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AN APPROACH TO CAPABLE GROUPS AND SCHUR'S THEOREM

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Abstract

Podoski and Szegedy ['On finite groups whose derived subgroup has bounded rank', *Israel J. Math.* **178** (2010), 51–60] proved that for a finite group G with rank r, the inequality $[G : Z_2(G)] \le |G'|^{2r}$ holds. In this paper we omit the finiteness condition on G and show that groups with finite derived subgroup satisfy the same inequality. We also construct an n-capable group which is not $(n + 1)$-capable for every $n \in \mathbf{N}$.

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1. Introduction

A famous theorem of Schur asserts that for a group G, if $G/Z(G)$ is finite then G' is finite. This theorem was extended to the terms of the upper and lower central series by Baer [1]. The converse does not hold in general, although Hall in [4] proved that if G' is finite, then $[G : Z_2(G)]$ is finite. Here, $Z_2(G)$ denotes the second centre of G, defined by $Z_2(G)/Z(G) = Z(G/Z(G))$. There are upper bounds for the index of the second centre of G in terms of the order of the derived subgroup. For instance, Podoski and Szegedy [9] proved that for a finite group G, if $\mathrm{rank}(G') = r$, then

$$[G : Z_2(G)] \le |G'|^{2r}. \tag{1.1}$$

The rank of a group is defined as follows.

Definition 1.1. Let G be a group. Suppose that r is a positive integer such that every subgroup of G is generated by r elements and there is a subgroup which is not generated by fewer than r elements. The smallest such r is called the rank of G and denoted by $\mathrm{rank}(G)$.

In Section 2, we verify the inequality (1.1) when the condition 'G is finite' is replaced by the weaker condition 'G' is finite'. Furthermore, we introduce an infinite group with finite derived subgroup which satisfies the inequality (1.1).

Burns and Ellis [2] and Moghaddam and Kayvanfar [7] independently introduced the concept of an n-capable group, as follows.

Definition 1.2. Let n be positive integer. A group G is called n-capable if there is a group K such that $G \cong K/Z_n(K)$. A 1-capable group is simply called a capable group.

Burns and Ellis in [2] showed that there exists a 1-capable group which is not 2-capable. In the final section, by invoking a theorem of Fernández-Alcober and Morigi [3], we show that for every capable group G, if $\gamma_{n+1}(G)$ is finite, then so is $G/Z_n(G)$. Using this theorem together with an example from [4], we extend the result of Burns and Ellis [2] by constructing n-capable groups which are not $(n+1)$-capable for every positive integer n. Our method is completely different from that in [2].

2. Some results for groups with finite derived subgroup

Hall in [4] showed that if the derived subgroup G' of a group G is finite, then $[G : Z_2(G)] < \infty$. The first explicit bound was stated by Macdonald [6]. Podoski and Szegedy [8] improved Macdonald's bound and obtained

$$[G : Z_2(G)] \leq |G'|^{c \log_2 |G'|},$$

where c is a constant. For finite groups, they also replaced the logarithm by the rank of the derived subgroup (see [9]). The principal results of [9] are the following theorems.

Theorem 2.1. *If G is a finite group and $\mathrm{rank}(G') = r$, then $[G : Z_2(G)] \leq |G'|^{2r}$.*

Theorem 2.2. *For a finite group G, if $\mathrm{rank}(G'/G' \cap Z(G)) = r$, then $|G/Z_2(G)| \leq |G'/G' \cap Z(G)|^{4r}$.*

Theorem 2.3. *Let G be a finite group with $Z(G) = 1$. Then $|G| \leq |G'|^{d(G')+1}$, where $d(X)$ denotes the minimal number of generators of the group X.*

Our aim in this paper is to obtain such results without assuming that the group is finite. We will need the following lemma.

Lemma 2.4. *A finitely generated nilpotent group G has a normal torsion-free subgroup with finite index.*

Proof. Since G is a finitely generated nilpotent group, it has a central series with cyclic factors, say

$$1 = H_0 \leq H_1 \leq \cdots \leq H_k = G.$$

Set $I = \{i \mid 1 \leq i \leq k,\ H_i/H_{i-1} \text{ is a finite cyclic group}\}$ and $d = \prod_{i \in I} |H_i/H_{i-1}|$. If $N = \langle x^d \mid x \in G \rangle$, then obviously N is a normal subgroup of G in which G/N is a finitely generated nilpotent torsion group and hence finite. To complete the proof, it is enough to show that N is a torsion-free subgroup of G. Let x^d be a nontrivial generator of N. Assume that j is the smallest positive number such that $x \in H_j$ and $x \notin H_{j-1}$. If j is not a member of I, then, for every $n \in \mathbf{N}$, we have $x^n \notin H_{j-1}$, since H_j/H_{j-1} is torsion-free. Therefore, x^n is nontrivial for every $n \in \mathbf{N}$. Now let $j \in I$ and $|H_j/H_{j-1}| = n_j$. Then $x^{n_j} \in H_{j-1}$. If $x^{n_j} = 1$, then $x^d = 1$, since $n_j \mid d$, which is a contradiction. Otherwise, we can replace x^{n_j} by x and repeat the process. This terminates by the definition of d. □

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The following definition will be used in the proof of Theorem 2.6.

Definition 2.5. For a group G, the normal core of a subgroup H, denoted by $\mathrm{Cor}_G(H)$, is the largest normal subgroup of G that is contained in H. Equivalently, the normal core of H in G is the intersection of all conjugates of H in G.

Theorem 2.6. *Let G be a group. If $|G'|$ is finite and $\mathrm{rank}(G') = r$, then $[G : Z_2(G)] \le |G'|^{2r}$.*

Proof. Since $|G'|$ is finite, so is $[G : Z_2(G)]$. Suppose that $G/Z_2(G)$ is generated by $\{x_1Z_2(G), x_2Z_2(G), \ldots, x_kZ_2(G)\}$ and set $H = \langle x_1, x_2, \ldots, x_k\rangle$. It is clear that $G/Z_2(G) \cong H/Z_2(H)$. Since H is finitely generated, $Z_2(H)$ is a finitely generated nilpotent group. By Lemma 2.4, $Z_2(H)$ has a torsion-free normal subgroup N such that $Z_2(H)/N$ is finite. Set $M = \mathrm{Cor}_H(N)$. Clearly, H/M is a finite group. Since H' is finite, we have $\gamma_3(H) \cap M = 1$ and therefore $Z_2(H/M) = Z_2(H)/M$. Thus,

$$\frac{H/M}{Z_2(H/M)} \cong \frac{G}{Z_2(G)}$$

and $(H/M)' \cong H'$. By [8],

$$[H/M : Z_2(H/M)] \le |(H/M)'|^{2r'},$$

where $r' = \mathrm{rank}((H/M)')$. Since $r' \le r$, we have $[G : Z_2(G)] \le |G'|^{2r}$. □

The argument in the proof of Theorem 2.6 also yields the following theorem.

Theorem 2.7. *Suppose that $G = HZ_2(G)$ for some subgroup H of G with H' finite and set $r = \mathrm{rank}(H')$. Then $[G : Z_2(G)] \le |H'|^{2r}$.*

Clearly, $\mathrm{rank}(H') \le \mathrm{rank}(G')$ and so the last result may give a better bound. Also note that the condition $|H'| < \infty$ is weaker than $|G'| < \infty$. The following example illustrates this fact.

Example 2.8. *Consider $G = S_3 \times \prod_{i\in I}(E_1)_i$, where I is an infinite set and E_1 is the extra special p-group of order p^3 and exponent p. Note that G' is infinite and $Z_2(G) = \prod_{i\in I}(E_1)_i$, but S_3' is finite and $[G : Z_2(G)] = |S_3| \le |S_3'|^2$.*

Using Theorem 2.2, we have the following corollary.

Corollary 2.9. *For a capable group H, if H' is finite, then $[H : Z(H)] \le |H'|^{4r}$, where $r = \mathrm{rank}(H')$.*

In the sequel, we need the notion of isoclinism.

Definition 2.10. Let G and H be two groups. An isoclinism from G to H is a pair of homomorphisms (α, β) with $\alpha : G/Z(G) \to H/Z(H)$ and $\beta : G' \to H'$ such that the following diagram is commutative:

$$\begin{array}{ccc} G/Z(G) \times G/Z(G) & \xrightarrow{\alpha^2} & H/Z(H) \times H/Z(H) \\ \downarrow & & \downarrow \\ G' & \xrightarrow{\beta} & H' \end{array}$$

In this situation, G and H are called isoclinic and we use the notation $G \sim H$.

Theorem 2.11. *Suppose that G is a group with trivial Frattini subgroup and finite derived subgroup. Then $[G : Z(G)] \leq |G'|^{d(G')+1}$.*

Proof. For the group G, there is a stem group H, that is, $Z(H) \leq H'$, where $H \sim G$. Observe that H' is finite and thus $[H : Z_2(H)]$ is finite. Furthermore, since $\phi(G) = 1$ and H is a stem group, $Z(H) = 1$ by [5, 3.12]. These facts imply that H is finite and the result follows from Theorem 2.3. □

3. On n-capable groups which are not $(n + 1)$-capable

It is obvious that if G is an $(n + 1)$-capable group, then it is n-capable for every $n \geq 1$. The following question arises naturally: is there an n-capable group which is not $(n + 1)$-capable? An affirmative answer for the case $n = 1$ was given by Burns and Ellis [2]. In this section, we construct an n-capable group which is not $(n + 1)$-capable for every positive integer n. To prove this claim, we need the following theorems.

Theorem 3.1 [3, Theorem A]. *If G is a group such that $[\gamma_{i+1}(G) : \gamma_{i+1}(G) \cap Z_i(G)]$ is finite, then $[G : Z_{2i}(G)]$ is finite.*

Theorem 3.2. *Let G be an n-capable group. If $\gamma_{n+1}(G)$ is finite, then $[G : Z_n(G)]$ is finite.*

Proof. By the n-capability of G, there is a group H such that $G \cong H/Z_n(H)$ and thus

$$\gamma_{n+1}(G) \cong \frac{\gamma_{n+1}(H)}{\gamma_{n+1}(H) \cap Z_n(H)}.$$

Since $\gamma_{n+1}(G)$ is finite, so is $H/Z_{2n}(H)$ and therefore also $[G : Z_n(G)]$ is finite. □

Now we are ready to give an example of an n-capable group which is not $(n + 1)$-capable for every $n \in \mathbf{N}$.

Theorem 3.3. *For every $n \in \mathbf{N}$, there exists an n-capable group which is not $(n + 1)$-capable.*

Proof. If $p = 2$, let F be the central product of a countably infinite number of copies of the quaternion group. For $p > 2$, take F to be the central product of a countably infinite number of copies of the nonabelian group of order p^3 and exponent p. Put $G = F \wr \mathbf{Z}_p$, where '$\wr$' denotes the standard wreath product. Hall [4] proved that G is a nilpotent group of class $2p$ with the following properties.

(i) $Z_i(G) = \gamma_{2p-i+1}(G)$ for every i, $0 \leq i \leq 2p$.
(ii) $Z_i(G)/Z_{i-1}(G)$ is of order p if $1 \leq i \leq p$ and of infinite order for $p + 1 \leq i \leq 2p$.

Now we prove the claim. Let $n \in \mathbf{N}$ and consider a prime number $p > n$. We define $H = G/Z_n(G)$. Obviously, H is an n-capable group. Suppose that H is also an $(n + 1)$-capable group. Then there exists a group K such that $H \cong K/Z_{n+1}(K)$. Assume that

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$p - n = t$. Then $G/Z_{p-1}(G) \cong H/Z_{t-1}(H) \cong K/Z_p(K)$. Since $G/Z_{p-1}(G)$ is p-capable and $\gamma_{p+1}(G/Z_{p-1}(G))$ is finite, it follows that $G/Z_{2p-1}(G)$ is finite by Theorem 3.2. But this contradicts (ii) above. □

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