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Frédéric Campana and Mihai Păun

Abstract

Let X be a compact Kähler manifold, endowed with an effective reduced divisor $B = \sum Y_k$ having simple normal crossing support. We consider a closed form of $(1,1)$-type α on X whose corresponding class $\{\alpha\}$ is nef, such that the class $c_1(K_X+B)+\{\alpha\} \in H^{1,1}(X,\mathbb{R})$ is pseudo-effective. A particular case of the first result we establish in this short note states the following. Let m be a positive integer, and let L be a line bundle on X, such that there exists a generically injective morphism $L \to \bigotimes^m T^\star_X\langle B\rangle$, where we denote by $T^\star_X\langle B\rangle$ the logarithmic cotangent bundle associated to the pair (X,B). Then for any Kähler class $\{\omega\}$ on X, we have the inequality

$$\int_X c_1(L) \wedge \{\omega\}^{n-1} \leqslant m \int_X (c_1(K_X+B)+\{\alpha\}) \wedge \{\omega\}^{n-1}.$$

If X is projective, then this result gives a generalization of a criterion due to Y. Miyaoka, concerning the generic semi-positivity: under the hypothesis above, let Q be the quotient of $\bigotimes^m T^\star_X\langle B\rangle$ by L. Then its degree on a generic complete intersection curve $C \subset X$ is bounded from below by

$$\left(\frac{n^m-1}{n-1} - m\right) \int_C (c_1(K_X+B)+\{\alpha\}) - \frac{n^m-1}{n-1} \int_C \alpha.$$

As a consequence, we obtain a new proof of one of the main results of our previous work [F. Campana and M. Păun, *Orbifold generic semi-positivity: an application to families of canonically polarized manifolds*, Ann. Inst. Fourier (Grenoble) **65** (2015), 835–861].

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1. Introduction

Let X be a compact Kähler manifold of dimension n. Let $B = \sum_{j\in J} Y_j$ be a reduced 'boundary' divisor on X, whose support has simple normal crossings.

We recall that the logarithmic cotangent bundle $T^\star_X\langle B\rangle$ associated to (X, B) is defined as follows. Let $U \subset X$ be a coordinate open set, and let $(z^1, \ldots, z^n)$ be a coordinate system of X defined on U, such that, say, $B \cap U = (z^1 z^2 \ldots z^p = 0)$. Such a coordinate system will be called *adapted* to the pair (X, B). When restricted to U, the bundle $T^\star_X\langle B\rangle$ corresponds to the locally free sheaf generated by

$$\frac{dz^1}{z^1}, \ldots, \frac{dz^p}{z^p}, dz^{p+1}, \ldots, dz^n.$$

The main topic we will explore in this paper can be formulated as follows. Let m be a positive integer, and let $\mathcal{F}$ be a subsheaf of the bundle $\bigotimes^m T^\star_X\langle B\rangle$. Let ω be a Kähler class on X; we would like to obtain an upper bound of the intersection number

$$\int_X c_1(\mathcal{F}) \wedge \omega^{n-1} \tag{1.1}$$

in terms of the first Chern class of the bundle $T^\star_X\langle B\rangle$.

We observe that this kind of question has a meaning in an 'abstract' context. Indeed, we consider the following data. Let (E, h_E) be a holomorphic Hermitian vector bundle, and let (L, h_L) be a Hermitian line bundle. We assume that $H^0(X, E \otimes L^{-1}) \neq 0$, and let u be a non-identically zero holomorphic section of $E \otimes L^{-1}$.

We denote by $|u|^2$ the pointwise norm of u, measured with respect to the metrics we have on E and L, respectively. Then the analogue of the Poincaré–Lelong formula (cf. [Dem82]) gives

$$dd^c \log |u|^2 \geqslant \Theta_{h_L}(L) - \frac{\langle \Theta_{h_E}(E)u, u\rangle}{|u|^2}. \tag{1.2}$$

On the other hand, the quantity we are interested in is equal to $\int_X \Theta_{h_L}(L) \wedge \omega^{n-1}$; by inequality (1.2) combined with Stokes formula we obtain

$$0 \geqslant \int_X \Theta_{h_L}(L) \wedge \omega^{n-1} - \int_X \frac{\langle \Theta_{h_E}(E)u, u\rangle}{|u|^2} \wedge \omega^{n-1}.$$

In conclusion, the degree of L with respect to ω is bounded from above as follows:

$$\int_X \Theta_{h_L}(L) \wedge \omega^{n-1} \leqslant \int_X \frac{\langle \Theta_{h_E}(E)u, u\rangle}{|u|^2} \wedge \omega^{n-1}. \tag{1.3}$$

We denote by Λ_ω the contraction operator corresponding to ω; its action on the curvature tensor is given by the formula

$$\frac{\Theta_{h_E}(E) \wedge \omega^{n-1}}{\omega^n} = \Lambda_\omega \Theta_{h_E}(E)$$

so that $\Lambda_\omega \Theta_{h_E}(E)$ is an endomorphism of E, called the *mean curvature* of E.

If the bundle E is stable with respect to ω, then it is known (cf. [Don85]) that we can construct a Hermite–Einstein metric h_E so that the endomorphism $\Lambda_\omega \Theta_{h_E}(E)$ is a multiple of the identity. In this case, the expression on the right-hand side of (3) is quickly computed in terms of the ω-degree of E. But in general, the mean curvature term seems to be very difficult to control.

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In the article [Eno88], Enoki found a very elegant way of dealing with this question. He observes that if the bundle E coincides with the cotangent bundle of X, then the mean curvature of E coincides with $-\mathrm{Ric}_\omega$, that is, the curvature of the canonical bundle. In conclusion, via the Monge–Ampère equation, it would be enough to assume that the canonical bundle of X has a (weak) positivity property in order to derive a bound for the degree of L with respect to ω.

The approach sketched above, combined with some recent results in the framework of metrics with conic singularities, allows us to establish the following statement.

Theorem 1.1. *Let m be a positive integer, and let $\mathcal{F}$ be a torsion free subsheaf of the bundle $\bigotimes^m T^\star_X\langle B\rangle$. We consider a closed $(1,1)$-form α on X whose associated class $\{\alpha\}$ is nef, and such that the adjoint class*

$$c_1(K_X+B)+\{\alpha\}$$

is pseudo-effective. Let ω_0 be a Kähler class on X; then

$$\frac{1}{\mathrm{rk}(\mathscr{F})}\int_X c_1(\mathscr{F})\wedge\{\omega_0\}^{n-1}\leqslant m\int_X (c_1(K_X+B)+\{\alpha\})\wedge\{\omega_0\}^{n-1}. \tag{1.4}$$

As a consequence, we obtain a new proof of the following result. Our arguments here are inspired by the original ones; cf. [CP15].

Theorem 1.2 [CP15]. *Let X be a projective manifold; we assume that there exist a positive integer m and a big line bundle L such that*

$$H^0\bigg(X,\bigotimes^m T^\star_X\langle B\rangle\otimes L^{-1}\bigg)\neq 0. \tag{1.5}$$

Then K_X+B is big.

In the first part of our note we will briefly recall the notion of pseudo-effective classes in the Kähler context, as well as a few results concerning the Monge–Ampère equations. The later topic concerns the so-called *metrics with conic singularities*; specifically, the result we need was established in [CGP13]. We remark here that the uniformity of the estimates obtained in [CGP13] will allow us a 'smooth' presentation of the proof.

Roughly speaking, the proof of Theorem 1.1 follows by using the approach initiated by Enoki mentioned above, combined with the estimates we have at our disposal in the Monge–Ampère theory. As for Theorem 1.2, we basically follow the 'second proof' in [CP15], and use Theorem 1.1 as main technical support, instead of the generic semi-positivity of orbifold cotangent bundles in the original argument.

Parts of the techniques we will use here are employed by Guenancia in [Gue12] to establish the stability of the tangent bundle of the canonically polarized manifolds in a singular setting. Finally, we mention here that in order to obtain a stronger version of Theorem 1.1, the best one can expect would be to replace ω_0^{n-1} with a mobile class. Our hope is that the methods we develop here might be useful in this context.

2. Pseudo-effective classes and Monge–Ampère equations

Let (X,ω) be a compact complex manifold endowed with a metric ω, and let $\{\gamma\}\in H^{1,1}(X,\mathbb{R})$ be a real cohomology class of type $(1,1)$. By convention, we assume that γ is a closed, non-singular $(1,1)$-form on X; we recall the following definitions.

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DEFINITION 1. We say that the class $\{\gamma\}$ is nef (in the metric sense) if for each $\varepsilon > 0$ there exists a function $f_\varepsilon \in \mathcal{C}^\infty(X)$ such that

$$\gamma_\varepsilon := \gamma + dd^c f_\varepsilon \geqslant -\varepsilon\omega. \tag{2.1}$$

The definition above was introduced by Demailly [Dem92] as an analytic counterpart of the notion of the nef divisor in algebraic geometry.

DEFINITION 2. We say that the class $\{\gamma\}$ is pseudo-effective if it contains a closed positive current. This is equivalent to the existence of a function $f \in L^1(X)$ such that

$$T := \gamma + dd^c f \geqslant 0. \tag{2.2}$$

Let $\{\gamma\}$ be a pseudo-effective class. If some multiple of it equals the first Chern class of a $\mathbb{Q}$-line bundle L and if X is projective, then the bundle L is pseudo-effective in the sense of algebraic geometry: given p a positive integer and an ample line bundle A, there exists a constant $C = C(p, A) > 0$ such that

$$h^0(X, k(pL + A)) \geqslant Ck^n$$

as $k \to \infty$. In other words, the Kodaira dimension of $pL + A$ is maximal. This statement is a consequence of [Dem92], and can be seen as a generalization of the Kodaira embedding theorem. Even if the manifold X is projective, the property of pseudo-effective line bundles (in the sense of algebraic geometry) revealed by the previous definition is important: such a line bundle may have negative Kodaira dimension, but nevertheless, its first Chern class carries a closed positive current.

While dealing with pseudo-effective classes on compact Kähler manifolds, one seldom works directly with the current T above, mainly because it may be too singular. The regularization theorem we will state next is therefore very useful. Before that, we recall the following notion.

DEFINITION 3. A function ψ on X has logarithmic poles if for each coordinate set $U \subset X$ there exists a family of holomorphic functions $\sigma_j \in \mathcal{O}(U)$ such that we have

$$\psi \equiv c \log\left(\sum_j |\sigma_j|^2\right) \bmod \mathcal{C}^\infty(U) \tag{2.3}$$

where c is a positive constant. In other words, the function ψ is locally equivalent with the logarithm of a sum of squares of absolute values of holomorphic functions.

The following regularization result will play an important role here.

THEOREM 2.1 [Dem92]. *Let $T = \gamma + dd^c f \geqslant 0$ be a closed positive current. Then there exists a sequence $(f_\eta)_{\eta>0}$ of functions with logarithmic poles, such that*

$$T_\eta := \gamma + dd^c f_\eta \geqslant -\eta\omega$$

for all η, and such that $f_\eta \to f$ in $L^1(X)$.

We recall next a few results in the theory of the Monge–Ampère equations. For the rest of our paper, the manifold X will be assumed to be compact Kähler, and let ω_0 be a fixed, reference metric on X.

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Theorem 1.1 states that a certain numerical positivity property of the bundle $\bigotimes^m T_X^\star\langle B\rangle$ holds, provided that the negative part of the Chern class of K_X+B is bounded.

The requirement concerning the bundle K_X+B in Theorem 1.1 is equivalent to the existence of a singular volume element on X, whose associated curvature current is greater than $-\alpha$. The link between such positivity (or negativity) properties of the canonical bundle and the differential geometry of X is given by the Monge–Ampère theory, or, more precisely, by the Aubin–Calabi–Yau theorem. In its original formulation [Yau78], this result states that given a non-singular volume element on X, whose total mass is equal to $\int_X \omega_0^n$, there exists a Kähler metric $\omega \in \{\omega_0\}$ whose determinant is equal to the said volume element.

Partly motivated by questions arising from algebraic geometry, many results generalizing the Calabi–Yau theorem for singular volume elements have recently been established. The following statement will be the main technical tool in our proof of Theorem 1.1.

Theorem 2.2 [CGP13]. *Let (X,ω_0) be a compact Kähler manifold, and let*

$$B=\sum_j Y_j$$

be a reduced divisor, whose support has simple normal crossings. We consider a finite set of functions $(\psi_r)_{r=1,\dots,A}\subset L^1(X)$ with logarithmic poles, normalized such that $\int_X \psi_r\,dV_0=0$ for each r, where dV_0 is the volume element on X induced by the metric ω_0. Let C_ψ be a positive constant, such that

$$C_\psi\omega_0+dd^c\psi_r\geqslant 0.$$

Then for each set of parameters $\Lambda=(\lambda,\varepsilon,\delta)\in\,]\,0,1\,]^A\times[0,1]\times\,]\,0,1/2\,]$ there exist a positive (normalization) constant C_Λ and a Kähler metric $\omega_\Lambda=\omega_0+dd^c\varphi_\Lambda$, such that

$$\omega_\Lambda^n=C_\Lambda\frac{\prod_r(\lambda_r^2+\exp(\psi_r))}{\prod_j(\varepsilon^2+|s_j|^2)^{1-\delta}}\omega_0^n, \tag{MA}$$

and such that the following uniform estimates hold.

(i) *There exists a constant $C_\delta>0$ depending on δ, C_ψ and (X,ω_0) but uniform with respect to $\lambda,\varepsilon\in[0,1/2]^{A+1}$ such that $\sup_X|\varphi_\Lambda|\leqslant C_\delta$.*

(ii) *For each coordinate system $(z^1,\dots,z^n)$ adapted to (X,B) on U we have*

$$\omega_\Lambda\leqslant C_\delta\bigg(\sum_{j=1}^p\frac{\sqrt{-1}\,dz^j\wedge dz^{\bar j}}{(\varepsilon^2+|z^j|^2)^{1-\delta}}+\sum_{j=p+1}^n\sqrt{-1}dz^j\wedge dz^{\bar j}\bigg).$$

In other words, the solution ω_Λ is bounded from above by the standard conic metric on each coordinate set.

(iii) *There exists a constant $C_{\lambda,\delta}$ uniform with respect to ε such that*

$$\omega_\Lambda\geqslant C_{\lambda,\delta}\bigg(\sum_{j=1}^p\frac{\sqrt{-1}dz^j\wedge dz^{\bar j}}{(\varepsilon^2+|z^j|^2)^{1-\delta}}+\sum_{j=p+1}^n\sqrt{-1}dz^j\wedge dz^{\bar j}\bigg).$$

In the statement above we denote by s_j the section of the bundle $\mathcal{O}(Y_j)$ whose zero set equals Y_j, and we denote by $|s_j|$ the (pointwise) norm of s_j measured with a non-singular metric h_j on $\mathcal{O}(Y_j)$. We denote by θ_j the curvature of the bundle $\mathcal{O}(Y_j)$ with respect to h_j.

Remark 2.3. We see that (unlike in all the other recent articles concerning the metrics with conic singularities) in the statement above we have obtained uniform the estimates with respect to the parameter $\varepsilon \geqslant 0$. This will be play an important role in the next section.

In the remainder of this section, we will evaluate the curvature of the canonical bundle K_X endowed with the determinant of the metric given by equation (MA). The formula reads

$$\Theta_{\omega_\Lambda}(K_X) = \Theta(K_X) + \sum_r dd^c \log(\lambda^2 + \exp(\psi_r)) - (1-\delta)\sum_j dd^c \log(\varepsilon^2 + |s_j|^2), \tag{2.4}$$

where $\Theta(K_X) := \Theta_{\omega_0}(K_X)$ is the curvature of the canonical bundle with respect to the volume element induced by the reference metric ω_0. We next expand the Hessian terms in (2.3):

$$\begin{aligned} dd^c \log(\lambda^2 + \exp(\psi)) &= \frac{\exp(\psi)}{\lambda^2 + \exp(\psi)} dd^c\psi + \frac{\lambda^2 \exp(\psi)}{(\lambda^2 + \exp(\psi))^2}\sqrt{-1}\partial\psi \wedge \overline{\partial}\psi \\ &= dd^c\psi + \frac{\lambda^2 \exp(\psi)}{(\lambda^2 + \exp(\psi))^2}\sqrt{-1}\partial\psi \wedge \overline{\partial}\psi - \frac{\lambda^2}{\lambda^2 + \exp(\psi)} dd^c\psi \end{aligned}$$

and

$$\begin{aligned} dd^c \log(\varepsilon^2 + |s_j|^2) &= \frac{\varepsilon^2}{(\varepsilon^2 + |s_j|^2)^2}\sqrt{-1}\partial s_j \wedge \overline{\partial s_j} - \frac{|s_j|^2}{\varepsilon^2 + |s_j|^2}\theta_j \\ &= -\theta_j + \frac{\varepsilon^2}{(\varepsilon^2 + |s_j|^2)^2}\sqrt{-1}\partial s_j \wedge \overline{\partial s_j} + \frac{\varepsilon^2}{\varepsilon^2 + |s_j|^2}\theta_j, \end{aligned}$$

respectively. Rearranging the terms, we obtain the identity

$$\begin{aligned} &\Theta_{\omega_\Lambda}(K_X) + (1-\delta)\sum_j \frac{\varepsilon^2}{(\varepsilon^2 + |s_j|^2)^2}\sqrt{-1}\partial s_j \wedge \overline{\partial s_j} + \delta \sum_j \theta_j \\ &\quad + \sum_r \frac{\lambda^2}{\lambda^2 + \exp(\psi_r)} dd^c\psi_r - (1-\delta)\sum_j \frac{\varepsilon^2}{\varepsilon^2 + |s_j|^2}\theta_j \\ &= \Theta(K_X) + \sum_j \theta_j + \sum_r dd^c\psi_r + \sum_r \frac{\lambda^2 \exp(\psi_r)}{(\lambda^2 + \exp(\psi_r))^2}\sqrt{-1}\partial\psi_r \wedge \overline{\partial}\psi_r. \end{aligned}$$

The relationship between this expression and Theorem 1.1 is easy to understand: if we add the semi-positive form α, the right-hand-side term can be assumed to be positive by choosing ψ in an appropriate manner (according to the pseudo-effectivity hypothesis). Thus, the curvature term $\Theta_{\omega_\Lambda}(K_X)$ is naturally written as the *difference of two semi-positive forms*, modulo some undesirable 'error' terms such as

$$\sum_r \frac{\lambda^2}{\lambda^2 + \exp(\psi_r)} dd^c\psi_r + \sum_j \left(\delta - \frac{(1-\delta)\varepsilon^2}{\varepsilon^2 + |s_j|^2}\right)\theta_j \tag{2.5}$$

which will converge to zero, provided that we are choosing the set of parameters Λ properly. It is at this point that the estimates in Theorem 2.2 are playing a determinant role.

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3. Proof of Theorem 1.1

In this section we will unfold our arguments for the proof of Theorem 1.1; we proceed in several steps. A first standard observation is that it is enough to prove this statement for $\mathcal{F} := L$ a line bundle on X. The reduction to this case is classical, as follows. If we denote by p the rank of $\mathcal{F}$ at the generic point of X, then we have a sheaf injection

$$0 \to \Lambda^p \mathcal{F} \to \bigotimes^{mp} T_X^\star \langle B \rangle$$

and our claim follows by passing to the bi-dual; we refer to [Kob87] for further details.

Step 1. By hypothesis, the class $c_1(K_X + B) + \{\alpha\}$ is pseudo-effective. When combined with Theorem 2.1, we infer the existence of a sequence of functions $(f_\eta)_{\eta>0}$ such that the following statements hold.

(1) For each $\eta > 0$, the function f_η has logarithmic poles, and $\int_X f_\eta \, dV_0 = 0$.

(2) We have $\Theta(K_X) + \sum_j \theta_j + \alpha_\eta + dd^c f_\eta \geqslant -\eta\omega_0$ in the sense of currents on X (we are using here the same notation as in the preceding section). We denote by $\alpha_\eta \in \{\alpha\}$ a non-singular representative such that $\alpha_\eta \geqslant -\eta\omega_0$ on X.

Relation (2) above is equivalent to the positivity of the current

$$T_\eta = \Theta(K_X) + \sum_j \theta_j + \alpha_\eta + dd^c f_\eta + \eta\omega_0;$$

for some technical reasons which will appear later on in the proof, we have to 'separate' the codimension-1 singularities of T_η from the rest. For a general closed positive current this is indeed possible thanks to a result due to Siu in [Siu87]. In the present situation it is much simpler, since f_η has logarithmic poles; in any case, we write

$$T_\eta = \sum_{r=1}^{A_\eta} b_\eta^r [W_{r,\eta}] + R, \tag{3.1}$$

where the singularities of the closed positive current R are in codimension 2 or higher. Let $g_{r,\eta}$ be an arbitrary, non-singular metric on the line bundle $\mathcal{O}(W_{r,\eta})$ associated to the hypersurface $W_{r,\eta}$, and let $\beta_{r,\eta}$ be the associated curvature form. We denote by $\sigma_{r,\eta}$ the tautological section of $\mathcal{O}(W_{r,\eta})$, whose set of zeros is precisely $W_{r,\eta}$, and we define

$$\widehat{f}_\eta := f_\eta - \sum_{r=1}^{A_\eta} b_\eta^r \log|\sigma_{r,\eta}|^2. \tag{3.2}$$

Since the current R is positive we have

$$\Theta(K_X) + \sum_{j=1}^N \theta_j + \alpha_\eta - \sum_{r=1}^{A_\eta} b_\eta^r \beta_{r,\eta} + dd^c \widehat{f}_\eta + \eta\omega_0 \geqslant 0 \tag{3.3}$$

in the sense of currents on X.

We next apply Theorem 2.2: for each set of parameters

$$\Lambda := (\varepsilon, \lambda, \eta, \rho, \delta) \in]\,0,1] \times]\,0,1] \times]\,0,1] \times]\,0,1] \times]\,0,1/2],$$

there exists a Kähler metric

$$\omega_\Lambda := \omega_0 + dd^c \varphi_\Lambda, \quad \int_X \varphi_\Lambda \, dV_0 = 0,$$

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such that

$$\omega_\Lambda^n = C_\Lambda \frac{(\lambda^2 + \exp(\widehat{f}_\eta)) \prod_r (\rho^2 + |\sigma_{r,\eta}|^2)^{b_\eta^r}}{\prod_j (\varepsilon^2 + |s_j|^2)^{1-\delta}} \omega_0^n, \tag{3.4}$$

where C_Λ is a normalization constant. By estimate (ii) of Theorem 2.2,

$$\frac{1}{C_{\delta,\eta}} \omega_\Lambda \leqslant \sum_{j=1}^{p} \frac{\sqrt{-1} dz^j \wedge dz^{\bar{j}}}{(\varepsilon^2 + |z^j|^2)^{1-\delta}} + \sum_{j=p+1}^{n} \sqrt{-1} dz^j \wedge dz^{\bar{j}} \tag{3.5}$$

on a small coordinate set, where (z^i) are local coordinates adapted to (X, B).

Remark 3.1. Actually, the constant $C_{\delta,\eta}$ as above can be assumed to be independent of η, provided that λ and ρ are small enough (depending on η). This amounts to taking $\rho \to 0$ and $\lambda \to 0$ before taking $\eta \to 0$ in the limiting process at the end of the proof. However, the estimate as stated in (3.5) will be sufficient for us.

Also, by the computations performed at the end of the previous section, the curvature of K_X with respect to the metric ω_Λ can be expressed as

$$\begin{aligned}
&\Theta_{\omega_\Lambda}(K_X) + (1-\delta) \sum_{j=1}^{N} \frac{\varepsilon^2}{(\varepsilon^2 + |s_j|^2)^2} \sqrt{-1} \partial s_j \wedge \overline{\partial s_j} + \alpha_\eta + \eta \omega_0 \\
&\quad + \frac{\lambda^2}{\lambda^2 + \exp(\widehat{f}_\eta)} dd^c \widehat{f}_\eta + \sum_{j=1}^{N} \left(\delta - \frac{(1-\delta)\varepsilon^2}{\varepsilon^2 + |s_j|^2} \right) \theta_j + \sum_{r=1}^{A_\eta} \frac{b_\eta^r \rho^2}{\rho^2 + |\sigma_{r,\eta}|^2} \beta_{r,\eta} \\
&= \Theta(K_X) + \sum_{j=1}^{N} \theta_j - \sum_{r=1}^{A_\eta} b_\eta^r \beta_{r,\eta} + dd^c \widehat{f}_\eta + \alpha_\eta + \eta \omega_0 \\
&\quad + \sum_{r=1}^{A_\eta} \frac{b_\eta^r \rho^2}{(\rho^2 + |\sigma_{r,\eta}|^2)^2} \sqrt{-1} \partial \sigma_{r,\eta} \wedge \overline{\partial \sigma_{r,\eta}} + \frac{\lambda^2 \exp(\widehat{f}_\eta)}{(\lambda^2 + \exp(\widehat{f}_\eta))^2} \sqrt{-1} \partial \widehat{f}_\eta \wedge \overline{\partial} \widehat{f}_\eta.
\end{aligned} \tag{3.6}$$

In the relation above, we remark that the right-hand side is positive definite.

Step 2. As we have seen at the beginning of this section, the sheaf $\mathcal{F}$ can be assumed to be a line bundle, say L. Let

$$u \in H^0 \left(X, \bigotimes^m T_X^\star \langle B \rangle \otimes L^{-1} \right)$$

be the holomorphic section corresponding to the injection $L \mapsto \bigotimes^m T_X^\star \langle B \rangle$ whose existence is ensured by hypothesis. We consider an arbitrary, non-singular metric h_L on L.

Let $M := X \setminus \operatorname{Supp}(B)$, and let $\Omega \subset M$ be an open set, such that

$$\overline{\Omega} \subset M.$$

The restriction bundle $\bigotimes^m T_X^\star \langle B \rangle|_M$ identifies with the tensor power of the usual cotangent bundle $\bigotimes^m T_X^\star$, and we can endow it with the metric induced by ω_Λ. Another way of presenting this is that we endow the bundle $\bigotimes^m T_X^\star \langle B \rangle$ with a singular metric given by ω_Λ. Let μ be a positive real number; we have the inequality

$$dd^c \log(\mu^2 + |s|^{2m\delta} |u|_\Lambda^2) \geqslant \frac{|s|^{2m\delta} |u|_\Lambda^2}{\mu^2 + |s|^{2m\delta} |u|_\Lambda^2} (\Theta_{h_L}(L) - \delta m \Theta_B) - |s|^{2m\delta} \frac{\langle \Theta_\Lambda(E_m) u, u \rangle}{\mu^2 + |s|^{2m\delta} |u|_\Lambda^2}, \tag{3.7}$$

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where E_m stands for $\bigotimes^m T^\star_X\langle B\rangle$, and the symbol $\Theta_\Lambda(E_m)$ denotes the curvature of E_m with respect to the metric induced by ω_Λ. The quantity $|u|_\Lambda$ represents the norm of u measured with respect to the metric induced by h_L and ω_Λ. We denote by s the canonical section corresponding to the boundary divisor B, and by $|s|$ its norm, measured with respect to a non-singular metric; the associated curvature form is Θ_B. We remark that this inequality (valid also if $\mu = 0$) is nothing but a version of the Poincaré–Lelong formula for vector bundles. On the other hand, we stress that inequality (3.7) only holds on sets $\Omega \subseteq M$, since the quantity $|s|^{2m\delta}|u|^2_\Lambda$ becomes meromorphic across the support of B.

Let ξ be a cut-off function, which equals 1 in the complement of an open set containing $\mathrm{Supp}(B)$ and which vanishes in a smaller open set containing $\mathrm{Supp}(B)$. Multiplying (3.7) by $\xi\omega_\Lambda^{n-1}$ and integrating over

$$X_0 := X\backslash\mathrm{Supp}(B),$$

we have

$$\int_{X_0}\xi dd^c\log(\mu^2+|s|^{2m\delta}|u|^2_\Lambda)\wedge\omega_\Lambda^{n-1}+\int_{X_0}\xi\frac{|s|^{2m\delta}}{\mu^2+|s|^{2m\delta}|u|^2_\Lambda}\langle\Theta_\Lambda(E_m)u,u\rangle\wedge\omega_\Lambda^{n-1}$$
$$\geqslant\int_{X_0}\xi\frac{|s|^{2m\delta}|u|^2_\Lambda}{\mu^2+|s|^{2m\delta}|u|^2_\Lambda}(\Theta_{h_L}(L)-m\delta\Theta_B)\wedge\omega_\Lambda^{n-1}. \tag{3.8}$$

Step 3. In the following lines we will evaluate the curvature term $\langle\Theta_\Lambda(E_m)u,u\rangle\wedge\omega_\Lambda^{n-1}$ in (3.8). To this end, we consider a point $x_0\in\Omega$, and we will do a pointwise computation by using geodesic coordinates at x_0 (here we ignore completely the log structure induced by B).

For any set of parameters Λ the metric ω_Λ is Kähler, so there exists a geodesic coordinate system $(z_1,\dots,z_n)$ centered at x_0; that is, near x_0 we write

$$\omega_\Lambda=\sum_{q=1}^n\sqrt{-1}dz_q\wedge d\overline{z}_q+\sum_{j,k,\alpha,\beta}c^k_{j\alpha\overline{\beta}}z_j\overline{z}_k\sqrt{-1}dz_\alpha\wedge d\overline{z}_\beta+\mathcal{O}(|z|^3). \tag{3.9}$$

In the expression above, the complex numbers $(c^k_{j\alpha\overline{\beta}})$ are defined as

$$\Theta_\Lambda(T^\star_X)_{x_0}=\sum_{j,\alpha,\beta}c^k_{j\alpha\overline{\beta}}dz_\alpha\wedge d\overline{z}_\beta\otimes dz_j\otimes\frac{\partial}{\partial z_k},$$

that is to say, they are the coefficients of the curvature tensor of $(T^\star_X,\omega_\Lambda)$.

Let

$$u=\sum_{J=(j_1,\dots,j_m)}u_J dz_{j_1}\otimes\cdots\otimes dz_{j_m}$$

be the local expression of the section u. By definition, the curvature of $\bigotimes^m T^\star_X\langle B\rangle$ acts on u as follows:

$$\Theta_\Lambda(E_m)u=\sum_{r,J=(j_1,\dots,j_m)}u_J dz_{j_1}\otimes\cdots\otimes\Theta_\Lambda(T^\star_X)dz_{j_r}\otimes\cdots\otimes dz_{j_m}$$
$$=\sum_{J=(j_1,\dots,j_m)}\sum_{r,p,\alpha,\beta}u_J c^r_{p\alpha\overline{\beta}}dz_{j_1}\otimes\cdots\otimes dz_p\otimes\cdots\otimes dz_{j_m}\otimes dz_\alpha\wedge d\overline{z}_\beta.$$

Therefore at the point x_0 we have

$$\frac{\langle\Theta_\Lambda(E_m)u,u\rangle\wedge\omega_\Lambda^{n-1}}{\omega_\Lambda^n}=\frac{1}{n}\sum_{J,p,d}u_J\overline{u}_{J_{\widehat{p}d}}\theta^p_d \tag{3.10}$$

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where for any index $J = (j_1, \ldots, j_m)$ we denote by $J_{\widehat{p}d}$ the index obtained by replacing j_p with d in the expression of J (the other elements are unchanged), and

$$\theta_d^p := \sum_\alpha c_{d\alpha\overline{\alpha}}^p$$

are the coefficients of the Hermitian form on $T_X^\star$ induced by the curvature of the canonical bundle $\Theta_\Lambda(K_X)$ by contraction with the metric (at this point we are using the fact that the metric ω_Λ is Kähler).

Given any $(1,1)$-form γ, we have an associate Hermitian form on $T_X^\star$, say Ψ_γ, obtained by contraction with the metric ω_Λ. This is obtained by 'raising the indexes' as follows. Locally near x_0 we write

$$\gamma = \sum_{p,q} \gamma_{p\overline{q}} dz_p \wedge d\overline{z}_q$$

and then the induced form Ψ_γ on $T_X^\star$ is given by the expression

$$\Psi_\gamma = \sum_{p,q,r,s} \gamma_{p\overline{q}} \omega^{p\overline{s}} \omega^{r\overline{q}} \frac{\partial}{\partial z_r} \wedge \frac{\partial}{\partial \overline{z}_s}. \tag{3.11}$$

We consider formula (3.6), and we introduce the following notation:

- Ψ_{Θ_λ} is the form induced by $\Theta_{\omega_\Lambda}(K_X)$;
- $\Psi_{\alpha,\varepsilon}$ is the form induced by $(1-\delta)\sum_{j=1}^N (\varepsilon^2/(\varepsilon^2 + |s_j|^2)^2)\sqrt{-1}\partial s_j \wedge \overline{\partial s_j} + \alpha_\eta + \eta\omega_0$;
- Ψ_Λ is the form induced by

$$\tau_\Lambda := \frac{\lambda^2}{\lambda^2 + \exp(\widehat{f}_\eta)} dd^c \widehat{f}_\eta + \sum_{j=1}^N \left(\delta - \frac{(1-\delta)\varepsilon^2}{\varepsilon^2 + |s_j|^2}\right)\theta_j + \sum_{r=1}^{A_\eta} \frac{b_\eta^r \rho^2}{\rho^2 + |\sigma_{r,\eta}|^2} \beta_{r,\eta}.$$

As we can see from formula (3.10), the curvature term in (3.8) is expressed as

$$\int_{X_0} \xi_\mu \frac{\langle \Theta_\Lambda(E_m) u, u\rangle}{|u|_\Lambda^2} \wedge \omega_\Lambda^{n-1} = \frac{1}{n} \int_{X_0} \xi_\mu \frac{\Psi_{\Theta_\lambda}(u, \overline{u})}{|u|_\Lambda^2} \omega_\Lambda^n, \tag{3.12}$$

where we denote

$$\xi_\mu := \xi \frac{|s|^{2m\delta} |u|_\Lambda^2}{\mu^2 + |s|^{2m\delta}|u|_\Lambda^2} \leqslant 1.$$

We observe that the form $\Psi_{\alpha,\varepsilon}$ is positive definite (since $\alpha_\eta \geqslant -\eta\omega_0$ by property (1) of Step 1), and then we have the inequalities

$$\begin{aligned}
\frac{\xi_\mu \Psi_{\Theta_\lambda}(u,\overline{u})}{m|u|_\Lambda^2} &\leqslant \frac{\xi_\mu(\Psi_{\Theta_\lambda} + \Psi_{\alpha,\varepsilon})(u,\overline{u})}{m|u|_\Lambda^2} \\
&= \frac{\xi_\mu(\Psi_{\Theta_\lambda} + \Psi_{\alpha,\varepsilon} + \Psi_\Lambda)(u,\overline{u}) - \xi_\mu \Psi_\Lambda(u,\overline{u})}{m|u|_\Lambda^2} \\
&\leqslant \operatorname{tr}_{\omega_\Lambda}\left(\Theta_{\omega_\Lambda}(K_X) + (1-\delta)\sum_{j=1}^N \frac{\varepsilon^2}{(\varepsilon^2 + |s_j|^2)^2}\sqrt{-1}\partial s_j \wedge \overline{\partial s_j} + \alpha_\eta + \eta\omega_0 + \tau_\Lambda\right) \\
&\quad - \frac{\xi_\mu \Psi_\Lambda(u,\overline{u})}{m|u|_\Lambda^2}.
\end{aligned}$$

The fact that the third inequality of the preceding relations holds true can be seen as a consequence of the following elementary result, combined with the fact that the form $\Psi_{\Theta_\lambda} + \Psi_{\alpha,\varepsilon} + \Psi_\Lambda$ is definite positive, as shown by formula (3.10).

Lemma 1. *Let $\Theta = (\theta_d^p)$ be a positive definite Hermitian form; then*

$$\sum_{J,r,d} u_J \overline{u}_{J(j_r,d)} \theta_{j_r}^d \leqslant m \left(\sum_J |u_J|^2\right)\left(\sum_j \theta_j^j\right)$$

for any set of complex numbers u_J, where $J = (j_1, \dots, j_m) \in \{1, \dots, n\}^m$ is the set of indices. In the relation above we denote by $J(j_r, d)$ the index $(j_1, \dots j_{r-1}, d, j_{r+1}, \dots, j_m)$ having the same components as J except that we replace j_r with d.

Proof. For $m = 1$, the inequality to be proved is

$$\sum_{j,r} u_j \overline{u}_r \theta_j^r \leqslant \left(\sum_j |u_j|^2\right)\left(\sum_j \theta_j^j\right)$$

and the easy verification will not be detailed here. For a general $m \geqslant 2$, we just observe that the corresponding sum for $r = 1$ can be written as

$$\sum_{J',p,d} u_{pJ'} \overline{u}_{dJ'} \theta_p^d$$

where $J' = (j_2, \dots, j_m)$. For each fixed index J', the inequality corresponding to $m = 1$ shows that

$$\sum_{J',p,d} u_{pJ'} \overline{u}_{dJ'} \theta_p^d \leqslant \left(\sum_j |u_{jJ'}|^2\right)\left(\sum_j \theta_j^j\right)$$

and, summing over all the indices J', and then over $r = 1, \dots m$, we infer the result. □

These considerations and (3.12) show that

$$\begin{aligned}\int_{X_0} \xi_\mu \frac{\langle \Theta_\Lambda(E_m)u, u\rangle}{|u|_\Lambda^2} \wedge \omega_\Lambda^{n-1} \leqslant{}& m \int_{X_0} (\Theta_{\omega_\Lambda}(K_X) + \alpha) \wedge \omega_\Lambda^{n-1} \\ &+ m(1-\delta) \int_{X_0} \left(\sum_{j=1}^N \frac{\varepsilon^2}{(\varepsilon^2 + |s_j|^2)^2} \sqrt{-1}\partial s_j \wedge \overline{\partial s_j}\right) \wedge \omega_\Lambda^{n-1} \\ &+ m \int_{X_0} \tau_\Lambda \wedge \omega_\Lambda^{n-1} + \eta m \int_{X_0} \omega_0^n - \int_{X_0} \xi_\mu \Psi_\Lambda(u, \overline{u}) \omega_\Lambda^n. \quad (3.13)\end{aligned}$$

We can remove the subscript 'η' for the first term on the right-hand side of (3.13) simply because $\alpha_\eta \in \{\alpha\}$.

By the computations in Step 2, we have the identity

$$(1-\delta) \frac{\varepsilon^2}{(\varepsilon^2 + |s_j|^2)^2} \sqrt{-1}\partial s_j \wedge \overline{\partial s_j} + \delta\theta_j = (1-\delta) dd^c \log(\varepsilon^2 + |s_j|^2) + \theta_j - \frac{(1-\delta)\varepsilon^2}{\varepsilon^2 + |s_j|^2} \theta_j$$

for each index j, so by integration over $X \backslash \mathrm{Supp}(B)$ and Stokes's formula (which indeed holds on the open manifold, since the functions/forms we are dealing with are smooth), inequality (3.13) becomes

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$$\int_{X_0} \xi_\mu \frac{\langle \Theta_\Lambda(E_m)u, u\rangle}{|u|^2_\Lambda} \wedge \omega_\Lambda^{n-1} \leqslant m \int_X \left(\Theta(K_X) + \alpha + \sum_{j=1}^N \theta_j \right) \wedge \omega_0^{n-1}$$
$$+ m \int_X \frac{\lambda^2}{\lambda^2 + \exp(\widehat{f}_\eta)} dd^c \widehat{f}_\eta \wedge \omega_\Lambda^{n-1}$$
$$- m \sum_{j=1}^N \int_X \frac{(1-\delta)\varepsilon^2}{\varepsilon^2 + |s_j|^2} \theta_j \wedge \omega_\Lambda^{n-1}$$
$$+ m \sum_{r=1}^{A_\eta} \int_X \frac{b_\eta^r \rho^2}{\rho^2 + |\sigma_{r,\eta}|^2} \beta_{r,\eta} \wedge \omega_\Lambda^{n-1} + \eta \int_X \omega_0^n$$
$$- \int_{X_0} \xi_\mu \Psi_\Lambda(u, \overline{u}) \omega_\Lambda^n. \tag{3.14}$$

In (3.14) we have used the fact that the singularities of $\widehat{f}_\eta$ are in codimension 2 or higher, so that we have the equality

$$\int_X \frac{\lambda^2}{\lambda^2 + \exp(\widehat{f}_\eta)} dd^c \widehat{f}_\eta \wedge \omega_\Lambda^{n-1} = \int_{X_0} \frac{\lambda^2}{\lambda^2 + \exp(\widehat{f}_\eta)} dd^c \widehat{f}_\eta \wedge \omega_\Lambda^{n-1}.$$

We next derive an upper bound for the last term in (3.14). By the expression of the form τ_Λ we see that for each $\eta > 0$ there exist a constant C_η depending on η and a constant C which is uniform with respect to the set of parameters Λ such that

$$\tau_\Lambda \geqslant -C_\eta \left(\frac{\lambda^2}{\lambda^2 + \exp(\widehat{f}_\eta)} + \sum_r \frac{\rho^2}{\rho^2 + |\sigma_{r,\eta}|^2} \right) \omega_0 - C \left(\delta + \sum_j \frac{\varepsilon^2}{\varepsilon^2 + |s_j|^2} \right) \omega_0,$$

and therefore we infer that

$$- \int_{X_0} \xi_\mu \Psi_\Lambda(u, \overline{u}) \omega_\Lambda^n \leqslant C_\eta \int_X \left(\delta + \frac{\lambda^2}{\lambda^2 + \exp(\widehat{f}_\eta)} + \sum_r \frac{\rho^2}{\rho^2 + |\sigma_{r,\eta}|^2} \right) \omega_0 \wedge \omega_\Lambda^{n-1}$$
$$+ C\delta + C \sum_j \frac{\varepsilon^2}{\varepsilon^2 + |s_j|^2}. \tag{3.15}$$

Step 4. We summarize here the conclusion of the computations of the preceding steps. By (3.8), the quantity

$$\int_{X_0} \xi \frac{|s|^{2m\delta} |u|^2_\Lambda}{\mu^2 + |s|^{2m\delta} |u|^2_\Lambda} \Theta_{h_L}(L) \wedge \omega_\Lambda^{n-1} \tag{3.16}$$

whose limit as ξ tends to the characteristic function of $X \backslash \mathrm{Supp}(B)$ and μ tends to zero respectively is smaller than

$$m \int_X \left(\Theta(K_X) + \alpha + \sum_{j=1}^N \theta_j \right) \wedge \omega_0^{n-1}$$

which is the bound we hope to obtain, modulo the terms

$$\int_X \frac{\varepsilon^2}{\varepsilon^2 + |s_j|^2} \omega_0 \wedge \omega_\Lambda^{n-1}, \tag{3.17}$$

$$\int_{X_0} \xi dd^c \log(\mu^2 + |s|^{2m\delta} |u|^2_\Lambda) \wedge \omega_\Lambda^{n-1}, \tag{3.18}$$

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as well as

$$C_\eta \int_X \frac{\lambda^2}{\lambda^2 + \exp(\widehat{f}_\eta)} \omega_0 \wedge \omega_\Lambda^{n-1}, \quad C_\eta \int_X \frac{\rho^2}{\rho^2 + |\sigma_{r,\eta}|^2} \omega_0 \wedge \omega_\Lambda^{n-1}. \tag{3.19}$$

and

$$\int_X \frac{\lambda^2}{\lambda^2 + \exp(\widehat{f}_\eta)} dd^c \widehat{f}_\eta \wedge \omega_\Lambda^{n-1}. \tag{3.20}$$

In conclusion, Theorem 1.1 will be proved if we are able to show that by some choice of the cut-off function ξ and the parameters $\Lambda = (\varepsilon, \lambda, \rho, \eta, \delta)$ respectively, the quantities (3.17), (3.19) and (3.20) tend to zero, and (3.18) is negative. This will be a consequence of the estimates provided by Theorem 2.2.

Step 5. We first let $\varepsilon \to 0$, while the other parameters are unchanged. The effects of this first operation are evaluated in what follows.

To start with, we recall that by (3.5) we have

$$\frac{1}{C_{\delta,\eta}} \omega_\Lambda \leqslant \sum_{j=1}^{p} \frac{\sqrt{-1} dz^j \wedge dz^{\bar{j}}}{(\varepsilon^2 + |z^j|^2)^{1-\delta}} + \sum_{j=p+1}^{n} \sqrt{-1} dz^j \wedge dz^{\bar{j}}$$

locally at each point of X, where (z_j) are coordinates adapted to the pair (X, B). As a consequence we infer that

$$\lim_{\varepsilon \to 0} \int_X \frac{\varepsilon^2}{\varepsilon^2 + |s_j|^2} \omega_0 \wedge \omega_\Lambda^{n-1} = 0 \tag{3.21}$$

by a quick computation which will not be detailed here.

We next analyze the quantity (3.18) as $\varepsilon \to 0$. Let $\Lambda_0 := (0, \lambda, \rho, \eta, \delta)$ be the new set of parameters. We know that

$$\omega_\Lambda \to \omega_{\Lambda_0}$$

uniformly on compact sets of X_0. Therefore

$$\lim_{\varepsilon \to 0} \int_{X_0} \xi dd^c \log(\mu^2 + |s|^{2m\delta} |u|^2_\Lambda) \wedge \omega_\Lambda^{n-1} = \int_{X_0} \xi dd^c \log(\mu^2 + |s|^{2m\delta} |u|^2_{\Lambda_0}) \wedge \omega_{\Lambda_0}^{n-1}. \tag{3.22}$$

The important difference between $|u|_\Lambda$ and $|u|_{\Lambda_0}$ is that the pole order of the latter quantity along the components of B is smaller than $m\delta$. Indeed,

$$\sum_{j=1}^{p} \frac{\sqrt{-1} dz^j \wedge dz^{\bar{j}}}{|z^j|^{2(1-\delta)}} + \sum_{j=p+1}^{n} \sqrt{-1} dz^j \wedge dz^{\bar{j}} \leqslant \frac{1}{C_{\lambda,\rho,\delta,\eta}} \omega_{\Lambda_0} \tag{3.23}$$

and

$$\frac{1}{C_{\delta,\eta}} \omega_{\Lambda_0} \leqslant \sum_{j=1}^{p} \frac{\sqrt{-1} dz^j \wedge dz^{\bar{j}}}{|z^j|^{2(1-\delta)}} + \sum_{j=p+1}^{n} \sqrt{-1} dz^j \wedge dz^{\bar{j}}, \tag{3.24}$$

by combining (3.5) with the Monge–Ampère equation verified by ω_Λ. The upshot is that the function

$$\log(\mu^2 + |s|^{2m\delta} |u|^2_{\Lambda_0})$$

is bounded by a constant depending on μ, Λ_0, but completely independent with respect to the size of the support of ξ.

At this stage or our proof, we choose a sequence of cut-off functions

$$\xi := \chi_\tau$$

as in [CGP13] converging to the characteristic function of the set $X \setminus \mathrm{Supp}(B)$, so that (3.18) becomes

$$\int_{X_0} \log(\mu^2 + |s|^{2m\delta}|u|^2_{\Lambda_0}) dd^c \chi_\tau \wedge \omega^{n-1}_{\Lambda_0}. \tag{3.25}$$

Indeed, the integration by parts is legitimate, since for every positive τ, the relevant quantities are non-singular on the support of χ_τ.

If we let $\tau \to 0$, then (3.25) tends to zero, thanks to the computations in, for example, [CGP13], together with the fact that the bound of the absolute value of the function $\log(\mu^2 + |s|^{2m\delta}|u|^2_{\Lambda_0})$ is independent of τ.

Also, we remark that the term (3.16) becomes simply

$$\int_X \frac{|s|^{2m\delta}|u|^2_{\Lambda_0}}{\mu^2 + |s|^{2m\delta}|u|^2_{\Lambda_0}} \Theta_{h_L}(L) \wedge \omega^{n-1}_{\Lambda_0}. \tag{3.26}$$

As a conclusion of this step, by first letting $\varepsilon \to 0$ and then $\xi \to \chi_{X_0}$ as indicated above, we infer that

$$\begin{aligned}\int_X \frac{|s|^{2m\delta}|u|^2_{\Lambda}}{\mu^2 + |s|^{2m\delta}|u|^2_{\Lambda_0}} \Theta_{h_L}(L) \wedge \omega^{n-1}_{\Lambda_0} \leqslant{}& m \int_X \left(\Theta(K_X) + \alpha + \sum_{j=1}^N \theta_j\right) \wedge \omega_0^{n-1} \\ &+ C_\eta \left(\int_X \frac{\lambda^2}{\lambda^2 + \exp(\widehat{f}_\eta)} \omega_0 \wedge \omega^{n-1}_{\Lambda_0}\right) \\ &+ C_\eta \left(\sum_r \int_X \frac{\rho^2}{\rho^2 + |\sigma_{r,\eta}|^2} \omega_0 \wedge \omega^{n-1}_{\Lambda_0}\right) \\ &+ C \int_X \frac{\lambda^2}{\lambda^2 + \exp(\widehat{f}_\eta)} dd^c \widehat{f}_\eta \wedge \omega^{n-1}_{\Lambda_0} + (\delta + \eta) C. \end{aligned} \tag{3.27}$$

We let $\mu \to 0$; the left-hand-side term in (3.27) becomes

$$\int_X \Theta_{h_L}(L) \wedge \omega_0^{n-1},$$

as we see by the dominated convergence theorem.

Step 6. This is the last step of our proof. We first establish the equalities

$$\lim_{\lambda \to 0} \int_X \frac{\lambda^2}{\lambda^2 + \exp(\widehat{f}_\eta)} \omega_0 \wedge \omega^{n-1}_{\Lambda_0} = 0 \tag{3.28}$$

and

$$\lim_{\rho \to 0} \int_X \frac{\rho^2}{\rho^2 + |\sigma_{r,\eta}|^2} \omega_0 \wedge \omega^{n-1}_{\Lambda_0} = 0 \tag{3.29}$$

for any set of parameters δ and η. This is quite easy: by the relation (3.24) we have

$$\int_X \frac{\lambda^2}{\lambda^2 + \exp(\widehat{f}_\eta)} \omega_0 \wedge \omega^{n-1}_{\Lambda_0} \leqslant C_{\delta,\eta} \int_X \frac{\lambda^2}{\lambda^2 + \exp(\widehat{f}_\eta)} \frac{dV}{\prod_j |s_j|^{2-2\delta}},$$

and (3.28) and (3.29) follow by the dominated convergence theorem.

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We next treat the remaining term. We claim that

$$\lim_{\lambda\to 0}\int_X \frac{\lambda^2}{\lambda^2+\exp(\widehat{f}_\eta)} dd^c\widehat{f}_\eta\wedge\omega_{\Lambda_0}^{n-1}=0 \tag{3.30}$$

holds for any set of parameters (η,δ) within the range fixed at the beginning of the proof. This is established as follows:

$$\int_X \frac{\lambda^2}{\lambda^2+\exp(\widehat{f}_\eta)} dd^c\widehat{f}_\eta\wedge\omega_{\Lambda_0}^{n-1}\leqslant C_{\eta,\delta}\int_X \frac{\lambda^2}{\lambda^2+\exp(\widehat{f}_\eta)}(dd^c\widehat{f}_\eta+C_\eta\omega_{\delta,\mathrm{st}})\wedge\omega_{\delta,\mathrm{st}}^{n-1}, \tag{3.31}$$

where C_η is a constant such that the $(1,1)$-form $dd^c\widehat{f}_\eta+C_\eta\omega_{\delta,\mathrm{st}}$ is definite positive. We use the same notation as before, namely $\omega_{\delta,\mathrm{st}}$ is the standard metric with conic singularities along B, with all cone angles equal to $2\pi(1-\delta)$.

We define

$$d\mu_{\eta,\delta}:=(dd^c\widehat{f}_\eta+C_\eta\omega_{\delta,\mathrm{st}})\wedge\omega_{\delta,\mathrm{st}}^{n-1},$$

a positive measure of finite mass. Let $p: X_\eta\to X$ be a birational map such that

$$p^\star(dd^c\widehat{f}_\eta+C_\eta\omega_{\delta,\mathrm{st}})=[E]+\gamma_\eta$$

where E is a divisor and γ_η is smooth. The important point about the singularities of $\widehat{f}_\eta$ is that the E is p-contractible. Thus

$$p^\star d\mu_{\eta,\delta}=\gamma_\eta\wedge p^\star\omega_{\delta,\mathrm{st}}^{n-1}:=d\widehat{\mu}_{\eta,\delta},$$

that is to say, a measure with mild singularities on X_η. In fact, an immediate calculation shows that the measure $d\widehat{\mu}_{\eta,\delta}$ is smaller than the determinant of a standard metric with conic singularities of angles $2\pi(1-\delta)$ along the support of the divisor $p^\star(B)$, modulo some constant *independent of* λ. The quantity we have to analyze becomes

$$\int_{X_\eta}\frac{\lambda^2}{\lambda^2+\exp(\widehat{f}_\eta\circ p)}d\widehat{\mu}_{\eta,\delta},$$

and indeed it converges to zero if $\lambda\to 0$ by the arguments previously invoked (the dominated convergence theorem). The assertion is therefore established.

Remark 3.2. We remark that the hypothesis concerning the codimension of the singularities of $\widehat{f}_\eta$ is essential. Indeed, if v is a holomorphic section of an ample line bundle and if we take, for example, $\log|v|^2$ instead of $\widehat{f}_\eta$, we see that (3.30) does not hold!

After this last operation, inequality (3.27) becomes

$$\int_X\Theta_{h_L}(L)\wedge\omega_0^{n-1}\leqslant(\delta+\eta)C+m\int_X\left(\Theta(K_X)+\alpha+\sum_{j=1}^N\theta_j\right)\wedge\omega_0^{n-1} \tag{3.32}$$

and Theorem 1.1 is proved, by taking $\delta,\eta\to 0$.

Remark 3.3. It is possible to adapt the previous argument to the case where α is possibly singular, that is, the class $\{\alpha\}$ is pseudo-effective rather than Hermitian semi-positive. However, we have to impose the condition that the generic Lelong number of α along each component of B is equal to zero. We leave the details to the interested reader.

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Remark 3.4. An important application of our arguments was obtained in [Gue12], where the stability of the log-tangent bundle with respect to $K_X + B$ is proved, under the hypothesis that the latter bundle is nef and big.

Remark 3.5. By the same arguments it is possible to obtain a more general version of Theorem 1.1 for pairs (X, B) whose boundary $B = \sum_j (1 - \nu_j) Y_j$ has rational coefficients belonging to the interval $]\,0, 1\,]$ and snc support. The only change required is to replace the definition of the logarithmic cotangent bundle $T^\star_X\langle B\rangle$ with the one defined in [CP15].

Let N be a positive integer, which is divisible by all the denominators of the coefficients (ν_j) of B. There exist a non-singular manifold X_B and a finite map $\pi : X_B \to X$ such that $\pi^\star Y_j$ is divisible by N, and such that $\pi^\star(\sum_j Y_j)$ is snc. If $(z_j = 0)$ is a local equation of the hypersurface Y_j, then we remark that the multi-valued function $z_j^{1-\nu_j}$ becomes holomorphic when pulled back via π. This observation allows us to define the cotangent bundle (and any holomorphic tensor) corresponding to (X, B). We obtain the analogue of Theorem 1.1 by using the full force of Theorem 2.2 (and its generalization to the case of arbitrary coefficients; cf. [GP16]).

4. Proof of Theorem 1.2

In this part of our paper we will prove Theorem 1.2. As we have already mentioned, the result itself is already known: its proof in [CP15] is based on a the generic semi-positivity result for the cotangent of the orbifold pairs whose canonical bundle is pseudo-effective. The proof we present here follows essentially the same ideas, modulo the fact that we are using Theorem 1.1 instead of the aforementioned result concerning the orbifold cotangent bundle.

4.1 Continuity method

For the rest of this paper the manifold X is assumed to be projective. Let A be ample on X such that $K_X + B + A$ is ample. We consider the interval

$$J := \{t \in [0, 1] : K_X + B + tA \text{ is big}\}. \tag{4.1}$$

It is clear that J is non-empty and open. We next show that J is closed, provided that there exists an injection

$$L \to \bigotimes^m T^\star_X\langle B\rangle$$

for some $m > 0$, where L is a big line bundle on X.

Let $(t_k) \subset J$, converging to a real number t_∞. We have to show that the limit $\mathbb{R}$-bundle $K_X + B + t_\infty A$ is big. The first observation is that this bundle is at least pseudo-effective, given that it is a limit of big bundles.

As a warm-up, we first discuss a very particular case, namely *we assume that the limit* $K_X + B + t_\infty A$ *is nef*. Then we argue as follows: for each k the bundle $K_X + B + t_k A$ is ample, and by Theorem 1.1 we have the numerical inequality

$$\int_X c_1(L) \wedge c_1(K_X + B + t_k A)^{n-1} \leqslant m \int_X c_1(K_X + B + t_k A)^n, \tag{4.2}$$

where we stress on the fact that m is a purely numerical constant, and in particular it is independent of k.

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Since L is big, we certainly have

$$\int_X c_1(L) \wedge c_1(K_X + B + t_k A)^{n-1} \geqslant C(L) \int_X c_1(A) \wedge c_1(K_X + B + t_k A)^{n-1}$$

for some constant $C(L)$ depending exclusively on L.

On the other hand, by the Hovanski–Teissier concavity inequality,

$$\int_X c_1(A) \wedge c_1(K_X + B + t_k A)^{n-1} \geqslant \left(\int_X c_1(A)^n\right)^{1/n} \left(\int_X c_1(K_X + B + t_k A)^n\right)^{(n-1)/n}. \quad (4.3)$$

By inequalities (4.2) and (4.3) we infer that

$$\int_X c_1(K_X + B + t_k A)^n \geqslant C_0 > 0$$

for some constant C_0 which is independent of k; hence, the same will be true for the limit, so the $\mathbb{R}$ bundle $K_X + B + t_\infty A$ is big, and the proof of this particular case is finished.

We will prove the general case of Theorem 1.2 by means of arguments along the same lines. The bundle $K_X + B + t_\infty A$ is no longer assumed to be nef, but it is nonetheless at least pseudo-effective, and the idea is to decompose it into two orthogonal pieces, a nef part and an effective part. For an arbitrary big line bundle, this cannot be done in a sufficiently accurate way so as to be relevant for us (the approximate Zariski decomposition is not useful here, as we will comment at the end of this paper). However, the bundle we are dealing with is an *adjoint bundle* and the additional techniques required to treat this case are explained in the following paragraphs.

4.2 Desingularization and Zariski decomposition

In this section we collect a few important facts concerning the principalization of the ideals of holomorphic functions. They will be used in conjunction with the finite generation result in [BCHM10].

Let $\mathcal{I} \subset \mathcal{O}_X$ be an ideal sheaf, and let $x \in X$ be a point. The *vanishing order* of $\mathcal{I}$ at x is defined as

$$\mathrm{ord}_x(\mathcal{I}) := \max\{r : \mathcal{I} \subset m_x^r\},$$

where m_x is the ideal sheaf of x. Given $Z \subset X$ a submanifold, the *order of $\mathcal{I}$ along Z* is defined as the vanishing order of $\mathcal{I}$ at the generic point of Z. Finally, the *maximal order of $\mathcal{I}$ along Z* is equal to the maximum of the numbers $\mathrm{ord}_x(\mathcal{I})$ for all $x \in Z$.

A *marked ideal* $(\mathcal{I}, m)$ is a couple consisting of an ideal sheaf $\mathcal{I}$ together with a positive integer m. Let Z be a smooth subvariety of X such that the order of $\mathcal{I}$ along Z is at least m. The inverse image of the ideal $\mathcal{I}$ with respect to the blow-up $\pi : \widehat{X} \to X$ of X along Z can be written as the product of the principal ideal corresponding to the exceptional divisor to the power m, multiplied by an ideal of holomorphic functions on $\widehat{X}$. The latter is called the *proper transform of the marked ideal* $(\mathcal{I}, m)$.

Let $E := (E_1, \ldots, E_s)$ be a set of non-singular hypersurfaces of X, such that $\sum_j E_j$ is a simple normal crossing divisor. Following [Kol07], we call $(X, \mathcal{I}, m, E)$ a *triple*.

A *smooth blow-up sequence of order at least m* of the triple $(X, \mathcal{I}, m, E)$ is a sequence $(X_j, \mathcal{I}_j, m, E_{(j)})_{j=0,\ldots,r}$ satisfying the following requirements.

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(1) For each $j = 0, \dots, r$ we have $\mathcal{I}_j \subset \mathcal{O}_{X_j}$.
(2) We have $(X_0, \mathcal{I}_0, m, E_{(0)}) := (X, \mathcal{I}, m, E)$ and for each $j = 1, \dots, r$ the map $X_j \to X_{j-1}$ is the blow-up of a smooth center $Z_{j-1} \subset X_{j-1}$ such that $\mathrm{ord}_{Z_{j-1}}(\mathcal{I}_{j-1}) \geqslant m$, and such that Z has only simple normal crossings with the components of $E_{(j)}$.
(3) For each $j = 1, \dots, r$ the ideal $\mathcal{I}_j$ is the proper transform of the marked ideal $(\mathcal{I}_{j-1}, m)$ with respect to the map $X_j \to X_{j-1}$.
(4) For each $j = 1, \dots, r$, the set of hypersurfaces $E_{(j)}$ is the birational transform of E, together with the exceptional divisor of the map $X_j \to X_{j-1}$.

In this context, we quote a result from [Kol07, p. 41].

Theorem 4.1 [Kol07]. *Let X be a smooth projective manifold, and let $0 \neq \mathcal{I} \subset \mathcal{O}_X$ be an ideal sheaf. Let m be a positive integer assumed to be smaller that the maximal order of $\mathcal{I}$ along X, and let E be a simple normal crossing divisor. Then there exists a smooth blow-up sequence of order at least m of the triple $(X, \mathcal{I}, m, E)$ such that the proper transform of $\mathcal{I}$ by the resulting map $\widehat{X} \to X$ is an ideal whose maximal order along $\widehat{X}$ is strictly smaller than m.*

As a consequence of the algorithm described in the previous theorem (with $m = 1$) we have the following result.

Theorem 4.2. *Let (X, B) be a smooth log-canonical pair, and let $0 \neq \mathcal{I} \subset \mathcal{O}_X$ be an ideal sheaf. There exists a birational map $p : X_1 \to X$ such that the inverse image of $\mathcal{I}$ with respect to p is a principal ideal, whose zero loci plus the inverse image of the boundary divisor B defines a normal crossing divisor. Moreover, p can be chosen so that the next property is satisfied.*

Let $x \in X$ be an arbitrary point, and let $(f_1, \dots, f_g)$ be the local generators of the ideal $\mathcal{I}$ on an open set U centered at x. Then the support of the relative canonical bundle $K_{X_1/X}$ intersected with the inverse image of U is contained in the set $\bigcap_{j=1}^{g}(f_j \circ p = 0)$.

We next consider the context of the adjoint bundles, in which we have the following important result. This represents the second main technical point in the proof of Theorem 1.2.

Theorem 4.3 [BCHM10]. *For each $k \geqslant 1$, the algebra $\mathcal{R}_k$ of pluricanonical sections corresponding to $K_X + B + t_k A$ is finitely generated. In particular, the $\mathbb{Q}$-bundle $K_X + B + t_k A$ admits a Zariski decomposition obtained by considering a desingularization of the ideal defined by the generators of $\mathcal{R}_k$.*

We consider the generators $(u_j^{(k)})_{j=1,\dots,g_k}$ of the algebra $\mathcal{R}_k$; we assume that they are sections of the bundle $m_k(K_X + B + t_k A)$. Let T_k be the curvature current of the metric on $K_X + B + t_k A$ induced by the generators above. There exists a modification of X, say $p_k : X_k \to X$, such that

$$p_k^\star(T_k) = \omega_k + [N_k]$$

where ω_k is a semi-positive $(1,1)$-form on X_k corresponding to the $\mathbb{Q}$-bundle P_k on X_k, and N_k is an effective divisor, such that

$$\mathrm{Vol}(K_X + B + t_k A) = \int_{X_k} \omega_k^n.$$

The finiteness of the algebra $\mathcal{R}_k$ is equivalent to the equality above; in general, we only have an approximation of the volume by the top power of ω_k.

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Combined with Theorem 4.2 above, we obtain the next statement.

Corollary 1. *For each $k \geqslant 1$ there exist a non-singular manifold X_k and a birational map $p_k : X_k \to X$ such that the following statements hold.*

(a) *We have $p_k^\star(K_X + B + t_k A) = P_k + N_k$, where N_k is effective, P_k is big, without base points, and such that*
$$P_k^n = \mathrm{Vol}(K_X + B + t_k A).$$
(b) *The orthogonality relation holds true: $P_k^{n-1} \cdot N_k = 0$.*
(c) *We have $\mathrm{Supp}(K_{X_k/X}) \subset \mathrm{Supp}(N_k)$.*

For the crucial point (b) we refer to the orthogonality lemma in [BDPP13]. Point (c) is a direct consequence of Theorem 4.2 discussed above.

4.3 Conclusion of the proof

We denote by $(\Xi_k^{(j)})_{j=1,\dots,i_k}$ the support of the exceptional divisor of the map p_k. By the inclusion (c) above, together with the orthogonality relation (b) of Corollary 1, we obtain
$$P_k^{n-1} \cdot \Xi_k^{(j)} = 0 \tag{4.4}$$
for each $j = 1, \dots, i_k$. This is an important fact, since it will allow us to 'ignore' the presence of the exceptional divisors, and argue in what follows basically as in the nef case explained at the beginning of our proof.

Let $B = Y_1 + \cdots + Y_N$. We have the change of variables formula
$$p_k^\star(K_X + B) = K_{X_k} + B_k - E_k^2 \tag{4.5}$$
where $B_k := \sum_{j=1}^N \overline{Y}_j + E_k^1$. Here we denote by $\overline{Y}_j$ the proper transform of Y_j and E_k^1, E_k^2 are effective divisors, such that their support is contained in the exceptional divisor of p_k. The divisor E_k^1 is reduced, thanks to the fact that (X, B) is an lc pair. Moreover, we have
$$\mathrm{Supp}(B_k) \subset \mathrm{Supp}(p_k^\star(B)). \tag{4.6}$$

By hypothesis, we have an injection
$$L \to \bigotimes^m T_X^\star\langle B\rangle$$
where L is a big line bundle. We consider the p_k-inverse image of this injection and we get
$$p_k^\star(L) \to \bigotimes^m T_{X_k}^\star\langle \overline{B}_k\rangle, \tag{4.7}$$
where $\overline{B}_k$ is the reduced part of the inverse image of the divisor $p_k^\star(B)$.

The bundle $K_{X_k} + \overline{B}_k + t_\infty p_k^* A$ is pseudo-effective, as it follows from (4.5) combined with (4.6). Indeed, as a consequence of these relations we infer that the difference $\overline{B}_k - B_k$ is effective and p_k-exceptional.

Moreover, the bundle P_k is nef, so by Theorem 1.1 we obtain the inequality
$$\int_{X_k} p_k^\star(L) \cdot P_k^{n-1} \leqslant m \int_{X_k} (K_{X_k} + \overline{B}_k + t_0 p_k^* A) \cdot P_k^{n-1}. \tag{4.8}$$

By (4.5) and (4.6), together with the definition of B_k, we infer that
$$(K_{X_k} + \overline{B}_k + t_\infty p_k^* A) \cdot P_k^{n-1} = p_k^\star(K_X + B + t_\infty A) \cdot P_k^{n-1}. \tag{4.9}$$

By statements (a) and (b) in Corollary 1,

$$p_k^\star(K_X+B+t_\infty A)\cdot P_k^{n-1}=P_k^n-(t_k-t_\infty)p_k^\star(A)\cdot P_k^{n-1}. \tag{4.10}$$

Since the bundle L is big, we certainly have $p_k^\star(L)\cdot P_k^{n-1}\geqslant \varepsilon_0 p_k^\star(A)\cdot P_k^{n-1}$ for some positive ε_0, so all in all we obtain

$$P_k^n\geqslant mp_k^\star(A)\cdot P_k^{n-1}. \tag{4.11}$$

We use the Khovanskii–Teissier inequality as in the nef case discussed previously, and as a consequence we get

$$p_k^\star(A)\cdot P_k^{n-1}\geqslant (A^n)^{1/n}\cdot (P_k^n)^{(n-1)/n}. \tag{4.12}$$

The conjunction of (4.11) and (4.12) gives

$$P_k^n\geqslant C(m)>0 \tag{4.13}$$

uniformly with respect to k, so Theorem 1.2 is proved. □

5. Further remarks

We consider the exact sequence

$$0\to L\to \bigotimes^m T_X^\star\langle B\rangle\to Q\to 0 \tag{5.1}$$

where (X,B) is a pair with the properties stated in Theorem 1.1, and L is a line bundle. Let $\{\alpha\}$ be a semi-positive class of $(1,1)$-type, such that $c_1(K_X+B)+\{\alpha\}$ is pseudo-effective.

As a consequence of Theorem 1.1, we infer

$$\int_X c_1(Q)\wedge\omega^{n-1}\geqslant\left(\frac{n^m-1}{n-1}-m\right)\int_X(c_1(K_X+B)+\{\alpha\})\wedge\omega^{n-1}-\frac{n^m-1}{n-1}\int_X\alpha\wedge\omega^{n-1} \tag{5.2}$$

for any class ω belonging to the closure of the Kähler cone of X.

We remark that even if $\alpha=0$, so that K_X+B is pseudo-effective, inequality (5.2) is more precise than the one derived from Miyaoka's result in [Miy87] as soon as $m\geqslant 2$.

The next result is an easy consequence of (5.2).

Theorem 5.1. *Under the hypothesis above, we assume that*

$$\int_X c_1(Q)\wedge\omega^{n-1}=0$$

for some Kähler metric ω. Then K_X+B is numerically trivial.

If L in (5.1) is a sheaf of arbitrary rank, say r, then we obtain a weaker inequality,

$$\int_X c_1(Q)\wedge\omega^{n-1}\geqslant -\frac{n^m-1}{n-1}\int_X\alpha\wedge\omega^{n-1}. \tag{5.3}$$

Other applications will be discussed in a forthcoming publication.

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