J. Fluid Mech. (2014), *vol.* 751, *pp.* 749–751.
doi:10.1017/jfm.2014.336

CORRIGENDUM

Reflections on dissipation associated with thermal convection – CORRIGENDUM

Thierry Alboussière and Yanick Ricard

doi:10.1017/jfm.2013.241, Published by Cambridge University Press, 27 May 2013

Key words: convection, general fluid mechanics, geophysical and geological flows, corrigendum

There is a flaw in the reasoning in the original paper, just before equation (2.9). Everything before that point is correct, essentially that $-\langle \boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{P} \rangle$ is an exact expression for dissipation in thermal convection. The conclusions obtained after that point are essentially unfounded, first that $\gamma - 1 \ll 1$ is the condition of validity for the anelastic liquid approximation, second that the scaling $P' \sim K_{T0}\alpha_0 T'$ applies within the general anelastic approximation. Other conclusions are derived here, in this corrected version, on the continuity equation in the anelastic approximation.

We first summarize the results that can be derived from the complete equations of convection. The dissipation is exactly

$$\langle \tau : \dot{\epsilon} \rangle = -\langle \boldsymbol{u} \cdot \nabla P \rangle, \tag{0.1}$$

Alboussière & Ricard (2013, equation (1.6)). Decomposing $P = P_0 + P'$ and $\rho = \rho_0 + \rho'$, where P_0 and ρ_0 are the adiabatic, hydrostatic pressure and density, we have

$$\langle \tau : \dot{\epsilon} \rangle = -\langle \boldsymbol{u} \cdot \nabla P_0 \rangle - \langle \boldsymbol{u} \cdot \nabla P' \rangle, \tag{0.2}$$

$$= \langle \rho_0 g u_z \rangle + \langle P' \nabla \cdot \boldsymbol{u} \rangle. \tag{0.3}$$

The second line (0.3) has been obtained using the hydrostatic equation for the reference adiabatic profile and a Gauss integration. The dissipation appears therefore to have two contributions. The first one $\langle \rho_0 g u_z \rangle$ is exactly $-\langle \rho' g u_z \rangle$ by global mass conservation. Because ρ' and u_z are generally correlated in a convective system (light material rises, dense material sinks), this term is positive, i.e. $\langle u_z \rangle > 0$. The second term can be estimated using also the continuity equation $\partial \rho'/\partial t + \nabla \cdot [(\rho_0 + \rho')\boldsymbol{u}] = 0$ which implies that $\rho_0 \nabla \cdot \boldsymbol{u} = -\rho' \nabla \cdot \boldsymbol{u} - u_z \mathrm{d}\rho_0/\mathrm{d}z - \boldsymbol{u} \cdot \nabla \rho' - \partial \rho'/\partial t$. In the limit of vanishing viscosity and thermal diffusivity, it is expected that the state variables will become close to the adiabatic hydrostatic profile, in particular $\rho'/\rho_0 \longrightarrow 0$, which implies that $\rho_0 \nabla \cdot \boldsymbol{u} \approx -u_z \mathrm{d}\rho_0/\mathrm{d}z$ at leading order. Dissipation (0.3) can be written:

$$\langle \tau : \dot{\epsilon} \rangle \approx -\langle \rho' g u_z \rangle - \left\langle P' u_z \frac{1}{\rho_0} \frac{\mathrm{d}\rho_0}{\mathrm{d}z} \right\rangle, \tag{0.4}$$

Then the density perturbations can be written in terms of entropy and pressure perturbations, as a linear expansion about the adiabatic profile:

$$\rho' \approx -\frac{\alpha_0 \rho_0 T_0}{c_{p0}} s' - \frac{1}{\rho_0 g} \frac{\mathrm{d}\rho_0}{\mathrm{d}z} P', \tag{0.5}$$

(cf our equation (2.6)), leading to the more familiar

$$\langle \tau : \dot{\epsilon} \rangle \approx \left\langle \frac{\alpha_0 \rho_0 T_0 g}{c_{p0}} u_z s' \right\rangle . \tag{0.6}$$

If we now start from the Navier Stokes equation in the anelastic approximation

$$\rho_0 \frac{D\boldsymbol{u}}{Dt} = -\rho_0 \nabla \left(\frac{P'}{\rho_0} \right) + \frac{\alpha_0 \rho_0 T_0 g}{c_{p0}} s' \hat{\boldsymbol{e}}_z + \nabla \cdot \tau \tag{0.7}$$

Alboussière & Ricard (2013, equation (2.8)), we get exactly

$$\langle \tau : \dot{\epsilon} \rangle = \left\langle \frac{\alpha_0 \rho_0 T_0 g}{c_{p0}} u_z s' \right\rangle , \tag{0.8}$$

and using (2.6),

$$\langle \tau : \dot{\epsilon} \rangle = -\langle \rho' g u_z \rangle - \left\langle P' u_z \frac{1}{\rho_0} \frac{\mathrm{d}\rho_0}{\mathrm{d}z} \right\rangle . \tag{0.9}$$

The expressions of dissipation in the exact case and in the anelastic approximation, in terms of entropy perturbation, (0.6) and (0.8), or in term of density and pressure perturbations (0.4) and (0.9) are very similar. However, and this is the weak point of our previous reasoning and the cause of this corrigendum, we used the anelastic mass conservation $\nabla \cdot (\rho_0 \boldsymbol{u}) = 0$ together with the exact dissipation expression (0.3) to conclude that $\langle \rho_0 g u_z \rangle = 0$ and that dissipation in the anelastic approximation is simply $-\langle \boldsymbol{u} \cdot \nabla P' \rangle$. In doing so, we have unduly mixed equations of two different formalisms. In the framework of anelasticity, we can only prove (0.9), and the anelastic mass conservation says nothing about $\langle \rho' g u_z \rangle$. Therefore, we must consider unproven that the dissipation in the anelastic formalism can be expressed in terms of pressure variations only, i.e. we have no arguments to neglect the term involving $\langle \rho' u_z \rangle$ in (0.9) and to retain that involving $\langle P' u_z \rangle$. The rest of our paper, comparing the relative amplitudes of terms in $\langle s' u_z \rangle$, $\langle P' u_z \rangle$ and $\langle T' u_z \rangle$ in the anelastic approximation is most probably wrong and, in any case, not demonstrated.

The treatment of continuity is at the origin why the expression of dissipation $-\langle \boldsymbol{u} \cdot \nabla P \rangle$ does not translate into $-\langle \boldsymbol{u} \cdot \nabla P' \rangle$ when using the anelastic approximation model. If we want that property to hold in the anelastic model, and pressure fluctuations to be a faithful image of that in the full model, one must substitute the continuity equation $\nabla \cdot [(\rho_0 + \rho')\boldsymbol{u}] = 0$ for the equation usually used in the anelastic model $\nabla \cdot (\rho_0 \boldsymbol{u}) = 0$. The fact that mass conservation is not well treated at first order in the perturbations from the adiabatic state can be seen from the original anelastic equations. Energy conservation (Alboussière & Ricard 2013, (2.4)) provides s', while the momentum equation (2.8) or (0.7) provides P' and $\boldsymbol{u}$ (with continuity (2.2)). From s' and P', it is possible to evaluate ρ' from the linearized equation of state. Then there is no equation ensuring mass conservation, i.e. $\langle \rho' \boldsymbol{u} \rangle$ for instance is not constrained to be zero.

In the original anelastic equations, the horizontal average of u_z is zero and $\langle u_z \rangle = 0$. However, the analysis in this corrigendum (in particular (0.3) stresses the importance of the mean upward vertical velocity $\langle u_z \rangle \neq 0$. Finally, the combination of that horizontally averaged vertical velocity $\overline{u_z}$ and the adiabatic hydrostatic pressure gradient produces a significant contribution to the expression of dissipation $-\langle \boldsymbol{u} \cdot \nabla P \rangle$, of the form $\langle \overline{u_z} \rho_0 g \rangle$. That contribution is not present in the original anelastic equations and is well accounted for in a modified version of the anelastic equations when continuity is written $\nabla \cdot [(\rho_0 + \rho')\boldsymbol{u}] = 0$.

To conclude, we have shown that the approximated continuity equation $\nabla \cdot (\rho_0 \boldsymbol{u}) = 0$ is not compatible with the property that energy dissipation is equal to $-\langle \boldsymbol{u} \cdot \nabla P \rangle$. Hence either dissipation and/or pressure are likely to be evaluated incorrectly in the original anelastic approximation. This inconsistency is resolved when the modified continuity equation $\nabla \cdot [(\rho_0 + \rho')\boldsymbol{u}] = 0$ is used, without affecting the anelastic nature of the approximation.

REFERENCE

ALBOUSSIÈRE, T. & RICARD, Y. 2013 Reflections on dissipation associated with thermal convection. *J. Fluid Mech.* **725**, R1.