $(X \otimes k)^{\mathrm{sch}}$ which is based on the notion of equivariant affine stacks that we just introduced.

The main idea of the construction is the following observation. Let $G := \pi_1(X, x)^{\mathrm{alg}}$ be the pro-algebraic completion of $\pi_1(X, x)$ over k. By Lemma 3.20 and Corollary 3.16, the natural morphism $(X \otimes k)^{\mathrm{sch}} \longrightarrow BG$ corresponds to a G-equivariant affine stack. Furthermore, the universal property of the schematization suggests that the corresponding G-equivariant cosimplicial algebra is the cosimplicial algebra of cochains of X with coefficients in the local system $\mathcal{O}(G)$. We will show that this guess is actually correct.

Let $\pi_1(X, x) \longrightarrow G$ be the universal morphism, and let $X \longrightarrow B(G(k))$ be the corresponding morphism of simplicial sets. This latter morphism is well-defined up to homotopy, and we choose once and for all a representative for it. Let $p : P \longrightarrow X$ be the corresponding G-torsor in $\mathsf{SPr}(k)$. More precisely, P is the simplicial presheaf sending an affine scheme $\operatorname{Spec} A \in \mathsf{Aff}$ to the simplicial set $P(A) := (EG(A) \times_{BG(A)} X)$, where $EG(A) \longrightarrow BG(A)$ is the natural projection. The morphism $p : P \longrightarrow X$ is then a well-defined morphism in $\operatorname{Ho}(G\text{-}\mathsf{SPr}(k))$. Here the group G is acting on $P = (EG \times_{BG} X)$ by its action on EG, and trivially on X. Alternatively, we can describe P by the formula

$$P \simeq (\widetilde{X} \times G)/\pi_1(X, x),$$

where $\widetilde{X}$ is the universal covering of X and $\pi_1(X, x)$ acts on $\widetilde{X} \times G$ by the diagonal action (our convention here is that $\pi_1(X, x)$ acts on G by left translation). We assume at this point that $\widetilde{X}$ is chosen to be cofibrant in the model category of $\pi_1(X, x)$-equivariant simplicial sets. For example, we may assume that $\widetilde{X}$ is a $\pi_1(X, x)$-equivariant cell complex.

We now consider the G-equivariant affine stack $\mathbb{R}\operatorname{Spec}_G \mathcal{O}_G(P) \in \operatorname{Ho}(G\text{-}\mathsf{SPr}(k))$, which comes naturally equipped with its adjunction morphism $P \longrightarrow \mathbb{R}\operatorname{Spec}_G \mathbb{L}\mathcal{O}_G(P)$. This induces a well-defined morphism in $\operatorname{Ho}(\mathsf{SPr}(k))$:

$$X \simeq [P/G] \longrightarrow [\mathbb{R}\operatorname{Spec}_G \mathbb{L}\mathcal{O}_G(P)/G].$$

Furthermore, since X is pointed, this morphism induces a natural morphism in $\operatorname{Ho}(\mathsf{SPr}_*(k))$,

$$f : X \longrightarrow [\mathbb{R}\operatorname{Spec}_G \mathbb{L}\mathcal{O}_G(P)/G].$$

With this notation, we have the following important theorem.

Theorem 3.30. *The natural morphism* $f : X \longrightarrow [\mathbb{R}\operatorname{Spec}_G \mathbb{L}\mathcal{O}_G(P)/G]$ *is a model for the schematization of* X.

Proof. Since $P \simeq (\widetilde{X} \times G)/\pi_1(X, x)$, the algebra $\mathbb{L}\mathcal{O}_G(P)$ can be identified with the cosimplicial algebra of cochains on X with coefficients in the local system of algebras $\mathcal{O}(G)$. More precisely, P is equivalent to the homotopy colimit of $\widetilde{X} \times G$ viewed as a $\pi_1(X, x)$-diagram in $G\text{-}\mathsf{SPr}(k)$. As $\mathcal{O}_G$ is left Quillen, one has equivalences

$$\mathbb{L}\mathcal{O}_G(P) \simeq \operatorname{holim}_{\pi_1(X,x)} \mathbb{L}\mathcal{O}_G(\widetilde{X} \times G) \simeq \operatorname{holim}_{\pi_1(X,x)} \mathbb{L}\mathcal{O}_G(G)^{\widetilde{X}},$$

where $(-)^{\widetilde{X}}$ is the exponential functor (which is part of the simplicial structure on $G\text{-}\mathsf{Alg}^{\Delta}$). In particular, we have an isomorphism

$$\mathbb{L}\mathcal{O}_G(P) \simeq (\mathcal{O}(G)^{\widetilde{X}})^{\pi_1(X,x)}$$

of $\mathbb{L}\mathcal{O}_G(P)$ with the $\pi_1(X, x)$-invariant G-equivariant cosimplicial algebra of $\mathcal{O}(G)^{\widetilde{X}}$. Note that the identification

$$\operatorname{holim}_{\pi_1(X,x)}(\mathcal{O}(G)^{\widetilde{X}}) \simeq \lim_{\pi_1(X,x)}(\mathcal{O}(G)^{\widetilde{X}}) = (\mathcal{O}(G)^{\widetilde{X}})^{\pi_1(X,x)}$$

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uses the fact that $\widetilde{X}$ is cofibrant as a $\pi_1(X,x)$-simplicial set. The underlying cosimplicial algebra of $\mathbb{L}\mathcal{O}_G(P)$ is therefore augmented and cohomologically connected and belongs to $\mathbb{U}$. Therefore, by Proposition 3.26, we conclude that $[\mathbb{R}\,\mathrm{Spec}_G\ \mathbb{L}\mathcal{O}_G(P)/G]$ is a pointed schematic homotopy type.

In order to finish the proof of the theorem, it remains to show that the morphism f is a P-equivalence. We start with an algebraic group H and a finite-dimensional linear representation V of H. Let F denote the pointed schematic homotopy type $K(H,V,n)$, and let $F \longrightarrow BH$ be the natural projection. It is instructive to observe that F is naturally isomorphic to $[\mathbb{R}\,\mathrm{Spec}_H\ B(V,n)/H]$, where $B(V,n)$ is the cosimplicial cochain algebra of $K(V,n)$ taken together with the natural action of H.

There is a commutative diagram

$$\begin{array}{ccc} \mathbb{R}\,\underline{\mathrm{Hom}}_*([\mathbb{R}\,\mathrm{Spec}_G\ \mathbb{L}\mathcal{O}_G(P)/G], F) & \xrightarrow{f^*} & \mathbb{R}\,\underline{\mathrm{Hom}}_*(X,F) \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ \mathrm{Hom}_{\mathsf{GAff}}(G,H) & \longrightarrow & \mathrm{Hom}_{\mathsf{Gp}}(\pi_1(X,x),H) \end{array}$$

in which the horizontal morphism at the bottom is an isomorphism due to the universal property of the map $\pi_1(X,x) \longrightarrow G$. Therefore it suffices to check that f^* induces an equivalence on the homotopy fibres of the two projections p and q. Let $\rho : G \longrightarrow H$ be a morphism of groups, and consider the homotopy fiber of $F \longrightarrow BH$ together with the G-action induced from ρ. This is a G-equivariant affine stack that will be denoted by F_G, and whose underlying stack is isomorphic to $K(V,n)$.

The homotopy fibers of p and q at ρ are isomorphic to $\mathbb{R}\,\underline{\mathrm{Hom}}_G(\mathbb{R}\,\mathrm{Spec}_G\,\mathbb{L}\mathcal{O}_G(P), F_G)$ and $\mathbb{R}\,\underline{\mathrm{Hom}}_G(P,F_G)$, respectively; but since F_G is a G-equivariant affine stack, Proposition 3.24 implies that the natural morphism

$$\mathbb{R}\,\underline{\mathrm{Hom}}_G(\mathbb{R}\,\mathrm{Spec}_G\,\mathbb{L}\mathcal{O}_G(P), F_G) \longrightarrow \mathbb{R}\,\underline{\mathrm{Hom}}_G(P,F_G)$$

is an equivalence. □

Now, consider $\mathcal{O}(G)$ as a locally constant sheaf of algebras on X via the natural action of $\pi_1(X,x)$, and let

$$C^\bullet(X,\mathcal{O}(G)) := (\mathcal{O}(G)^{\widetilde{X}})^{\pi_1(X,x)}$$

be the cosimplicial algebra of cochains on X with coefficients in $\mathcal{O}(G)$. This cosimplicial algebra is equipped with a natural G-action, induced by the regular representation of G. One can thus consider $C^\bullet(X,\mathcal{O}(G))$ as a object in $G\text{-}\mathsf{Alg}^\Delta$. So the previous theorem and the natural identification $\mathbb{L}\mathcal{O}_G(P) \simeq C^\bullet(X,\mathcal{O}(G))$ as objects in $\mathrm{Ho}(G\text{-}\mathsf{Alg}^\Delta)$ immediately yield the following.

Corollary 3.31. *With the previous notation, one has*

$$(X \otimes k)^{\mathrm{sch}} \simeq [\mathbb{R}\,\mathrm{Spec}_G\, C^\bullet(X,\mathcal{O}(G))/G].$$

The case of characteristic zero. Suppose that k is of characteristic zero. In this case, Corollary 3.31 can be reformulated in terms of the pro-reductive completion of the fundamental group.

Let $x \in X$ be a pointed $\mathbb{U}$-small simplicial set, and let G^{red} be the pro-reductive completion of $\pi_1(X,x)$ over k. By definition, G^{red} is the universal reductive affine group scheme over k equipped with a morphism from $\pi_1(X,x)$ with Zariski dense image. We consider $\mathcal{O}(G^{\mathrm{red}})$ as a local system of k-algebras on X, and consider the cosimplicial algebra $C^*(X,\mathcal{O}(G^{\mathrm{red}}))$ as an object in $G^{\mathrm{red}} - \mathsf{Alg}^\Delta$. The proof of Theorem 3.30 can be adapted to obtain the following result.

Corollary 3.32. *There exists a natural isomorphism in* $\mathrm{Ho}(\mathsf{SPr}_*(k))$*:*

$$(X \otimes k)^{\mathrm{sch}} \simeq [\mathbb{R}\,\mathrm{Spec}_{G^{\mathrm{red}}}\, C^{\bullet}(X, \mathcal{O}(G^{\mathrm{red}}))/G^{\mathrm{red}}].$$

Proof. The natural morphism

$$f : X \longrightarrow [\mathbb{R}\,\mathrm{Spec}_{G^{\mathrm{red}}}\, C^{\bullet}(X, \mathcal{O}(G^{\mathrm{red}}))/G^{\mathrm{red}}] =: \mathcal{X}$$

is defined in the same way as in the proof of Theorem 3.30. Let H be a reductive linear algebraic group, and V a linear representation of H. We set $F := K(H, V, n)$ and let $\mathrm{Hom}^{\mathrm{zd}}_{\mathsf{Gp}}(\pi_1(X, x), H)$ be the subset of morphisms $\pi_1(X, x) \longrightarrow H$ with a Zariski dense image. We also let $\mathbb{R}\,\underline{\mathrm{Hom}}^{\mathrm{zd}}(X, F)$ be defined by the following homotopy pull-back diagram.

$$\begin{array}{ccc} \mathbb{R}\,\underline{\mathrm{Hom}}^{\mathrm{zd}}(X, F) & \longrightarrow & \mathbb{R}\,\underline{\mathrm{Hom}}(X, F) \\ \downarrow & & \downarrow \\ \mathrm{Hom}^{\mathrm{zd}}_{\mathsf{Gp}}(\pi_1(X, x), H) & \longrightarrow & \mathrm{Hom}_{\mathsf{Gp}}(\pi_1(X, x), H) \end{array}$$

In the same way, we define $\mathrm{Hom}^{\mathrm{zd}}_{\mathsf{Gp}}(\pi_1(\mathcal{X}, x), H)$ to be the subset of morphisms $\pi_1(\mathcal{X}, x) \longrightarrow H$ with a Zariski dense image. Notice that these are precisely the morphisms $\pi_1(\mathcal{X}, x) \longrightarrow H$ that factor through the natural quotient $\pi_1(\mathcal{X}, x) \longrightarrow G^{\mathrm{red}}$, that is,

$$\mathrm{Hom}^{\mathrm{zd}}_{\mathsf{Gp}}(\pi_1(\mathcal{X}, x), H) = \mathrm{Hom}_{\mathsf{Gp}}(G^{\mathrm{red}}, H).$$

Finally, we define $\mathbb{R}\,\underline{\mathrm{Hom}}^{\mathrm{red}}(\mathcal{X}, F)$ by the homotopy pull-back diagram below.

$$\begin{array}{ccc} \mathbb{R}\,\underline{\mathrm{Hom}}^{\mathrm{red}}_*(\mathcal{X}, F) & \longrightarrow & \mathbb{R}\,\underline{\mathrm{Hom}}_*(\mathcal{X}, F) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{\mathsf{Gp}}(G^{\mathrm{red}}, H) & \longrightarrow & \mathrm{Hom}_{\mathsf{Gp}}(\pi_1(\mathcal{X}, x), H) \end{array}$$

We the have a commutative diagram as follows.

$$\begin{array}{ccc} \mathbb{R}\,\underline{\mathrm{Hom}}^{\mathrm{red}}_*(\mathcal{X}, F) & \xrightarrow{f^*} & \mathbb{R}\,\underline{\mathrm{Hom}}^{\mathrm{zd}}_*(X, F) \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ \mathrm{Hom}_{\mathsf{GAff}}(G^{\mathrm{red}}, H) & \longrightarrow & \mathrm{Hom}^{\mathrm{zd}}_{\mathsf{Gp}}(\pi_1(X, x), H) \end{array}$$

As in the proof of Theorem 3.30, the induced morphisms on the homotopy fibers of the vertical morphisms p and q are all weak equivalences. Since the bottom horizontal morphism is bijective, this shows that the top horizontal morphism

$$f^* : \mathbb{R}\,\underline{\mathrm{Hom}}^{\mathrm{red}}_*(\mathcal{X}, F) \longrightarrow \mathbb{R}\,\underline{\mathrm{Hom}}^{\mathrm{zd}}_*(X, F)$$

is a weak equivalence.

Because this is true for any H, V and n, we find that the natural morphism

$$\pi_1(X, x) \longrightarrow \pi_1(\mathcal{X}, x)$$

induces an equivalence of the categories of reductive linear representations, and that for any such reductive representation V, the induced morphism

$$f^* : H^*(\mathcal{X}, V) \longrightarrow H^*(X, V)$$

is an isomorphism. This, in turn, implies that the morphism f is a P-equivalence. □

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4. Some properties of the schematization functor

In this section we collect some basic properties of the schematization functor. All of these are purely topological, but are geared toward the schematization of algebraic varieties.

For a topological space X, we will write $(X \otimes k)^{\mathrm{sch}}$ instead of $(S(X) \otimes k)^{\mathrm{sch}}$, where $S(X)$ is the singular simplicial set of X.

4.1 Schematization of smooth manifolds

The purpose of this subsection is to give a formula for the schematization of a smooth manifold in terms of differential forms.

Let X be a topological space that is assumed to be locally contractible and for which any open subset is paracompact. For example, X could be any CW complex, including any covering of a compact and smooth manifold. Let G be an affine group scheme, and consider $\mathsf{Rep}(G)(X)$, the abelian category of sheaves on X with values in the category $\mathsf{Rep}(G)$ of linear representations of G. The category $\mathsf{Rep}(G)(X)$ is, in particular, the category of sheaves on a (small) site with values in a Grothendieck abelian category. Thus, $\mathsf{Rep}(G)(X)$ has enough injectives. This implies that the opposite category possesses enough projectives, and therefore [Qui67, II.4.11,Remark 5] can be applied to endow the category $C^+(\mathsf{Rep}(G)(X))$ of positively graded complexes in $\mathsf{Rep}(G)(X)$ with a model structure. Recall that the equivalences in $C^+(\mathsf{Rep}(G)(X))$ are morphisms inducing isomorphisms on cohomology sheaves (i.e. quasi-isomorphisms of complexes of sheaves), and that the fibrations are epimorphisms whose kernel K is such that each K_n is an injective object in $\mathsf{Rep}(G)(X)$. One checks immediately that $C^+(\mathsf{Rep}(G)(X))$ is a cofibrantly generated model category.

The Dold–Kan correspondence yields an equivalence of categories

$$D : C^+(\mathsf{Rep}(G)(X)) \longrightarrow \mathsf{Rep}(G)(X)^{\Delta}, \qquad C^+(\mathsf{Rep}(G)(X)) \longleftarrow \mathsf{Rep}(G)(X)^{\Delta} : N,$$

where N is the normalization functor and D the denormalization functor. Through this equivalence, we can transplant the model structure of $C^+(\mathsf{Rep}(G)(X))$ to a model structure on $\mathsf{Rep}(G)(X)^{\Delta}$. As explained in [GJ99, II, Example 2.8], the category $\mathsf{Rep}(G)(X)^{\Delta}$ possesses a natural simplicial structure, and it is easy to check that this simplicial structure is compatible with the model structure. Therefore, $\mathsf{Rep}(G)(X)^{\Delta}$ is a simplicial and cofibrantly generated model category.

The two categories $\mathsf{Rep}(G)(X)^{\Delta}$ and $C^+(\mathsf{Rep}(G)(X))$ have natural symmetric monoidal structures induced by the usual tensor product on $\mathsf{Rep}(G)$. The monoidal structures turn these categories into symmetric monoidal model categories. However, the functors D and N are not monoidal functors, but are related to the monoidal structure via the usual Alexander–Whitney and shuffle products

$$\mathrm{aw}_{X,Y} : N(X) \otimes N(Y) \longrightarrow N(X \otimes Y), \qquad \mathrm{sp}_{X,Y} : D(X) \otimes D(Y) \longrightarrow D(X \otimes Y).$$

The morphisms $\mathrm{aw}_{X,Y}$ are unital and associative, while the $\mathrm{sp}_{X,Y}$ are unital, associative and commutative.

We will denote the categories of commutative monoids in the categories $\mathsf{Rep}(G)(X)^{\Delta}$ and $C^+(\mathsf{Rep}(G)(X))$ by

$$G\text{-}\mathsf{Alg}^{\Delta}(X) \text{ and } G\text{-}\mathsf{CDGA}(X),$$

respectively, and call them the categories of G-equivariant cosimplicial algebras on X and of G-equivariant commutative differential algebras on X, respectively.

The following proposition is standard.

Proposition 4.1.

(i) *There exists a unique simplicial model structure on* $G\text{-}\mathsf{Alg}^{\Delta}$ *such that a morphism is an equivalence (respectively, a fibration) if and only if the induced morphism in* $\mathsf{Rep}(G)(X)^{\Delta}$ *is an equivalence, i.e. induces a quasi-isomorphism on the normalized complexes of sheaves (respectively a fibration, i.e. induces an epimorphism on the normalized complexes of sheaves whose kernel is levelwise injective).*

(ii) *There exist a unique simplicial model structure on* $G\text{-}\mathsf{CDGA}(X)$ *such that a morphism is an equivalence (respectively a fibration) if and only if the induced morphism in* $C^{+}(\mathsf{Rep}(G)(X))$ *is an equivalence, i.e. a quasi-isomorphism of complexes of sheaves (respectively, a fibration, i.e. a epimorphism of complexes of sheaves whose kernel is levelwise injective).*

By naturality, the functor Th of Thom–Sullivan cochains (see [HS87, 4.1]) extends to a functor

$$\mathrm{Th}:\mathrm{Ho}(G\text{-}\mathsf{CDGA}(X))\longrightarrow \mathrm{Ho}(G\text{-}\mathsf{Alg}^{\Delta}(X)).$$

This functor is an equivalence, and its inverse is the denormalization functor

$$D:\mathrm{Ho}(G\text{-}\mathsf{Alg}^{\Delta}(X))\longrightarrow \mathrm{Ho}(G\text{-}\mathsf{CDGA}(X)).$$

Let $f:X\longrightarrow Y$ be a continuous map of topological spaces (again paracompact and locally contractible). The inverse and direct image functors of sheaves induce Quillen adjunctions

$$G\text{-}\mathsf{CDGA}(Y)\underset{f_*}{\overset{f^*}{\rightleftarrows}}G\text{-}\mathsf{CDGA}(X)\quad\text{and}\quad G\text{-}\mathsf{Alg}^{\Delta}(Y)\underset{f_*}{\overset{f^*}{\rightleftarrows}}G\text{-}\mathsf{Alg}^{\Delta}(X).$$

Furthermore, one checks immediately that the diagram

$$\begin{array}{ccc}\mathrm{Ho}(G\text{-}\mathsf{CDGA}(X)) & \xrightarrow{D} & \mathrm{Ho}(G\text{-}\mathsf{Alg}^{\Delta}(X))\\ {\scriptstyle\mathbb{R}f_*}\downarrow & & \downarrow{\scriptstyle\mathbb{R}f_*}\\ \mathrm{Ho}(G\text{-}\mathsf{CDGA}(Y)) & \xrightarrow[D]{} & \mathrm{Ho}(G\text{-}\mathsf{Alg}^{\Delta}(Y))\end{array}$$

commutes. As usual, in the case where $Y=*$, the functor f_* will be denoted simply by $\Gamma(X,-)$. The previous diagram should be understood as a functorial isomorphism $D\mathbb{R}\Gamma(X,A)\simeq\mathbb{R}\Gamma(X,D(A))$, for any $A\in\mathrm{Ho}(G\text{-}\mathsf{CDGA}(X))$.

The category $\mathsf{Rep}(G)(X)^{\Delta}$ is naturally enriched over the category of sheaves of simplicial sets on X. Indeed, for $F\in\mathsf{SSh}(X)$ a simplicial sheaf and $E\in\mathsf{Rep}(G)(X)^{\Delta}$, one can define $F\otimes E$ to be the sheaf associated to the presheaf defined by the formula

$$\begin{aligned}\mathsf{Open}(X)^{\mathrm{op}} &\longrightarrow \mathsf{Rep}(G)^{\Delta}\\ U &\longmapsto F(U)\otimes E(U),\end{aligned}$$

where $\mathsf{Open}(X)$ denotes the category of open sets in X and $F(U)\otimes E(U)$ is viewed as an object in $\mathsf{Rep}(G)^{\Delta}$ via the natural simplicial structure on the model category $\mathsf{Rep}(G)^{\Delta}$. It is straightforward to check that this structure makes the model category $\mathsf{Rep}(G)(X)^{\Delta}$ a $\mathsf{SSh}(X)$-model category in the sense of [Hov98, § 4.2], where $\mathsf{SSh}(X)$ is taken with the injective model structure defined in [Jar87]. In particular, one can define functors

$$\begin{aligned}\mathsf{SSh}(X)^{\mathrm{op}}\otimes\mathsf{Rep}(G)(X)^{\Delta} &\to \mathsf{Rep}(G)(X)^{\Delta}\\ (F,E) &\longmapsto E^{F}\end{aligned}$$

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and

$$(\mathsf{Rep}(G)(X)^{\Delta})^{\mathrm{op}} \otimes \mathsf{Rep}(G)(X)^{\Delta} \longrightarrow \mathsf{SSh}(X)$$
$$(E, E') \longmapsto \underline{\mathrm{Hom}}_{\mathsf{SSh}(X)}(E, E'),$$

which are related by the usual adjunction isomorphisms

$$\underline{\mathrm{Hom}}_{\mathsf{SSh}(X)}(F \otimes E, E') \simeq \underline{\mathrm{Hom}}_{\mathsf{SSh}(X)}(E, (E')^F) \simeq \underline{\mathrm{Hom}}_{\mathsf{SSh}(X)}(F, \underline{\mathrm{Hom}}_{\mathsf{SSh}(X)}(E, E')),$$

where $\underline{\mathrm{Hom}}_{\mathsf{SSh}(X)}$ denotes the internal Hom in $\mathsf{SSh}(X)$.

Lemma 4.2. *Let $A \in \mathsf{Rep}(G)(X)$ be an injective object. Then the presheaf of abelian groups on X underlying A is acyclic.*

Proof. As the space X is paracompact, its sheaf cohomology coincides with its Čech cohomology. Thus it is enough to show that for any open covering $\{U_i\}_{i\in I}$, the Čech complex

$$A(X) \longrightarrow \prod_{i\in I} A(U_i) \longrightarrow \cdots \longrightarrow \prod_{(i_0,\dots,i_p)\in I^{p+1}} A(U_{i_0} \cap \cdots \cap U_{i_p}) \longrightarrow \cdots$$

is exact.

Since A is an injective object, it is fibrant as a constant cosimplicial object $A \in \mathsf{Rep}(G)(X)^{\Delta}$. Let $\{U_i\}_{i\in I}$ be an open covering of X, and let $N(U/X)$ be its nerve; this is the simplicial sheaf on X whose sheaf of n-simplices is defined by the formula

$$N(U/X)_n : \quad \mathsf{Open}(X)^{\mathrm{op}} \longrightarrow \mathsf{SSet}$$
$$V \longmapsto \textstyle\prod_{(i_0,\dots,i_n)\in I^{n+1}} \mathrm{Hom}_X(V, U_{i_0} \cap \cdots \cap U_{i_n}).$$

The simplicial sheaf $N(U/X)$ is contractible (i.e. equivalent to $*$ in $\mathsf{SSh}(X)$), as can easily be seen on stalks. Furthermore, since every object in $\mathsf{SSh}(X)$ is cofibrant and $A \in \mathsf{Rep}(G)(X)^{\Delta}$ is fibrant, the natural morphism

$$A^{\bullet} \simeq A \longrightarrow A^{N(U/X)}$$

is an equivalence of fibrant objects in $\mathsf{Rep}(G)(X)^{\Delta}$ (here $A^{N(U/X)}$ is part of the $\mathsf{SSh}(X)$-module structure on $\mathsf{Rep}(G)(X)^{\Delta}$)). Therefore, since the global sections functor

$$\Gamma(X, -) : \mathsf{Rep}(G)(X)^{\Delta} \longrightarrow \mathsf{Rep}(G)^{\Delta}$$

is right Quillen, we conclude that the induced morphism $\Gamma(X, A) \longrightarrow \Gamma(X, A^{N(U/X)})$ is a quasi-isomorphism. Since $\Gamma(X, A^{N(U/X)})$ is just the Čech complex of A for the covering $\{U_i\}_{i\in I}$, this implies that A is acyclic on X. □

Lemma 4.3. *Let $A \in G\text{-}\mathsf{CDGA}(X)$ be a G-equivariant commutative differential graded algebra over X such that for every $n \geq 0$, the sheaf of abelian groups A_n is an acyclic sheaf on X. Then the natural morphism in* $\mathrm{Ho}(G\text{-}\mathsf{CDGA})$,

$$\Gamma(X, A) \longrightarrow \mathbb{R}\Gamma(X, A),$$

is an isomorphism.

Proof. Let $A \longrightarrow RA$ be a fibrant model of A in $G\text{-}\mathsf{CDGA}(X)$. Then, by the definition of a fibration in $G\text{-}\mathsf{CDAG}(X)$, each A_n is an injective object in $\mathsf{Rep}(G)(X)$ and is therefore acyclic by Lemma 4.2. The morphism $A \longrightarrow RA$ is thus a quasi-isomorphism of complexes of acyclic sheaves of abelian groups on X, which implies that the induced morphism on global sections

$$\Gamma(X, A) \longrightarrow \Gamma(X, RA)$$

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is a quasi-isomorphism of complexes. By the definition of an equivalence in G-CDGA ,this implies that the morphism

$$\Gamma(X, A) \longrightarrow \Gamma(X, RA) =: \mathbb{R}\Gamma(X, A)$$

is actually an isomorphism in Ho(G-CDGA). □

Now, let (X, x) be a pointed connected compact smooth manifold, and let X^{top} denote the underlying topological space of X. Let $L_B(X)$ be the category of semi-simple local systems of finite-dimensional $\mathbb{C}$-vector spaces on X^{top}. It is a rigid $\mathbb{C}$-linear tensor category which is naturally equivalent to the category of finite-dimensional semi-simple representations of the fundamental group $\pi_1(X^{\mathrm{top}}, x)$.

The category of semi-simple C^∞ complex vector bundles with flat connections on X will be denoted by $L_{DR}(X)$, as before. Recall that the category $L_{DR}(X)$ is a rigid $\mathbb{C}$-linear tensor category, and that the functor which maps a flat bundle to its monodromy representations at x induces an equivalence of tensor categories $L_B(X) \simeq L_{DR}(X)$ (this is again the Riemann–Hilbert correspondence).

Let $G_X := \pi_1(X^{\mathrm{top}}, x)^{\mathrm{red}}$ be the pro-reductive completion of the group $\pi_1(X^{\mathrm{top}}, x)$. Note that it is the Tannaka dual of the category $L_B(X)$. The algebra $\mathcal{O}(G_X)$ can be viewed as the left regular representation of G_X. Through the universal morphism $\pi_1(X^{\mathrm{top}}, x) \longrightarrow G_X$, we can also consider $\mathcal{O}(G_X)$ as a linear representation of $\pi_1(X^{\mathrm{top}}, x)$. This linear representation is not finite-dimensional, but it is admissible in the sense that it equals the union of its finite-dimensional sub-representations. Therefore, the algebra $\mathcal{O}(G_X)$ corresponds to an object in the $\mathbb{C}$-linear tensor category $T_B(X)$ of Ind-local systems on X^{top}. By convention, all of our Ind-objects are labelled by $\mathbb{U}$-small index categories.

Furthermore, the algebra structure on $\mathcal{O}(G_X)$ gives rise to a morphism

$$\mu : \mathcal{O}(G_X) \otimes \mathcal{O}(G_X) \longrightarrow \mathcal{O}(G_X),$$

which is easily checked to be a morphism in $T_B(X)$. This means that if $\mathcal{O}(G_X)$ is written as the colimit of local systems $\{V_i\}_{i\in I}$, then the product μ is given by a compatible system of morphisms in $L_B(X)$,

$$\mu_{i,k} : V_i \otimes V_i \longrightarrow V_k,$$

for some index $k \in I$ with $i \le k \in I$. The morphism $\mu = \{\mu_{i,k}\}_{i,k\in I}$ endows the object $\mathcal{O}(G_X) \in T_B(X)$ with the structure of a commutative unital monoid. Through the Riemann–Hilbert correspondence $T_B(X) \simeq T_{DR}(X)$, the algebra $\mathcal{O}(G_X)$ can also be considered as a commutative monoid in the tensor category $T_{DR}(X)$ of Ind-objects in $L_{DR}(X)$.

Let $\{(V_i, \nabla_i)\}_{i\in I} \in T_{DR}(X)$ be the object corresponding to $\mathcal{O}(G_X)$. For any $i \in I$, one can form the de Rham complex of C^∞-differential forms

$$(A^\bullet_{DR}(V_i), \nabla_i) := A^0(V_i) \xrightarrow{\nabla_i} A^1(V_i) \xrightarrow{\nabla_i} \cdots \xrightarrow{\nabla_i} A^n(V_i) \xrightarrow{D_i} \cdots.$$

In this way, we obtain an inductive system of complexes $\{(A^\bullet_{DR}(V_i), D_i)\}_{i\in I}$ whose colimit complex is defined to be the de Rham complex of the local system $\mathcal{O}(G_X)$ on X:

$$(A^\bullet_{DR}(\mathcal{O}(G_X)), \nabla) := \mathrm{colim}_{i\in I}(A^\bullet_{DR}(V_i), \nabla_i).$$

The complex $(A^\bullet_{DR}(\mathcal{O}(G_X)), \nabla)$ has the natural structure of a commutative differential graded algebra, coming from the commutative monoid structure on $\{(V_i, \nabla_i)\}_{i\in I} \in T_{DR}(X)$. Using wedge products of differential forms and the monoidal structure, we obtain in the usual fashion morphisms of complexes

$$(A^\bullet_{DR}(V_i), \nabla_i) \otimes (A^\bullet_{DR}(V_j), \nabla_j) \longrightarrow (A^\bullet_{DR}(V_k), \nabla_k)$$

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which, after passing to the colimit along I, turn $(A^\bullet_{DR}(\mathcal{O}(G_X)), \nabla)$ into a commutative differential graded algebra.

The affine group scheme G_X acts via the right regular representation on the Ind-local system $\mathcal{O}(G_X)$, and this action is compatible with the algebra structure. By functoriality, this action gives rise to an action of G_X on the corresponding objects in $T_{DR}(X)$. Furthermore, if G_X acts on an inductive system of flat bundles (V_i, ∇_i), then it acts naturally on its de Rham complex $\mathrm{colim}_{i\in I}(A^\bullet_{DR}(V_i), \nabla_i)$ by acting on the spaces of differential forms with coefficients in the various V_i. Indeed, if the action of G_X is given by a comodule structure

$$\{(V_i, \nabla_i)\}_{i\in I} \longrightarrow \{\mathcal{O}(G_X) \otimes (V_i, \nabla_i)\}_{i\in I},$$

then one obtains a morphism of Ind- C^∞-bundles by tensoring with the sheaf A^n of differential forms on X:

$$\{V_i \otimes A^n\}_{i\in I} \longrightarrow \{\mathcal{O}(G_X) \otimes (V_i \otimes A^n)\}_{i\in I}.$$

Upon taking global sections on X, one has a morphism

$$\mathrm{colim}_{i\in I}\, A^n(V_i) \longrightarrow \mathrm{colim}_{i\in I}\, A^n(V_i) \otimes \mathcal{O}(G_X),$$

which defines an action of G_X on the space of differential forms with values in the Ind- C^∞-bundle $\{V_i\}_{i\in I}$. Since this action is compatible with the differentials ∇_i, one obtains an action of G_X on the de Rham complex $\mathrm{colim}_{i\in I}(A^\bullet_{DR}(V_i), \nabla_i)$. Furthermore, since the action is compatible with the algebra structure on $\mathcal{O}(G_X)$, it follows that G_X acts on $\mathrm{colim}_{i\in I}(A^\bullet_{DR}(V_i), \nabla_i)$ by algebra automorphisms. Thus, the group scheme G_X acts in a natural way on the complex $(A^\bullet_{DR}(\mathcal{O}(G_X)), \nabla)$, turning it into a well-defined G_X-equivariant commutative differential graded algebra.

Applying the denormalization functor $D : \mathrm{Ho}(G_X\text{-}\mathsf{CDGA}) \longrightarrow \mathrm{Ho}(G_X\text{-}\mathsf{Alg}^\Delta)$, we obtain a well-defined G_X-equivariant cosimplicial algebra, denoted by

$$C^\bullet_{DR}(X, \mathcal{O}(G_X)) := D(A^\bullet_{DR}(\mathcal{O}(G_X)), \nabla) \in \mathrm{Ho}(G_X\text{-}\mathsf{Alg}^\Delta).$$

To summarize, for any pointed connected smooth manifold (X, x) we let $G_X := \pi_1(X, x)^{\mathrm{red}}$ be the pro-reductive completion of its fundamental group; the G_X-equivariant commutative differential graded algebra of the de Rham cohomology of X with coefficients in $\mathcal{O}(G_X)$ is denoted by

$$(A^\bullet_{DR}(\mathcal{O}(G_X)), \nabla) \in \mathrm{Ho}(G_X\text{-}\mathsf{CDGA}),$$

and its denormalization is denoted by

$$C^\bullet_{DR}(X, \mathcal{O}(G_X)) := D(A^\bullet_{DR}(\mathcal{O}(G_X)), \nabla) \in \mathrm{Ho}(G_X\text{-}\mathsf{Alg}^\Delta).$$

Any smooth map $f : (Y, y) \longrightarrow (X, x)$ of pointed connected smooth manifolds induces a morphism $G_Y := \pi_1(Y, y)^{\mathrm{red}} \longrightarrow G_X := \pi_1(X, x)^{\mathrm{red}}$ and therefore a well-defined functor

$$f^* : \mathrm{Ho}(G_X\text{-}\mathsf{Alg}^\Delta) \longrightarrow \mathrm{Ho}(G_Y\text{-}\mathsf{Alg}^\Delta).$$

It is not difficult to check that the pull-back of differential forms via f induces a well-defined morphism in $\mathrm{Ho}(G_Y\text{-}\mathsf{Alg}^\Delta)$,

$$f^* : f^* C^\bullet_{DR}(X, \mathcal{O}(G_X)) \longrightarrow C^\bullet_{DR}(Y, \mathcal{O}(G_Y)).$$

Furthermore, this morphism depends functorially (in an obvious fashion) on the morphism f.

The preceding discussion allows us to construct a functor

$$(X, x) \mapsto [\mathbb{R}\,\mathrm{Spec}_{G_X}\ C^\bullet_{DR}(X, \mathcal{O}(G_X))/G_X]$$

from the category of pointed connected smooth manifolds to the category of pointed schematic homotopy types. To simplify notation, we will denote this functor by $(X, x) \mapsto (X \otimes \mathbb{C})^{\mathrm{diff}}$.

The following corollary is a generalization of the Poincaré lemma.

Corollary 4.4. *Let X be a pointed connected compact and smooth manifold. Then there exists an isomorphism in* $\mathrm{Ho}(G\text{-}\mathsf{CDGA})$,

$$(A_{DR}(\mathcal{O}(G_X)), D) \simeq \mathbb{R}\Gamma(X, \mathcal{O}(G_X)),$$

which is functorial in X.

Proof. This follows from Lemma 4.3 and the fact that the sheaves $A^n(\mathcal{O}(G_X))$ of differential forms with coefficients in the Ind-flat bundle associated to $\mathcal{O}(G_X)$ are filtered colimits of soft sheaves on X. In particular, they are acyclic since X is compact.

The functoriality is straightforward. □

As a consequence of Corollary 4.4 we obtain the following.

Corollary 4.5. *Let X be a pointed connected compact and smooth manifold. Then there exists an isomorphism in* $\mathrm{Ho}(G\text{-}\mathsf{Alg}^{\Delta})$,

$$D(A_{DR}(\mathcal{O}(G_X)), D) \simeq \mathbb{R}\Gamma(X, \mathcal{O}(G_X)),$$

where D is the denormalization functor and $\mathcal{O}(G_X)$ is considered as an object in $G\text{-}\mathsf{Alg}^{\Delta}(X)$. This isomorphism is, moreover, functorial in X.

Let (S, s) be a pointed connected simplicial set, $\pi := \pi_1(S, s)$ its fundamental group, and G_S the pro-reductive completion of π. The conjugation action of π on G_S is an action by group scheme automorphism and therefore gives rise to a natural action on the model category $G_S\text{-}\mathsf{Alg}^{\Delta}$. More precisely, if $\gamma \in \pi$ and if $A \in G_S\text{-}\mathsf{Alg}^{\Delta}$ corresponds to a cosimplicial $\mathcal{O}(G_S)$-comodule E, then one defines $\gamma \cdot E$ to be the comodule structure

$$E \longrightarrow E \otimes \mathcal{O}(G_S) \xrightarrow{\mathrm{Id} \otimes \gamma} E \otimes \mathcal{O}(G_S)\ .$$

We will be concerned with the fixed-point model category of $G_S\text{-}\mathsf{Alg}^{\Delta}$ under the group π, denoted by $(G_S\text{-}\mathsf{Alg}^{\Delta})^{\pi}$ and described in [KPT08].

The category $(G_S\text{-}\mathsf{Alg}^{\Delta})^{\pi}$ is naturally enriched over the category $\pi\text{-}\mathsf{SSet}$ of π-equivariant simplicial sets. Indeed, for $X \in \pi\text{-}\mathsf{SSet}$ and $A \in (G_S\text{-}\mathsf{Alg}^{\Delta})^{\pi}$, one can define $X \otimes A$ whose underlying G_S-equivariant cosimplicial algebra is $X^{\mathrm{for}} \otimes A^{\mathrm{for}}$ (where X^{for} and A^{for} are the underlying simplicial set and the G_S-equivariant cosimplicial algebra of X and A, respectively, and $X^{\mathrm{for}} \otimes A^{\mathrm{for}}$ uses the simplicial structure of $G_S\text{-}\mathsf{Alg}^{\Delta}$). The action of π on $X^{\mathrm{for}} \otimes A^{\mathrm{for}}$ is then defined diagonally. In particular, we can use the exponential product

$$A^X \in (G_S\text{-}\mathsf{Alg}^{\Delta})^{\pi}, \quad \text{where } X \in \pi\text{-}\mathsf{SSet} \text{ and } A \in (G_S\text{-}\mathsf{Alg}^{\Delta})^{\pi}.$$

The functor

$$\begin{array}{c}(\pi\text{-}\mathsf{SSet})^{\mathrm{op}} \times (G_S\text{-}\mathsf{Alg}^{\Delta})^{\pi} \to (G_S\text{-}\mathsf{Alg}^{\Delta})^{\pi} \\ (X, A) \longmapsto A^X\end{array}$$

is a bi-Quillen functor (see [Hov98, § 4]) and can be derived into a functor

$$\begin{array}{c}\mathrm{Ho}(\pi\text{-}\mathsf{SSet})^{\mathrm{op}} \times \mathrm{Ho}((G_S\text{-}\mathsf{Alg}^{\Delta})^{\pi}) \to \mathrm{Ho}((G_S\text{-}\mathsf{Alg}^{\Delta})^{\pi}) \\ (X, A) \longmapsto A^{\mathbb{R}X}.\end{array}$$

Recall that, by definition,

$$A^{\mathbb{R}X} := (RA)^{QX},$$

where RA is a fibrant model for A in $(G_S\text{-}\mathsf{Alg}^{\Delta})^{\pi}$ and QX is a cofibrant model for X in $\pi\text{-}\mathsf{SSet}$.

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In the following definition, $\mathcal{O}(G_S)$ is considered together with its π and G_S actions, i.e. it is viewed as an object in $(G_S\text{-}\mathsf{Alg}^{\Delta})^{\pi}$.

Definition 4.6. The G_S-equivariant cosimplicial algebra of cochains of S with coefficients in the local system $\mathcal{O}(G_S)$ is defined to be

$$C^{\bullet}(S, \mathcal{O}(G_S)) := \mathcal{O}(G_S)^{\mathbb{R}\widetilde{S}} \in \mathrm{Ho}(G_S\text{-}\mathsf{Alg}^{\Delta}),$$

where $\widetilde{S} \in \mathrm{Ho}(\pi\text{-}\mathsf{SSet})$ is the universal covering of S.

Since $\mathcal{O}(G_S)$ is always a fibrant object in $(G_S\text{-}\mathsf{Alg}^{\Delta})^{\pi}$, one has

$$C^{\bullet}(S, \mathcal{O}(G_S)) \simeq \mathcal{O}(G_S)^{\widetilde{S}} \in \mathrm{Ho}(G_S\text{-}\mathsf{Alg}^{\Delta})$$

as soon as $\widetilde{S}$ is chosen to be cofibrant in π-SSet (e.g. is chosen to be a π-equivariant cell complex).

Going back to our space X, which is assumed to be pointed and connected, let $S := S(X)$ be its singular simplicial set, naturally pointed by the image $s \in S$ of $x \in X$. Observe that in this case one has a natural isomorphism $G_S \simeq G_X$, induced by the natural isomorphism $\pi_1(X, x) \simeq \pi_1(S, s)$.

Proposition 4.7. *There exists an isomorphism in* $\mathrm{Ho}(G_S\text{-}\mathsf{Alg}^{\Delta})$*:*

$$C^{\bullet}(S, \mathcal{O}(G_S)) \simeq \mathbb{R}\Gamma(X, \mathcal{O}(G_S)),$$

where $\mathcal{O}(G_S)$ *is considered as an object in* $G_S\text{-}\mathsf{Alg}^{\Delta}(X)$*. Furthermore, this isomorphism is functorial in* X*.*

Proof. We prove only the existence of the isomorphism. The functoriality statement is straightforward and is left to the reader.

Let $\pi = \pi_1(X, x) \simeq \pi_1(S, s)$, let $\widetilde{X} \longrightarrow X$ be the universal cover of X, let $\widetilde{S} \longrightarrow S(\widetilde{X})$ be a cofibrant replacement of $S(\widetilde{X}) \in \pi$-SSet, and let $p : \widetilde{S} \longrightarrow S(X)$ be the natural π-equivariant projection. For any open subset $U \subset X$, denote by $\widetilde{S}_U$ the fiber product

$$\widetilde{S}_U := \widetilde{S} \times_{S(X)} S(U) \in \pi\text{-}\mathsf{SSet}.$$

Consider the presheaf $C^{\bullet}(-, \mathcal{O}(G_S))$ of G_S-equivariant cosimplicial algebras on X, defined by

$$C^{\bullet}(-, \mathcal{O}(G_S)) : \quad \mathsf{Open}(X)^{\mathrm{op}} \longrightarrow G_S\text{-}\mathsf{Alg}^{\Delta}$$

$$U \longmapsto C^{\bullet}(U, \mathcal{O}(G_S)) := \mathcal{O}(G_S)^{\widetilde{S}_U},$$

where $\mathcal{O}(G_S)^{\widetilde{S}_U}$ is the exponentiation of $\mathcal{O}(G_S) \in (G_S\text{-}\mathsf{Alg}^{\Delta})^{\pi}$ by $\widetilde{S}_U \in \pi$-SSet. We denote by $aC^{\bullet}(-, \mathcal{O}(G_S)) \in G_S\text{-}\mathsf{Alg}^{\Delta}(X)$ the associated sheaf.

We have a natural morphism in $G_S\text{-}\mathsf{Alg}^{\Delta}$,

$$C^{\bullet}(X, \mathcal{O}(G_S)) \longrightarrow \Gamma(X, aC^{\bullet}(-, \mathcal{O}(G_S))) \longrightarrow \Gamma(X, RaC^{\bullet}(-, \mathcal{O}(G_S))) \longleftarrow \Gamma(X, R\mathcal{O}(G)_S),$$

where $RaC^{\bullet}(-, \mathcal{O}(G_S)))$ is a fibrant replacement of $aC^{\bullet}(-, \mathcal{O}(G_S)))$ and $R\mathcal{O}(G)_S$ is a fibrant replacement of $\mathcal{O}(G)_S$. Since X is assumed to be locally contractible, the natural morphism

$$\mathcal{O}(G_S) \longrightarrow C^{\bullet}(-, \mathcal{O}(G_S))$$

in $G_S\text{-}\mathsf{Alg}^{\Delta}(X)$, induced over every open subset $U \subset X$ by the projection $\widetilde{S}_U \longrightarrow *$, is an equivalence. Therefore, it only remains to show that the morphism

$$C^{\bullet}(X, \mathcal{O}(G_S)) \longrightarrow \Gamma(X, RaC^{\bullet}(-, \mathcal{O}(G_S))$$

is an equivalence in $G_S\text{-}\mathsf{Alg}^{\Delta}$.

For this, let $U_\bullet \longrightarrow X$ be an open hyper-cover of X such that each U_n is the disjoint union of contractible open subsets of X (such a hyper-cover exists owing, again, to local contractibility of X). The simplicial sheaf represented by $U_\bullet$ is equivalent to $*$ in $\mathsf{SSh}(X)$. As $RaC^\bullet(-, \mathcal{O}(G_S))$ is fibrant, we obtain a natural equivalence in $G_S\text{-}\mathsf{Alg}^\Delta$:

$$\Gamma(X, RaC^\bullet(-, \mathcal{O}(G_S)) \simeq \Gamma(X, RaC^\bullet(-, \mathcal{O}(G_S))^{U_\bullet}) \simeq \mathrm{holim}_{[n]\in\Delta}\, \Gamma(U_n, RaC^\bullet(-, \mathcal{O}(G_S))).$$

Furthermore, it is shown in [Toe02, Lemma 2.10] that the natural morphism

$$\mathrm{hocolim}_{[n]\in\Delta^{\mathrm{op}}}\, U_\bullet \longrightarrow X$$

is a weak equivalence; therefore

$$\mathrm{hocolim}_{[n]\in\Delta^{\mathrm{op}}}\, \widetilde{S}_{U_\bullet} \longrightarrow \widetilde{S}_X = \widetilde{S}$$

is an equivalence in π-SSet. This implies that the natural morphism in $G_S\text{-}\mathsf{Alg}^\Delta$,

$$C^\bullet(X, \mathcal{O}(G_S)) \longrightarrow \mathrm{holim}_{[n]\in\Delta}\, C^\bullet(U_n, \mathcal{O}(G_S)),$$

is an equivalence. Moreover, there exists a commutative diagram as follows.

$$\begin{array}{ccc} C^\bullet(X, \mathcal{O}(G_S)) & \longrightarrow & \mathrm{holim}_{[n]\in\Delta}\, C^\bullet(U_n, \mathcal{O}(G_S)) \\ \downarrow & & \downarrow \\ \Gamma(X, RaC^\bullet(-, \mathcal{O}(G_S)) & \longrightarrow & \mathrm{holim}_{[n]\in\Delta}\, \Gamma(U_n, RaC^\bullet(-, \mathcal{O}(G_S))) \end{array}$$

Thus, without loss of generality, we may assume that X is contractible. Under this assumption, it only remains to check that the natural morphism

$$\mathcal{O}(G_S) \longrightarrow \mathbb{R}\Gamma(X, \mathcal{O}(G_S))$$

is an isomorphism in $\mathrm{Ho}(G_S\text{-}\mathsf{Alg}^\Delta)$. But, by Lemma 4.2 and the description of fibrant objects in $G_S\text{-}\mathsf{Alg}^\Delta(X)$, one has

$$H^n(\mathbb{R}\Gamma(X, \mathcal{O}(G_S))) \simeq H^n(X, \mathcal{O}(G_S)).$$

Since the space X is paracompact and contractible, its sheaf cohomology with coefficients in the constant local system $\mathcal{O}(G_S)$ is trivial; this completes the proof of the proposition. □

As an immediate consequence of Corollary 4.5 and Proposition 4.7, one obtains the following description of the schematization of a smooth manifold in terms of its differential forms.

Proposition 4.8. *Let X be a pointed connected compact and smooth manifold. Then there exists an isomorphism in* $\mathrm{Ho}(G\text{-}\mathsf{Alg}^\Delta)$:

$$D(A_{DR}(\mathcal{O}(G_X)), D) \simeq C^\bullet(S(X), \mathcal{O}(G_X)),$$

where D is the denormalization functor and $\mathcal{O}(G_X)$ is considered as an object in $G\text{-}\mathsf{Alg}^\Delta(X)$. In other words, one has a natural isomorphism

$$(X \otimes k)^{\mathrm{sch}} \simeq [\mathbb{R}\,\mathrm{Spec}_{G_X}\, D(A_{DR}(\mathcal{O}(G_X)), D)/G_X].$$

4.2 A schematic Van Kampen theorem

Let X be a pointed and connected ($\mathbb{U}$-small) topological space, and let $\{U_i\}_{i\in I}$ be a finite open covering. We will assume that each U_i contains the base point, and that each of the $(p+1)$-tuple intersections $U_{i_0,\dots,i_{p+1}} := U_{i_0} \cap \cdots \cap U_{i_p}$ is connected. Here we have in mind the example of a smooth projective complex variety X covered by Zariski open subsets containing the base point. We form the poset $N(U)$ (the nerve of $\{U_i\}_{i\in I}$), whose objects are strings of indices

$(i_0, \ldots, i_p) \in I^{p+1}$ (for various lengths $p \geq 0$) and where $(i_0, \ldots, i_p) \leq (j_0, \ldots, j_p)$ if and only if $U_{i_0,\ldots,i_{p+1}} \subset U_{j_0,\ldots,j_{p+1}}$. There exists a natural functor

$$\begin{aligned} N(U) &\longrightarrow \mathsf{Top}_*^{\mathrm{con}} \\ (i_0, \ldots, i_p) &\longmapsto U_{i_0,\ldots,i_p}, \end{aligned}$$

where $\mathsf{Top}_*^{\mathrm{con}}$ is the category of pointed and connected $\mathbb{U}$-topological spaces. This diagram is, moreover, augmented to the constant diagram X. Applying the schematization functor, we get a diagram of pointed simplicial presheaves,

$$\begin{aligned} N(U) &\longrightarrow \mathsf{SPr}_*(k) \\ (i_0, \ldots, i_p) &\longmapsto (U_{i_0,\ldots,i_p} \otimes k)^{\mathrm{sch}}, \end{aligned}$$

which is naturally augmented towards $(X \otimes k)^{\mathrm{sch}}$. (Here we use a version of the schematization functor which is defined on the level of simplicial sets and not only on its homotopy category. For example, one can use the functor $Z \mapsto \boldsymbol{B}\,\mathrm{Spec}\,\mathcal{O}(GZ^{\mathrm{alg}})^P$, where $(-)^P$ is a cofibrant replacement functor for the P-local model structure on Hopf^Δ; see Corollary 3.19.)

Proposition 4.9. *For any psht F, the natural morphism*

$$\mathbb{R}\,\underline{\mathrm{Hom}}((X \otimes k)^{\mathrm{sch}}, F) \longrightarrow \mathrm{Holim}_{\alpha \in N(U)}\, \mathbb{R}\,\underline{\mathrm{Hom}}((U_\alpha \otimes k)^{\mathrm{sch}}, F)$$

induced by the augmentation is an equivalence of simplicial sets.

Proof. Using the universal property of the schematization functor it is enough to prove that the natural morphism

$$\mathrm{Hocolim}_{\alpha \in N(U)}\, U_\alpha \longrightarrow X$$

is a weak equivalence of topological spaces. When X is a CW complex, this is well-known (see, for example, [Toe02, Lemma 2.10]). In the general case, it suffices consider the commutative square

$$\begin{array}{ccc} \mathrm{Hocolim}_{\alpha \in N(U)}\, |S(U_\alpha)| & \longrightarrow & |S(X)| \\ \downarrow & & \downarrow \\ \mathrm{Hocolim}_{\alpha \in N(U)}\, U_\alpha & \longrightarrow & X \end{array}$$

where $|S(-)|$ is the geometric realization of the singular functor. Since the functors $|-|$ and $S(-)$ form a Quillen equivalence, the vertical morphisms are both weak equivalences. The top horizontal morphism being an equivalence, the proof of the proposition is complete. □

Remark 4.10. Another interpretation of Proposition 4.9 is to say that $(X \otimes k)^{\mathrm{sch}}$ is the homotopy colimit *in the category of psht* of the diagram $\alpha \mapsto U_\alpha$. However, this point of view is somewhat trickier since, when appropriately defined, the homotopy colimits of psht are not the same as homotopy colimits of pointed simplicial presheaves. Such subtleties go beyond the scope of this paper, so we will not discuss them here.

Proposition 4.9 is a Van Kampen type result, as it states that the schematization of a space X is uniquely determined by the diagram $\alpha \mapsto (U_\alpha \otimes k)^{\mathrm{sch}}$. This property will be useful only if the space X behaves well locally with respect to the schematization functor. This is the case, for instance, when X is a smooth projective complex variety. Indeed, locally for the Zariski topology X is a $K(\pi, 1)$, where π is a successive extension of free groups of finite type. In Theorem 4.16 below, we will show that such groups π are *algebraically good*, and therefore $(K(\pi, 1) \otimes k)^{\mathrm{sch}}$ is itself 1-truncated. So, in a way, the schematization of a smooth projective complex variety is well-understood locally for the Zariski topology, and Proposition 4.9 tells us that, globally, the schematization is obtained by the homotopy colimits of the schematization of a covering.

4.3 Good groups

Recall that a discrete group Γ in $\mathbb{U}$ is algebraically good (relative to the field k) if the natural morphism

$$(K(\Gamma,1)\otimes k)^{\mathrm{sch}} \longrightarrow K(\Gamma^{\mathrm{alg}},1)$$

is an isomorphism. The term *algebraically good* mimicks the corresponding pro-finite notion introduced by J. P. Serre in [Ser94]. It is justified by the following lemma. Let $H^{\bullet}_{H}(\Gamma^{\mathrm{alg}},V)$ denote the Hochschild cohomology of the affine group scheme Γ^{alg} with coefficients in a linear representation V (as defined in [DG70b, I]).

Lemma 4.11. *Let Γ be a group in $\mathbb{U}$ and Γ^{alg} its pro-algebraic completion. Then, Γ is an algebraically good group if and only if for every finite-dimensional linear representation V of Γ^{alg}, the natural morphism*

$$H^{\bullet}_{H}(\Gamma^{\mathrm{alg}},V) \longrightarrow H^{\bullet}(\Gamma,V)$$

is an isomorphism.

Proof. Using [Toe06, § 1.5] and [Toe06, Corollary 3.3.3], the fact that $H^{\bullet}_{H}(\Gamma^{\mathrm{alg}},V)\cong H^{\bullet}(\Gamma,V)$ for all V is just a reformulation of the fact that $K(\Gamma,1)\longrightarrow K(\Gamma^{\mathrm{alg}},1)$ is a P-equivalence. Since $K(\Gamma^{\mathrm{alg}},1)$ is a pointed schematic homotopy type, this implies the lemma. ☐

We will also use the following very general fact.

Proposition 4.12. *Let Γ be a group in $\mathbb{U}$, Γ^{alg} its pro-algebraic completion, and $n>1$ an integer. The following are equivalent.*

(1) *For any linear representation V of Γ^{alg}, the induced morphism*

$$H^{i}_{H}(\Gamma^{\mathrm{alg}},V) \longrightarrow H^{i}(\Gamma,V)$$

is an isomorphism for $i<n$ and injective for $i=n$.

(2) *For any linear representation V of Γ^{alg}, the induced morphism*

$$H^{i}_{H}(\Gamma^{\mathrm{alg}},V) \longrightarrow H^{i}(\Gamma,V)$$

is surjective for $i<n$.

Proof. Let $\mathsf{Rep}(\Gamma)$ be the category of linear complex representations of Γ (possibly infinite-dimensional), and let $\mathsf{Rep}(\Gamma^{\mathrm{alg}})$ be the category of linear representations of the affine group scheme Γ^{alg} (again possibly of infinite dimension). The proposition follows from [And74, Lemma 11] applied to the inclusion functor $\mathsf{Rep}(\Gamma^{\mathrm{alg}})\longrightarrow \mathsf{Rep}(\Gamma)$. ☐

As an immediate corollary, we obtain that finite groups are algebraically good over k. Furthermore, for any finitely generated group Γ, the universal property of the pro-algebraic completion implies that the natural map $\Gamma\to\Gamma^{\mathrm{alg}}$ induces an isomorphism on cohomology in degrees zero and one. Therefore, the previous proposition implies that the natural map $H^{2}_{H}(\Gamma^{\mathrm{alg}},V)\longrightarrow H^{2}(\Gamma,V)$ is always injective. In particular, any free group of finite type will be algebraically good over k. In the same vein, we have the following proposition.

Proposition 4.13. *The following groups are algebraically good groups:*

(1) *abelian groups of finite type;*

(2) *fundamental groups of a compact Riemann surface when $k=\mathbb{C}$.*

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Proof. (1) For any abelian group M of finite type, there exists a morphism of exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H & \longrightarrow & M & \longrightarrow & M_0 & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & H & \longrightarrow & M^{\mathrm{alg}} & \longrightarrow & M_0^{\mathrm{alg}} & \longrightarrow & 0 \end{array}$$

where H is a finite group and M_0 is torsion-free. The comparison of the associated Leray spectral sequences shows that, without loss of generality, one may assume M to be torsion-free. Then, again by comparing the two Leray spectral sequences for the rows of the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z}^{n-1} & \longrightarrow & \mathbb{Z}^n & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & (\mathbb{Z}^{n-1})^{\mathrm{alg}} & \longrightarrow & (\mathbb{Z}^n)^{\mathrm{alg}} & \longrightarrow & \mathbb{Z}^{\mathrm{alg}} & \longrightarrow & 0 \end{array}$$

and proceeding by induction on the rank of M, one reduces the proof to the case of $M = \mathbb{Z}$. But, as $\mathbb{Z}$ is a free group, this is algebraically good.

(2) Note that, by (1), one can suppose that $g > 2$. As Γ_g is the fundamental group of a compact Riemann surface of genus g, one has $H^i(\Gamma_g, V) = 0$ for any $i > 2$ and any linear representation V. By Proposition 4.12, it is enough to prove that for any finite-dimensional linear representation V of Γ, the natural morphism $H^2_H(\Gamma_g^{\mathrm{alg}}, V) \longrightarrow H^2(\Gamma_g, V)$ is an isomorphism. We need the following lemma.

Lemma 4.14. *Let V be any finite-dimensional linear representation of Γ_g, and let $x \in H^2(\Gamma_g, V)$. Then there exist a finite-dimensional linear representation V' and an injective morphism $j : V \hookrightarrow V'$ such that $j(x) = 0 \in H^2(\Gamma_g, V')$.*

Proof. Using Poincare duality, the assertion of the lemma can be restated as follows: given any finite-dimensional linear representation E of Γ_g and any invariant vector $e \in H^0(\Gamma_g, E) = E^{\Gamma_g}$, there exist a finite-dimensional linear representation E' and a Γ_g-equivariant surjection $q : E' \to E$ such that $e \notin \mathrm{im}[H^0(\Gamma_g, E') \to H^0(\Gamma_g, E)]$. Let $\mathbb{I}$ denote the one-dimensional trivial representation of Γ_g; then the element $e \in H^0(\Gamma_g, E)$ can be viewed as an injective homomorphism $e : \mathbb{I} \to E$ of finite-dimensional Γ_g-modules. Let $F := E/\mathbb{I}$ be the corresponding quotient module and let Σ be an irreducible r-dimensional representation of Γ_g, for some $r \geq 1$. The short exact sequence of Γ_g-modules

$$0 \to \mathbb{I} \xrightarrow{e} E \to F \to 0$$

induces a long exact sequence of Ext in the category of Γ_g-modules:

$$\ldots \longrightarrow \mathrm{Ext}^1(F, \Sigma) \longrightarrow \mathrm{Ext}^1(E, \Sigma) \xrightarrow{ev_e} \mathrm{Ext}^1(\mathbb{I}, \Sigma) \longrightarrow \mathrm{Ext}^2(F, \Sigma) \longrightarrow \ldots.$$

Now observe that $\dim \mathrm{Ext}^1(\mathbb{I}, \Sigma) = \dim H^1(\Gamma_g, \Sigma) \geq \chi(\Gamma_g, \Sigma) = 2r(g-1)$. Furthermore, by Poincaré duality we have $\mathrm{Ext}^2(F, \Sigma) \cong \mathrm{Hom}(\Sigma, F)^\vee$, and since Σ contains only finitely many irreducible subrepresentations of Γ_g, it follows that for the generic choice of Σ, the natural map

$$\mathrm{Ext}^1(E, \Sigma) \xrightarrow{ev_e} \mathrm{Ext}^1(\mathbb{I}, \Sigma)$$

will be surjective. Choose such a Σ and let $\epsilon \in \mathrm{Ext}^1(E, \Sigma)$ be an element such that $ev_e(\epsilon) \neq 0 \in \mathrm{Ext}^1(\mathbb{I}, \Sigma)$. Now let

$$0 \to \Sigma \to E' \xrightarrow{q} E \to 0$$

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be the extension of Γ_g modules corresponding to ϵ. Then the image of $e \in H^0(\Gamma_g, E)$ under the first edge homomorphism $H^0(\Gamma_g, E) \to H^1(\Gamma_g, \Sigma) \cong \mathrm{Ext}^1(\mathbb{I}, \Sigma)$ is precisely $ev_e(\epsilon) \neq 0$. The lemma is thus proven. □

Returning to the proof of Proposition 4.13, let V by any finite-dimensional linear representation of Γ_g, let $x \in H^2(\Gamma_g, V)$, and let $V \hookrightarrow V'$ as in the previous lemma. Then, the morphism of long exact sequences

$$\begin{array}{ccccccc} H^1_H(\Gamma_g^{\mathrm{alg}}, V'/V) & \longrightarrow & H^2_H(\Gamma_g^{\mathrm{alg}}, V) & \longrightarrow & H^2_H(\Gamma_g^{\mathrm{alg}}, V') & \longrightarrow & \cdots \\ \downarrow & & \downarrow & & \downarrow & & \\ H^1(\Gamma_g, V'/V) & \longrightarrow & H^2(\Gamma_g, V) & \longrightarrow & H^2(\Gamma_g, V') & \longrightarrow & \cdots \end{array}$$

shows that x can be lifted to $H^2(\Gamma_g^{\mathrm{alg}}, V)$. This implies that $H^2_H(\Gamma_g^{\mathrm{alg}}, V) \longrightarrow H^2(\Gamma_g, V)$ is surjective, and finishes the proof of the proposition. □

Before constructing more examples of algebraically good groups, we describe a convenient criterion (which was already used implicitly in the proof of Proposition 4.13) for checking if a given group is good. For a finitely generated group Γ we denote by $\mathsf{Rep}(\Gamma)$ the category of all representations of Γ in k-vector spaces (possibly infinite-dimensional) and by $\mathsf{Rep}(\Gamma^{\mathrm{alg}})$ the category of all algebraic representations of the affine pro-algebraic group Γ^{alg}. Recall also that a group Γ is of type $(\boldsymbol{F})$ (over k) if: (a) for every n and every finite-dimensional complex representation V of Γ, the group $H^n(\Gamma, V)$ is finite-dimensional; and (b) $H^\bullet(\Gamma, -)$ commutes with inductive limits of finite-dimensional complex representations of Γ. With this notation, we now have the following lemma.

Lemma 4.15. *Let Γ be a finitely generated group of type $(\boldsymbol{F})$. The following properties of Γ are equivalent.*

(a) *Γ is algebraically good over k.*

(b) *For every positive integer n, every finite-dimensional representation $L \in \mathsf{Rep}(\Gamma)$ and every class $\alpha \in H^n(\Gamma, L)$, there exists an injection $\imath : L \hookrightarrow W_\alpha$ in some finite-dimensional Γ-module $W_\alpha \in \mathsf{Rep}(\Gamma)$ such that the induced map $\imath^* : H^n(\Gamma, L) \to H^n(\Gamma, W_\alpha)$ annihilates α, i.e. $\imath^*(\alpha) = 0$.*

(c) *For every positive integer n and every finite-dimensional representation $L \in \mathsf{Rep}(\Gamma)$, there exists an injection $\imath : L \hookrightarrow W_\alpha$ in some finite-dimensional Γ-module $W_\alpha \in \mathsf{Rep}(\Gamma)$ such that the induced map $\imath^* : H^n(\Gamma, L) \to H^n(\Gamma, W_\alpha)$ is identically zero, i.e. $\imath \equiv 0$.*

Proof. First, we show that (a) ⇒ (b). Suppose that Γ is good over k. Fix an integer $n > 0$, a representation $L \in \mathsf{Rep}(\Gamma)$ with $\dim(L) < +\infty$ and some class $\alpha \in H^n(\Gamma, L)$.

Consider the regular representation of Γ^{alg} on the algebra $\mathcal{O}(\Gamma^{\mathrm{alg}})$ of k-valued regular functions on the affine group Γ^{alg}. The natural map $s : \Gamma \to \Gamma^{\mathrm{alg}}$ allows us to view $\mathcal{O}(\Gamma^{\mathrm{alg}})$ as a Γ-module, and so the tensor product

$$L' := L \otimes_{\mathbb{C}} \mathcal{O}(\Gamma^{\mathrm{alg}})$$

can be interpreted both as a Γ-module and as a Γ^{alg}-module.

Note that when viewed as an object in $\mathsf{Rep}(\Gamma^{\mathrm{alg}})$, the module L' is injective. In particular,

$$H^i_H(\Gamma^{\mathrm{alg}}, L') = 0 \quad \text{for all } i > 0.$$

Now write L' as an inductive limit $L' = \mathrm{colim}\, L'_i$ of finite-dimensional Γ-modules L'_i. We consider the natural inclcusion $L \hookrightarrow L'$, which induces inclusions $L' \hookrightarrow L'_i$ for all $i \gg 0$. Since Γ is

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algebraically good, we have

$$H^n(\Gamma, L) \cong H^n_H(\Gamma^{\mathrm{alg}}, L),$$

and so we may view α as an element in $H^n_H(\Gamma^{\mathrm{alg}}, L)$. Furthermore, since Γ is of type $(\boldsymbol{F})$, we get

$$H^n_H(\Gamma^{\mathrm{alg}}, L') = H^n_H(\Gamma^{\mathrm{alg}}, \mathrm{colim}_{\to} L'_i) = \mathrm{colim}_{\to} H^n_H(\Gamma^{\mathrm{alg}}, L'_i).$$

Since L' was chosen so that $H^n_H(\Gamma^{\mathrm{alg}}, L') = 0$, it follows that

$$\alpha \to 0 \in H^n_H(\Gamma^{\mathrm{alg}}, L'_i)$$

for all sufficiently large i. Combined with the fact that $L \hookrightarrow L'_i$ for all $i \gg 0$, this yields the implication (a) $\Rightarrow$ (b).

We prove the implication (b) $\Rightarrow$ (a) by induction on n. More precisely, for every integer $n > 0$, consider the condition

($*_n$) for every finite-dimensional represenation $V \in \mathsf{Rep}(\Gamma)$, the natural map $s : \Gamma \to \Gamma^{\mathrm{alg}}$ induces an isomorphism

$$s^* : H^k_H(\Gamma^{\mathrm{alg}}, V) \xrightarrow{\cong} H^k(\Gamma, V)$$

for all $k < n$.

We need to show that ($*_n$) holds for all integers $n > 0$. By the universal property of pro-algebraic completions, we know that ($*_1$) holds. This provides the base for the induction. Assume next that ($*_n$) holds. This automatically implies that

$$s^* : H^n_H(\Gamma^{\mathrm{alg}}, L) \to H^n(\Gamma, L)$$

is injective, and so we only have to show that $s^* : H^n_H(\Gamma^{\mathrm{alg}}, L) \to H^n(\Gamma, L)$ is also surjective.

Fix $\alpha \in H^n(\Gamma, L)$ and let $W \in \mathsf{Rep}(\Gamma)$ be a finite-dimensional representation for which we can find an injection $\imath : L \hookrightarrow W$ so that $\imath(\alpha) = 0$. In particular, we can find a class $\beta \in H^{n-1}(\Gamma, W/L)$ which is mapped to α by the edge homomorphism of the long exact sequence in cohomology associated to the sequence of Γ-modules:

$$0 \to L \xrightarrow{\imath} W \to W/L \to 0.$$

However, by the inductive hypothesis ($*_n$), we have an isomorphism

$$H^{n-1}(\Gamma, W/L) \cong H^{n-1}_H(\Gamma^{\mathrm{alg}}, W/L);$$

so from the commutative diagram

$$\begin{array}{ccc} H^n_H(\Gamma^{\mathrm{alg}}, L) & \xrightarrow{\;s^*\;} & H^n(\Gamma, L) \\ \uparrow & & \uparrow \\ H^{n-1}_H(\Gamma^{\mathrm{alg}}, W/L) & \underset{s^*}{\xrightarrow{\;\cong\;}} & H^{n-1}(\Gamma, W/L) \end{array}$$

it follows that α comes from $H^n_H(\Gamma^{\mathrm{alg}}, L)$.

The implication (c) $\Rightarrow$ (b) is obvious. For the implication (b) $\Rightarrow$ (c) we need to construct a finite-dimensional representation $W \in \mathsf{Rep}(L)$ and a monomorphism $\imath : L \to W$ so that $\imath(H^n(\Gamma, L)) = \{0\} \subset H^n(\Gamma, W)$. Choose a basis $e_1, e_2, \ldots, e_m$ of $H^n(\Gamma, L)$. By (b), we can find monomorphisms $\imath_1 : L \hookrightarrow W_1$, $\imath_2 : L \hookrightarrow W_2$, $\ldots$, $\imath_m : L \hookrightarrow W_m$ so that $\imath_i(e_i) = 0 \in H^n(\Gamma, W_i)$ for $i = 1, \ldots, m$. Consider now the sequence of finite-dimensional representations $V_k \in \mathsf{Rep}(\Gamma)$ and monomorphisms $\jmath_k : L \hookrightarrow V_k$ constructed inductively as follows.

- $V_1 := W_1$ and $\jmath_1 := \imath_1$.

- Assuming that $\jmath_{k-1} : L \hookrightarrow V_{k-1}$ has already been constructed, define V_k as the pushout

$$\begin{array}{ccc} L & \xrightarrow{-\imath_k} & W_k \\ {\scriptstyle \jmath_{k-1}}\big\downarrow & & \big\downarrow \\ V_{k-1} & \longrightarrow & V_k \end{array}$$

in $\mathsf{Rep}(\Gamma)$. Explicitly, $V_k = (V_{k-1} \oplus W_k)/\operatorname{im}[L \xrightarrow{\jmath_{k-1}\times(-\imath_k)} V_{k-1} \oplus W_k]$. Moreover, the natural map $\jmath_{k-1} \times \imath_k : L \to V_{k-1} \oplus W_k$ induces a monomorphism $\jmath_k : L \hookrightarrow V_k$, and this completes the induction step.

Now let $W := V_m$ and $\imath := \jmath_m$. By construction, we have inclusions $W_k \subset W$ for all $k = 1, \ldots, m$ and, for each k, the map $\jmath$ factors as

$$\begin{array}{ccc} L & \xrightarrow{\quad \imath \quad} & W \\ & {\scriptstyle \imath_k}\searrow \quad \nearrow & \\ & W_k & \end{array}$$

In particular, $\imath(L) \subset \cap_{k=1}^m W_k \subset W$ and so $\imath(H^n(\Gamma, L)) = 0$. The lemma is therefore proven. □

Using the basic good groups (e.g. free, finite, abelian, surface groups) as building blocks and the criterion from Lemma 4.15, we can construct more good groups as follows.

Theorem 4.16. *Suppose that*

$$1 \to F \to \Gamma \to \Pi \to 1 \tag{2}$$

*is a short exact sequence of finitely generated groups of type (**F**) such that Π is algebraically good over k and F is free. Then Γ is algebraically good over k.*

Proof. The proof is essentially contained in an argument of Beilinson which appears in [Bei87, Lemmas 2.2.1 and 2.2.2] in a slightly different guise. Since the statement of the theorem is an important ingredient in the localization technique for computing schematic homotopy types, we have decided to present the proof in detail in our context.

Let $p : X \to S$ be the Serre fibration corresponding to the short exact sequence (2), and let Y denote the homotopy fiber of p. Note that $X = K(\Gamma, 1)$, $S = K(\Pi, 1)$ and $Y = K(F, 1)$. Given a representation $L \in \mathsf{Rep}(\Gamma)$ (respectively, in $\mathsf{Rep}(\Pi)$ or $\mathsf{Rep}(F)$) we write $\mathbb{L}$ for the corresponding local system of k-vector spaces on X (respectively, on S or Y).

To prove that Γ is good, it suffices (see Lemma 4.15) to show that for every positive integer n and any finite-dimensional representation $L \in \mathsf{Rep}(\Gamma)$ we can find an injection $\imath : L \to W$ into a finite-dimensional $W \in \mathsf{Rep}(\Gamma)$ such that the induced map $\imath : H^n(X, \mathbb{L}) \to H^n(X, \mathbb{W})$ is identically zero.

The representation W will be constructed in three steps.

Step 1. Fix n and L as above; then we can find a finite-dimensional $W \in \mathsf{Rep}(\Gamma)$ and an injection $\imath : L \hookrightarrow W$ such that the induced map

$$H^n(S, p_* \mathbb{L}) \to H^n(S, p_* \mathbb{W})$$

is identically zero.

Indeed, since Π is assumed to be algebraically good and $p_*\mathbb{L}$ is a finite-dimensional local system on S, we can find (see Lemma 4.15) a finite-dimensional local system $\mathbb{V}$ on S and an injection $p_*\mathbb{L} \hookrightarrow \mathbb{V}$ which induces the zero map

$$H^n(S, p_*\mathbb{L}) \xrightarrow{0} H^n(S, \mathbb{V})$$

on cohomology.

Now define a local system $\mathbb{W} \to X$ as the pushout

$$\begin{array}{ccc} p^*p_*\mathbb{L} & \hookrightarrow & p^*\mathbb{V} \\ \downarrow & & \downarrow \\ \mathbb{L} & \hookrightarrow & \mathbb{W} \end{array}$$

i.e. $\mathbb{W} := (p^*\mathbb{V} \oplus \mathbb{L})/p^*p_*\mathbb{L}$. By pushing this cocartesian square down to S and using the projection formula $p_*p^*p_*\mathbb{L} \cong p_*\mathbb{L}$, we get the following commutative diagram of local systems on S.

$$\begin{array}{ccc} & & p_*p^*\mathbb{V} = \mathbb{V} \\ & \nearrow & \downarrow \\ p_*\mathbb{L} & & \\ & \searrow & \downarrow \\ & & p_*\mathbb{W} \end{array}$$

In particular, the natural map

$$H^n(S, p_*\mathbb{L}) \to H^n(S, p_*\mathbb{W})$$

factors through

$$H^n(S, p_*\mathbb{L}) \xrightarrow{0} H^n(S, \mathbb{V})$$

and so is identically zero. This finishes the proof of Step 1. In fact, the same reasoning can be used to prove the following enhancement of Step 1.

Step 2. Let n be a positive integer and let $L, P \in \mathsf{Rep}(\Gamma)$ be finite-dimensional representations. Then we can find a finite-dimensional representation $W \in \mathsf{Rep}(\Gamma)$ and an injection of Γ-modules $L \hookrightarrow W$ such that the induced map $H^2(S, p_*(\mathbb{P}^\vee \otimes \mathbb{L})) \to H^2(S, p_*(\mathbb{P}^\vee \otimes \mathbb{W}))$ vanishes identically.

Indeed, the goodness of Π together with the fact that $p_*(\mathbb{P}^\vee \otimes \mathbb{L})$ is finite-dimensional again implies the existence of an injection $p_*(\mathbb{P}^\vee \otimes \mathbb{L}) \hookrightarrow \mathbb{M}$ into a finite-dimensional local system $\mathbb{M}$ on S, for which the induced map

$$H^2(S, p_*(\mathbb{P}^\vee \otimes \mathbb{L})) \to H^2(S, p_*\mathbb{M})$$

is identically zero. Now define a local system $\mathbb{W} \to X$ as the pushout

$$\begin{array}{ccc} p^*p_*(\mathbb{P}^\vee \otimes \mathbb{L}) \otimes \mathbb{P} & \hookrightarrow & p^*\mathbb{M} \otimes \mathbb{P} \\ \downarrow & & \downarrow \\ \mathbb{L} & \hookrightarrow & \mathbb{W} \end{array}$$

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where $p^*p_*(\mathbb{P}^\vee \otimes \mathbb{L}) \otimes \mathbb{P} \to \mathbb{L}$ is the natural morphism corresponding to $\mathrm{id}_{p_*(\mathbb{P}^\vee \otimes \mathbb{L})}$ under the identifications

$$\mathrm{Hom}(p_*(\mathbb{P}^\vee \otimes \mathbb{L}), p_*(\mathbb{P}^\vee \otimes \mathbb{L})) = \mathrm{Hom}(p^*p_*(\mathbb{P}^\vee \otimes \mathbb{L}), \mathbb{P}^\vee \otimes \mathbb{L}) = \mathrm{Hom}(p^*p_*(\mathbb{P}^\vee \otimes \mathbb{L}) \otimes \mathbb{P}, \mathbb{L}).$$

Note that the definition of $\mathbb{W}$ implies that the natural map $p^*p_*(\mathbb{P}^\vee \otimes \mathbb{L}) \hookrightarrow \mathbb{P}^\vee \otimes \mathbb{L}$ factors through $p^*\mathbb{M}$; so, by pushing forward to S and using the projection formula, we get a commutative diagram of finite-dimensional local systems on S as follows.

$$\begin{array}{ccc} p_*(\mathbb{P}^\vee \otimes \mathbb{L}) & \hookrightarrow & \mathbb{M} \\ & \searrow & \downarrow \\ & & p_*(\mathbb{P}^\vee \otimes \mathbb{W}) \end{array}$$

Thus, the inclusion $\mathbb{L} \hookrightarrow \mathbb{W}$ will induce the zero map

$$H^2(S, p_*(\mathbb{P}^\vee \otimes \mathbb{L})) \xrightarrow{0} H^2(S, p_*(\mathbb{P}^\vee \otimes \mathbb{W}))$$

as claimed. Step 2 is therefore proven.

Step 3. Let $L \in \mathsf{Rep}(\Gamma)$ be a finite-dimensional representation, and let

$$\begin{array}{ccc} Y & \hookrightarrow & X \\ & & \downarrow{\scriptstyle p} \\ & & S \end{array}$$

be a fibration corresponding to the sequence (2). Then there exists an injection $L \hookrightarrow W$ into a finite-dimensional $W \in \mathsf{Rep}(\Gamma)$ that induces the zero map

$$H^1(Y, \mathbb{L}_{|Y}) \xrightarrow{0} H^1(Y, \mathbb{W}_{|Y})$$

on the fiberwise cohomology.

In order to construct $\mathbb{W}$, we begin by choosing an injection $\imath_Y : \mathbb{L}_{|Y} \hookrightarrow \mathbb{E}_Y$ into a suitable local system $\mathbb{E}_Y \to Y$ so that the natural map

$$H^1(Y, \mathbb{L}_{|Y}) \to H^1(Y, \mathbb{E}_Y)$$

which $\imath_Y$ induces on cohomology is identically zero.

We will choose $\mathbb{E}_Y$ as follows. Start with the trivial rank-one local system $\mathbb{I}_Y \to Y$ and consider the trivial local system

$$\mathbb{P}_Y := H^1(Y, \mathbb{L}_{|Y}) \otimes_{\mathbb{C}} \mathbb{I}_Y$$

on Y. Let

$$(e)\; 0 \longrightarrow \mathbb{L}_{|Y} \xrightarrow{\imath_Y} \mathbb{E}_Y \longrightarrow \mathbb{P}_Y \longrightarrow 0$$

be the associated tautological extension of local systems on Y (the universal extension of $\mathbb{P}_Y$ by $\mathbb{L}_{|Y}$). The definition of $\mathbb{E}_Y$ ensures that the pushforward of any extension $0 \to \mathbb{L}_{|Y} \to \mathbb{L}' \to \mathbb{I}_Y \to 0$ via the map $\imath_Y : \mathbb{L}_{|Y} \to \mathbb{E}_Y$ will be split. In particular, the map $H^1(Y, \mathbb{L}_{|Y}) \to H^1(Y, \mathbb{E}_Y)$ induced by $\imath_Y$ is identically zero.

Observe also that we can view $\mathbb{P}_Y$ as the restriction $\mathbb{P}_Y = \mathbb{P}_{|Y}$ of the global local system $\mathbb{P} := p^*(R^1p_*\mathbb{L})$. If it happens that the extension (e) is also a restriction from a global extension of local systems on X, then we can take as W the monodromy representation of this global local system, and this would complete the proof of Step 2.

By universality, we know that $e \in \mathrm{Ext}^1_Y(\mathbb{P}_Y, \mathbb{L}_{|Y})$ will be fixed by the monodromy action of Π and so

$$e \in \mathrm{Ext}^1_Y(\mathbb{P}_{|Y}, \mathbb{L}_{|Y})^{\Pi} = H^0(S, R^1 p_*(\mathbb{P}^\vee \otimes \mathbb{L})).$$

The group $H^0(S, R^1 p_*(\mathbb{P}^\vee \otimes \mathbb{L}))$ fits in the following chunk of the Leray spectral sequence for the map $p : X \to S$:

$$0 \to H^1(S, p_*(\mathbb{P}^\vee \otimes \mathbb{L})) \to H^1(X, \mathbb{P}^\vee \otimes \mathbb{L}) \to H^0(S, R^1 p_*(\mathbb{P}^\vee \otimes \mathbb{L})) \xrightarrow{\delta} H^2(S, p_*(\mathbb{P}^\vee \otimes \mathbb{L})).$$

The obstruction to lifting $\mathbb{E}_Y$ to a global local system on X is precisely the class

$$\delta(e) \in H^2(S, p_*(\mathbb{P}^\vee \otimes \mathbb{L})).$$

We can kill this obstruction by using Step 2. Indeed, by Step 2 we can find an injection $L \hookrightarrow V$ of finite-dimensional Γ modules inducing the zero map

$$H^2(S, p_*(\mathbb{P}^\vee \otimes \mathbb{L})) \xrightarrow{0} H^2(S, p_*(\mathbb{P}^\vee \otimes \mathbb{V})).$$

Since the morphism $\mathbb{L} \hookrightarrow \mathbb{V}$ induces a morphism of long exact sequences

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & H^1(X, \mathbb{P}^\vee \otimes \mathbb{L}) & \longrightarrow & H^0(S, R^1 p_*(\mathbb{P}^\vee \otimes \mathbb{L})) & \xrightarrow{\delta} & H^2(S, p_*(\mathbb{P}^\vee \otimes \mathbb{L})) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle 0} & & \\
\cdots & \longrightarrow & H^1(X, \mathbb{P}^\vee \otimes \mathbb{V}) & \longrightarrow & H^0(S, R^1 p_*(\mathbb{P}^\vee \otimes \mathbb{V})) & \xrightarrow{\delta} & H^2(S, p_*(\mathbb{P}^\vee \otimes \mathbb{V})) & \longrightarrow & \cdots
\end{array}$$

we conclude that the pushforward

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{L}_{|Y} & \longrightarrow & \mathbb{E}_Y & \longrightarrow & \mathbb{P}_{|Y} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \mathbb{V}_{|Y} & \longrightarrow & \mathbb{W}_Y & \longrightarrow & \mathbb{P}_{|Y} & \longrightarrow & 0
\end{array}$$

of the extension (e) via the map $\mathbb{L}_{|Y} \hookrightarrow \mathbb{V}_{|Y}$ is the restriction to Y of some global extension

$$0 \to \mathbb{V} \to \mathbb{W} \to \mathbb{P} \to 0$$

of finite-dimensional local systems on X. Now, by construction, the natural map $H^1(Y, \mathbb{L}_{|Y}) \to H^1(Y, \mathbb{W}_{|Y})$ induced by the injection $\mathbb{L} \hookrightarrow \mathbb{W}$ will vanish identically, which completes the proof of Step 3.

By putting together the existence results established in Steps 1–3, we can now finish the proof of Theorem 4.16. Fix a positive integer n and let $L \in \mathsf{Rep}(\Gamma)$ be a finite-dimensional representation

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of Γ. By Step 3, we can find an injection $L \hookrightarrow W$ in a finite-dimensional representation $W \in \mathsf{Rep}(\Gamma)$ so that the induced map $R^1p_*\mathbb{L} \to R^1p_*\mathbb{W}$ vanishes identically. The Leray spectral sequence for p then yields the following commutative diagram with exact rows.

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & H^n(S, p_*\mathbb{L}) & \longrightarrow & H^n(X, \mathbb{L}) & \longrightarrow & H^{n-1}(S, R^1p_*\mathbb{L}) & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow{\scriptstyle 0} & & \\ \cdots & \longrightarrow & H^n(S, p_*\mathbb{W}) & \longrightarrow & H^n(X, \mathbb{W}) & \longrightarrow & H^{n-1}(S, R^1p_*\mathbb{W}) & \longrightarrow & \cdots \end{array}$$

Therefore

$$H^n(X, \mathbb{L}) \to \mathrm{im}[H^n(S, p_*\mathbb{W}) \to H^n(X, \mathbb{W})] \subset H^n(X, \mathbb{W}).$$

Furthermore, by Step 1, we can find a finite-dimensional local system $\mathbb{W}'$ and an injection $\mathbb{W} \hookrightarrow \mathbb{W}'$ inducing

$$H^n(S, p_*\mathbb{W}) \xrightarrow{0} H^n(S, p_*\mathbb{W}').$$

By functoriality of the Leray spectral sequence, this implies that the natural map

$$H^n(X, \mathbb{L}) \to H^n(X, \mathbb{W})$$

induced by the injection $\mathbb{L} \hookrightarrow \mathbb{W} \subset \mathbb{W}'$ is identically zero. The theorem is proven. □

Remark 4.17.

(i) The only property of free groups that was used in the proof of Theorem 4.16 is the fact that free groups have cohomological dimension less than or equal to one. Therefore, over $\mathbb{C}$ we can use the same reasoning as in the proof of the theorem (in fact, one only needs Step 1) to argue that the extension of a good group by a finite group will also be good.

(ii) An easy application of the Leray spectral sequence shows that any successive extension of free groups of finite type will be of type $(\boldsymbol{F})$, and so the previous theorem again implies that such successive extensions are algebraically good.

By a classical result of M. Artin, every point x on a smooth complex algebraic variety X admits a Zariski neighborhood $x \in U \subset X$ which is a tower of smooth affine morphisms of relative dimension one. In particular, the underlying topological space of U (for the classical topology) is a $K(\pi, 1)$ with π being a successive extension of free groups of finite type. Since π is manifestly of type $(\boldsymbol{F})$, the previous theorem implies that π is algebraically good over $\mathbb{C}$. For future reference, we record this as the next corollary.

Corollary 4.18. *Artin neighborhoods have good fundamental groups. Therefore every smooth complex variety admits a covering by Zariski open subvarieties, each of whose schematization is a schematic $K(\pi, 1)$.*

4.4 Lefschetz exactness

Let $f : X \longrightarrow Y$ be a morphism of connected topological spaces, and let $n \geq 1$ be an integer. We will say that f is an n-epimorphism if the induced morphism $\pi_i(X) \longrightarrow \pi_i(Y)$ is an isomorphism for $i < n$ and an epimorphism for $i = n$. In the same way, one can define the notion of an n-epimorphism of psht. A morphism $f : F \longrightarrow F'$ of psht is an n-epimorphism if the induced morphism of sheaves $\pi_i(F) \longrightarrow \pi_i(F')$ is an isomorphism for $i < n$ and an epimorpism for $i = n$.

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Proposition 4.19. *Let $f : X \longrightarrow Y$ be an n-epimorphism of pointed and connected topological spaces (in $\mathbb{U}$); then the induced morphism $f : (X \otimes k)^{\mathrm{sch}} \longrightarrow (Y \otimes k)^{\mathrm{sch}}$ is an n-epimorphism of psht.*

Proof. For $n = 1$ the result is clear, because $\pi_1((X \otimes k)^{\mathrm{sch}}) \simeq \pi_1(X)^{\mathrm{alg}}$ and the functor $G \mapsto G^{\mathrm{alg}}$ preserves epimorphisms. So let us assume that $n > 1$.

First, observe that f is an n-epimorphism if and only if it satisfies the following two conditions:

- the induced morphism $\pi_1(X) \longrightarrow \pi_1(Y)$ is an isomorphism;
- for any local system of abelian groups M on Y, the induced morphism $H^i(Y, M) \longrightarrow H^i(X, f^*M)$ is an isomorphism for $i < n$ and a monomorphism for $i = n$.

In the same way, $f : (X \otimes k)^{\mathrm{sch}} \longrightarrow (Y \otimes k)^{\mathrm{sch}}$ is an n-epimorphism if and only if the two conditions below are satisfied:

- the induced morphism $\pi_1((X \otimes k)^{\mathrm{sch}}) \longrightarrow \pi_1((Y \otimes k)^{\mathrm{sch}})$ is an isomorphism;
- for any local system of finite-dimensional k-vector spaces L on Y, the induced morphism $H^i(Y, L) \longrightarrow H^i(X, f^*L)$ is an isomorphism for $i < n$ and a monomorphism for $i = n$.

The proposition now follows immediately from the fundamental property of the schematization [Toe06, Definition 3.3.1]. □

Corollary 4.20. *Let Y be a smooth projective (connected) complex variety of dimension $n + 1$, and let $X \hookrightarrow Y$ be a smooth hyperplane section. Then, for any base point $x \in X$, the induced morphism*

$$\pi_i((X \otimes k)^{\mathrm{sch}}, x) \longrightarrow \pi_i((Y \otimes k)^{\mathrm{sch}}, f(x))$$

is an isomorphism for $i < n$ and an epimorphism for $i = n$.

Proof. This follows from Proposition 4.19 and the Lefschetz theorem on hyperplane sections. □

4.5 Homotopy fibers of schematizations

Let $X \longrightarrow B$ be a morphism of pointed and connected $\mathbb{U}$-simplicial sets, and suppose that its homotopy fiber Z is also connected. We want to relate the schematization of Z with the homotopy fiber F of $(X \otimes k)^{\mathrm{sch}} \longrightarrow (Y \otimes k)^{\mathrm{sch}}$. The universal property of the schematization induces a natural morphism of psht

$$(Z \otimes k)^{\mathrm{sch}} \longrightarrow F,$$

which in general is far from being an isomorphism. The following proposition gives a sufficient condition for this morphism to be an isomorphism.

Proposition 4.21. *Assume that Z is 1-connected and of $\mathbb{Q}$-finite type. Then the natural morphism $(Z \otimes k)^{\mathrm{sch}} \longrightarrow F$ is an equivalence.*

Proof. Note first that $(Z \otimes k)^{\mathrm{sch}} \simeq (Z \otimes k)^{\mathrm{uni}}$, as Z is simply connected (see [Toe06, Corollary 3.3.8]).

The fibration sequence $Z \to X \to B$ is classified by a morphism $B \longrightarrow B\mathbb{R}\underline{\mathrm{Aut}}(Z)$ (well-defined in the homotopy category), where $\mathbb{R}\underline{\mathrm{Aut}}(Z)$ is the space of auto-equivalences of Z. We consider the morphism $B\mathbb{R}\underline{\mathrm{Aut}}(Z) \longrightarrow B\mathbb{R}\underline{\mathrm{Aut}}((Z \otimes k)^{\mathrm{uni}})$ and observe that $B\mathbb{R}\underline{\mathrm{Aut}}((Z \otimes k)^{\mathrm{uni}})$ is the pointed simplicial set of global sections of the pointed stack $B\mathbb{R}\mathcal{AUT}((Z \otimes k)^{\mathrm{uni}})$. The finiteness assumption on Z implies that $B\mathbb{R}\mathcal{AUT}((Z \otimes k)^{\mathrm{uni}})$ is a psht, and therefore the morphism $B \longrightarrow B\mathbb{R}\underline{\mathrm{Aut}}((Z \otimes k)^{\mathrm{uni}})$ can be inserted into a (homotopy) commutative square of pointed simplicial sets, as follows.

$$\begin{array}{ccc} B & \longrightarrow & B\mathbb{R}\underline{\mathrm{Aut}}((Z\otimes k)^{\mathrm{uni}}) \\ \downarrow & & \downarrow \\ (B\otimes k)^{\mathrm{sch}} & \longrightarrow & B\mathbb{R}\mathcal{AUT}((Z\otimes k)^{\mathrm{uni}}) \end{array}$$

By passing to the associated fibration sequences, one gets a morphism of fibration sequences of pointed simplicial presheaves, as shown in the diagram below.

$$\begin{array}{ccccc} Z & \longrightarrow & X & \longrightarrow & B \\ \downarrow & & \downarrow & & \downarrow \\ (Z\otimes k)^{\mathrm{sch}} & \longrightarrow & \widetilde{F} & \longrightarrow & (B\otimes k)^{\mathrm{sch}} \end{array}$$

We deduce from this diagram the following morphism between the Leray spectral sequences, for any local system L on B of finite-dimensional k-vector spaces.

$$\begin{array}{ccc} H^p(B, H^q(Z,L)) & \Longrightarrow & H^{p+q}(X,L) \\ \downarrow & & \downarrow \\ H^p((B\otimes k)^{sch}, H^q((Z\otimes k)^{sch}, L)) & \Longrightarrow & H^{p+q}(\widetilde{F}, L) \end{array}$$

The finiteness condition on Z ensures that this morphism is an isomorphism on the E_2 term, showing that $X \longrightarrow \widetilde{F}$ is a P-equivalence. Therefore, the induced morphism $(X\otimes k)^{\mathrm{sch}} \longrightarrow \widetilde{F}$ is an equivalence of psht, which yields the proposition. □

An important consequence of Proposition 4.21 is that, under these conditions, the schematization of the base $(B\otimes k)^{\mathrm{sch}}$ *acts* on the schematization of the fiber $(Z\otimes k)^{\mathrm{sch}}$. By this we mean that the fibration sequence $(Z\otimes k)^{\mathrm{sch}} \longrightarrow (Z\otimes k)^{\mathrm{sch}} \longrightarrow (B\otimes k)^{\mathrm{sch}}$ gives rise to a classification morphism

$$(B\otimes k)^{\mathrm{sch}} \longrightarrow B\mathbb{R}\mathcal{AUT}((Z\otimes k)^{\mathrm{sch}}),$$

where $\mathbb{R}\mathcal{AUT}$ denotes the stack of aut-equivalences. Using [Toe06, Theorem 1.4.3], this morphism can also be considered as a morphism of H_∞-stacks

$$\Omega_*(B\otimes k)^{\mathrm{sch}} \longrightarrow \mathbb{R}\mathcal{AUT}((Z\otimes k)^{\mathrm{sch}})$$

or, in other words, as an action loop stack $\Omega_*(B\otimes k)^{\mathrm{sch}}$ on $(Z\otimes k)^{\mathrm{sch}}$. This action contains, of course, the monodromy action of $\pi_1(B)$ on the homotopy groups of $(Z\otimes k)^{\mathrm{sch}}$, but it also contains higher homotopical invariants, such as the higher monodromy maps

$$\pi_i((B\otimes k)^{\mathrm{sch}}) \otimes \pi_n((Z\otimes k)^{\mathrm{sch}}) \longrightarrow \pi_{n+i-1}((Z\otimes k)^{\mathrm{sch}}).$$

A typical situation to which Proposition 4.21 can be applied is the case where $X \longrightarrow B$ is a smooth projective familly of simply connected complex varieties over a smooth and projective base. In this situation, the Hodge decomposition constructed in [KPT08] is compatible with the monodromy action of $(B\otimes\mathbb{C})^{\mathrm{sch}}$ on the schematization of the fiber. In other words, if Z is the homotopy type of the fiber, then $(Z\otimes\mathbb{C})^{\mathrm{sch}}$ has an action of $\Omega_*(B\otimes\mathbb{C})^{\mathrm{sch}}$ which is $\mathbb{C}^*$-equivariant with respect to the Hodge decomposition. This can also be taken to mean that $(Z\otimes\mathbb{C})^{\mathrm{sch}}$ forms a *variation of non-abelian schematic Hodge structures on B*. Of course, this variation contains more information than the variations of Hodge structures on the rational homotopy groups of the fiber, since it captures higher homotopical data. An example of such non-trivial higher invariants is given in [Sim97].

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