Appendix C

Useful formulae

C–1 Numerics

Conversion factors ($\hbar = c = k_B = 1$):

$$\begin{aligned} 1\ \text{GeV}^{-1} &= 6.582122 \times 10^{-25}\ \text{s} & 1\ \text{GeV} &= 1.16 \times 10^{13}\ \text{K} \\ &= 0.197327\ \text{fm} & &= 1.78 \times 10^{-24}\ \text{g}. \end{aligned}$$

Physical constants ($\hbar = c = 1$):

$$\begin{aligned} G_\mu &= 1.1663787(6) \times 10^{-5}\ \text{GeV}^{-2} & G_N^{-1/2} &= M_{\text{Pl}} = 1.2 \times 10^{19}\ \text{GeV} \\ \alpha^{-1} &= 137.035999074(44) & \sin^2\theta_{\text{w}}^{\overline{\text{MS}}}(M_Z) &= 0.23125(16) \\ m_W &= 80.385(15)\ \text{GeV} & m_Z &= 91.1876(21)\ \text{GeV} \\ m_e &= 0.510998928(11)\ \text{MeV} & m_p &= 938.272046(21)\ \text{MeV} \\ F_\pi &= 92.2(2)\text{MeV} & F_K &= 110.4(8)\ \text{MeV} \\ |\eta_{+-}| &= 2.232(11) \times 10^{-3} & |\eta_{00}| &= 2.220(11) \times 10^{-3}. \end{aligned}$$

CKM matrix elements:

$$\begin{aligned} |V_{\text{ud}}| &= 0.97427(15) & |V_{\text{us}}| &= 0.22534(65) & |V_{\text{ub}}| &= 0.00351^{+0.00015}_{-0.00014} \\ |V_{\text{cd}}| &= 0.22520(65) & |V_{\text{cs}}| &= 0.97344(16) & |V_{\text{cb}}| &= 0.0412^{+0.0011}_{-0.0005} \\ |V_{\text{td}}| &= 0.00867^{+0.00029}_{-0.00031} & |V_{\text{ts}}| &= 0.0404^{+0.0011}_{-0.0005} & |V_{\text{tb}}| &= 0.999146^{+0.000021}_{-0.000046}. \end{aligned}$$

C–2 Notations and identities

Metric tensor:

$$g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \qquad g^{\mu}{}_{\mu} = 4. \tag{2.1}$$

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Totally antisymmetric four-tensor:

$$\epsilon^{\mu\nu\alpha\beta} = \begin{cases} +1 & \{\mu, \nu, \alpha, \beta\} \text{ even permutation of } \{0, 1, 2, 3\} \\ -1 & \text{odd permutation} \\ 0 & \text{otherwise} \end{cases}$$

$$\epsilon^{\mu\nu\alpha\beta}\epsilon_{\mu}{}^{\nu'\alpha'\beta'} = g^{\nu\alpha'}g^{\alpha\nu'}g^{\beta\beta'} + g^{\nu\nu'}g^{\alpha\beta'}g^{\beta\alpha'} + g^{\nu\beta'}g^{\alpha\alpha'}g^{\beta\nu'} - g^{\nu\nu'}g^{\alpha\alpha'}g^{\beta\beta'} - g^{\nu\beta'}g^{\alpha\nu'}g^{\beta\alpha'} - g^{\nu\alpha'}g^{\alpha\beta'}g^{\beta\nu'}. \quad (2.2)$$

Totally antisymmetric three-tensor:

$$\epsilon_{ijk} = \begin{cases} +1 & \{i, j, k\} \text{ even permutation of } \{1, 2, 3\} \\ -1 & \text{odd permutation} \\ 0 & \text{otherwise} \end{cases}$$

$$\epsilon^{0ijk} = -\epsilon_{0ijk} = \epsilon^{ijk} = \epsilon_{ijk}$$

$$\epsilon_{ijk}\epsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}. \quad (2.3)$$

Pauli matrices:

$$\sigma^j\sigma^k = \delta^{jk}I + i\epsilon^{jkl}\sigma^l \qquad (j, k, l = 1, 2, 3)$$

$$\sigma^j_{ab}\sigma^j_{cd} = 2\delta_{ad}\delta_{bc} - \delta_{ab}\delta_{cd} \qquad (a, b, c, d = 1, 2). \quad (2.4)$$

Dirac matrices:

$$\gamma_5 = -i\gamma^0\gamma^1\gamma^2\gamma^3$$

$$\sigma^{\mu\nu} = \frac{i}{2}\left[\gamma^\mu, \gamma^\nu\right]$$

$$\gamma^\mu\gamma^\nu\gamma^\alpha = g^{\mu\nu}\gamma^\alpha + g^{\nu\alpha}\gamma^\mu - g^{\alpha\mu}\gamma^\nu - i\epsilon^{\mu\nu\alpha\beta}\gamma_\beta\gamma_5$$

$$\gamma^0\Gamma_i^\dagger\gamma^0 = \Gamma_i \qquad (\Gamma_i = 1, \gamma^\mu, \gamma^\mu\gamma_5, \sigma^{\mu\nu})$$

$$\gamma^0\Gamma_i^\dagger\gamma^0 = -\Gamma_i \qquad (\Gamma_i = \gamma_5). \quad (2.5)$$

Trace relations:

$$\mathrm{Tr}\ (\gamma^\mu) = 0$$

$$\mathrm{Tr}\ (\gamma_5) = 0$$

$$\mathrm{Tr}\ (\gamma^\mu\gamma^\nu) = 4g^{\mu\nu}$$

$$\mathrm{Tr}\ (\gamma^\mu\gamma^\nu\gamma_5) = 0$$

$$\mathrm{Tr}\ \left(\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta\right) = 4\left(g^{\mu\nu}g^{\alpha\beta} - g^{\mu\alpha}g^{\nu\beta} + g^{\mu\beta}g^{\nu\alpha}\right)$$

$$\mathrm{Tr}\ \left(\gamma_5\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta\right) = 4i\epsilon^{\mu\nu\alpha\beta}$$

$$\mathrm{Tr}\ \left(\not{a}_1 \ldots \not{a}_{2n+1}\right) = 0$$

$$\mathrm{Tr}\ \left(\not{a}_1 \ldots \not{a}_{2n}\right) = \mathrm{Tr}\ \left(\not{a}_{2n} \ldots \not{a}_1\right). \quad (2.6)$$

Plane wave solutions:

The Dirac spinor $u(\mathbf{p}, s)$ is a positive-energy eigenstate of the momentum $\mathbf{p}$ and energy $E = \sqrt{\mathbf{p}^2 + m^2}$. Antifermions are described in terms of the Dirac spinor $v(\mathbf{p}, s)$. The adjoint solutions are denoted by $\bar{u} \equiv u^\dagger \gamma^0$ and $\bar{v} \equiv v^\dagger \gamma^0$. Note that our normalization of Dirac spinors behaves smoothly in the massless limit.

$$
\begin{aligned}
(\not{p} - m)u(\mathbf{p}, s) &= 0 \\
\bar{u}(\mathbf{p}, s)(\not{p} - m) &= 0 \\
(\not{p} + m)v(\mathbf{p}, s) &= 0 \\
\bar{v}(\mathbf{p}, s)(\not{p} + m) &= 0 \\
\bar{u}(\mathbf{p}, r)u(\mathbf{p}, s) &= 2m\delta_{rs} \\
\bar{v}(\mathbf{p}, r)v(\mathbf{p}, s) &= -2m\delta_{rs} \\
u^\dagger(\mathbf{p}, r)u(\mathbf{p}, s) &= 2E\delta_{rs} \\
v^\dagger(\mathbf{p}, r)v(\mathbf{p}, s) &= 2E\delta_{rs} \\
\sum_s u(\mathbf{p}, s)\bar{u}(\mathbf{p}, s) &= \not{p} + m \\
\sum_s v(\mathbf{p}, s)\bar{v}(\mathbf{p}, s) &= \not{p} - m.
\end{aligned}
\tag{2.7}
$$

Gordon decomposition for a fermion of mass m:

$$
\bar{u}\left(\mathbf{p}', r\right)\gamma^\mu u(\mathbf{p}, s) = \bar{u}(\mathbf{p}', r)\left(\frac{\left(p' + p\right)^\mu}{2m} + \frac{i\sigma^{\mu\nu}\left(p' - p\right)_\nu}{2m}\right)u(\mathbf{p}, s). \tag{2.8}
$$

Dirac representation:

$$
\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \boldsymbol{\gamma} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ -\boldsymbol{\sigma} & 0 \end{pmatrix} \quad \gamma_5 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \tag{2.9}
$$

$$
u(\mathbf{p}, s) = \sqrt{E + m}\begin{pmatrix} \chi_s \\ \dfrac{\boldsymbol{\sigma} \cdot \mathbf{p}}{E + m}\chi_s \end{pmatrix} \qquad v(\mathbf{p}, s) = \sqrt{E + m}\begin{pmatrix} \dfrac{\boldsymbol{\sigma} \cdot \mathbf{p}}{E + m}\chi_s \\ \chi_s \end{pmatrix}. \tag{2.10}
$$

Fierz relations:

The anticommutativity of fermion fields and the algebra of Dirac matrices imply the (particularly useful) Fierz relations,

$$
\begin{aligned}
\bar{\psi}_1\gamma^\mu(1 + \gamma_5)\psi_2\bar{\psi}_3\gamma_\mu(1 + \gamma_5)\psi_4 &= \bar{\psi}_1\gamma^\mu(1 + \gamma_5)\psi_4\bar{\psi}_3\gamma_\mu(1 + \gamma_5)\psi_2 \\
\bar{\psi}_1\gamma^\mu(1 + \gamma_5)\psi_2\bar{\psi}_3\gamma_\mu(1 - \gamma_5)\psi_4 &= -2\bar{\psi}_1(1 - \gamma_5)\psi_4\bar{\psi}_3(1 + \gamma_5)\psi_2.
\end{aligned}
\tag{2.11}
$$

Propagators:
The propagators associated with fields $\varphi(x)$, $\psi(x)$, $W_\lambda(x)$ having spins 0, 1/2, 1 and masses μ, m, M are, respectively,

$$i\Delta_F(x) = \langle 0|T\left(\varphi(x)\varphi^\dagger(0)\right)|0\rangle = \int \frac{d^4p}{(2\pi)^4}\, e^{-ip\cdot x}\, \frac{i}{p^2-\mu^2+i\epsilon}$$

$$iS_{F\beta\alpha}(x) = \langle 0|T\left(\psi_\beta(x)\bar{\psi}_\alpha(0)\right)|0\rangle = \int \frac{d^4p}{(2\pi)^4}\, e^{-ip\cdot x}\, \frac{i\,(\not p+m)_{\beta\alpha}}{p^2-m^2+i\epsilon}$$

$$iD_{F\lambda\nu}(x) = \langle 0|T\left(W^\lambda(x)W^{\dagger\nu}(0)\right)|0\rangle$$

$$= \int \frac{d^4p}{(2\pi)^4}\, e^{-ip\cdot x}\, \frac{i\,\left(-g_{\lambda\nu}+(1-\xi)p_\lambda p_\nu/\left(p^2-\xi M^2+i\epsilon\right)\right)}{p^2-M^2+i\epsilon}, \quad (2.12)$$

where ξ is a gauge-dependent parameter.

Feynman parameterization:

$$\frac{1}{a^n b^m} = \frac{\Gamma(n+m)}{\Gamma(n)\Gamma(m)} \int_0^1 dx\, \frac{x^{n-1}(1-x)^{m-1}}{[ax+b(1-x)]^{n+m}} \qquad (n,m>0)$$

$$\frac{1}{abc} = 2\int_0^1 x\,dx \int_0^1 dy\, \frac{1}{[a(1-x)+bxy+cx(1-y)]^3}. \quad (2.13)$$

C–3 Decay lifetimes and cross sections

Parameters of choice for quantum fields:
The literature reveals a variety of conventions employed in quantum field theory. We can characterize all of these with certain parameters of choice, J_i, K_i, L_i ($i = B, F$ distinguishes bosons from fermions), occurring in the normalization of spin zero and spin one-half fields,

$$\varphi(x) = \int \frac{d^3k}{J_B}\, \left(a(\mathbf{k})e^{-ik\cdot x} + a^\dagger(\mathbf{k})e^{ik\cdot x}\right)$$

$$\psi(x) = \sum_s \int \frac{d^3p}{J_F}\, \left(b(\mathbf{p},s)u(\mathbf{p},s)e^{-ip\cdot x} + d^\dagger(\mathbf{p},s)v(\mathbf{p},s)e^{ip\cdot x}\right), \quad (3.1)$$

in momentum space algebraic relations, e.g.,

$$\left[a(\mathbf{k}), a^\dagger(\mathbf{k}')\right] = K_B\delta^3(\mathbf{k}-\mathbf{k}'),$$

$$\left\{b(\mathbf{p},r), b^\dagger(\mathbf{p}',s)\right\} = K_F\delta_{rs}\delta^3(\mathbf{p}-\mathbf{p}'), \quad (3.2)$$

and in the normalization of single-particle states

$$|\mathbf{k}\rangle_B = L_B a^\dagger(\mathbf{k})|0\rangle, \qquad |\mathbf{p},s\rangle_F = L_F b^\dagger(\mathbf{p},s)|0\rangle. \quad (3.3)$$

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It is convenient to introduce an additional parameter N_F to characterize the choice of fermion spinor normalization,

$$u^\dagger(\mathbf{p}, r)u(\mathbf{p}, s) = N_F 2E_\mathbf{p}\delta_{rs}. \tag{3.4}$$

For uniformity of notation, we also define $N_B \equiv 1$. The constants J_i, K_i, N_i are constrained by the canonical commutation or anticommutation relations to obey

$$\frac{K_i N_i}{J_i^2} = \frac{1}{(2\pi)^3 2E} \qquad (i = B, F). \tag{3.5}$$

Using the above, one can express the single-particle expectation value of the quantum mechanical probability density as

$$\rho_i = \frac{K_i L_i^2}{(2\pi)^3} \qquad (i = B, F). \tag{3.6}$$

The conventions employed in this book, together with the implied normalization for boson or fermion single-particle states, are

$$\begin{aligned} &L_B = L_F = N_B = N_F = 1, \qquad J_B = J_F = K_B = K_F = 2E(2\pi)^3, \\ &\langle \mathbf{p}', s|\mathbf{p}, r\rangle = 2E_\mathbf{p}\delta_{rs}(2\pi)^3\delta^{(3)}(\mathbf{p}' - \mathbf{p}), \end{aligned} \tag{3.7}$$

where r, s are spin labels. This choice, although somewhat unconventional for fermions,[1] has the advantages that bosons and fermions are treated symmetrically throughout the formalism, the zero-mass limit presents no difficulty, and matrix elements are free of cumbersome kinematic factors.

Lifetimes:
From the decay law $N(t) = N(0)e^{-t/\tau}$, the inverse mean life τ^{-1} is seen to be the transition rate per decaying particle, $\Gamma = \tau^{-1} = -\dot{N}/N$. For decay of a particle of energy E_1 into a total of $n-1$ bosons and/or fermions, the $\mathcal{S}$-matrix amplitude can be written in terms of a reduced (or invariant) amplitude $\mathcal{M}_{\rm fi}$ as

$$\begin{aligned} \langle f|\mathcal{S} - 1|i\rangle &= -i(2\pi)^4\delta^{(4)}(p_1 - p_2 \cdots - p_n)\prod_{k=1}^{n}\left(\frac{K_k L_k}{J_k}\right)\mathcal{M}_{\rm fi} \\ &= -i(2\pi)^4\delta^{(4)}(p_1 - p_2 \cdots - p_n)\prod_{k=1}^{n}\left(\frac{\rho_k}{2E_k N_k}\right)^{1/2}\mathcal{M}_{\rm fi}, \end{aligned} \tag{3.8}$$

where the index k labels the individual particles as to whether they are bosons or fermions. The inverse lifetime is computed from the squared S-matrix amplitude per spacetime volume VT and incident particle density ρ_1, integrated over final-state phase space. The choice of phase space is already fixed by our analysis. Thus,

[1] Another book sharing this convention is [ChL 84].

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defining a parameter of choice $A(\mathbf{p})$ for the (momentum) phase space per particle,

$$\text{Phase space per particle} \equiv \int \frac{d^3\mathbf{k}}{A(\mathbf{k})}, \tag{3.9}$$

the application of completeness to Eq. (3.7) yields

$$\langle \mathbf{p}'|\mathbf{p}\rangle = \int \frac{d^3k}{A} \langle \mathbf{p}'|\mathbf{k}\rangle\langle \mathbf{k}|\mathbf{p}\rangle \;\; \Rightarrow \;\; A = KL^2 = (2\pi)^3\rho. \tag{3.10}$$

The inverse lifetime (or decay width) is then given by

$$\begin{aligned} \tau^{-1} = \Gamma &= \frac{1}{\rho_1}\frac{1}{\mathcal{Z}} \int \left(\prod_{k=2}^{n} \frac{d^3 p_k}{(2\pi)^3 \rho_k} \right) \frac{|\mathcal{S}-1|^2_{\text{fi}}}{VT} \\ &= \frac{1}{2E_1 N_1}\frac{1}{\mathcal{Z}} \int \left(\prod_{k=2}^{n} \frac{d^3 p_k}{(2\pi)^3 2E_k N_k} \right) (2\pi)^4 \delta^4(p_1 - \cdots - p_n) \sum_{\text{int}} |\mathcal{M}_{\text{fi}}|^2, \end{aligned} \tag{3.11}$$

where $\mathcal{Z} = \prod_j n_j!$ is a statistical factor accounting for the presence of n_j identical particles of type j in the final state, and the sum 'int' is over internal degrees of freedom such as spin and color.

Cross sections:
For the reaction $1 + 2 \to 3 + \ldots n$, the cross section σ is the transition rate per incident flux. The incident flux f_{inc} can be represented as

$$f_{\text{inc}} = \rho_1\rho_2|\mathbf{v}_1 - \mathbf{v}_2| = \frac{\rho_1\rho_2}{E_1E_2}[(p_1\cdot p_2)^2 - m_1^2 m_2^2]^{1/2}, \tag{3.12}$$

and the cross section becomes

$$\begin{aligned} \sigma = &\frac{1}{\mathcal{Z}} \frac{1}{4\left((p_1\cdot p_2)^2 - m_1^2 m_2^2\right)^{1/2}} \\ &\times \int \left(\prod_{k=3}^{n} \frac{d^3 p_k}{(2\pi)^3 2E_k N_k} \right) (2\pi)^4 \delta^4(p_1 + p_2 - \cdots - p_n) \sum_{\text{int}} |\mathcal{M}_{\text{fi}}|^2. \end{aligned} \tag{3.13}$$

Watson's theorem:
The scattering operator S is unitary, $S^\dagger S = 1$. Thus, the transition operator $\mathcal{T}$, defined by $S = 1 - i\mathcal{T}$, obeys $i(\mathcal{T} - \mathcal{T}^\dagger) = \mathcal{T}^\dagger\mathcal{T}$. With the aid of the relation $\langle f|\mathcal{T}^\dagger|i\rangle = \langle i|\mathcal{T}|f\rangle^*$, we obtain the unitarity constraint for matrix elements,

$$i\left(\mathcal{T}_{\text{fi}} - \mathcal{T}^*_{\text{if}}\right) = \sum_n \mathcal{T}^*_{\text{nf}}\mathcal{T}_{\text{ni}}, \tag{3.14}$$

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where $\mathcal{T}_{\text{fi}} \equiv \langle f|\mathcal{T}|i\rangle$. This constraint implies the existence of phase relations between the various intermediate-state amplitudes. For example, consider a weak transition followed by a strong final-state interaction for which there is a unique intermediate state identical to the final state,

$$A \underset{\text{weak}}{\longrightarrow} BC \underset{\text{strong}}{\longrightarrow} BC, \tag{3.15}$$

i.e., $i = A, n = f = BC$. In this circumstance, time-reversal invariance of the hamiltonian implies $\mathcal{T}_{\text{fi}} = \mathcal{T}_{\text{if}}$, so the left-hand side of the unitarity relation reduces to $-2\text{Im}\mathcal{T}_{\text{if}}$ and both sides of Eq. (3.14) are real-valued. Denoting the weak and strong matrix elements as $|T_{\text{w}}|e^{i\delta_{\text{w}}}$ and $|T_{\text{s}}|e^{i\delta_{\text{s}}}$, it then follows that $\delta_{\text{w}} = \delta_{\text{s}}$.

C–4 Field dimension

We consider a limit in which the theory is invariant under the set of scale transformations $x^\mu \to \lambda x^\mu$ $(\lambda > 0)$ of the spacetime coordinates. Associate with each such coordinate transformation a unitary operator $U(\lambda)$ whose effect on a generic quantum field Φ is given by $U(\lambda)\Phi(x)U^\dagger(\lambda) = \lambda^{d_\Phi}\Phi(\lambda x)$, where d_Φ is the *dimension* of the field Φ. From the canonical commutation relation obeyed by a boson field φ or the canonical anticommutation relation obeyed by a fermion field ψ_α,

$$[\varphi(0,\mathbf{x}), \dot{\varphi}(0)] = i\delta^3(\mathbf{x}), \qquad \left\{\psi_\alpha(0,\mathbf{x}), \psi^\dagger_\beta(0)\right\} = \delta_{\alpha\beta}\delta^3(\mathbf{x}), \tag{4.1}$$

it follows that the *canonical* field dimensions are $d_\varphi = 1$ and $d_\psi = 3/2$. Composites built from products of these fields carry a dimension of their own, e.g., all fermion bilinears $\overline{\psi}\Gamma\psi$ (Γ is a 4×4 matrix) have canonical dimension 3. Unless protected by some kind of algebraic relation, a field dimension will generally be modified from the canonical value by interaction-dependent *anomalous* dimensions. Field dimensions are particularly useful in ordering the terms contained in a short-distance expansion,

$$A(x)B(0) \underset{x\to 0}{\longrightarrow} \sum_n c_n(x) O_n, \tag{4.2}$$

where A, B, O_n are local quantum fields. From the scale invariance of the short-distance limit, it follows that $c_n(x) \sim x^{d_{O_n} - d_A - d_B}$. Thus, the fields O_n of lowest dimension have the most singular coefficient functions.

C–5 Mathematics in *d* dimensions

Dirac algebra:

The following set of rules, generally referred to as NDR (*naive dimensional regularization*), is the one most commonly used in the literature. We employ a metric

$g_{\mu\nu}$ corresponding to a spacetime of continuous dimension d and maintain certain $d = 4$ properties of the Dirac matrices such as the trace relations of Eq. (2.6). In the following, I_d is a diagonal d-dimensional matrix with Tr $I_d = 4$ and $\epsilon \equiv (4-d)/2$.

$$\begin{aligned} g_\mu^{\ \mu} &= d \\ \{\gamma_\mu, \gamma_\nu\} &= 2g_{\mu\nu} I_d \\ \gamma_\mu \gamma^\mu &= d\ I_d \\ \gamma_\mu\ \not{p}\ \gamma^\mu &= (2\epsilon - 2)\ \not{p} \\ \gamma_\mu\ \not{p}\ \not{q}\ \gamma^\mu &= 4p\cdot q I_d - 2\epsilon\ \not{p}\ \not{q} \\ \gamma_\mu\ \not{p}\ \not{q}\ \not{r}\ \gamma^\mu &= -2\ \not{r}\ \not{q}\ \not{p} + 2\epsilon\ \not{p}\ \not{q}\ \not{r} \\ \not{p}\not{q}\not{r} + \not{r}\not{q}\not{p} &= 2p\cdot q\not{r} + 2q\cdot r\not{p} - 2p\cdot r\not{q} \\ \{\gamma_\mu, \gamma_5\} &= 0. \end{aligned} \tag{5.1}$$

Note that in NDR, γ_5 anticommutes with the gamma matrices. This will suffice for the calculations appearing in this book, but is not valid for all amplitudes (e.g. closed odd-parity fermion loops).

Integrals:
For the following integrals, we define the denominator function

$$\mathcal{D} \equiv m_1^2 x + m_2^2(1-x) - q^2 x(1-x) - i\epsilon, \tag{5.2}$$

take $n_1, n_2 \geq 1$, and denote $i\epsilon$ as the infinitesimal Feynman parameter.

$$\begin{aligned} &\int \frac{d^d p}{(2\pi)^d} \frac{1}{\left[(p-q)^2 - m_1^2 + i\epsilon\right]^{n_1}\left[p^2 - m_2^2 + i\epsilon\right]^{n_2}} \\ &\quad = (-1)^{n_1+n_2} \frac{i}{(4\pi)^{d/2}} \frac{\Gamma(n_1+n_2-d/2)}{\Gamma(n_1)\Gamma(n_2)} \int_0^1 dx\ \frac{x^{n_1-1}(1-x)^{n_2-1}}{\mathcal{D}^{n_1+n_2-d/2}}, \end{aligned} \tag{5.3a}$$

$$\begin{aligned} &\int \frac{d^d p}{(2\pi)^d} \frac{p^\mu}{\left[(p-q)^2 - m_1^2 + i\epsilon\right]^{n_1}\left[p^2 - m_2^2 + i\epsilon\right]^{n_2}} \\ &\quad = (-1)^{n_1+n_2} q^\mu \frac{i}{(4\pi)^{d/2}} \frac{\Gamma(n_1+n_2-d/2)}{\Gamma(n_1)\Gamma(n_2)} \int_0^1 dx\ \frac{x^{n_1}(1-x)^{n_2-1}}{\mathcal{D}^{n_1+n_2-d/2}}, \end{aligned} \tag{5.3b}$$

$$\begin{aligned} &\int \frac{d^d p}{(2\pi)^d} \frac{p^\mu p^\nu}{\left[(p-q)^2 - m_1^2 + i\epsilon\right]^{n_1}\left[p^2 - m_2^2 + i\epsilon\right]^{n_2}} \\ &\quad = \frac{i}{(4\pi)^{d/2}} \frac{(-1)^{n_1+n_2}}{\Gamma(n_1)\Gamma(n_2)} \left[q^\mu q^\nu \Gamma(n_1+n_2-d/2) \int_0^1 dx\ \frac{x^{n_1+1}(1-x)^{n_2-1}}{\mathcal{D}^{n_1+n_2-d/2}} \right. \\ &\qquad \left. - \frac{g^{\mu\nu}}{2}\Gamma(n_1+n_2-1-d/2) \int_0^1 dx\ \frac{x^{n_1-1}(1-x)^{n_2-1}}{\mathcal{D}^{n_1+n_2-1-d/2}} \right], \end{aligned} \tag{5.3c}$$

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$$\int \frac{d^d p}{(2\pi)^d} \frac{p^\mu p^\nu p^\lambda}{\left[(p-q)+i\epsilon\right]^{n_1}\left[p^2-m_2^2+i\epsilon\right]^{n_2}}$$
$$= \frac{i}{(4\pi)^{d/2}} \frac{(-1)^{n_1+n_2}}{\Gamma(n_1)\Gamma(n_2)} \Bigg[q^\mu q^\nu q^\lambda \Gamma(n_1+n_2-d/2) \int_0^1 dx \, \frac{x^{n_1+2}(1-x)^{n_2-1}}{\mathcal{D}^{n_1+n_2-d/2}}$$
$$- \frac{1}{2}\left(g^{\mu\nu}q^\lambda + g^{\mu\lambda}q^\nu + g^{\nu\lambda}q^\mu\right)\Gamma(n_1+n_2-1-d/2)\int_0^1 dx \, \frac{x^{n_1}(1-x)^{n_2-1}}{\mathcal{D}^{n_1+n_2-1-d/2}}\Bigg], \tag{5.3d}$$

$$\int \frac{d^d p}{(2\pi)^d} \frac{p^\mu p^\nu p^\lambda p^\sigma}{\left[(p-q)^2-m_1^2+i\epsilon\right]^{n_1}\left[p^2-m_2^2+i\epsilon\right]^{n_2}}$$
$$= \frac{i}{(4\pi)^{d/2}} \frac{(-1)^{n_1+n_2}}{\Gamma(n_1)\Gamma(n_2)} \Bigg[q^\mu q^\nu q^\lambda q^\sigma \Gamma(n_1+n_2-d/2) \int_0^1 dx \, \frac{x^{n_1+3}(1-x)^{n_2-1}}{\mathcal{D}^{n_1+n_2-d/2}}$$
$$- \frac{1}{2}\left(g^{\mu\nu}q^\lambda q^\sigma + g^{\mu\lambda}q^\nu q^\sigma + 4\,\text{perm.}\right)\Gamma(n_1+n_2-1-d/2)\int_0^1 dx \, \frac{x^{n_1+1}(1-x)^{n_2-1}}{\mathcal{D}^{n_1+n_2-1-d/2}}$$
$$+\frac{1}{4}\left(g^{\mu\nu}g^{\lambda\sigma} + g^{\mu\lambda}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\lambda}\right)\Gamma(n_1+n_2-2-d/2)\int_0^1 dx \, \frac{x^{n_1-1}(1-x)^{n_2-1}}{\mathcal{D}^{n_1+n_2-2-d/2}}\Bigg]. \tag{5.3e}$$

Solid angle:

$$\Omega_d = \int_0^\pi d\theta_{d-1}\sin^{d-2}\theta_{d-1}\ldots\int_0^\pi d\theta_2 \sin\theta_2 \int_0^{2\pi} d\theta_1 = \frac{2\pi^{d/2}}{\Gamma(d/2)}.$$
$$\Omega_2 = 2\pi, \quad \Omega_3 = 4\pi, \quad \Omega_4 = 2\pi^2, \ldots. \tag{5.4}$$

Gamma, psi, beta, and hypergeometric functions:

$$\Gamma(z) = \int_0^\infty dt\; e^{-t}\, t^{z-1} \qquad (\text{Re}\, z > 0),$$
$$\Gamma(z+1) = z\Gamma(z) = z(z-1)\Gamma(z-1) = \cdots = z!,$$
$$\Gamma(-n+\epsilon) = \frac{(-)^n}{n!}\left[\frac{1}{\epsilon} + \psi(n+1) + \mathcal{O}(\epsilon)\right] \quad (n \text{ integer}),$$
$$d\Gamma(z)/dz = \Gamma(z)\psi(z) \;\text{ where }\; \psi(z+1) = \psi(z) + 1/z,$$
$$\psi(1) = -\gamma = -\lim_{n\to\infty}\left(1 + \frac{1}{2} + \cdots + \frac{1}{n} - \ln n\right) \simeq -0.5772,$$
$$d\psi(z+1)/dz \equiv \psi'(z+1) = \psi'(z) - 1/z^2 \;\text{ with }\; \psi'(1) = \pi^2/6,$$
$$B(z,w) = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)} = 2\int_0^\infty dt\, \frac{t^{2z-1}}{(t^2+1)^{z+w}} \quad (\text{Re}\, z,\ \text{Re}\, w > 0),$$

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$$F(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 dt\, t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}$$
$$(\mathrm{Re}\, c > \mathrm{Re}\, b > 0),$$
$$F(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)},$$
$$\frac{dF(a, b; c; z)}{dz} = \frac{ab}{c} F(a+1, b+1; c+1; z). \tag{5.5}$$

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