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## Higher symmetries of powers of the Laplacian and rings of differential operators

T. Levasseur and J. T. Stafford

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# Higher symmetries of powers of the Laplacian and rings of differential operators

T. Levasseur and J. T. Stafford

## ABSTRACT

We study the interplay between the minimal representations of the orthogonal Lie algebra  $\mathfrak{g} = \mathfrak{so}(n + 2, \mathbb{C})$  and the algebra of symmetries  $\mathcal{S}(\square^r)$  of powers of the Laplacian  $\square$  on  $\mathbb{C}^n$ . The connection is made through the construction of a highest-weight representation of  $\mathfrak{g}$  via the ring of differential operators  $\mathcal{D}(X)$  on the singular scheme  $X = (\mathbf{F}^r = 0) \subset \mathbb{C}^n$ , for  $\mathbf{F} = \sum_{j=1}^n X_j^2 \in \mathbb{C}[X_1, \dots, X_n]$ . In particular, we prove that  $U(\mathfrak{g})/K_r \cong \mathcal{S}(\square^r) \cong \mathcal{D}(X)$  for a certain primitive ideal  $K_r$ . Interestingly, if (and only if)  $n$  is even with  $r \geq n/2$ , then both  $\mathcal{S}(\square^r)$  and its natural module  $\mathcal{A} = \mathbb{C}[\partial/\partial X_n, \dots, \partial/\partial X_n]/(\square^r)$  have a finite-dimensional factor. The same holds for the  $\mathcal{D}(X)$ -module  $\mathcal{O}(X)$ . We also study higher-dimensional analogues  $M_r = \{x \in A : \square^r(x) = 0\}$  of the module of harmonic elements in  $A = \mathbb{C}[X_1, \dots, X_n]$  and of the space of ‘harmonic densities’. In both cases we obtain a minimal  $\mathfrak{g}$ -representation that is closely related to the  $\mathfrak{g}$ -modules  $\mathcal{O}(X)$  and  $\mathcal{A}$ . Essentially all these results have real analogues, with the Laplacian replaced by the d’Alembertian  $\square_p$  on the pseudo-Euclidean space  $\mathbb{R}^{p,q}$  and with  $\mathfrak{g}$  replaced by the real Lie algebra  $\mathfrak{so}(p + 1, q + 1)$ .

## Contents

<b>1</b>	<b>Introduction</b>	<b>679</b>
<b>2</b>	<b>Notation</b>	<b>682</b>
<b>3</b>	<b>Differential operators</b>	<b>684</b>
<b>4</b>	<b>The <math>\mathfrak{so}(n + 2, \mathbb{C})</math>-module structure of <math>A/(\mathbf{F}^r)</math></b>	<b>686</b>
<b>5</b>	<b>Rings in category <math>\mathcal{O}</math></b>	<b>692</b>
<b>6</b>	<b>Modules of <math>\mathfrak{so}(n + 2, \mathbb{C})</math>-finite vectors</b>	<b>694</b>
<b>7</b>	<b>Differential operators and primitive factor rings of <math>U(\mathfrak{so}(n + 2, \mathbb{C}))</math></b>	<b>696</b>
<b>8</b>	<b>Higher symmetries of powers of the Laplacian</b>	<b>698</b>
<b>9</b>	<b>The real case</b>	<b>700</b>
<b>10</b>	<b>Harmonic polynomials and <math>\mathcal{O}</math>-duality</b>	<b>703</b>
<b>11</b>	<b>Conformal densities and the ambient construction</b>	<b>707</b>
	<b>Appendix</b>	<b>713</b>
	<b>References</b>	<b>714</b>

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1. Introduction

In this paper we explain a close relationship between two extensively studied subjects: the construction of ‘minimal representations’ of the orthogonal Lie algebra  $\mathfrak{so}(p + 1, q + 1)$  and the ‘algebra of symmetries’ of powers of the Laplacian on the pseudo-Euclidean space  $\mathbb{R}^{p,q}$ . The connection is made through the construction of a highest-weight representation of  $\mathfrak{so}(n + 2, \mathbb{C})$  via differential operators on a singular subscheme of  $\mathbb{C}^n$ . This approach, and its relation to Howe duality, goes back to [LSS88, LS89] (see also [Jos88a]).

Let  $\mathbb{R}[X_1, \dots, X_n]$  be a polynomial ring in  $n \geq 3$  variables over the real numbers, with ring of differential operators (also known as the Weyl algebra)  $A_n(\mathbb{R}) = \mathbb{R}[X_j, \partial_{X_k}; 1 \leq j, k \leq n]$ , where  $\partial_{X_k} = \partial/\partial X_k$ . An important technique for the group-theoretical analysis of the differential equations associated to a differential operator  $\Theta \in A_n(\mathbb{R})$  is to determine the higher symmetries of  $\Theta$ . Here, an operator  $P \in A_n(\mathbb{R})$  is a (higher) symmetry of  $\Theta$  provided that  $\Theta P = P' \Theta$  for some second operator  $P' \in A_n(\mathbb{R})$ . The operator  $P$  is a trivial symmetry when  $P \in A_n(\mathbb{R}) \Theta$ , and factoring the space of symmetries by the trivial ones gives the algebra of symmetries  $\mathcal{S}(\Theta)$  (see § 8 for more details). These symmetries are important to differential equations through exact integrability and separation of variables (see, for example, [BG90, BKM76, Mil77]) and have further applications in general relativity and the theory of higher spin fields (see [Eas05] and the references therein).

Finding these symmetries is, however, nontrivial and has been the focus of considerable research. In this paper we are concerned with the case where  $\Theta = \square_p^r$  is some power of the d’Alembertian  $\square_p = \sum_{j=1}^p \partial_{X_j}^2 - \sum_{j=p+1}^{p+q} \partial_{X_j}^2$  defined on the pseudo-Euclidean space  $\mathbb{R}^{p,q}$ , where  $n = p + q$ , equipped with the metric  $F_p = \sum_{j=1}^p X_j^2 - \sum_{j=p+1}^n X_j^2$ . In this case the algebra of symmetries is known and, moreover, forms a factor algebra of the enveloping algebra  $U(\mathfrak{so}(p + 1, q + 1))$  of the real orthogonal Lie algebra  $\mathfrak{so}(p + 1, q + 1)$ ; see [Eas05] for the first definitive study for  $r = 1$  and [EL08, GS12, BG13, Mic14] for more general results proved using a variety of methods. Earlier results, albeit without full proofs, appear in [ShS92].

We develop a more algebraic approach to this problem that also permits a much more detailed description of the  $\mathfrak{so}(p + 1, q + 1)$ -module structure of the space of symmetries, which in turn has some interesting consequences. In particular, in Theorem 9.8 and Corollary 9.10 we prove the following results.

THEOREM 1.1. (i) For an appropriate primitive ideal  $K_r$  there exists an isomorphism

$$\frac{U(\mathfrak{so}(p + 1, q + 1))}{K_r} \xrightarrow{\sim} \mathcal{S}(\square_p^r),$$

where  $n = p + q$ . In particular,  $\mathcal{S}(\square_p^r)$  is a finitely generated, prime noetherian ring of Goldie rank  $r$  (see Remark 1.4).

(ii) Consider  $\mathcal{A} = \mathbb{R}[\partial_{X_1}, \dots, \partial_{X_n}]/(\square_p^r)$  under its natural  $\mathcal{S}(\square_p^r)$ -module structure. If  $n$  is even with  $r \geq n/2$ , then  $\mathcal{A}$  has a unique proper factor  $\mathcal{S}(\square_p^r)$ -module  $L$ , which is finite-dimensional. Otherwise  $\mathcal{A}$  is an irreducible  $\mathcal{S}(\square_p^r)$ -module.

Remark 1.2. Theorem 9.8 also gives explicit sets of generators for the algebra  $\mathcal{S}(\square_p^r)$ . For the weights of the corresponding highest-weight  $\mathfrak{so}(n + 2, \mathbb{C})$ -module  $\mathcal{A}_{\mathbb{C}} = \mathcal{A} \otimes_{\mathbb{R}} \mathbb{C}$  and its factor  $L_{\mathbb{C}}$ , see Theorems 1.5 and 4.13.

The significance of the finite-dimensional module  $L$  in this theorem (and of its annihilator, which will be an ideal of finite codimension in  $\mathcal{S}(\square_p^r)$ ) needs further study.

Let us now describe our approach. First, results for the real algebras follow easily from the corresponding results for complex algebras, so in the rest of this introduction we concentrate on the latter case; thus we work with the polynomial ring  $A = \mathbb{C}[X_1, \dots, X_n]$  and the Weyl algebra  $A_n(\mathbb{C}) = \mathbb{C}[X_j, \partial_{X_k}; 1 \leq j, k \leq n]$ , and consider the complex algebra of symmetries  $\mathcal{S}(\Delta_1^r)$  for the Laplacian  $\Delta_1 = \sum_{j=1}^n \partial_{X_j}^2$ . Secondly, let  $\mathcal{F}$  be the Fourier transform of the Weyl algebra  $A_n = A_n(\mathbb{C})$  which interchanges the  $X_j$  and  $\partial_{X_j}$ . This maps the Laplacian  $\Delta_1$  onto  $F = \sum_{j=1}^n X_j^2$ . A quick examination of the definitions shows that  $\mathcal{F}$  gives an anti-isomorphism from  $\mathcal{S}(\Delta_1^r)$  onto  $\mathbb{I}(\mathbb{F}^r A_n)/\mathbb{F}^r A_n$ , where  $\mathbb{I}(\mathbb{F}^r A_n) = \{\theta \in A_n : \theta \mathbb{F}^r A_n \subseteq \mathbb{F}^r A_n\}$  is the *idealiser* of  $\mathbb{F}^r A_n$ . Moreover, it is standard that  $\mathbb{I}(\mathbb{F}^r A_n)/\mathbb{F}^r A_n \cong \mathcal{D}(R_r)$ , the ring of differential operators on the commutative ring  $R_r = A/\mathbb{F}^r A$  (see § 3).

Therefore, we need to understand the ring  $\mathcal{D}(R_r)$  and its canonical module  $R_r$ . A remarkable property of the algebra  $R_1$  is that it carries a representation of the orthogonal Lie algebra  $\mathfrak{g} = \mathfrak{so}(n + 2, \mathbb{C})$ , and the starting point of this paper is a result proved in [LSS88, Theorem 4.6]: there exists a maximal ideal  $J_1$ , called the Joseph ideal, in the enveloping algebra  $U(\mathfrak{g})$  such that  $\mathcal{D}(R_1) \cong U(\mathfrak{g})/J_1$ . Via the Fourier transform, this also gives an alternative proof of Eastwood’s result [Eas05] on  $\mathcal{S}(\square_p)$ .

Now suppose that  $r \geq 1$ . The algebra  $R_r$  also carries a representation of  $\mathfrak{g}$  (see § 4), and the main aim of this paper is to use this to describe  $\mathcal{D}(R_r)$ .

**THEOREM 1.3** (See Theorems 7.6 and 8.6). *There exist a primitive ideal  $J_r \subset U(\mathfrak{g})$  and isomorphisms*

$$U(\mathfrak{g})/J_r \xrightarrow{\sim} \mathcal{D}(R_r) \xrightarrow{\sim} \mathcal{S}(\Delta_1^r).$$

*Remark 1.4.* (1) The primitive ideal  $J_r \subset U(\mathfrak{g})$  from Theorem 1.3 is also the annihilator of the  $\mathfrak{g}$ -module  $R_r$ . This representation of  $\mathfrak{g}$  is *minimal* in the sense that the associated variety of  $J_r$  is the closure of the minimal nonzero nilpotent orbit  $\mathbf{O}_{\min}$  in  $\mathfrak{g}$ . At least for  $n > 3$ , the Joseph ideal  $J_1$  is the unique completely prime ideal with this associated variety, which explains why it necessarily appears in this theorem. In contrast, for  $r > 1$  the ideal  $J_r$  is not completely prime; in fact  $U(\mathfrak{g})/J_r$  has Goldie rank  $r$ , in the sense that its simple artinian ring of fractions is an  $r \times r$  matrix ring over a division ring.

(2) The ideal  $K_r$  of Theorem 1.1 is the intersection  $K_r = J_r \cap U(\mathfrak{so}(p + 1, q + 1))$ .

(3) Results like Theorems 1.1 and 1.3 are very sensitive to the precise operator  $\Delta_1$  or  $\square_p$ . For example, if  $F$  is replaced by  $\varphi = \sum X_j^3 \in A = \mathbb{C}[X_1, X_2, X_3]$ , then  $\mathcal{D}(A/\varphi A)$  is neither finitely generated nor noetherian, and even has an infinite ascending chain of ideals [BGG72]. By applying the Fourier transform  $\mathcal{F}$ , the same properties hold for the algebra of symmetries  $\mathcal{S}(\Theta)$ , for  $\Theta = \sum_{j=1}^3 \partial_{X_j}^3 \in A_3$ .

The idea behind the proof of Theorem 1.3 is to relate  $\mathcal{D}(R_r)$  to  $\mathcal{D}(R_1)$  and hence reduce the theorem to the known case of  $U(\mathfrak{g})/J_1 \cong \mathcal{D}(R_1)$ . This is achieved by showing that the set of differential operators  $\mathcal{D}(R_r, R_1)$  from  $R_r$  to  $R_1$  (which naturally relates  $\mathcal{D}(R_r)$  to  $\mathcal{D}(R_1)$ ) equals the module of  $\mathfrak{g}$ -finite vectors  $\mathcal{L}(R_r, R_1)$  that connects  $U(\mathfrak{g})/J_r$  to  $U(\mathfrak{g})/J_1$ . Showing that, in fact,  $\mathcal{D}(R_r, R_1) = \mathcal{L}(R_r, R_1)$  is the key step in the proof of the theorem, since it reduces the problem to understanding algebras in the category  $\mathcal{O}$  of highest-weight  $\mathfrak{g}$ -modules; see §§ 5 and 6 in particular. (We remark that, here and elsewhere, we use the Lie-theoretic notation from [Bou81, Bou90] and [Jan83], although the relevant terms are also defined in the body of the paper.)

Along the way we also obtain a detailed understanding of the  $\mathfrak{g}$ -module structure of  $R_r$ , and this forms the next main result.

**THEOREM 1.5** (See Theorem 4.13). *The  $\mathfrak{g}$ -module  $N(\lambda) = R_r$  is a highest-weight module, with highest weight  $\lambda = (n/2 - r)\varpi_1$  where  $\varpi_1$  is the first fundamental weight of  $\mathfrak{g}$ . Furthermore:*

- (i) *if  $n$  is even with  $r < n/2$  or if  $n$  is odd, then  $N(\lambda) = L(\lambda)$  is irreducible;*
- (ii) *if  $n$  is even with  $r \geq n/2$ , then  $N(\lambda)$  has an irreducible socle  $S$  isomorphic to  $L(\mu)$ , while the quotient  $N(\lambda)/S = L(\lambda)$  is irreducible and finite-dimensional. (The formula for  $\mu$  is given in (4.14).)*

One significant consequence of this proof is that

$$\mathcal{D}(R_r) \text{ is a maximal order in its simple ring of fractions}$$

(this concept is the natural noncommutative analogue of being integrally closed; see also Definition 5.6).

The space of harmonic polynomials  $H = \{p \in A : \Delta_1(p) = 0\}$  and its real analogues are fundamental objects with many applications; see, for example, [HSS12, KØ03] for applications to minimal representations of the conformal group  $O(p+1, q+1)$  and [Bek09, Bek11] for applications in physics. Given that we have been interested in symmetries of powers of the Laplacian, it is therefore natural to consider solutions of the  $r$ th power of the Laplacian  $\Delta_1^r$ . In other words, we are interested in the space of ‘harmonics of level  $r$ ’, defined as

$$M_r = \{f \in A : \Delta_1^r(f) = 0\}.$$

This is the topic of § 10, where we show that  $M_r$  is a  $\mathfrak{g}$ -module, indeed an  $\mathcal{S}(\Delta_1^r)$ -module, which is closely related to the  $\mathcal{D}(R_r)$ -module  $R_r$  through the isomorphism of Theorem 1.3.

**THEOREM 1.6** (See Corollary 10.12).

- (i) *In the category  $\mathcal{O}$  of  $\mathfrak{g}$ -modules,  $M_r$  is isomorphic to the dual  $N(\lambda)^\vee$  of  $N(\lambda) = R_r$ .*
- (ii) *Consequently, if  $n$  is even with  $r < n/2$  or if  $n$  is odd, then  $M_r \cong N(\lambda) \cong L(\lambda)$  is simple.*
- (iii) *If  $n$  is even with  $r \geq n/2$ , then  $M_r$  has an irreducible finite-dimensional socle  $E \cong L(\lambda)$ . The quotient  $M_r/E \cong L(\mu)$  is an irreducible highest-weight module.*

*Remark 1.7.* As was true for Theorem 1.1, there are also analogues of this result for solutions of powers of the d’Alembertian  $\square_p$ , and these are described in Corollary 10.13.

One should observe that the action of the Lie algebra  $\mathfrak{g}$  on  $R_r$  and  $M_r$  is not given by linear vector fields. For instance, one needs differential operators  $P_j$  of order 2 for  $R_r$  and their Fourier transforms  $\mathcal{F}(P_j)$  for the action on  $M_r$ ; see (3.7) and Theorem 9.8. This is similar to the action on the minimal representation of  $\mathfrak{so}(p+1, q+1)$  in its Schrödinger model, as described in [KM11].

The  $\mathfrak{g}$ -module  $M_1$  of harmonic polynomials is also an incarnation of the scalar singleton module introduced by Dirac through the ambient method [Dir35, Dir36, EG91, Bek11]. In this approach one studies the Laplacian on  $\mathbb{R}^n$  via a conformal compactification of  $\mathbb{R}^n$  which appears as a projective quadric  $\mathcal{Q} = \{Q = 0\} \subset \mathbb{RP}^{n+1}$  together with the action of the Laplacian from  $\mathbb{R}^{n+2}$  on densities of a particular weight. Similar questions arise for densities on the ‘generalised light cone’  $\{Q^r = 0\}$  and, by [EG91] and [GJMS92], the ambient method also works here, in this

case for densities of weight  $-n/2 + r$ . This produces an  $\mathfrak{so}(p + 1, q + 1)$ -module, which in the case of the Minkowski space–time corresponds to the higher-order singleton as defined in [BG13].

A more detailed review of this technique is given in the final section, § 11, where we give an algebraic version of the ambient construction and relate it to the  $\mathfrak{g}$ -modules  $R_r$  and  $M_r$  described above. Roughly speaking, in our setting the generalised light cone  $\{Q^r = 0\}$  is replaced by a factor  $B/Q^r B$  of a polynomial ring  $B$  in  $n + 2$  variables, equipped with an appropriate Laplacian  $\Delta \in \mathcal{D}(B)$ , while the densities are replaced by homogeneous polynomials in a finite extension  $S/Q^r S$  of  $B/Q^r B$ .

As such, the Laplacian  $\Delta$  acts on the space  $(S/Q^r S)(-n/2 + r)$  of densities of weight  $-n/2 + r$  and gives the  $\mathfrak{g}$ -module of *harmonic densities*

$$N_\lambda = \left\{ \bar{f} \in (S/Q^r S)\left(-\frac{n}{2} + r\right) : \Delta(f) = 0 \right\}$$

(see Definition 11.7 and Proposition 11.9). With this notation and using the ideas of the ambient method, we are able to relate all the earlier constructions by proving the following result.

**THEOREM 1.8** (See Corollary 11.13). *There are  $\mathfrak{g}$ -module isomorphisms  $N_\lambda \cong M_r \cong N(\lambda)^\vee$ .*

We remark that one consequence of this result is that the algebra of symmetries  $\mathcal{S}(\square_{n-1})$  is isomorphic to the ‘on-shell higher-spin algebra’ of [Bek09, § 3.1.3, Corollary 3]. See Remark 11.14 for the details.

## 2. Notation

Fix an integer  $n \geq 3$  and set  $N = n + 2$ . In this section we fix some notation about the complex simple Lie algebra  $\mathfrak{g} = \mathfrak{so}(n + 2, \mathbb{C})$ . The rank of  $\mathfrak{g}$  will be denoted by  $\text{rk } \mathfrak{g} = \ell = \ell' + 1$ ; thus,  $\mathfrak{g}$  is of type  $B_\ell$  (with  $\ell = n/2 + 1/2$ ) when  $n$  is odd and of type  $D_\ell$  (with  $\ell = n/2 + 1$ ) when  $n$  is even. Note that  $\mathfrak{g} \cong \mathfrak{sl}(4, \mathbb{C})$  when  $n = 4$ .

A convenient presentation of  $\mathfrak{g}$  is given by derivations on the polynomial algebra in  $N$  variables. Hence, let  $U_{\pm 1}, \dots, U_{\pm \ell}$  be  $2\ell$  indeterminates over  $\mathbb{C}$ . If  $n$  is odd we let  $U_0$  denote another indeterminate, and we set  $U_0 = 0$  when  $n$  is even; thus in each case  $B = \mathbb{C}[U_{\pm 1}, \dots, U_{\pm \ell}, U_0]$  is a polynomial ring in  $N$  variables. Write  $\partial_{U_j} = \partial/\partial U_j$  with the convention that  $\partial_{U_0} = 0$  if  $n$  is even.

As in [GW98, Corollaries 1.2.7 and 1.2.9], we identify  $\mathfrak{g}$  with the Lie subalgebra of elements in  $\mathfrak{gl}(N, \mathbb{C}) = \text{End } \mathbb{C}^N$  which preserve the quadratic form

$$Q = \sum_{j=1}^{\ell} U_j U_{-j} + \frac{1}{2} U_0^2.$$

Define differential operators on  $B$  by

$$E = U_0 \partial_{U_0} + \sum_{j=1}^{\ell} (U_j \partial_{U_j} + U_{-j} \partial_{U_{-j}}), \quad \Delta = \partial_{U_0}^2 + 2 \sum_{j=1}^{\ell} \partial_{U_j} \partial_{U_{-j}}, \tag{2.1}$$

and define derivations of  $B$  by setting

$$E_{ij} = U_i \partial_{U_j} - U_{-j} \partial_{U_{-i}} \quad \text{for } i, j = 0, \pm 1, \dots, \pm \ell. \tag{2.2}$$

Observe that  $E_{-i, -j} = -E_{j, i}$ , while  $E_{-i, i} = 0$  and  $E_{ij}(Q) = 0$ . The following result is classical.

PROPOSITION 2.3. (1) The  $\mathbb{C}$ -vector space spanned by the derivations  $E_{ij}$  is a subalgebra of the Lie algebra  $(\text{End}_{\mathbb{C}} B, [\cdot, \cdot])$  that is isomorphic to the orthogonal Lie algebra  $\mathfrak{so}(n + 2, \mathbb{C})$ .

(2) The Lie subalgebra  $\mathfrak{sp}$  of  $(\text{End}_{\mathbb{C}} B, [\cdot, \cdot])$  spanned by  $\Delta, \mathbf{Q}$  and  $-(\mathbf{E} + N/2)$  is isomorphic to  $\mathfrak{sl}(2, \mathbb{C})$ .

*Proof.* Under the identification of  $\mathfrak{gl}(N, \mathbb{C})$  with the space of vector fields spanned by  $\{U_i \partial_{U_j}\}$ , assertion (1) follows from [GW98, § 2.3.1, p. 70], while (2) is in [How85, § 6].  $\square$

We can and therefore will identify  $\mathfrak{g} = \mathfrak{so}(n + 2, \mathbb{C})$  with the Lie algebra spanned by  $\{E_{ij}\}$ . Notice that each of the subalgebras  $\mathfrak{sp}$  and  $\mathfrak{g}$  is contained in the commutant of the other. This is an infinitesimal version of the existence of the dual pair  $(\text{O}(n + 2, \mathbb{C}), \text{Sp}(2, \mathbb{C}))$ ; see [How85, § 6] or [How89, § 3].

The space  $\mathfrak{h} = \bigoplus_{j=1}^{\ell} \mathbb{C}E_{jj}$  is a Cartan subalgebra of  $\mathfrak{g}$  and we set  $\varepsilon_j = E_{jj}^* \in \mathfrak{h}^*$ . Let  $\Phi$  denote the set of roots of  $\mathfrak{h}$  in  $\mathfrak{g}$  and write  $\mathfrak{g}^\alpha$  for the space of root vectors of weight  $\alpha \in \mathfrak{h}^*$ . The set  $\Phi$  is then given by

$$\{\pm(\varepsilon_a \pm \varepsilon_b) : 1 \leq a < b \leq \ell\} \sqcup \{\pm\varepsilon_b : 1 \leq b \leq \ell\} \quad \text{when } n \text{ is odd}$$

and  $\{\pm(\varepsilon_a \pm \varepsilon_b) : 1 \leq a < b \leq \ell\}$  when  $n$  is even. We choose positive roots by setting

$$\Phi^+ = \{\varepsilon_1 \pm \varepsilon_{b+1}\}_{1 \leq b \leq \ell'} \sqcup \{\varepsilon_{a+1} \pm \varepsilon_{b+1}\}_{1 \leq a < b \leq \ell'} \sqcup \{\varepsilon_1\} \sqcup \{\varepsilon_{b+1}\}_{1 \leq b \leq \ell'} \quad \text{if } n \text{ is odd}$$

and  $\Phi^+ = \{\varepsilon_1 \pm \varepsilon_{b+1}\}_{1 \leq b \leq \ell'} \sqcup \{\varepsilon_{a+1} \pm \varepsilon_{a+1}\}_{1 \leq a < b \leq \ell'}$  if  $n$  is even. For more details, see [GW98, § 2.3.1]. In general we will use [Bou81, Bou90] as our basic reference for Lie-theoretic concepts. In particular, the set  $\Phi^+$  is taken from [Bou81, Planches II and IV], and we fix the same basis  $\mathbf{B} = \{\alpha_1, \dots, \alpha_\ell\}$  as described there. Set  $\mathfrak{n}^+ = \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}^\alpha$  and  $\mathfrak{b}^+ = \mathfrak{h} \oplus \mathfrak{n}^+$ , which is a Borel subalgebra of  $\mathfrak{g}$ . A more refined Chevalley system in  $\mathfrak{g}$  is then given in the next proposition; see [Bou90, ch. VIII, § 13] for further details.

PROPOSITION 2.4. The following set  $\{y_\alpha \in \mathfrak{g}^\alpha\}_{\alpha \in \Phi}$  of root vectors is a Chevalley system in  $\mathfrak{g}$  (recall that  $[y_\alpha, y_{-\alpha}] = h_\alpha$ ):

- (a)  $y_{\varepsilon_a - \varepsilon_b} = E_{ab}$  and  $y_{\varepsilon_a + \varepsilon_b} = E_{a,-b}$  for  $1 \leq a < b \leq \ell$ , while  $y_{\varepsilon_b} = \sqrt{2}E_{b,0}$  for  $1 \leq b \leq \ell$ , if  $n$  is odd;
- (b)  $y_{-(\varepsilon_a - \varepsilon_b)} = E_{ba}$  and  $y_{-(\varepsilon_a + \varepsilon_b)} = E_{-b,a}$  for  $1 \leq a < b \leq \ell$ , while  $y_{-\varepsilon_b} = \sqrt{2}E_{0,b}$  for  $1 \leq b \leq \ell$ , if  $n$  is even;
- (c)  $h_{\varepsilon_j \pm \varepsilon_k} = E_{jj} \pm E_{kk}$  for  $1 \leq j < k \leq \ell$ , while  $h_{\varepsilon_j} = 2E_{jj}$  for  $1 \leq j \leq \ell$ , if  $n$  is odd.

In particular,  $\mathfrak{g}$  has a triangular decomposition  $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$  with

$$\mathfrak{n}^+ = \bigoplus_{1 \leq i < j \leq \ell} \mathbb{C}E_{ij} \oplus \bigoplus_{0 \leq i < j \leq \ell} \mathbb{C}E_{i,-j} \quad \text{and} \quad \mathfrak{n}^- = \bigoplus_{1 \leq i < j \leq \ell} \mathbb{C}E_{ji} \oplus \bigoplus_{0 \leq i < j \leq \ell} \mathbb{C}E_{-j,i}.$$

The subspace  $\mathfrak{k} \subset \text{Der}_{\mathbb{C}}(A)$  generated by the derivations  $E_{ij}$  with  $i, j \in \{0, \pm 2, \dots, \pm \ell\}$  is a Lie subalgebra of  $\mathfrak{g}$  isomorphic to  $\mathfrak{so}(n, \mathbb{C})$  and is of type  $\mathbf{B}_{\ell-1} = \mathbf{B}_{\ell'}$  ( $n$  odd) or  $\mathbf{D}_{\ell-1} = \mathbf{D}_{\ell'}$  ( $n$  even). Furthermore,  $\mathfrak{m} = \mathbb{C}E_{11} \oplus \mathfrak{k} \subset \mathfrak{g} \cong \mathfrak{so}(2, \mathbb{C}) \times \mathfrak{so}(n, \mathbb{C})$  and  $\mathfrak{g}$  decomposes as

$$\mathfrak{g} = \mathfrak{r}^- \oplus \mathfrak{p} \quad \text{for } \mathfrak{p} = \mathfrak{m} \oplus \mathfrak{r}^+ \text{ and } \mathfrak{k} = [\mathfrak{m}, \mathfrak{m}],$$

where  $\mathfrak{r}^- = \bigoplus_{p \neq \pm 1} \mathbb{C}E_{p1} \subset \mathfrak{n}^-$  and  $\mathfrak{r}^+ = \bigoplus_{p \neq \pm 1} \mathbb{C}E_{1p} \subset \mathfrak{n}^+$ . Here  $\mathfrak{p}$  is a maximal parabolic subalgebra of  $\mathfrak{g}$  with abelian nilradical  $\mathfrak{r}^+ \cong \mathbb{C}^n$  and Levi subalgebra  $\mathfrak{m}$ ; for more details, see



[LSS88, §3]. In the notation of [LSS88, Table 3.1],  $\mathfrak{p}$  is the parabolic  $\mathfrak{p}_1$  (of type  $B_\ell$  or  $D_\ell$  when  $n$  is odd or even, respectively) except that when  $n = 4$ ,  $\mathfrak{p}$  is the parabolic  $\mathfrak{p}_2$  of type  $A_3$ .

Let  $\varpi_1, \dots, \varpi_\ell$  denote the fundamental weights of  $\mathfrak{g}$  and  $P = \bigoplus_{j=1}^\ell \mathbb{Z}\varpi_j$  the lattice of weights. Set  $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$  and  $P_{++} = \bigoplus_{j=1}^\ell \mathbb{N}^*\varpi_j \subset P_+ = \bigoplus_{j=1}^\ell \mathbb{N}\varpi_j$ . In order to quote results from [Jan83], we will often need to shift by  $\rho = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha = \sum_{j=1}^\ell \varpi_j$ , and we define

$$P^{++} = -\rho + P_{++} = \{\mu \in P : \langle \mu + \rho, \alpha^\vee \rangle > 0, \alpha \in B\} = \{\mu \in P : \langle \mu, \alpha^\vee \rangle \geq 0, \alpha \in B\},$$

$$P^+ = -\rho + P_+ = \{\mu \in P : \langle \mu + \rho, \alpha^\vee \rangle \geq 0, \alpha \in B\} = \{\mu \in P : \langle \mu, \alpha^\vee \rangle \geq -1, \alpha \in B\}.$$

The definition of a Verma module  $M(\mu)$  and its unique simple quotient  $L(\mu)$ , for  $\mu \in \mathfrak{h}^*$ , will be relative to our given triangular decomposition  $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ . Recall that  $L(\mu)$  is a finite-dimensional simple module if and only if  $\mu \in P^{++}$ . Finally, we let  $\mathcal{O}$  denote the category of highest-weight modules as defined in, for example, [Jan83, 4.3].

### 3. Differential operators

We continue the discussion of the Lie algebra  $\mathfrak{so}(n+2, \mathbb{C})$  and its presentation as differential operators, most especially those on the ring  $\mathbb{C}[X_1, \dots, X_n]/(\mathbb{F}^r)$  from the introduction. It is worth emphasising that, in contrast to the previous section, here we start with a polynomial ring in  $n$  rather than  $n+2$  variables. Both approaches will be needed in the paper, but the present approach is more subtle since  $\mathfrak{so}(n+2, \mathbb{C})$  can no longer be presented as an algebra of derivations on  $\mathbb{C}[X_1, \dots, X_n]$ ; one needs to add certain second-order operators. Many of the results in this section come from [Lev86, KM11].

For the moment, let  $\mathbb{K}$  be any field of characteristic 0 and write  $\mathcal{D}(R)$  for the ring of  $\mathbb{K}$ -linear differential operators on a commutative  $\mathbb{K}$ -algebra  $R$ , as defined in [EGA67] for example. More generally, given  $R$ -modules  $M$  and  $N$ , define  $\mathcal{D}(M, N) = \bigcup_{k \geq 0} \mathcal{D}_k(M, N)$  where  $\mathcal{D}_{-1}(M, N) = 0$  and, for  $k \geq 0$ ,

$$\mathcal{D}_k(M, N) = \{\theta \in \text{Hom}_{\mathbb{C}}(M, N) : [x, \theta] \in \mathcal{D}_{k-1}(M, N) \text{ for all } x \in R\}. \tag{3.1}$$

The elements of  $\mathcal{D}_k(M, N)$  are called *differential operators of order at most  $k$* . Set  $\mathcal{D}(M) = \mathcal{D}(M, M)$ .

Fix an integer  $n \geq 3$  and set  $A = \mathbb{K}[X_1, \dots, X_n]$  for commuting indeterminates  $X_j$ . Recall that  $\mathcal{D}(A)$  identifies with the  $n$ th Weyl algebra  $A_n(\mathbb{K}) = \mathbb{K}[X_1, \dots, X_n, \partial_{X_1}, \dots, \partial_{X_n}]$  where, as before,  $\partial_{X_i} = \partial/\partial X_i$ . Given a factor ring  $R = A/\mathfrak{a}$ , the *idealiser* of  $\mathfrak{a}\mathcal{D}(A)$  is defined to be

$$\mathbb{I}_{\mathcal{D}(A)}(\mathfrak{a}\mathcal{D}(A)) = \{P \in \mathcal{D}(A) : P\mathfrak{a}\mathcal{D}(A) \subseteq \mathfrak{a}\mathcal{D}(A)\}. \tag{3.2}$$

As noted in [SS88, Proposition 1.6], this provides the following useful description of  $\mathcal{D}(R)$ :

$$\mathcal{D}(R) \cong \mathbb{I}_{\mathcal{D}(A)}(\mathfrak{a}\mathcal{D}(A))/\mathfrak{a}\mathcal{D}(A). \tag{3.3}$$

Let  $r$  be a positive integer and set

$$\mathbb{F} = \sum_{i=1}^n X_i^2 \in A \quad \text{and} \quad R = R_r = A/\mathbb{F}^r A. \tag{3.4}$$

Define the *Laplacian*  $\Delta_1 \in \mathcal{D}(A)$ , the *Euler operator*  $E_1 \in \mathcal{D}(A)$  and derivations  $D_{ij}$  for  $1 \leq i, j \leq n$  by

$$\Delta_1 = \frac{1}{2} \sum_{i=1}^n \partial_{X_i}^2, \quad E_1 = \sum_{i=1}^n X_i \partial_{X_i}, \quad D_{ij} = -D_{ji} = X_i \partial_{X_j} - X_j \partial_{X_i}. \tag{3.5}$$



(Note the minor differences between these operators and the Euler  $\mathbf{E}$  and Laplacian  $\Delta$  from (2.1).) The next lemma follows from straightforward calculations; see, for example, [Lev86, Proposition 1.2.2].

LEMMA 3.6. For  $1 \leq i, j, k, l \leq n$  one has:

- (i)  $[D_{kl}, X_i] = \delta_{il}X_k - \delta_{ik}X_l$  while  $[D_{ij}, \mathbf{F}] = D_{ij}(\mathbf{F}) = 0$ ;
- (ii)  $[D_{ij}, D_{kl}] = \delta_{jk}D_{il} + \delta_{jl}D_{ki} + \delta_{ik}D_{lj} + \delta_{il}D_{jk}$ .

Let  $d \in \mathbb{K}$  and define second-order differential operators in  $\mathcal{D}(A)$  by

$$P_j(d) = X_j\Delta_1 - (\mathbf{E}_1 + d)\partial_{X_j} = X_j\Delta_1 - \partial_{X_j}(\mathbf{E}_1 + d - 1) \quad \text{for } 1 \leq j \leq n. \tag{3.7}$$

Direct computations (see, in particular, [Lev86, Proposition 1.2.2 and Théorème 2.1.3] or [KM11, Theorem 2.4.1]) yield the following proposition.

PROPOSITION 3.8. (1) For  $1 \leq i, j, k \leq n$ , the elements  $P_j(d)$  satisfy:

- (a)  $[\mathbf{E}_1 + d, P_j(d)] = -P_j(d)$  and  $[P_j(d), \partial_{X_k}] = \partial_{X_j}\partial_{X_k} - \delta_{j,k}\Delta_1$ ;
- (b)  $[X_i, P_j(d)] = D_{i,j} + \delta_{i,j}(\mathbf{E}_1 + d)$  and  $[D_{k,l}, P_j(d)] = \delta_{l,j}P_k(d) - \delta_{k,j}P_l(d)$ ;
- (c)  $[P_i(d), P_j(d)] = 0$ .

- (2) Assume that  $\mathbb{K}$  is algebraically closed. The subspace  $\tilde{\mathfrak{g}}_d$  of  $(\mathcal{D}(A), [,])$  spanned by the elements  $X_i, \mathbf{E}_1 + d, D_{ij}$  and  $P_j(d)$  (for  $1 \leq i, j \leq n$ ) is isomorphic to the Lie algebra  $\mathfrak{so}(n+2, \mathbb{K})$ .
- (3) The subspace of  $\tilde{\mathfrak{g}}_d$  spanned by the  $D_{ij}$  is isomorphic to  $\mathfrak{so}(n, \mathbb{K})$ .

Notation 3.9. For the rest of the section assume (for simplicity) that  $\mathbb{K} = \mathbb{C}$  and fix  $r \in \mathbb{N}^* = \mathbb{N} \setminus \{0\}$ . Recall from § 2 that the (abstract) Lie algebra  $\mathfrak{so}(n+2, \mathbb{C})$  is denoted by  $\mathfrak{g}$ . Let  $\tilde{\psi} = \tilde{\psi}_d : \mathfrak{g} \rightarrow \tilde{\mathfrak{g}}_d \subset \mathcal{D}(A)$  be an isomorphism defined by Proposition 3.8(2); an explicit map can be found in Corollary 4.4.

We now want to choose  $d \in \mathbb{C}$  such that  $P_j(d)(\mathbf{F}^r A) \subseteq \mathbf{F}^r A$ , and hence such that  $P_j(d) \in \mathbb{I}_{\mathcal{D}(A)}(\mathbf{F}^r \mathcal{D}(A))$ .

LEMMA 3.10. Let  $d \in \mathbb{C}$  and  $1 \leq j \leq n$ . Then for any  $k \geq 1$  and function  $u$  of class  $\mathcal{C}^2$  on  $\mathbb{R}^n$  or  $\mathbb{C}^n$ , we have

$$P_j(d)(u\mathbf{F}^k) = P_j(2k + d)(u)\mathbf{F}^k + k(n - 2(k + d))X_j u\mathbf{F}^{k-1}. \tag{3.11}$$

Fix  $r \in \mathbb{N}^*$  and set  $d = d(r) = n/2 - r$ . Then for any  $u \in A$  we have

$$P_j(d)(u\mathbf{F}^r) = P_j(2r + d)(u)\mathbf{F}^r. \tag{3.12}$$

Proof. We first show that for  $k \geq 0$ ,

$$\Delta_1(u\mathbf{F}^k) = \Delta_1(u)\mathbf{F}^k + 2k\mathbf{E}_1(u)\mathbf{F}^{k-1} + k(n + 2(k - 1))u\mathbf{F}^{k-1}. \tag{3.13}$$

To see this, note that  $\partial_{X_j}(u\mathbf{F}^k) = \partial_{X_j}(u)\mathbf{F}^k + 2kX_j u\mathbf{F}^{k-1}$  and hence

$$\partial_{X_j}^2(u\mathbf{F}^k) = \partial_{X_j}^2(u)\mathbf{F}^k + 2kX_j\partial_{X_j}(u)\mathbf{F}^{k-1} + 2k(u\mathbf{F}^{k-1} + X_j\partial_{X_j}(u)\mathbf{F}^{k-1} + 2(k - 1)X_j^2 u\mathbf{F}^{k-2}).$$

Equation (3.13) follows by summing over  $j$ , and (3.11) then follows by an elementary computation. Taking  $d = n/2 - r$  and  $k = r$  in (3.11) immediately gives (3.12).  $\square$

COROLLARY 3.14. Assume that  $r \in \mathbb{N}^*$  and set  $d = n/2 - r$ . Set  $R = R_r = A/(\mathbb{F}^r)$  and let  $\tilde{\mathfrak{g}}_d$  be the Lie subalgebra of  $\mathcal{D}(A)$  defined by Proposition 3.8. Then the elements of  $\tilde{\mathfrak{g}}_d$  induce differential operators on  $R_r$ . This therefore defines an algebra morphism  $\psi_r : U(\mathfrak{g}) \rightarrow \mathcal{D}(R)$ .

*Proof.* Set  $\mathcal{B} = \mathbb{I}_{\mathcal{D}(A)}(\mathbb{F}^r \mathcal{D}(A))$  in the notation of (3.2). Combining Lemmas 3.6 and 3.10 shows that  $\tilde{\mathfrak{g}}_d \subset \mathcal{B}$ ; equivalently, by (3.3), the elements of  $\tilde{\mathfrak{g}}_d$  induce differential operators on  $R_r$ . Now apply Proposition 3.8(2).  $\square$

Remark 3.15. Write  $\text{Ker } \psi_r = J_r$ ; then  $U(\mathfrak{g})/J_r \xrightarrow{\sim} \text{Im}(\psi_r) \subset \mathcal{D}(R)$ . It is easily seen that  $\psi_r(\xi) \neq 0$  for all  $\xi \in \mathfrak{g}$ . We therefore can, and frequently will, identify  $\mathfrak{g}$  with  $\mathfrak{g}_r = \psi_r(\mathfrak{g}) \subset \mathcal{D}(R)$ . In this case the morphism constructed in Corollary 3.14 will be denoted by  $\psi_r : U(\mathfrak{g}) \rightarrow \mathcal{D}(R)$ .

#### 4. The $\mathfrak{so}(n + 2, \mathbb{C})$ -module structure of $A/(\mathbb{F}^r)$

We keep the notation of the last section, in particular (3.4), Notation 3.9 and Remark 3.15. Set  $d = n/2 - r$  where  $n \geq 3$  and  $r \geq 1$ , and write  $P_j = P_j(d)$  in the notation of (3.7). Clearly  $R$  is a module over  $\mathfrak{g} = \mathfrak{so}(n + 2, \mathbb{C})$  via the morphism  $\psi_r$ , and the aim of the current section is to examine this module structure. In particular, we show that  $R$  either is a simple  $\mathfrak{g}$ -module or has a unique finite-dimensional factor module, with the latter occurring if and only if  $n$  is even with  $d \leq 0$ . This proves Theorem 1.6 from the introduction.

We first show that the  $\mathfrak{g}$ -module  $R$  is a highest-weight module for our choice of Cartan and Borel subalgebras of  $\mathfrak{g}$ . The action of an element  $u \in U(\mathfrak{g})$  on  $f \in R$  will be denoted by  $u.f = \psi_r(u)(f)$ . For the rest of this section we will identify  $\mathfrak{g}$  with  $\mathfrak{g}_r \subset \mathcal{D}(R)$  through  $\psi_r$  or, equivalently, identify  $\mathfrak{g}$  with  $\tilde{\mathfrak{g}}_d \subset \mathcal{D}(A)$  through  $\tilde{\psi}_d$ . We begin by making this identification more precise (see Lemma 4.3 and Corollary 4.4).

Let  $i = \sqrt{-1} \in \mathbb{C}$  and define elements of  $\tilde{\mathfrak{g}}_d$  by

$$I(d) = E_1 + d, \quad h_1 = -I(d) \quad \text{and} \quad h_{j+1} = iD_{j,j+\ell'} \quad \text{for } 1 \leq j \leq \ell'.$$

Thus  $\bigoplus_{j=1}^{\ell'} \mathbb{C}h_j \equiv \mathfrak{h}$  is a Cartan subalgebra of  $\tilde{\mathfrak{g}}_d \equiv \mathfrak{g}$ .

Next, observe that the approaches of the last two sections are related by the changes of variables

$$U_{a+1} = X_a - iX_{a+\ell'}, \quad U_{-(a+1)} = X_a + iX_{a+\ell'} \quad \text{for } 1 \leq a \leq \ell'$$

and

$$U_0 = \sqrt{2}X_n \quad \text{when } n \text{ is odd.} \tag{4.1}$$

In this notation,  $F = \sum_{j=2}^{\ell'} U_j U_{-j} + \frac{1}{2}U_0^2$  and  $\Delta_1 = 2 \sum_{j=2}^{\ell'} \partial_{U_j} \partial_{U_{-j}} + \partial_{U_0}^2$  while  $E_1 = U_0 \partial_{U_0} + \sum_{\pm k=2}^{\ell'} U_k \partial_{U_k}$ . As  $\mathfrak{r}^- = \bigoplus_{j=1}^n \mathbb{C}X_j$ , we can and will identify  $A = \mathbb{C}[X_1, \dots, X_n]$  with  $S(\mathfrak{r}^-)$  or, equivalently, with the algebra of polynomial functions on  $\mathfrak{r}^+$ .

We will also need to define a Chevalley system inside  $\tilde{\mathfrak{g}}_d$ . Set

$$V_{a+1} = \frac{1}{2}(P_a - iP_{a+\ell'}), \quad V_{-(a+1)} = \frac{1}{2}(P_a + iP_{a+\ell'}) \quad \text{for } 1 \leq a \leq \ell' \quad \text{and} \quad V_0 = \frac{1}{\sqrt{2}}P_n \quad \text{when } n \text{ is odd.}$$

Observe that  $\bigoplus_j \mathbb{C}P_j = \bigoplus_j \mathbb{C}V_j$ ; it is easy to check that

$$V_j = \frac{1}{2}U_j \Delta_1 - I(d) \partial_{U_{-j}} \quad \text{for } j \in \{0, \pm 2, \dots, \pm \ell'\}. \tag{4.2}$$

The next result follows from straightforward computations.

LEMMA 4.3. *The following set  $\{x_\alpha\}_{\alpha \in \Phi}$  of root vectors is a Chevalley system in  $\tilde{\mathfrak{g}}_d$  or  $\mathfrak{g}_r$  (recall that  $[x_\alpha, x_{-\alpha}] = h_\alpha$ ):*

- (i)  $x_{\varepsilon_{a+1} \pm \varepsilon_{b+1}} = U_{a+1} \partial_{U_{\mp(b+1)}} - U_{\pm(b+1)} \partial_{U_{-(a+1)}}$  for  $1 \leq a < b \leq \ell'$ ;
- (ii)  $x_{-(\varepsilon_{a+1} \pm \varepsilon_{b+1})} = U_{\mp(b+1)} \partial_{U_{a+1}} - U_{-(a+1)} \partial_{U_{\pm(b+1)}}$  for  $1 \leq a < b \leq \ell'$ ;
- (iii)  $x_{\varepsilon_1 \pm \varepsilon_{a+1}} = V_{\pm(a+1)}$  for  $1 \leq a \leq \ell'$ , while  $x_{\varepsilon_1} = \sqrt{2}V_0$ , if  $n$  is odd;
- (iv)  $x_{-(\varepsilon_1 \pm \varepsilon_{a+1})} = U_{\mp(a+1)}$  for  $1 \leq a \leq \ell'$ , while  $x_{-\varepsilon_1} = \sqrt{2}U_0$ , if  $n$  is odd;
- (v)  $x_{\varepsilon_{b+1}} = \sqrt{2}(U_{b+1} \partial_{U_0} - U_0 \partial_{U_{-(b+1)}})$  and  $x_{-\varepsilon_{b+1}} = \sqrt{2}(U_0 \partial_{U_{b+1}} - U_{-(b+1)} \partial_{U_0})$  for  $1 \leq b \leq \ell'$ , if  $n$  is odd;
- (vi) then  $h_{\varepsilon_j \pm \varepsilon_k} = h_j \pm h_k$  for  $1 \leq j < k \leq \ell$ , while  $h_{\varepsilon_j} = 2h_j$  for  $1 \leq j \leq \ell$ , if  $n$  is odd.

In particular, one can identify  $\mathfrak{r}^+$  with  $\bigoplus_j \mathbb{C}P_j = \bigoplus_j \mathbb{C}V_j$ .

COROLLARY 4.4. *Retaining the notation of Proposition 2.4 and Lemma 4.3, the map  $\tilde{\psi}_d$  from  $\mathfrak{g} \subset \mathcal{D}(B)$  onto  $\tilde{\mathfrak{g}}_d \subset \mathcal{D}(A)$ , given by  $\tilde{\psi}_d(y_\alpha) = x_\alpha$  for  $\alpha \in \Phi$ , is an isomorphism of Lie algebras.*

Remark 4.5. Recall that a Chevalley system in a semi-simple Lie algebra defines a Chevalley anti-involution [Jan83, 2.1]. Let  $\vartheta$  (respectively  $\vartheta_d$ ) be the anti-involution of  $\mathfrak{g}$  (respectively  $\tilde{\mathfrak{g}}_d$ ) defined by  $\{y_\alpha\}_\alpha$  (respectively  $\{x_\alpha\}_\alpha$ ). Then, by definition, the isomorphism  $\tilde{\psi}_d$  satisfies  $\tilde{\psi}_d \circ \vartheta = \vartheta_d \circ \tilde{\psi}_d$ .

Let  $G = \text{SO}(n+2, \mathbb{C}) \supset M$  be connected algebraic groups such that  $\text{Lie}(G) = \mathfrak{g} \supset \text{Lie}(M) = \mathfrak{m}$ , with  $G$  acting on  $\mathfrak{g}$  through the adjoint action. The (reduced) quadratic cone  $Z = \{\mathbf{F} = 0\} \subset \mathbb{C}^n$  can be identified with the closure of the  $M$ -orbit of a highest-weight vector  $x_{\tilde{\alpha}} \in \mathfrak{r}^+ \cong \mathbb{C}^n$ , and this then identifies  $R_1 = A/FA$  with the algebra of regular functions on  $Z$ . Recall that the orbit  $\mathbf{O}_{\min} = G.x_{\tilde{\alpha}}$  is the nonzero nilpotent orbit of minimal dimension and that  $\frac{1}{2} \dim \mathbf{O}_{\min} = \dim M.x_{\tilde{\alpha}} = n - 1$ .

Recall that the Verma module  $M(\mu)$  with highest weight  $\mu \in \mathfrak{h}^*$  has a unique simple quotient  $L(\mu)$ , and set  $I(\mu) = \text{ann}_{U(\mathfrak{g})} L(\mu)$ . The associated variety  $\mathcal{V}(J) \subset \mathfrak{g}$  of  $J = I(\mu)$  is the closure of a nilpotent orbit; cf. [Jos85]. Let  $W$  be the associated Weyl group of  $\Phi$  and denote by  $w \cdot \lambda = w(\lambda + \rho) - \rho$  the ‘dot’ action of  $w \in W$  on  $\lambda \in \mathfrak{h}^*$ ; cf. [Jan83, 2.3].

The starting point for this paper is the following result from [LSS88] (see also [Lev86]), which gives a construction of the Joseph ideal. (See [Jos88a] for another proof of this theorem.)

THEOREM 4.6. *The ideal  $J_1 = \text{Ker } \psi_1$  is a completely prime maximal ideal such that  $\mathcal{V}(J_1) = \overline{\mathbf{O}}_{\min}$ . Moreover,  $U(\mathfrak{g})/J_1 \cong \mathcal{D}(R_1)$ .*

Recall (see [Jos76]) that if  $\mathfrak{s}$  is a complex simple Lie algebra not of type  $A_\ell$ , then there exists a unique completely prime ideal  $J \subset U(\mathfrak{s})$  such that  $\mathcal{V}(J) = \overline{\mathbf{O}}_{\min}$ . The primitive ideal  $J$  is called the *Joseph ideal*. Thus, when  $n \neq 4$ , the ideal  $J_1$  is the Joseph ideal. When  $n = 4$ , i.e. when  $\mathfrak{g}$  is of type  $D_3 = A_3$ , with a slight abuse of notation we will still call  $J_1$  the Joseph ideal.

Remark 4.7. The ideal  $K_1$  from Theorem 1.1 is the intersection of  $J_1$  with  $U(\mathfrak{so}(p+1, q+1))$ . Eastwood also notes this fact, but his argument relies on an explicit set of generators for the Joseph ideal (see [ESS05, Str08]). In contrast, in Theorem 4.6 this is nearly automatic; it follows almost immediately from the fact that  $R_1$  has Krull dimension  $n - 1$ ; see, for example, [LSS88, Proposition 3.5].

In the general case,  $r \geq 1$ , we begin with an easy lemma.

LEMMA 4.8. *The  $\mathfrak{g}$ -module  $R = R_r$  is a highest-weight module with highest weight  $\lambda = -d\varpi_1 = -d\varepsilon_1$ .*

Remark 4.9. We typically write  $R = N(\lambda)$  when thinking of  $R$  as a  $\mathfrak{g}$ -module.

Proof. Since  $R \subset \text{Im}(\psi_r) \subset \mathcal{D}(R)$ , one has  $R = U(\mathfrak{g}).1$ . By construction,  $D_{s,t}.1 = P_j.1 = 0$  and so the choice of Chevalley system in Lemma 4.3 implies that  $\mathfrak{n}^+.1 = 0$ . Moreover,  $h_j.1 = 0$  for  $2 \leq j \leq \ell$  whereas  $h_1.1 = -I(d).1 = -d.1 = -d\varepsilon_1(h_1).1$ . The choice of  $\mathcal{B} \subset \Phi^+$  yields  $\varpi_1 = \varepsilon_1$ ; see [Bou81, Planche II, pp. 252–253, and Planche IV, pp. 256–257]. Hence  $R$  is a highest-weight module with highest weight  $\lambda = -d\varpi_1$ .  $\square$

We need to analyse the structure of the  $\mathfrak{g}$ -module  $N(\lambda)$  in more detail, for which we need some notation. Grade  $A = \bigoplus_{m \geq 0} A(m)$  by total polynomial degree (denoted by  $\text{deg}$ ). Let  $H = \{p \in A : \Delta_1(p) = 0\}$  be the space of harmonic polynomials; it is graded and we set  $H(m) = A(m) \cap H$ . The algebraic group  $K = \text{SO}(\mathbb{C}^n, \mathbb{F}) \cong \text{SO}(n, \mathbb{C})$  acts naturally on  $A$  through the identification of  $A$  with the ring of regular functions of the standard representation  $\mathbb{C}^n$  of  $K$ . The differential of this  $K$ -action is given by the natural representation of

$$\mathfrak{k} \cong \bigoplus_{1 \leq s < t \leq n} \mathbb{C}D_{s,t} \cong \mathfrak{so}(n, \mathbb{C}) = \mathfrak{so}(2\ell', \mathbb{C}).$$

It is clear that  $\mathbb{C}\mathbb{F}^k$  is the trivial  $\mathfrak{k}$ -module for all  $k$  and that each  $H(m)$  is a  $\mathfrak{k}$ -module. Notice that  $\mathfrak{h}' = \bigoplus_{j=2}^{\ell} \mathbb{C}h_j$  is a Cartan subalgebra of  $\mathfrak{k}$  and set  $\mathfrak{n}' = \bigoplus_{1 \leq a < b \leq \ell'} \mathfrak{g}^{\varepsilon_{a+1} \pm \varepsilon_{b+1}} \subset \mathfrak{k}$ . Then  $\mathfrak{b}' = \mathfrak{h}' \oplus \mathfrak{n}'$  is a Borel subalgebra of  $\mathfrak{k}$ . The following result is classical; see, for example, [GW98, § 5.2.3].

THEOREM 4.10. *Each space  $H(k)$  is an irreducible  $\mathfrak{k}$ -module, of highest weight  $k\varpi_1$  (in Bourbaki’s notation for type  $B_{(n-1)/2}$  or  $D_{n/2}$ ). The  $\mathfrak{k}$ -module  $A(m)$  decomposes as the direct sum of irreducible modules:*

$$A(m) = \sum_{k=0}^{\lfloor m/2 \rfloor} H(m - 2k)\mathbb{F}^k \cong \bigoplus_{k=0}^{\lfloor m/2 \rfloor} H(m - 2k) \otimes \mathbb{C}\mathbb{F}^k.$$

Let  $B$  be the polar form of  $\mathbb{F}$  and let  $\xi \in \mathbb{C}^n$  be a nonzero isotropic vector with respect to  $\mathbb{F}$ . If  $\xi^k \in A(k)$  is the function given by  $v \mapsto B(v, \xi)^k$ , then  $H(k) = U(\mathfrak{k}).\xi^k$ .

If  $(v_1, \dots, v_n)$  is the canonical basis of  $\mathbb{C}^n$ , the vector  $\xi = v_1 - iv_\ell$  is isotropic and the associated polynomial function  $\xi^k$  is simply  $U_2^k = (X_1 - iX_\ell)^k$ . As observed in Lemma 4.3,  $\xi$  considered as an element of  $\mathfrak{g}$  has weight  $-\varepsilon_1 + \varepsilon_2$  under the adjoint action of  $\mathfrak{g}$ . It follows from the choice of the Borel subalgebra  $\mathfrak{b}' = \mathfrak{h}' \oplus \mathfrak{n}'$  of  $\mathfrak{k}$  that  $\xi^k \in H(k)$  is a highest-weight vector. We will choose this generator for the simple finite-dimensional  $\mathfrak{k}$ -module  $H(k) \cong L(k\varpi'_1)$  (where  $\varpi'_1 = \varepsilon_2|_{\mathfrak{b}'}$  is the first fundamental weight for  $\mathfrak{k}$ ). Observe the following consequence of Theorem 4.10.

COROLLARY 4.11. *Let  $0 \leq v \leq t$  and  $p = \sum_{j=v}^t p_j \mathbb{F}^j \in A(m)$ , where  $p_j \in H(m - 2j)$  for each  $j$ . Then, for each  $k \in \{v, \dots, t\}$ , there exists  $u \in U(\mathfrak{k})$  such that  $p_k \mathbb{F}^k = u.p$ .*

Proof. Since  $\bigoplus_{j=v}^t H(m - 2j)\mathbb{F}^j$  is a multiplicity-free, semi-simple  $U(\mathfrak{k})$ -module, the claim follows from Jacobson’s density theorem; see [Bou73, § 4, no. 2, Corollaire 2].  $\square$

Let  $0 \neq p \in A(m)$ . By Theorem 4.10 we may uniquely write  $p = \sum_{k=v}^t p_k \mathbf{F}^k$  with  $p_k \in H(m - 2k)$ ,  $v \leq t$  and  $p_t \neq 0 \neq p_v$ . The integer  $v = v(p)$  will be called the *harmonic valuation* of  $p$ . Let  $P_j(d')$  be as in (3.11) (with  $d' \in \mathbb{C}$  arbitrary). For  $a \in H(m - 2k)$  one has  $P_j(d')(a) = -(k + d')\partial_{X_j}(a) \in H(m - 2k - 1)$  (with the convention that  $H(j) = 0$  when  $j < 0$ ). Also,  $\Delta_1(X_j a) = [\Delta_1, X_j](a) + X_j \Delta_1(a) = \partial_{X_j}(a)$ . Therefore, by Lemma 3.10 with  $d = n/2 - r$ , we obtain

$$P_j(d)(a\mathbf{F}^k) = P_j(2k + d)(a)\mathbf{F}^k + k(n - 2k - 2d)X_j a\mathbf{F}^{k-1}$$

with  $P_j(2k + d)(a) \in H(m - 2k - 1)$  and  $\Delta_1(X_j a) = \partial_{X_j}(a)$ . In particular, this gives the following result.

LEMMA 4.12. (1) If  $\partial_{X_j}(a) = 0$  for  $a \in A$ , then  $P_j(d)(a\mathbf{F}^k) = P_j(2k + d)(a)\mathbf{F}^k + k(n - 2k - 2d)X_j a\mathbf{F}^{k-1}$ . If in addition  $a \in H(m - 2k)$ , then  $P_j(2k + d)(a) \in H(m - 2k - 1)$  and  $X_j a \in H(m - 2k + 1)$ .

(2) If  $a = \xi^{m-2k}$  with  $\xi = v_1 - iv_\ell$ , and  $j \neq 1, \ell$ , then the condition  $\partial_{X_j}(a) = 0$  is satisfied.

The structure of the  $\mathfrak{g}$ -module  $N(\lambda) = R_r$  is given by the next result. Recall that  $\lambda = -d\varpi_1$ .

THEOREM 4.13. (1) Assume that either  $n$  is even with  $r < n/2$  or  $n$  is odd. Then  $R_r = N(\lambda) \cong L(\lambda)$  is a simple  $\mathfrak{g}$ -module.

(2) Assume that  $n$  is even with  $r \geq n/2$ , so that  $d = n/2 - r \leq 0$ . Then, as a  $\mathfrak{g}$ -module,  $R_r = N(\lambda)$  has a simple socle  $Z_r \cong L(\mu)$ , where

$$\mu = \lambda + (d - 1)\alpha_1 = \begin{cases} (d - 2)\varpi_1 + (1 - d)\varpi_2 & \text{when } \ell \geq 4, \\ (d - 2)\varpi_1 + (1 - d)(\varpi_2 + \varpi_3) & \text{when } \ell = 3. \end{cases} \tag{4.14}$$

The quotient  $N(\lambda)/Z_r \cong L(\lambda)$  is an irreducible finite-dimensional  $\mathfrak{g}$ -module, isomorphic to the module of harmonic polynomials of degree  $-d$  in  $n + 2$  variables. Also,  $Z_r$  is the ideal of  $R_r$  generated by  $H(1 - d)$ .

*Proof.* Let  $0 \neq \overline{M} \subseteq N(\lambda)$  be a  $\mathfrak{g}$ -submodule. Since  $R_r \subset \text{Im}(\psi_r)$ , the module  $\overline{M}$  is also an ideal of  $R_r$ . Therefore  $\overline{M} = M/(\mathbf{F}^r)$  for some  $\tilde{\mathfrak{g}}_d$ -stable ideal  $M$  of  $A$ . In particular, by Proposition 3.8,  $(E_1 + d)(M) \subseteq M$ , whence  $E_1(M) \subseteq M$ , and so  $M = \bigoplus_m M \cap A(m)$  is homogeneous.

Let  $p \in M \cap A(m)$  with  $p \notin (\mathbf{F}^r)$ , and write  $p = \sum_{k=v}^t p_k \mathbf{F}^k$  for  $p_k \in H(m - 2k)$  with  $p_v \neq 0 \neq p_t$ . Note that  $t < r$ . We want to simplify our choice of  $p$ . Recall that  $\mathfrak{g} \supset \mathfrak{k} \cong \mathfrak{so}(n, \mathbb{C})$ . Thus, by Corollary 4.11,  $p_k \mathbf{F}^k \in M \cap H(m - 2k)\mathbf{F}^k$  for all  $k$  and so, as  $H(m - 2k)\mathbf{F}^k$  is an irreducible  $\mathfrak{k}$ -module,  $H(m - 2k)\mathbf{F}^k \subset M$  for all  $k$  such that  $p_k \neq 0$ . In particular,  $H(m - 2v)\mathbf{F}^v \subset M$  and we can replace  $p$  by  $p_v \mathbf{F}^v$ . There exists  $u \in U(\mathfrak{k})$  such that  $u.p_v = \xi^{m-2v} = (X_1 - iX_\ell)^{m-2v}$ . From  $D_{s,t}.\mathbf{F}^k = 0$  we then deduce that  $u.p = (u.p_v)\mathbf{F}^v \in M$  and, upon replacing  $p$  by  $u.p$ , we may therefore assume that  $p_v = \xi^{m-2v}$ . Finally, we assume that the harmonic valuation  $v = v(p)$  is as small as possible among such choices of  $p$ .

Now apply the operator  $P_{\ell+1} = P_{\ell+1}(d)$  to  $p$ ; from Lemma 4.12 and  $\partial_{X_{\ell+1}}(p_v) = 0$  we obtain

$$P_{\ell+1}.p = (P_{\ell+1}(2v + d).p_v)\mathbf{F}^v + v(n - 2v - 2d)X_{\ell+1}p_v\mathbf{F}^{v-1} \tag{4.15}$$

where  $P_{\ell+1}(2v + d).p_v \in H(m - 2v - 1)$  and  $X_{\ell+1}p_v \in H(m - 2v + 1)$ . Since  $P_{\ell+1}.p \in M \setminus (\mathbf{F}^r)$  and  $X_{\ell+1}p_v \neq 0$ , the minimality of  $v$ , combined with (4.15), forces  $v(n - 2v - 2d) = 0$ . In other words, either  $v = 0$  or  $n - 2v - 2d = 0$ . The latter is equivalent to  $v = r$ , which is excluded. Thus we have proved that  $M$  contains a nonzero element  $p = p_0 \in H(m)$ . In particular,  $H(m)A \subseteq M$ .

Now pick  $0 \neq p = \xi^m \in M \cap H(m)$  with  $m$  minimal. As  $p$  is homogeneous, if  $m = 0$  then  $1 \in M$  and  $\overline{M} = N(\lambda)$ . In the  $m \geq 1$  case, applying  $P_1 = X_1\Delta_1 - (\mathbf{E}_1 + d)\partial_{X_1}$  gives

$$P_1.\xi^m = -m(\mathbf{E}_1 + d)(\xi^{m-1}) = -m(m - 1 + d)\xi^{m-1} \in M \cap H(m - 1).$$

The minimality of  $m$  then implies that  $m(m - 1 + d) = 0$ , whence  $m - 1 + d = 0$ . Thus, we have shown that  $\overline{M} = N(\lambda)$  unless  $d$  satisfies  $d = -m + 1$  for some  $m \geq 1$ . In the latter case  $0 \geq d = n/2 - r$ . Since  $r, m \in \mathbb{N}$ , this is equivalent to  $r$  being even with  $r \geq n/2$ . This proves (1).

It remains to prove (2), where  $r = n/2 + m - 1$  for  $m = 1 - d \geq 1$ . Note that, in this case, we have shown that every nonzero  $\mathfrak{g}$ -submodule  $\overline{M}$  of  $R_r = N(\lambda)$  contains the image  $S$  of the ideal  $\mathfrak{J} = H(m)A$ .

Since  $\lambda = -d\varpi_1 = (m - 1)\varpi_1 \in \mathbf{P}^{++}$  is a dominant integral weight,  $L(\lambda)$  is finite-dimensional. Moreover, it is isomorphic to the  $\mathfrak{g}$ -module of harmonic polynomials of degree  $m - 1$  in  $n + 2$  variables (apply Theorem 4.10 to  $\mathfrak{so}(n + 2, \mathbb{C})$ ).

Since  $D_{s,t}(H(m)) \subseteq H(m)$  and  $\mathfrak{J}$  is homogeneous, certainly  $\Theta.\mathfrak{J} \subseteq \mathfrak{J}$  for  $\Theta = D_{s,t}, X_j$  and  $(\mathbf{E}_1 + d)$ . Therefore, in order to show that  $\mathfrak{J}$  is a  $\mathfrak{g}$ -module, it remains to prove that  $P_j.\mathfrak{J} \subseteq \mathfrak{J}$  or, equivalently,  $P_j.X^\alpha f \in \mathfrak{J}$  for any  $f \in H(m)$  and monomial  $X^\alpha = X_1^{\alpha_1} \cdots X_n^{\alpha_n}$ . We argue by induction on  $|\alpha| = \sum_i \alpha_i$ . If  $|\alpha| = 0$ , we have

$$P_j.f = -(m - 1 + d)\partial_{X_j}(f) = 0 \tag{4.16}$$

by the choice of  $m$ . If  $|\alpha| \geq 1$ , write  $X^\alpha = X_k X^\beta$  for  $|\beta| = |\alpha| - 1$ . Then

$$P_j.X^\alpha f = [P_j, X_k].X^\beta f + X_k P_j.X^\beta f.$$

By induction,  $P_j.X^\beta f \in \mathfrak{J}$ , while  $[P_j, X_k] = D_{j,k} - \delta_{j,k}(\mathbf{E}_1 + d)$  from Proposition 3.8(b), which implies that  $[P_j, X_k].X^\beta f \in \mathfrak{J}$ . Hence  $P_j.X^\alpha f \in \mathfrak{J}$  and  $\mathfrak{J}$  is indeed a  $\mathfrak{g}$ -module. Since  $H(m) \neq A$  and  $H(m) \not\subseteq (\mathbb{F}^r)$ , the image  $S$  of  $\mathfrak{J}$  in  $R_r$  is a nontrivial  $\mathfrak{g}$ -submodule of  $R_r = N(\lambda)$ . However, we have already noted that  $S$  is contained in every nonzero  $\mathfrak{g}$ -submodule  $\overline{M} \subseteq N(\lambda)$ ; in other words,  $S = \text{Soc}(N(\lambda)) = Z_r$ .

Recall that  $H(m)$  contains all the functions  $\zeta^m$  associated to the isotropic vectors  $\zeta \in \mathbb{C}^n$ . Since  $\mathbb{C}^n$  has a basis  $(\zeta_j)_j$  of such vectors, the ideal  $\sum_{j=1}^n \zeta^m A$  has finite codimension in  $A$ . Therefore,  $\dim R_r/Z_r \leq \dim A/H(m)A < \infty$ . Thus, on the one hand,  $N(\lambda)/Z_r$  is finite-dimensional and hence completely reducible; but on the other hand, as it is a factor of the Verma module  $M(\lambda)$ , it has a unique simple quotient. This forces  $N(\lambda)/Z_r \cong L(\lambda)$ .

Finally, we need to compute the highest weight of  $Z_r$ . Recall that the  $\mathfrak{k}$ -module  $H(m)$  is generated by the function  $\xi^m$  associated to the isotropic vector  $\xi = v_1 - iv_\ell \in \mathbb{C}^n$ . We have shown in (4.16) that  $P_j.\xi^m = 0$  for all  $j$  and hence  $\mathfrak{r}^+.\xi^m = 0$ . Since  $\xi^m = U_2^m$ , Lemma 4.3 shows that  $x_{\varepsilon_{a+1} \pm \varepsilon_{b+1}}.\xi^m = 0$  for  $1 \leq a < b \leq \ell'$ . Hence  $\mathfrak{n}'.\xi^m = 0$  (recall that  $n$  is even) and so  $\mathfrak{n}^+.\xi^m = 0$ . Moreover, from our choice of  $h_j = iD_{j,j+\ell'}$  one obtains

$$h_1.\xi^m = -\xi^m, \quad h_2.\xi^m = m\xi^m \quad \text{and} \quad h_j.\xi^m = 0 \text{ for } j \geq 3.$$

Thus  $\xi^m \in Z_r$  is a highest-weight vector, with weight  $\mu = -\varepsilon_1 + m\varepsilon_2$ . Hence  $Z_r \cong L(\mu)$ . That  $\mu$  equals  $\lambda - m\alpha_1$  and satisfies (4.14) are easy exercises using [Bou81, Planche IV].  $\square$

Recall that  $s_{\alpha_1} \in W$  is the transposition (1, 2). In the notation of Theorem 4.13, an easy calculation shows that  $\mu = \lambda + (d - 1)\alpha_1 = s_{\alpha_1} \cdot \lambda$ . We can therefore unify the cases in that result by defining

$$\omega = \begin{cases} s_{\alpha_1} \cdot \lambda = \lambda - \left(r - \frac{n}{2} + 1\right)\alpha_1 & \text{if } n \text{ is even with } r \geq \frac{n}{2}, \\ \lambda & \text{otherwise.} \end{cases}$$



Write  $\text{GKdim } M = \text{GKdim}_U M$  for the *Gelfand–Kirillov dimension* of a module  $M$  over an algebra  $U$ , whenever it is defined; see [MR00, § 8.1.11] for more details.

COROLLARY 4.17. (1) *The ideal  $J_r$  is equal to  $I(\omega) = \text{ann}_{U(\mathfrak{g})} L(\omega)$ .*

(2) *The associated variety of  $J_r$  is  $\mathcal{V}(J_r) = \overline{\mathbf{O}}_{\min}$ .*

*Proof.* (1) By Theorem 4.13,  $Z_r = \text{Soc } N(\lambda) \cong L(\omega)$  while  $N(\lambda)/Z_r$  is finite-dimensional (possibly zero). Hence, by [Jan83, Lemma 8.14],  $\text{ann}_{U(\mathfrak{g})} Z_r = \text{ann}_{U(\mathfrak{g})} N(\lambda)$ ; equivalently,  $J_r = I(\lambda) = I(\omega)$ .

(2) Since  $\text{GKdim } L(\omega) = \text{GKdim } R_r = n - 1$ , [Jan83, Satz 10.9] implies that  $\text{GKdim } U(\mathfrak{g})/J(\omega) = 2(n - 1)$ . Thus  $\mathcal{V}(J_r) = \overline{\mathbf{O}}$  for a nilpotent orbit  $\mathbf{O}$  with  $\dim \mathbf{O} = 2(n - 1)$ . This forces  $\mathbf{O} = \mathbf{O}_{\min}$ . □

The ideal  $J_r$  contains the kernel of the character  $\chi_\lambda$  in the centre of  $U(\mathfrak{g})$ . Since  $\lambda = -d\varpi_1 \in \mathbb{Q}\varpi_1$ , there exists a unique weight  $\nu$  such that  $\nu \in W(\lambda + \rho)$ , and  $\langle \nu, \alpha^\vee \rangle \geq 0$  for all simple roots  $\alpha$ ; equivalently,  $\nu$  is in the dominant chamber  $\overline{\mathbf{C}}$  for the given choice of positive roots  $\Phi^+$ . Moreover,  $\chi_\lambda = \chi_{\nu - \rho}$  since  $w \cdot \lambda = \nu - \rho$ . For these and related Lie-theoretic facts, see [Bou81, Bou90], especially [Bou81, ch. V.3.3, VI.1.10] and [Bou90, ch. VIII.8.5].

PROPOSITION 4.18. *Let  $\lambda = -d\varpi_1$ , so that  $\lambda + \rho = (1 - d)\varpi_1 + \varpi_2 + \cdots + \varpi_\ell$ .*

(1) *If  $d \leq 1$ , then the weight  $\nu$  in the dominant chamber is given by  $\nu = \lambda + \rho \in \overline{\mathbf{C}}$ .*

(2) *Assume that  $n$  is odd ( $\mathfrak{g}$  is of type  $B_\ell$ ) and  $d > 1$  (i.e.  $1 \leq r \leq \ell - 2 = (n - 3)/2$ ). Then*

$$\nu = \varpi_1 + \varpi_2 + \cdots + \varpi_{\ell-r-2} + \frac{1}{2}\varpi_{\ell-r-1} + \frac{1}{2}\varpi_{\ell-r} + \varpi_{\ell-r+1} + \cdots + \varpi_\ell \in \overline{\mathbf{C}} \cap W(\lambda + \rho).$$

(3) *Assume that  $n$  is even ( $\mathfrak{g}$  is of type  $D_\ell$ ) and  $d > 1$  (i.e.  $1 \leq r \leq \ell - 3 = (n - 4)/2$ ). Then*

$$\nu = \varpi_1 + \varpi_2 + \cdots + \varpi_{\ell-r-2} + \varpi_{\ell-r} + \varpi_{\ell-r+1} + \cdots + \varpi_\ell \in \overline{\mathbf{C}} \cap W(\lambda + \rho).$$

*Proof.* (1) This is obvious.

(2) Observe that  $1 - d < 0 \iff r < n/2 - 1 \iff r \leq \ell - 2 = (n - 3)/2$ . Let  $w^{-1}$  be the cycle  $(\ell - r, \ell - r - 1, \dots, 2, 1) \in W$ . Then  $\nu = w^{-1}(\lambda + \rho)$  has the desired form.

(3) Here,  $1 - d < 0 \iff r < \ell - 2$ . If  $w^{-1}$  is the cycle  $(\ell - r - 1, \ell - r - 2, \dots, 2, 1) \in W$ , the weight  $\nu = w^{-1}(\lambda + \rho)$  has the desired form. □

COROLLARY 4.19. (i) *If  $n$  is even with  $r \geq n/2$ , then  $J_r$  is not maximal. The unique primitive ideal containing  $J_r$  is the finite-codimensional (maximal) ideal  $I((r - n/2)\varpi_1)$ .*

(ii) *In all other cases,  $J_r$  is maximal.*

*Proof.* Since  $\mathcal{V}(J_r) = \overline{\mathbf{O}}_{\min}$ , any proper factor of  $U(\mathfrak{g})/J_r$  must be finite-dimensional, and that factor then has to be unique. On the other hand, if  $U(\mathfrak{g})/J_r$  is a simple ring, then it cannot have a nonzero finite-dimensional module. Thus part (i) already follows from Theorem 4.13.

Conversely, suppose that  $J_r = I(\lambda)$  is not maximal; then it has a finite-dimensional quotient. This implies that there exists  $\nu \in \mathbf{P}_{++} \subset \overline{\mathbf{C}}$  such that  $\nu \in W(\lambda + \rho)$  (see [Bou90, ch. VIII.7.2, Corollaire 1, and VIII.8.5, Corollaire 1]). Proposition 4.18 gives the unique weight  $\nu$  in  $W(\lambda + \rho) \cap \overline{\mathbf{C}}$ . By inspection,  $\nu \in \mathbf{P}_{++}$  if and only if  $1 - d \in \mathbb{N}^*$ . As usual, this forces  $n$  to be even and  $r - n/2 \geq 0$ . □



### 5. Rings in category $\mathcal{O}$

In this section we study rings  $R$  on which a reductive Lie algebra  $\mathfrak{g}$  acts as differential operators, abstracting the situation from the introduction. Although we need a considerable number of hypotheses, they do hold frequently and the consequences are surprisingly strong. In particular, they force a factor ring of  $U(\mathfrak{g})$  to be a ring of  $\mathfrak{g}$ -finite vectors (see Theorem 5.7 and Definition 5.3).

We are concerned with the situation described in the following hypotheses, for which we need a definition. A module  $M$  over a ring  $U$  of finite Gelfand–Kirillov dimension is said to be *quasi-simple* if its socle  $\text{Soc}(M)$  is simple, with  $\text{GKdim}_U(M/\text{Soc}(M)) < \text{GKdim}_U M$ .

**HYPOTHESES 5.1.** Let  $\mathfrak{g}$  be a finite-dimensional complex reductive Lie algebra with triangular decomposition  $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ , and assume that  $R$  is a commutative, finitely generated  $\mathbb{C}$ -algebra that is a  $\mathfrak{g}$ -module via a morphism  $\chi : U(\mathfrak{g}) \rightarrow \mathcal{D}(R)$ . Set  $I = \text{Ker}(\chi)$  and identify  $\bar{U} = U(\mathfrak{g})/I$  with its image  $\chi(U(\mathfrak{g}))$  in  $\mathcal{D}(R)$ . Assume moreover that:

- (1) under the above action,  $R$  is a quasi-simple highest-weight  $\mathfrak{g}$ -module;
- (2)  $R$  is generated by  $\chi(\mathfrak{r})$  for some Lie subalgebra  $\mathfrak{r} \subseteq \mathfrak{n}^-$ ;
- (3) if  $N = \text{Soc}_{U(\mathfrak{g})}(R)$ , then  $\text{End}_R(N) = R$ .

Although these hypotheses are clearly strong, they do occur for the rings relevant to this paper (see Lemma 5.2 for (3)). It is important, however, that we do not require that  $\mathfrak{g}$  act by derivations, and so, in that sense at least, our definition is weaker than Joseph’s concept of  $\mathcal{O}$ -rings from [Jos88b].

We remark that, by (2), the  $\mathfrak{g}$ -socle  $N$  of  $R$  is also an  $R$ -module and so (3) makes sense. Moreover,  $R$  has finite length as a  $U(\mathfrak{g})$ -module. A routine argument (see, for example, [Jan83, Lemma 8.14]) shows that  $I = \text{ann}_{U(\mathfrak{g})}(R) = \text{ann}_{U(\mathfrak{g})}(N)$ . Hence  $\bar{U}$  is primitive.

First we clarify the conditions under which Hypothesis 5.1(3) holds.

**LEMMA 5.2.** *Let  $C$  be a finitely generated commutative  $\mathbb{C}$ -algebra and let  $f \in C$ . Suppose that  $C$  is a Cohen–Macaulay domain and set  $R = C/(f)$ . Let  $M$  be an ideal of  $R$  such that  $\text{GKdim}(R/M) \leq \text{GKdim}(R) - 2$ . Then Hypothesis 5.1(3) holds for  $M$  in the sense that  $\text{End}_R(M) = R$ .*

*Proof.* The hypothesis on  $C$  implies that  $R$  is Cohen–Macaulay and that the grade of the  $R$ -module  $R/M$  is at least 2; see [Mat80, (16.A) and (16.B)]. In other words,  $\text{Hom}_R(R/M, R) = \text{Ext}_R^1(R/M, R) = 0$ . Applying  $\text{Hom}_R(-, R)$  to the short exact sequence  $0 \rightarrow M \rightarrow R \rightarrow R/M \rightarrow 0$  then gives  $R = \text{End}_R(R) = \text{Hom}_R(M, R)$ . It follows that  $\text{Hom}_R(M, R) = \text{End}_R(M)$  and so  $\text{End}_R(M) = R$ . □

**DEFINITION 5.3.** Let  $R$  and  $\mathfrak{g}$  satisfy Hypotheses 5.1, and let  $M$  and  $N$  be (left)  $\mathfrak{g}$ -modules. Define the adjoint action of  $x \in \mathfrak{g}$  on  $\theta \in \text{Hom}_{\mathbb{C}}(M, N)$  by  $\text{ad}(x)(\theta) = x\theta - \theta x$ . Set

$$\mathcal{A}(M, N) = \{\theta \in \text{Hom}_{\mathbb{C}}(M, N) : \mathfrak{r} \text{ acts ad-nilpotently on } \theta\}.$$

Analogously to the order filtration for  $\mathcal{D}(R)$  in (3.1), we filter the spaces  $\mathcal{A}(M, N)$  by  $\mathcal{A}(M, N) = \bigcup_{k \geq 0} \mathcal{A}_k(M, N)$  where  $\mathcal{A}_{-1}(M, N) = 0$  and, for  $k \geq 0$ ,

$$\mathcal{A}_k(M, N) = \{\theta \in \text{Hom}_{\mathbb{C}}(M, N) : [x, \theta] = x\theta - \theta x \in \mathcal{A}_{k-1}(M, N) \text{ for all } x \in \mathfrak{r}\}. \tag{5.4}$$

Recall, for example from [Jan83, §6.8], that one also has the set of  $\mathfrak{g}$ -finite elements  $\mathcal{L}(M, N) \subseteq \text{Hom}_{\mathbb{C}}(M, N)$ , on which  $\mathfrak{g}$  acts locally finitely under the adjoint action. If  $M = N$ , then [Jan83, §6.8(5)] implies that  $\mathcal{L}(M, M)$  contains  $U(\mathfrak{g})/\text{ann}(M)$  as a subalgebra.

PROPOSITION 5.5. Assume that  $(R, \mathfrak{g})$  satisfies Hypotheses 5.1 and let  $M$  and  $N$  be nonzero left  $\mathfrak{g}$ -modules.

- (i) Under the natural  $R$ -module action induced from Hypothesis 5.1(2), each  $\mathcal{A}_k(M, N)$  is an  $R$ -bimodule.
- (ii)  $\mathcal{D}(M, N) = \mathcal{A}(M, N) \supseteq \mathcal{L}(M, N)$ ; indeed,  $\mathcal{D}_k(M, N) = \mathcal{A}_k(M, N)$  for all  $k \geq 0$ .
- (iii) Assume that  $M \subseteq R$  with  $\text{GKdim}(R/M) < \text{GKdim}(R)$  and  $\text{End}_R(M) = R$ . Then  $\mathcal{L}(M, M) = \overline{U}$ .

*Proof.* (i) By induction, assume that  $\mathcal{A}_{k-1}(M, N)$  is an  $R$ -bimodule. Let  $\theta \in \mathcal{A}_k(M, N)$ ,  $y \in R$  and  $x \in \mathfrak{r}$ . Then, by induction and the fact that  $R$  is commutative,

$$[x, \theta y] = x\theta y - \theta x y = (x\theta - \theta x)y \in \mathcal{A}_{k-1}(M, N)R = \mathcal{A}_{k-1}(M, N).$$

Hence  $\theta y \in \mathcal{A}_k(M, N)$  and, by symmetry, the result follows.

(ii) As  $\mathfrak{n}^-$  acts ad-nilpotently on any finite-dimensional  $\mathfrak{g}$ -module and hence on  $\mathcal{L}(M, N)$ , Hypothesis 5.1(2) implies that  $\mathcal{A}(M, N) \supseteq \mathcal{L}(M, N)$ . Since the inclusion  $\mathcal{A}_k(M, N) \supseteq \mathcal{D}_k(M, N)$  is obvious, we are reduced to proving that  $\mathcal{A}_k(M, N) \subseteq \mathcal{D}_k(M, N)$  for each  $k$ . By induction and for some  $m$ , assume that the result holds for all  $k < m$  and fix  $\theta \in \mathcal{A}_m(M, N)$ .

Filter  $R = \bigcup \Lambda_p R$  by taking  $\Lambda_0 = \mathbb{C}$  and  $\Lambda_p R = \chi(\mathfrak{r})^p + \Lambda_{p-1} R$  for  $p > 0$ . For some  $q$ , assume that  $[\theta, z] \in \mathcal{D}_{m-1}(M, N)$  for all  $z \in \Lambda_s R$  and  $s < q$ . (The case  $s = 1$  holds automatically by the definition of  $\mathcal{A}(M, N)$ , so the induction does start.) Let  $y \in \Lambda_q R$  and suppose that  $y = x_1 x_2$  for  $x_1 \in \chi(\mathfrak{r})$  and  $x_2 \in \Lambda_{q-1} R$ . Then, by part (i) and induction on  $q$ ,

$$\begin{aligned} \theta y - y\theta &= \theta x_1 x_2 - x_1 x_2 \theta = \theta x_1 x_2 - x_1 \theta x_2 + x_1 \theta x_2 - x_1 x_2 \theta \\ &= (\theta x_1 - x_1 \theta)x_2 + x_1(\theta x_2 - x_2 \theta) \\ &\in \mathcal{A}_{m-1}(M, N)x_2 + x_1 \mathcal{A}_{m-1}(M, N) \subseteq \mathcal{A}_{m-1}(M, N). \end{aligned}$$

Hence  $[\theta, y] \in \mathcal{A}_{m-1}(M, N)$ . By linearity and our inductive hypotheses,  $[\theta, z] \in \mathcal{A}_{m-1}(M, N) = \mathcal{D}_{m-1}(M, N)$  for all  $z \in R$ . Hence  $\mathcal{A}_m(M, N) \subseteq \mathcal{D}_m(M, N)$ .

(iii) Since  $R$  is a quasi-simple  $U(\mathfrak{g})$ -module, [Jan83, Lemma 8.14] implies that  $\text{ann}_{U(\mathfrak{g})}(M) = \text{ann}_{U(\mathfrak{g})}(R)$  and so  $\overline{U} \hookrightarrow \mathcal{L}(M, M)$  by [Jan83, § 6.8(5)]. By definition,  $\mathcal{M} = \mathcal{L}(M, M)$  is a locally finite  $\mathfrak{g}$ -module under the adjoint action and hence is semi-simple. Thus, we may write  $\mathcal{M} = \overline{U} \oplus P$  for some complementary  $(\text{ad } \mathfrak{g})$ -module  $P$ . Assume that  $P \neq 0$ .

Pick  $p \in P \setminus \{0\}$  with  $p \in \mathcal{A}_k(M, M)$  for  $k$  as small as possible. Then, for  $x \in \mathfrak{r}$ , we have both  $[x, p] \in \mathcal{A}_{k-1}(M, M)$  by the definition of the order filtration (5.4) and  $[x, p] \in P$  as  $P$  is an  $(\text{ad } \mathfrak{r})$ -module. Hence, by the choice of  $k$  we have  $[x, p] = 0$  for all  $x \in \mathfrak{r}$ . Therefore, by hypothesis and part (ii),

$$p \in \mathcal{A}_0(M, M) = \mathcal{D}_0(M) = \text{End}_R(M) = R.$$

However, by Hypothesis 5.1(2),  $R \subseteq \overline{U}$  as subrings of  $\mathcal{D}(R)$ . Hence  $p \in \overline{U}$ , giving the required contradiction.  $\square$

DEFINITION 5.6. If  $A$  and  $B$  are prime Goldie rings with the same simple artinian ring of fractions  $Q$ , then  $A$  and  $B$  are *equivalent orders* if  $a_1 B a_2 \subseteq A$  and  $b_1 A b_2 \subseteq B$  for some regular elements  $a_i \in A$  and  $b_j \in B$ . The ring  $A$  is called a *maximal order* if it is not equivalent to any  $B \supsetneq A$ .

The property of being a maximal order is the appropriate noncommutative analogue of being an integrally closed commutative ring and has a number of useful consequences; see, for example, [JS84] or [MaR73].

THEOREM 5.7. *Let  $R$  and  $\mathfrak{g}$  satisfy Hypotheses 5.1. Then*

$$\mathcal{L}(R, R) = \mathcal{L}(N, N) = \overline{U}.$$

*In particular,  $\mathcal{L}(R, R)$  is a maximal order.*

Remark 5.8. In the sense of, say, [Jos80], this solves the Kostant problem for the ring  $\overline{U}$ .

Proof. Both  $M = R$  and  $M = N$  satisfy the hypotheses of Proposition 5.5 and so the displayed equation follows. The final assertion then follows from [JS84, Theorem 2.9].  $\square$

Here is a simple example that illustrates the restrictions of Hypothesis 5.1(3), even if one assumes that  $R/N(R)$  has pleasant properties.

Example. Take  $A = \mathbb{C}[x, y, xz, yz, z^2, z^3] \subset \mathbb{C}[x, y, z] = C$ ; then  $C = A + \mathbb{C}z$ . Set  $I = A \cap z^2C$  and  $R = A/I$ . Clearly the nilradical  $N(R)$  equals  $(zC \cap A)/I = ((xz, yz) + I)/I$ , and so  $R/N(R) \cong \mathbb{C}[x, y]$ . However,  $N(R)/I = (xz, yz) \cong (x, y)$  as modules over  $k[x, y] \cong R/N(R)$ .

Now let  $M = (x, y, zx, zy, z^2, z^3)/I$  be the augmentation ideal of  $R$ . The maximal ring of fractions  $\text{Quot}(R)$  clearly contains  $z = (zx)x^{-1}$ . However,  $z \notin R$ , yet by the computations of the last paragraph one has  $zM \subseteq R$ . Indeed,  $zM \subseteq M$  and so  $\text{End}_R(M) \supsetneq R$ .

### 6. Modules of $\mathfrak{so}(n + 2, \mathbb{C})$ -finite vectors

As we noted in Theorem 4.6,  $\mathcal{D}(R_1) = U(\mathfrak{so}(n + 2))/J$  for the Joseph ideal  $J$ . We would like to pass from  $\mathcal{D}(R_1)$  to  $\mathcal{D}(R_r)$  for any  $r$  to obtain the analogous result for  $\mathcal{D}(R_r)$ . There are two potential ways of creating such a passage: through differential operators  $\mathcal{D}(R_r, R_1)$  and through the  $\mathfrak{g}$ -finite vectors  $\mathcal{L}(R_r, R_1)$ . The first step, however, is to prove that the latter is nonzero. This we accomplish in the present section, and then we use it in the next section to prove the major part of Theorem 1.3 from the introduction.

There are a number of different rings involved in this discussion, so we need to refine the earlier notation. As in §3, set  $A = \mathbb{C}[X_1, \dots, X_n] \ni \mathbf{F} = \sum_{i=1}^n X_i^2$  with  $R_r = A/(\mathbf{F}^r)$  for any  $r \geq 1$ . Recall the elements  $\Delta_1, \mathbf{E}_1, D_{kl} \in \mathcal{D}(A)$  from (3.5) as well as the elements

$$P_{kj} = P_j \binom{n}{2} - k = X_j \Delta_1 - \left( \mathbf{E}_1 + \binom{n}{2} - k \right) \partial_j \in \mathcal{D}(A) \quad \text{for } 1 \leq j \leq n \text{ and } k \geq 1$$

defined by (3.7). By Corollary 3.14, the images of  $\{P_{rj}, D_{ij}, (\mathbf{E}_1 + n/2 - r), X_i : 1 \leq i, j \leq n\}$  span a copy  $\mathfrak{g}_r$  of  $\mathfrak{g} = \mathfrak{so}(n + 2, \mathbb{C})$  inside  $\mathcal{D}(R_r)$ . Moreover, for  $r, s \geq 1$ , Proposition 3.8 shows that the map  $P_{rj} \mapsto P_{sj}$  and  $\mathbf{E}_1 + n/2 - r \mapsto \mathbf{E}_1 + n/2 - s$ , with  $D_{ij} \mapsto D_{ij}$  and  $X_i \mapsto X_i$ , induces a Lie algebra isomorphism  $\mathfrak{g}_r \rightarrow \mathfrak{g}_s$ . We may therefore write  $\{\tilde{P}_j, D_{ij}, \tilde{\mathbf{E}}_1, X_i : 1 \leq i, j \leq n\}$  for the corresponding generators of the abstract copy  $\mathfrak{g}$  of  $\mathfrak{so}(n + 2, \mathbb{C})$  and rephrase Corollary 3.14 as saying that there is a Lie algebra isomorphism  $\psi_r : \mathfrak{g} \rightarrow \mathfrak{g}_r$  given by  $\tilde{P}_j \mapsto P_{rj}$  and  $\tilde{\mathbf{E}}_1 \mapsto \mathbf{E}_1 + n/2 - r$  etc.

LEMMA 6.1. *Fix  $r \geq 2$  and recall the notation  $\mathcal{D}(M, N)$  from (3.1). Then, regarded as differential operators acting on  $A$ , the elements  $\Delta_1, \partial_j$  and 1 induce differential operators in  $\mathcal{D}(R_r, R_{r-1})$ .*

Proof. Suppose first that  $\phi \in \mathcal{D}(A)$  satisfies  $\phi(\mathbf{F}^r A) \subseteq \mathbf{F}^{r-1} A$  and so induces a function  $\tilde{\phi} : R_r \rightarrow R_{r-1}$ . We claim that  $\tilde{\phi} \in \mathcal{D}(R_r, R_{r-1})$ . To see this, assume that  $\phi \in \mathcal{D}_t(A)$  and that the analogous

statement is true for all  $\phi' \in \mathcal{D}_{t-1}(A)$ . Then, for  $a \in A$ , the function  $\phi' = [\phi, a]$  also satisfies  $\phi'(\mathbb{F}^r A) \subseteq \mathbb{F}^{r-1}A$ , but now  $\phi' \in \mathcal{D}_{t-1}(A)$ . Hence  $\phi' \in \mathcal{D}_{t-1}(R_r, R_{r-1})$ . Thus  $\phi \in \mathcal{D}_t(R_r, R_{r-1})$ , proving the claim.

In order to prove the lemma, it therefore suffices to show that each of our elements  $\phi$  satisfies  $\phi(\mathbb{F}^r A) \subseteq \mathbb{F}^{r-1}A$ . This is obvious for 1 and  $\partial_i$ , while for  $\Delta_1$  it follows from (3.13).  $\square$

*Notation 6.2.* For a fixed value of  $r$ , the images of  $\Delta_1$ ,  $\partial_j$  and 1 in  $\mathcal{D}(R_r, R_{r-1})$  will be written, respectively, as  $\bar{\Delta}_1$ ,  $\bar{\partial}_j$  and  $\bar{1}$ . For concreteness, we note that the adjoint action  $\tilde{\psi}$  of  $\mathfrak{g} = \mathfrak{so}(n+2, \mathbb{C})$  on  $\mathcal{D} = \mathcal{D}(R_r, R_{r-1})$  is defined by  $\tilde{\psi}(x)(\theta) = \psi_{r-1}(x) \circ \theta - \theta \circ \psi_r(x)$  for  $x \in \mathfrak{g}$  and  $\theta \in \mathcal{D}$ .

**PROPOSITION 6.3.** Fix  $r \geq 2$  and let  $V = \mathbb{C}\bar{1} + \sum_i \mathbb{C}\bar{\partial}_i + \mathbb{C}\bar{\Delta}_1 \subset \mathcal{D}(R_r, R_{r-1})$ . Then, under the adjoint action  $\tilde{\psi}$ ,  $V$  is a  $\mathfrak{g}$ -submodule of  $\mathcal{D}(R_r, R_{r-1})$ . In particular,  $V \subseteq \mathcal{L}(R_r, R_{r-1})$ .

*Proof.* By the definition of  $\mathcal{L}(R_r, R_{r-1})$  it suffices to prove that  $V$  is a  $\mathfrak{g}$ -module. This requires some explicit computations, for which the following three formulæ in  $\mathfrak{g}$  will prove useful and will be used frequently without comment: for all  $1 \leq i, j \leq n$  we have  $[\Delta_1, \partial_j] = [\Delta_1, D_{ij}] = 0$ , while

$$[\Delta_1, X_j] = \left[ \frac{1}{2} \partial_j^2, X_j \right] = \partial_j \quad \text{and} \quad [\mathbb{E}_1, \Delta_1] = \frac{1}{2} \sum_j [\mathbb{E}_1, \partial_j^2] = -2\Delta_1.$$

First, computing in  $\mathcal{D}(R_r, R_{r-1})$  we have

$$\begin{aligned} \tilde{\psi}(\tilde{P}_j)(\bar{\Delta}_1) &= P_j \left( \frac{n}{2} - r + 1 \right) \bar{\Delta}_1 - \bar{\Delta}_1 P_j \left( \frac{n}{2} - r \right) \\ &= (X_j \Delta_1 \bar{\Delta}_1 - \bar{\Delta}_1 X_j \Delta_1) + (-\mathbb{E}_1 \partial_j \bar{\Delta}_1 + \bar{\Delta}_1 \mathbb{E}_1 \partial_j) \\ &\quad + \left( - \left( \frac{n}{2} - r + 1 \right) \partial_j \bar{\Delta}_1 + \left( \frac{n}{2} - r \right) \bar{\Delta}_1 \partial_j \right) \\ &= -\partial_j \bar{\Delta}_1 + 2\partial_j \bar{\Delta}_1 - \partial_j \bar{\Delta}_1 = 0. \end{aligned}$$

Next,

$$\tilde{\psi}(\tilde{\mathbb{E}}_1)(\bar{\Delta}_1) = \left( \mathbb{E}_1 + \frac{n}{2} - r + 1 \right) \bar{\Delta}_1 - \bar{\Delta}_1 \left( \mathbb{E}_1 + \frac{n}{2} - r \right) = -\bar{\Delta}_1 + 2\bar{\Delta}_1 = -\bar{\Delta}_1.$$

Similarly,  $\tilde{\psi}(D_{ij})(\bar{\Delta}_1) = [D_{ij}, \bar{\Delta}_1] = 0$  and  $\tilde{\psi}(X_j)(\bar{\Delta}_1) = [D_{ij}, \bar{\Delta}_1] = 0$ . Therefore,  $\tilde{\psi}(\mathfrak{g})(\bar{\Delta}_1) \subseteq V$ .

Now we compute that

$$\begin{aligned} \tilde{\psi}(\tilde{P}_i)(\bar{\partial}_i) &= P_i \left( \frac{n}{2} - r + 1 \right) \bar{\partial}_i - \bar{\partial}_i P_i \left( \frac{n}{2} - r \right) \\ &= (X_i \Delta_1 \bar{\partial}_i - \bar{\partial}_i X_i \Delta_1) + (-\mathbb{E}_1 \partial_i \bar{\partial}_i + \bar{\partial}_i \mathbb{E}_1 \partial_i) + \left( - \left( \frac{n}{2} - r + 1 \right) \partial_i \bar{\partial}_i + \left( \frac{n}{2} - r \right) \bar{\partial}_i \partial_i \right) \\ &= -\bar{\Delta}_1 - \partial_i \bar{\partial}_i + \bar{\partial}_i \partial_i = -\bar{\Delta}_1. \end{aligned}$$

If  $i \neq j$  we obtain

$$\begin{aligned} \tilde{\psi}(\tilde{P}_j)(\bar{\partial}_i) &= (X_j \Delta_1 \bar{\partial}_i - \bar{\partial}_i X_j \Delta_1) + (-\mathbb{E}_1 \partial_j \bar{\partial}_i + \bar{\partial}_i \mathbb{E}_1 \partial_j) + \left( - \left( \frac{n}{2} - r + 1 \right) \partial_j \bar{\partial}_i + \left( \frac{n}{2} - r \right) \bar{\partial}_i \partial_j \right) \\ &= -\partial_j \bar{\partial}_i + \bar{\partial}_i \partial_j = 0. \end{aligned}$$

On the other hand,  $\tilde{\psi}(\tilde{\mathbf{E}}_1)(\bar{\partial}_i) = (\mathbf{E}_1 + n/2 - r + 1)\bar{\partial}_i - \bar{\partial}_i(\mathbf{E}_1 + n/2 - r) = \bar{\partial}_i - \bar{\partial}_i = 0$ , while

$$\tilde{\psi}(D_{ab})\bar{\partial}_i = (X_a\partial_b - X_b\partial_a)\bar{\partial}_i - \bar{\partial}_i(X_a\partial_b - X_b\partial_a) = \sum_u \alpha_u \bar{\partial}_u$$

for any  $1 \leq a, b, i \leq n$  and appropriate scalars  $\alpha_u \in \mathbb{C}$ . Since  $\tilde{\psi}(X_k)\bar{\partial}_i = [X_k, \bar{\partial}_i] = -\delta_{k,i}\bar{1}$ , it follows that  $\tilde{\psi}(\mathfrak{g})(\bar{\partial}_i) \subseteq V$ . The easy fact that  $\tilde{\psi}(\mathfrak{g})(\bar{1}) \subseteq V$  is left to the reader and completes the proof.  $\square$

COROLLARY 6.4. *Fix  $r > s \geq 1$ . Then the following hold.*

- (1) *Both the projection map  $\pi_{rs} : R_r \rightarrow R_s$  and the operator  $\Delta_1^{r-s}$  belong to  $\mathcal{L}(R_r, R_s)$ .*
- (2) *Set  $Z_s = \text{Soc}_{U(\mathfrak{g})} R_s$ . Then  $\mathcal{L}(Z_r, Z_s) \neq 0 \neq \mathcal{L}(Z_s, Z_r)$ .*

*Proof.* (1) If  $\theta \in \mathcal{L}(A, B)$  and  $\phi \in \mathcal{L}(B, C)$  for  $\mathfrak{g}$ -modules  $A, B$  and  $C$ , then  $\phi\theta \in \mathcal{L}(A, C)$ ; see [Jan83, 6.8(3)]. Thus the result follows from Proposition 6.3 and induction on  $r - s$ .

(2) By Theorem 4.13 each  $R_s/Z_s$  is finite-dimensional (or zero), say with  $\mathfrak{a}_s = \text{ann}_{U(\mathfrak{g})}(R_s/Z_s)$ . Since  $Z_r$  is infinite-dimensional, the projection map  $\pi_{rs} \in \mathcal{L}(R_r, R_s)$  has an infinite-dimensional image, as does its restriction  $\pi'_{rs} : Z_r \rightarrow R_s$ . Hence  $\mathfrak{a}_s \pi'_{rs} \neq 0$ , and so for some  $a \in \mathfrak{a}_s$  one has  $0 \neq a \circ \pi'_{rs} \in \mathcal{L}(Z_r, Z_s)$ . Thus  $\mathcal{L}(Z_r, Z_s) \neq 0$ . By [Jan83, §6.9(5)], this also implies that  $\mathcal{L}(Z_s, Z_r) \neq 0$ .  $\square$

### 7. Differential operators and primitive factor rings of $U(\mathfrak{so}(n + 2, \mathbb{C}))$

As usual, we write  $R_r = A/(\mathbb{F}^r)$  for  $A = \mathbb{C}[X_1, \dots, X_n]$  and some  $r \geq 1$ . The aim of this section is to combine the earlier results to prove that  $\mathcal{D}(R_r) = \mathcal{L}(R_r, R_r) = U(\mathfrak{g})/J_r$ , where  $\mathfrak{g} = \mathfrak{so}(n + 2, \mathbb{C})$  and  $J_r = \text{ann}_{U(\mathfrak{g})}(R_r)$ . In particular, this proves the major part of Theorem 1.3 from the introduction.

We begin with some elementary results, the first of which is a minor generalisation of [JS84, Corollary 2.10].

LEMMA 7.1. *If  $M$  is a simple  $U(\mathfrak{g})$ -module with  $I = \text{ann}_{U(\mathfrak{g})}(M)$ , then  $\mathcal{L}(M, M)$  is the unique maximal object among orders containing and equivalent to  $U = U(\mathfrak{g})/I$ .*

*Proof.* By [JS84, Corollary 2.10],  $\mathcal{L}(M, M)$  is a maximal order and it is noetherian since it is finitely generated as both a left and a right  $U$ -module. So it remains to prove uniqueness. Suppose that  $U \subseteq T$  for some other order  $T$  equivalent to  $U$ . Then  $aTb \subseteq U$  for some regular elements  $a, b \in U$ . Let  $T' = UaT + U$ ; this is an overring of  $U$  with  $T'b \subseteq U$ . So, since  $b$  is regular,  $T'$  is certainly finitely generated as a left  $U$ -module. Therefore, by [JS84, Theorem 2.9],  $T' \subseteq \mathcal{L}(M, M)$ . In particular,  $T'$  is also finitely generated as a right  $U$ -module. Since  $aT \subseteq T'$ , it follows that  $T$  is also a finitely generated right  $U$ -module. Hence [JS84, Theorem 2.9] implies that  $T \subseteq \mathcal{L}(M, M)$ .  $\square$

LEMMA 7.2. *Let  $a, b, c \in \mathbb{N}^*$  and set  $Z_a = \text{Soc}_{U(\mathfrak{g})}(R_a)$ .*

- (i)  $\mathcal{L}(R_a, R_a) = \mathcal{L}(Z_a, Z_a) = U(\mathfrak{g})/J_a$  is a primitive ring.
- (ii)  $\mathcal{L}(R_a, R_b)$  and  $\mathcal{L}(Z_a, Z_b)$  are nonzero and torsion-free both as left  $U(\mathfrak{g})/J_b$ -modules and as right  $U(\mathfrak{g})/J_a$ -modules.
- (iii) Under composition of operators, both  $\mathcal{L}(R_b, R_a)\mathcal{L}(R_a, R_b)$  and  $\mathcal{L}(Z_b, Z_a)\mathcal{L}(Z_a, Z_b)$  are nonzero ideals of  $\mathcal{L}(R_a)$ .

*Proof.* (i) By Theorem 4.13, Hypotheses 5.1 are satisfied and so the result follows from Theorem 5.7.

(ii) By [Jan83, Lemma 8.14],  $\text{ann}_{U(\mathfrak{g})}(R_b) = \text{ann}_{U(\mathfrak{g})}(Z_b) = J_b$  and so  $R_b$  is *ann-homogeneous* in the sense of [JS84, § 2.7]. Thus, by [JS84, Lemma 2.8],  $\mathcal{L}(R_a, R_b)$  is *GK-homogeneous* in the sense that any nonzero left or right  $\mathfrak{g}$ -submodule  $N$  of  $\mathcal{L}(R_a, R_b)$  has  $\text{GKdim}(N) = \text{GKdim}(\mathcal{L}(R_a, R_b))$ . Since each  $U(\mathfrak{g})/J_c$  is a prime ring by (i), this is equivalent to  $\mathcal{L}(R_a, R_b)$  being torsion-free on both sides. Exactly the same argument works for  $\mathcal{L}(Z_a, Z_b)$ .

It therefore remains to prove that these modules are nonzero. For  $\mathcal{L}(Z_a, Z_b)$  this is Corollary 6.4. In particular,  $\mathcal{L}(Z_a, Z_b)$  is infinite-dimensional, and hence  $\mathcal{L}(Z_a, Z_b)\mathfrak{a}_a \neq 0$ , where  $\mathfrak{a}_a$  is the annihilator of the finite-dimensional  $\bar{U}_a$ -module  $R_a/Z_a$ . Thus

$$0 \neq \mathcal{L}(Z_a, Z_b)\mathfrak{a}_a \subseteq \mathcal{L}(R_a, Z_b) \subseteq \mathcal{L}(R_a, R_b).$$

(iii) This is automatic from (ii). □

We also have analogous elementary results about differential operators, although as we do not (yet) know that these rings are noetherian the proof is a little more involved.

LEMMA 7.3. *Set  $\{1, r\} = \{a, b\}$ . Then:*

- (i)  $\mathcal{D}(R_a)$  is a prime Goldie ring;
- (ii)  $\mathcal{D}(R_a)$  has Goldie rank  $a$ , and indeed the simple artinian ring of fractions  $\text{Quot}(\mathcal{D}(R_a))$  is even isomorphic to the  $a \times a$  matrix ring  $M_a(\text{Quot}(\mathcal{D}(R_1)))$ ;
- (iii)  $\mathcal{D}(R_a, R_b)$  is nonzero and torsion-free both as a left  $\mathcal{D}(R_b)$ -module and as a right  $\mathcal{D}(R_a)$ -module.

*Proof.* (i) Clearly  $R_a$  has a local artinian ring of fractions  $(R_a)_{(\mathbb{F})}$ . Now apply [Mus89, Corollary 2.6].

(ii) This follows from [Mus89, Lemma 2.4 and (2.5)].

(iii) The projection map  $\pi : R_r \rightarrow R_1$  is an  $A$ -module map and so belongs to  $\mathcal{D}(R_r, R_1)$ . Similarly, the  $A$ -module isomorphism  $\phi : R_1 \rightarrow \mathbb{F}^{r-1}R_r$  belongs to  $\mathcal{D}(R_1, R_r)$ . Thus,  $\mathcal{D}(R_r, R_1) \neq 0 \neq \mathcal{D}(R_1, R_r)$ .

As right  $\mathcal{D}(R_r)$ -modules,  $\mathcal{D}(R_r, R_1) \cong \mathcal{D}(R_r, \mathbb{F}^{r-1}R_r) \hookrightarrow \mathcal{D}(R_r, R_r)$ . Therefore, by (i), it is a torsion-free right  $\mathcal{D}(R_r)$ -module. For the left action, consider  $0 \neq \phi \in \mathcal{D}(R_r, R_1)$  and pick  $s$  maximal such that  $\phi(\mathbb{F}^s R_r) \neq 0$ . If  $\bar{r} \in R_1$ , write  $\bar{r} = \pi(r)$  for some  $r \in R_r$  and define  $\tilde{\phi}(\bar{r}) = \phi(\mathbb{F}^s r)$ . Since  $\phi(\mathbb{F}^{s+1}R_r) = 0$ , this is a well-defined morphism and hence a differential operator; thus  $\tilde{\phi} \in \mathcal{D}(R_1, R_1)$ . Moreover,  $\tilde{\phi} \neq 0$  by the choice of  $s$ , and so  $\tilde{\phi}$  is not torsion as an element of the left  $\mathcal{D}(R_1)$ -module  $\mathcal{D}(R_1)$ . Consequently,  $\phi$  is not torsion under the left  $\mathcal{D}(R_1)$  action.

Conversely, if  $0 \neq \theta \in \mathcal{D}(R_1, R_r)$ , then  $0 \neq \theta \circ \pi \in \mathcal{D}(R_r, R_r)$ . Thus neither  $\theta \circ \pi$  nor  $\theta$  is torsion under the left  $\mathcal{D}(R_r)$  action. Hence  $\mathcal{D}(R_1, R_r)$  is torsion-free as a left  $\mathcal{D}(R_r)$ -module. Finally, let  $0 \neq \phi \in \mathcal{D}(R_1, R_r)$  and choose  $s$  minimal such that  $\phi(R_1) \subseteq \mathbb{F}^s R_r$ . Thus,  $\phi \in \mathcal{D}(R_1, \mathbb{F}^s R_r)$ . Moreover, if  $\rho : \mathbb{F}^s R_r \rightarrow \mathbb{F}^s R_r / \mathbb{F}^{s+1} R_r \cong R_1$ , then  $0 \neq \rho \circ \phi \in \mathcal{D}(R_1, R_1)$ . As in the last paragraph, this ensures that  $\phi$  is not torsion under the right  $\mathcal{D}(R_1)$  action. Hence  $\mathcal{D}(R_1, R_r)$  is torsion-free on the right. □

By Lemma 4.3, we know that  $A = \mathbb{C}[X_1, \dots, X_n] \hookrightarrow U(\mathfrak{n}^-) \subset U(\mathfrak{g})$ . As before, set  $Z_a = \text{Soc}_{U(\mathfrak{g})}(R_a)$  for  $a \in \mathbb{N}^*$ . Then, by Proposition 5.5(ii) and Theorem 4.13, we have

$$\mathcal{L}(R_a, R_b) \subseteq \mathcal{D}(R_a, R_b) \quad \text{and} \quad \mathcal{L}(Z_a, Z_b) \subseteq \mathcal{D}(Z_a, Z_b) \quad \text{for } \{a, b\} = \{1, r\}. \tag{7.4}$$



PROPOSITION 7.5. *The rings  $\mathcal{D}(R_r)$  and  $\mathcal{L}(R_r, R_r)$  are equivalent orders with  $\mathcal{D}(R_r) \supseteq \mathcal{L}(R_r, R_r)$ .*

*Proof.* Consider

$$F = \mathcal{D}(R_1, R_r)\mathcal{L}(R_r, R_1) \quad \text{and} \quad G = \mathcal{L}(R_1, R_r)\mathcal{D}(R_r, R_1),$$

where multiplication is composition of functions. We claim that  $F \neq 0 \neq G$ . To see this, by (7.4) it suffices to prove that  $\mathcal{L}(R_a, R_b) \cdot \mathcal{L}(R_a, R_b) \neq 0$  (where  $\{a, b\} = \{1, r\}$ ). This follows from Lemma 7.2(ii).

Recall from Theorem 4.6 that  $\mathcal{D}(R_1) = \mathcal{L}(R_1, R_1)$ . Thus, repeating the argument of the last paragraph shows that

$$\begin{aligned} 0 \neq GF &= \mathcal{L}(R_1, R_r)\mathcal{D}(R_r, R_1) \cdot \mathcal{D}(R_1, R_r)\mathcal{L}(R_r, R_1) \\ &\subseteq \mathcal{L}(R_1, R_r)\mathcal{D}(R_1)\mathcal{L}(R_r, R_1) = \mathcal{L}(R_1, R_r)\mathcal{L}(R_1, R_1)\mathcal{L}(R_r, R_1) \\ &= \mathcal{L}(R_1, R_r)\mathcal{L}(R_r, R_1) \subseteq \mathcal{L}(R_r, R_r). \end{aligned}$$

Both  $F$  and  $G$  contain  $\mathcal{L}(R_1, R_r)\mathcal{L}(R_r, R_1)$ , which, by Lemma 7.2(i) and (iii), contains a regular element  $a \in \mathcal{L}(R_r, R_r)$ . Therefore, from the last display we have  $a\mathcal{D}(R_r)a \subseteq G\mathcal{D}(R_r)F = GF \subseteq \mathcal{L}(R_r, R_r)$ , as required.  $\square$

Finally, upon putting everything together we get the following theorem.

THEOREM 7.6. *One has  $\mathcal{D}(R_r) = \mathcal{L}(R_r, R_r) = \mathcal{L}(Z_r, Z_r) = U(\mathfrak{so}(n + 2, \mathbb{C}))/J_r$ . Moreover, this ring is a maximal order in its simple artinian ring of fractions and has Goldie rank  $r$ .*

*Proof.* By Theorem 4.13, Hypotheses 5.1 are satisfied and so  $U(\mathfrak{g})/J_r = \mathcal{L}(R_r, R_r) = \mathcal{L}(Z_r, Z_r)$  is a maximal order by Theorem 5.7. Thus, by Proposition 7.5,  $\mathcal{D}(R_r) = \mathcal{L}(R_r, R_r)$ . The fact that  $\mathcal{D}_r$  has Goldie rank  $r$  follows from Lemma 7.3.  $\square$

*Remark 7.7.* There are other ways to prove aspects of this theorem. For example, recall that the associated variety of the primitive ideal  $J_r$  is equal to  $\overline{\mathbf{O}}_{\min}$ ; cf. Corollary 4.17. Assume that  $n \neq 4$  and  $R_r \cong L(\lambda)$  is an irreducible  $\mathfrak{g}$ -module (so we are in case (1) of Theorem 4.13). Then the minimal orbit  $\mathbf{O}_{\min}$  is rigid, in the sense that it is not induced from any proper parabolic subalgebra; this implies that the  $\mathfrak{g}$ -module  $R_r$  is rigid in the sense of [Jos88a, 1.2]. Then, [Jos88a, 5.8] can be applied in this situation to show that  $\mathcal{L}(R_r, R_r) = \mathcal{D}(R_r)$ , and one deduces from Lemma 7.2 that  $U(\mathfrak{g})/J_r = \mathcal{D}(R_r)$ .

### 8. Higher symmetries of powers of the Laplacian

Recall that the starting point of this paper was to examine the symmetries of powers of the Laplacian  $\Delta_1 = \frac{1}{2} \sum \partial_{X_i}^2$ , as defined below, and thereby to extend the work of Eastwood and others [Eas05, EL08, Mic14] on this concept. In this section we translate Theorem 7.6 into a result about those symmetries. In particular, this proves the second half of Theorem 1.3 from the introduction, by identifying the ring of symmetries of  $\Delta_1^r$  with a factor ring of  $U(\mathfrak{so}(n + 2, \mathbb{C}))$ .

We briefly retain the notation of §3 for a general field  $\mathbb{K}$  of characteristic zero, in particular setting  $A = \mathbb{K}[X_1, \dots, X_n] \ni \mathbf{F} = \sum X_i^2$ . We begin with the definition of higher symmetries as described, for example, in [BS90, (1.2)], [ShS92, (4.3)] or [Eas05, Definition 1].



DEFINITION 8.1. Fix an operator  $P \in \mathcal{D}(A)$ . An operator  $Q \in \mathcal{D}(A)$  is a *symmetry* of  $P$  if there exists  $Q' \in \mathcal{D}(A)$  such that  $PQ = Q'P$ . Equivalently, in the notation of (3.2), the set of symmetries of  $P$  equals the idealiser  $\mathbb{I}(\mathcal{D}(A)P)$ . The elements  $Q \in \mathcal{D}(A)P$  are trivially symmetries of  $P$  and so one usually factors them out and defines *the algebra of symmetries of  $P$*  to be the factor algebra

$$\mathfrak{S}(P) = \mathbb{I}(\mathcal{D}(A)P)/\mathcal{D}(A)P.$$

The definition of  $\mathfrak{S}(\Delta_1^r)$  is of course similar in style to that of the ring of differential operators  $\mathcal{D}(A/\mathbb{F}^r) \cong \mathbb{I}(\mathbb{F}^r\mathcal{D}(A))/\mathbb{F}^r\mathcal{D}(A)$  from (3.3), except that one works with constant-coefficient differential operators rather than polynomials and with left rather than right ideals. However, as we show next, one can easily pass from the one to the other by taking a Fourier transform.

We now return to the field  $\mathbb{K} = \mathbb{C}$ . In our applications it will be convenient to use the Chevalley system from Lemma 4.3, so we will use the Fourier transform  $\mathcal{F}$  in  $\mathcal{D}(A)$  with respect to the variables  $U_{\pm j}$  from (4.1). Thus we write  $A = \mathbb{C}[U_{\pm 2}, \dots, U_{\pm \ell}, U_0]$  and define

$$\mathcal{F}(U_j) = \partial_{U_j}, \quad \mathcal{F}(\partial_{U_j}) = U_j \quad \text{for } j \in \{0, \pm 2, \dots, \pm \ell\}. \tag{8.2}$$

In terms of the variables  $X_j$  it is routine to check that

$$\mathcal{F}(X_j) = \begin{cases} \frac{1}{2}\partial_{X_j} & \text{if } 1 \leq j \leq \ell' \text{ or if } j = n \text{ when } n \text{ is odd,} \\ -\frac{1}{2}\partial_{X_j} & \text{if } \ell' + 1 \leq j \leq 2\ell'. \end{cases} \tag{8.3}$$

Further routine properties of  $\mathcal{F}$  are given in the following lemma. For this we recall that  $d = n/2 - r$  for some  $r \in \mathbb{N}^*$  and that  $P_j = P_j(d) = X_j\Delta_1 - (\mathbf{E}_1 + d)\partial_{X_j}$ .

LEMMA 8.4. *The map  $\mathcal{F}$  is an involutive anti-automorphism of  $\mathcal{D}(A)$ . Moreover,*

$$\mathcal{F}(\mathbf{F}) = \frac{1}{2}\Delta_1, \quad \mathcal{F}(\Delta_1) = 2\mathbf{F}, \quad \mathcal{F}(\mathbf{E}_1) = \mathbf{E}_1, \quad \mathcal{F}(D_{jk}) = \pm D_{jk}, \quad \mathcal{F}(P_j) = \pm(\mathbf{F}\partial_{X_j} - 2X_j(\mathbf{E}_1 + d)).$$

*Proof.* Use the formulæ  $\partial_{X_j} = (\partial_{U_{j+1}} + \partial_{U_{-(j+1)}})/2$  and  $\partial_{X_{j+\ell'}} = (\partial_{U_{j+1}} - \partial_{U_{-(j+1)}})/(2i)$  for  $1 \leq j \leq \ell'$ , and  $\partial_{X_n} = \sqrt{2}\partial_{U_0}$ . □

PROPOSITION 8.5. *Let  $r \geq 1$  and  $R_r = A/\mathbb{F}^r A$ . Then the Fourier transform  $\mathcal{F}$  induces an anti-isomorphism  $\mathcal{F} = \mathcal{F}_r : \mathcal{D}(R_r) \rightarrow \mathfrak{S}(\Delta_1^r)$ .*

*Proof.* Set  $\mathcal{D} = \mathcal{D}(A)$ . By Lemma 8.4,  $\mathcal{F}(\mathbb{F}^r) = 2^{-r}\Delta_1^r$ . Thus, in the notation of Definition 8.1,  $\Delta_1^r Q = Q'\Delta_1^r$  for some  $Q, Q' \in \mathcal{D} \iff \mathcal{F}(Q)\mathbb{F}^r = \mathbb{F}^r\mathcal{F}(Q')$ . Equivalently,  $Q \in \mathbb{I}_{\mathcal{D}}(\mathcal{D}\Delta_1^r) \iff \mathcal{F}(Q) \in \mathbb{I}_{\mathcal{D}}(\mathbb{F}^r\mathcal{D})$ . Hence  $\mathcal{F}$  induces an anti-isomorphism  $\mathbb{I}_{\mathcal{D}}(\mathcal{D}\Delta_1^r)/\mathcal{D}\Delta_1^r \rightarrow \mathbb{I}_{\mathcal{D}}(\mathbb{F}^r\mathcal{D})/\mathbb{F}^r\mathcal{D}$ . □

Recall from Theorem 7.6 that we have an isomorphism  $\psi_r : U(\mathfrak{g})/J_r \rightarrow \mathcal{D}(R_r)$  and hence an anti-isomorphism from  $U(\mathfrak{g})/J_r$  to  $\mathfrak{S}(\Delta_1^r)$ . We want to convert this into an automorphism. By [Jan83, 5.2(2)],  $\vartheta(J_r) = J_r$  for any Chevalley anti-involution  $\vartheta$  of  $\mathfrak{g}$ , and hence for the anti-involution  $\vartheta$  defined by the Chevalley system  $\{y_\alpha\}_\alpha \subset \mathfrak{g}$  described in Lemma 4.3. Notice that  $\tilde{\psi}_d \circ \vartheta = \vartheta_d \circ \tilde{\psi}_d$  in the notation of Remark 4.5. In particular, using Theorem 7.6,  $\vartheta_d$  induces an anti-automorphism  $\vartheta_r$  on  $\mathcal{D}(R_r)$ , and by combining the earlier results in the paper we obtain the main result of this section.

THEOREM 8.6. (1) Fix  $r \geq 1$ . Then there are algebra isomorphisms

$$\mathcal{F} \circ \vartheta_r \circ \psi_r : U(\mathfrak{so}(n + 2, \mathbb{C}))/J_r \xrightarrow{\sim} \mathcal{D}(R_r) \xrightarrow{\sim} \mathcal{S}(\Delta_1^r).$$

In particular, the algebra of symmetries  $\mathcal{S}(\Delta_1^r)$  is a noetherian maximal order in its simple artinian ring of fractions and has Goldie rank  $r$ . Moreover, it is generated by the elements

$$E_1 + d; \quad \partial_{X_k}, \quad 1 \leq k \leq n; \quad D_{j,k}, \quad 1 \leq j, k \leq n; \quad F\partial_{X_j} - 2X_j(E_1 + d), \quad 1 \leq j \leq n.$$

(2) Consider  $\mathcal{A} = \mathbb{C}[\partial_{X_1}, \dots, \partial_{X_n}]/(\Delta_1^r)$ . If  $n$  is even with  $r < n/2$  or if  $n$  is odd, then  $\mathcal{A}$  is a simple  $\mathcal{S}(\Delta_1^r)$ -module. If  $n$  is even with  $r \geq n/2$ , then  $\mathcal{A}$  has a unique proper factor module, which is finite-dimensional.

*Proof.* (1) Since  $\mathcal{S}(\Delta_1^r)$  is generated by the Fourier transforms of the generators of  $\mathcal{D}(R_r)$ , the assertion about generators is simply the translation of Proposition 3.8 and Lemma 8.4. For the remaining assertions, combine Proposition 8.5 and Theorem 7.6.

(2) Under the Fourier transform, this is a direct consequence of (1) and Theorem 4.13. Up to a change of Borel, the weights of these modules are also given by that theorem.  $\square$

### 9. The real case

In this section we show that Theorems 7.6 and 8.6 have natural analogues for differential operators with real coefficients, thereby proving Theorem 1.1 from the introduction. There are in fact a number of different real forms of  $\mathfrak{so}(n + 2, \mathbb{C})$ , each with their own Laplacian, and these are all covered by our results.

To make this precise, we will always write  $A_{\mathbb{R}} = \mathbb{R}[X_1, \dots, X_n] \subset A = \mathbb{C}[X_1, \dots, X_n]$  and set  $\mathcal{D}(A_{\mathbb{R}}) = \mathbb{R}[X_1, \dots, X_n, \partial_{X_1}, \dots, \partial_{X_n}]$ . Throughout this section we fix  $p, q \in \mathbb{N}$  with  $p + q = n \geq 3$ . Then the elements  $F = \sum X_j^2$  and  $\Delta_1 = \sum \frac{1}{2}\partial_{X_j}^2$  can be replaced by, respectively,

$$\tilde{F} = \tilde{F}_p = \sum_{j=1}^p X_j^2 - \sum_{j=p+1}^q X_j^2 \in A_{\mathbb{R}}$$

and

$$\square = \square_p = \frac{1}{2} \left( \sum_{j=1}^p \partial_{X_j}^2 - \sum_{j=p+1}^n \partial_{X_j}^2 \right) \in \mathcal{D}(A_{\mathbb{R}}).$$

(Of course,  $F = \tilde{F}_n$  and  $\Delta_1 = \square_n$ .) The signature of the quadratic form  $\tilde{F}$  will be denoted by  $(p, q)$  and the operator  $\square$  is called the d'Alembertian (or Laplacian) on the quadratic space  $\mathbb{R}^{p,q} = (\mathbb{R}^n, \tilde{F})$ .

For fixed  $r \geq 1$ , set  $S = S_{p,r} = A_{\mathbb{R}} / \tilde{F}^r A_{\mathbb{R}}$  and  $\tilde{S} = A / \tilde{F}^r A = S \otimes_{\mathbb{R}} \mathbb{C}$ . Define an automorphism  $\phi = \phi_{p,q}$  of  $\mathcal{D}(A)$ , with  $\phi(A) = A$ , by setting

$$\phi(X_j) = \begin{cases} X_j & \text{if } 1 \leq j \leq p, \\ iX_j & \text{if } p + 1 \leq j \leq n; \end{cases} \quad \text{and} \quad \phi(\partial_{X_j}) = \begin{cases} \partial_{X_j} & \text{if } 1 \leq j \leq p, \\ -i\partial_{X_j} & \text{if } p + 1 \leq j \leq n. \end{cases} \quad (9.1)$$

Note that  $\phi(F) = \tilde{F}$  and so  $\mathcal{D}(A / \tilde{F}^r A) \cong \mathcal{D}(A / F^r A)$ .

Fix  $d = d_n = n/2 - r$  and recall from Notation 3.9 that we have a copy  $\tilde{\mathfrak{g}} = \tilde{\mathfrak{g}}_d \subset \mathcal{D}(A)$  of  $\mathfrak{g} = \mathfrak{so}(n + 2, \mathbb{C})$ . We now want to describe a real analogue  $\mathfrak{s} \subset \mathcal{D}(A_{\mathbb{R}})$  of  $\tilde{\mathfrak{g}}$ . By analogy with (3.5) and (3.7), set:

- (a)  $\tilde{D}_{kl} = \tilde{D}_{lk} = X_k \partial_{X_l} + X_l \partial_{X_k}$  if  $1 \leq k \leq p < l \leq n$ , while  $\tilde{D}_{kk} = 0$ ;
- (b)  $\tilde{P}_j = \tilde{P}_j(d) = \begin{cases} X_j \square - (\mathbf{E}_1 + d) \partial_{X_j} & \text{if } 1 \leq j \leq p, \\ X_j \square + (\mathbf{E}_1 + d) \partial_{X_j} & \text{if } p + 1 \leq j \leq n. \end{cases}$

Observe that

$$\begin{aligned} X_j &= \phi(X_j) & \text{for } j \leq p, & & X_j &= -i\phi(X_j) & \text{for } j \geq p + 1, \\ \tilde{P}_j &= \phi(P_j) & \text{for } j \leq p, & & \tilde{P}_j &= -i\phi(P_j) & \text{for } j \geq p + 1, \\ \tilde{D}_{kl} &= i\phi(D_{kl}) = -i\phi(D_{lk}) & \text{for } 1 \leq k \leq p < l \leq n. & & & & \end{aligned} \tag{9.2}$$

The next result is standard.

LEMMA 9.3. Let  $\mathfrak{s} = \mathfrak{s}(p, r) \subset \mathcal{D}(A_{\mathbb{R}})$  be the real subspace spanned by the elements

$$\{\mathbf{E}_1 + d; X_k; \tilde{P}_j, 1 \leq j, k \leq n; D_{k,l}, 1 \leq k, l \leq p \text{ or } p + 1 \leq k, l \leq n; \tilde{D}_{kl}, 1 \leq k \leq p < l \leq n\}.$$

Then:

- (1)  $\mathfrak{s}$  is a Lie subalgebra of  $(\mathcal{D}(A_{\mathbb{R}}), [ , ])$  with  $\mathfrak{s} \otimes_{\mathbb{R}} \mathbb{C} \cong \tilde{\mathfrak{g}} \cong \mathfrak{so}(n + 2, \mathbb{C})$ ;
- (2) moreover,  $\mathfrak{s} \subset \mathbb{I}_{\mathcal{D}(A_{\mathbb{R}})}(\tilde{\mathbf{F}}_p^r \mathcal{D}(A_{\mathbb{R}}))$ , hence inducing a ring homomorphism  $\tilde{\varphi}_{\mathbb{R}} : U(\mathfrak{s}) \rightarrow \mathcal{D}(S_{p,r})$ .

*Proof.* (1) It is a routine exercise to see that  $\mathfrak{s}$  is a Lie algebra; indeed, one can use (9.2) to reduce this claim to the formulæ in Lemma 3.6 and Proposition 3.8. By construction, the generators of  $\phi(\mathfrak{s})$  also span  $\tilde{\mathfrak{g}}$ , whence  $\mathfrak{s} \otimes_{\mathbb{R}} \mathbb{C} \cong \tilde{\mathfrak{g}}$ .

(2) Mimic the proof of Lemma 3.10 and Corollary 3.14. □

PROPOSITION 9.4. Let  $d = d_n = n/2 - r$  and  $\mathfrak{s} = \mathfrak{s}(p, r)$ . Then  $\tilde{\varphi}_{\mathbb{R}}$  yields an isomorphism

$$\tilde{\psi}_{\mathbb{R}} : U(\mathfrak{s}) / (J_r \cap U(\mathfrak{s})) \xrightarrow{\sim} \mathcal{D}(S_{p,r}).$$

*Proof.* Recall that  $U(\mathfrak{s}) \otimes_{\mathbb{R}} \mathbb{C} \cong U(\tilde{\mathfrak{g}})$  by Lemma 9.3. Also, by construction  $\phi(\mathbf{F}^r) = \tilde{\mathbf{F}}^r$ , and hence it induces an isomorphism  $\phi : \mathcal{D}(R_r) \rightarrow \mathcal{D}(\tilde{S})$ . Thus one has a natural map  $\tilde{\varphi} = \phi \circ \psi_r$  from  $U(\tilde{\mathfrak{g}})$  to  $\mathcal{D}(\tilde{S})$  which, by Theorem 7.6, is surjective.

But if  $\tilde{\varphi}_{\mathbb{R}}$  is the map from Lemma 9.3, then  $\tilde{\varphi}_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C} = \tilde{\varphi}$ . By faithful flatness of the functor  $- \otimes_{\mathbb{R}} \mathbb{C}$ , this implies that  $\tilde{\varphi}_{\mathbb{R}}$  is surjective and  $\text{Ker}(\tilde{\varphi}_{\mathbb{R}}) = \text{Ker}(\tilde{\varphi}) \cap U(\mathfrak{s})$ . (Notice that this also proves that  $\mathcal{D}(R_r) \cong \mathcal{D}(\tilde{S}) = \mathcal{D}(S_{p,r}) \otimes_{\mathbb{R}} \mathbb{C}$ .) Finally, by Theorem 7.6 again,  $\text{Ker}(\tilde{\varphi}) = \text{Ker}(\psi) = J_r$ , as required. □

The real forms of  $\mathfrak{so}(n + 2, \mathbb{C})$  are classified, as in [Hel01, ch. X], and in Helgason’s notation they are the real Lie algebras  $\mathfrak{so}(p' + 1, q' + 1)$  for  $p' + q' = n$ . It is therefore not surprising that we have the following result.

PROPOSITION 9.5. The Lie algebra  $\mathfrak{s} = \mathfrak{s}(p, r)$  is isomorphic to  $\mathfrak{so}(p + 1, q + 1)$  for  $p + q = n$ .

*Proof.* The proof is omitted since it follows from straightforward, but not particularly illuminating, examination of the relations in  $\tilde{\mathfrak{g}}$ . □

Combined with Proposition 9.4 this gives the next theorem.

**THEOREM 9.6.** *Let  $p + q = n \geq 3$ ,  $r \geq 1$  and  $\tilde{F}_p = \sum_{j=1}^p X_j^2 - \sum_{j=p+1}^n X_j^2 \in A_{\mathbb{R}} = \mathbb{R}[X_1, \dots, X_n]$ . Then*

$$\mathcal{D}(A_{\mathbb{R}}/\tilde{F}_p^r A_{\mathbb{R}}) \cong \frac{U(\mathfrak{so}(p + 1, q + 1))}{J_r \cap U(\mathfrak{so}(p + 1, q + 1))},$$

where  $J_r$  is the primitive ideal of  $U(\mathfrak{so}(n + 2, \mathbb{C}))$  described in Corollary 4.17.

In §8 we showed that the higher symmetries of the Laplacian are equal to a factor of the enveloping algebra of  $\mathfrak{so}(n + 2, \mathbb{C})$ . However, in applications (see, for example, [BS90, Eas05]) one is interested in the real case. To end this section we show that those results from §8 also descend readily to the real case.

We are therefore interested in symmetries of powers of the d'Alembertian  $\square = \square_p$  for a fixed integer  $p$ . Now the basic results from §8 do restrict to the real field. In particular, the Fourier transform  $\mathcal{F}$  from (8.3) is well-defined on  $\mathcal{D}(A_{\mathbb{R}})$  and, by Lemma 8.4, satisfies  $\mathcal{F}(\square) = 2\tilde{F}_p$ . Therefore, the proof of Proposition 8.5 can be used mutatis mutandis to show that  $\mathcal{F}$  induces an anti-isomorphism

$$\mathcal{D}(S) \longrightarrow \mathcal{S}(\square^r) \quad \text{for } S = S_{p,r} = A_{\mathbb{R}}/\tilde{F}_p^r A_{\mathbb{R}}. \tag{9.7}$$

It is known (see, for example, [DFG13, 2.2 and 2.3]) that for each real form  $\mathfrak{so}(p + 1, q + 1)$  of  $\mathfrak{g}$  there exists a Chevalley anti-involution  $\varkappa$  on  $\mathfrak{g}$  which stabilises this real form. Moreover,  $\varkappa$  fixes  $J_r$  by [Jan83, 5.2(2)]. It then follows from Theorem 9.6 that  $\varkappa$  induces an anti-automorphism  $\varkappa$  on  $\mathcal{D}(S)$ . We therefore obtain the following analogue of Theorem 8.6.

**THEOREM 9.8.** *Let  $n = p + q \geq 3$  and  $r \geq 1$ , and let  $\square = \square_p$  be the d'Alembertian on  $\mathbb{R}^{p,q}$ . There exists a primitive ideal  $J_r$  of  $U(\mathfrak{so}(n + 2, \mathbb{C}))$  and algebra isomorphisms*

$$\mathcal{F} \circ \varkappa \circ \tilde{\psi}_{\mathbb{R}} : \frac{U(\mathfrak{so}(p + 1, q + 1))}{(J_r \cap U(\mathfrak{so}(p + 1, q + 1)))} \xrightarrow{\sim} \mathcal{D}(S_{p,r}) \xrightarrow{\sim} \mathcal{S}(\square^r).$$

Consequently,  $\mathcal{S}(\square^r)$  is a primitive noetherian ring that is a maximal order in its simple artinian quotient ring. Moreover, it is generated as an algebra by the elements

$$\{\mathbf{E}_1 + d; \partial_{X_k}; \tilde{Q}_j, 1 \leq j, k \leq n; D_{k,l}, 1 \leq k, l \leq p \text{ or } p + 1 \leq k, l \leq n; \tilde{D}_{kl}, 1 \leq k \leq p < l \leq n\},$$

thought of as differential operators on  $S_{p,r}$ . Here

$$\tilde{Q}_j = \mathcal{F}(\tilde{P}_j) = \begin{cases} \tilde{F} \partial_{X_j} - 2X_j(\mathbf{E}_1 + d) & \text{if } 1 \leq j \leq p, \\ \tilde{F} \partial_{X_j} + 2X_j(\mathbf{E}_1 + d) & \text{if } p + 1 \leq j \leq n. \end{cases}$$

*Proof.* The isomorphisms follow by combining Theorem 9.6 with (9.7). It follows immediately that  $\mathcal{S} = \mathcal{S}(\square^r)$  is noetherian. The fact that  $\mathcal{S}$  has the specified generators then follows from the Fourier transform applied to Lemma 9.3.

It remains to prove that  $\mathcal{S}$  (or equivalently  $\mathcal{D}(S)$ ) is a primitive maximal order. Recall from the proof of Proposition 9.4 that  $\mathcal{D}(S) \otimes_{\mathbb{R}} \mathbb{C} = \mathcal{D}(\tilde{S}) \cong \mathcal{D}(R_r)$ . Thus  $\mathcal{D}(\tilde{S})$  is primitive and it follows, for example from [MR00, 10.1.9], that  $\mathcal{D}(S)$  is primitive. If  $\mathcal{D}(S)$  is not a maximal order, there exists an overring  $\mathcal{D}(S) \subset T$  with  $aTb \subseteq \mathcal{D}(S)$ , for some regular elements  $a, b \in \mathcal{D}(S)$ . Tensoring with  $\mathbb{C}$  shows that  $a(T \otimes \mathbb{C})b \subseteq \mathcal{D}(\tilde{S})$  and hence, by Theorem 7.6, that  $T \otimes \mathbb{C} \subseteq \mathcal{D}(\tilde{S})$ . By the faithful flatness of  $- \otimes_{\mathbb{R}} \mathbb{C}$ , this forces  $T = \mathcal{D}(S)$ . Thus,  $\mathcal{D}(S)$  is a maximal order.  $\square$

*Remark 9.9.* The theorem recovers the generators for  $\mathcal{S}(\square^r)$  given in [ShS92, (4.8)].

COROLLARY 9.10. *Keeping the notation of the theorem, consider  $\mathcal{A} = \mathbb{R}[\partial_{x_1}, \dots, \partial_{x_n}]/(\square_p^r)$  under its natural  $\mathcal{S}(\square_p^r)$ -module structure. If  $n$  is even with  $r \geq n/2$ , then  $\mathcal{A}$  has a unique proper factor module, which is finite-dimensional. Otherwise  $\mathcal{A}$  is an irreducible module.*

*Proof.* Passing from the  $\mathcal{S}(\square_p^r)$ -module  $\mathcal{A}$  to the  $\mathcal{S}(\Delta_1^r)$ -module  $\mathcal{A}_{\mathbb{C}} = \mathbb{C}[\partial_{x_1}, \dots, \partial_{x_n}]/(\Delta_1^r)$  is given by  $\phi^{-1} \circ (- \otimes_{\mathbb{R}} \mathbb{C})$ , where the automorphism  $\phi$  of  $\mathcal{S}(\Delta_1^r)$  is induced by (9.1). We claim that the corollary follows by the faithful flatness of  $\phi^{-1} \circ (- \otimes_{\mathbb{R}} \mathbb{C})$  and Theorem 8.6(2).

In order to prove the claim, we first note that, as a  $\mathcal{S}(\Delta_1^r)$ -module,  $\mathcal{A}_{\mathbb{C}}$  satisfies the claimed results by Theorem 8.6(2). If  $\mathcal{A}$  is not a simple  $\mathcal{S}(\Delta_1^r)$ -module, then faithful flatness implies that the same is true of  $\mathcal{A}_{\mathbb{C}}$  as an  $\mathcal{S}(\square_p^r)$ -module and hence  $n$  must be even with  $r \geq n/2$ . Moreover, faithful flatness and Theorem 8.6(2) further imply that  $\mathcal{A}$  will then have a simple infinite-dimensional submodule with a simple finite-dimensional factor module.

Conversely, if  $\mathcal{A}_{\mathbb{C}}$  is not simple as an  $\mathcal{S}(\Delta_1^r)$ -module, then it has a finite-dimensional factor module, as an  $\mathcal{S}(\Delta_1^r)$ -module and hence as an  $\mathcal{S}(\square_p^r)$ -module. Since  $\mathcal{A}_{\mathbb{C}} \cong \mathcal{A}^{(2)}$  as  $\mathcal{S}(\square_p^r)$ -modules, this implies that  $\mathcal{A}^{(2)}$ , and hence  $\mathcal{A}$ , also has a finite-dimensional factor module and so is certainly not simple. □

### 10. Harmonic polynomials and $\mathcal{O}$ -duality

The set of harmonic polynomials  $H = \{p \in A : \Delta_1(p) = 0\}$  and its real analogues are fundamental objects with many applications, notably in Lie theory (see, for example, [GW98, HSS12, KØ03]) and physics (see, for example, [Bek09, Bek11]). In the latter subject, Dirac [Dir35, Dir36] constructed a remarkable unitary irreducible representation  $D(n/2 - 1, 0)$  of the Lie algebra  $\mathfrak{so}(n, 2)$ , known as the scalar singleton module. This module can be realised in different ways (see [Bek09, Bek11]): as a highest-weight module, as harmonic scalar fields  $\phi$  (in the sense that  $\square_{n-1} \cdot \phi = 0$ ) on the space  $\mathbb{R}^{n-1,1}$  which are preserved by  $\mathfrak{so}(n, 2)$ , or as harmonic distributions  $\varphi$  of weight  $1 - n/2$  on the ‘ambient space’  $\mathbb{R}^{n,2}$  (hence  $\square_{n,2} \cdot \varphi = 0$ ). The algebra of symmetries of the scalar singleton, as defined in [Bek09, 4.4], acts on  $D(n/2 - 1, 0)$ . This algebra can be identified, thanks to the results of [Eas05] and [Vas03], with the algebra  $\mathcal{S}(\square_{n-1,1})$  (see also Remark 11.14(2)).

Replacing the condition  $\square_{n-1} \cdot \phi = 0$  by  $\square_{n-1}^r \cdot \phi = 0$  generalises the singleton module to give *higher singleton modules*. These have been studied in [BG13], with applications to anti-de Sitter gauge fields and related topics.

It is therefore natural to give a mathematical description of these higher singletons or, more generally, of the space  $\mathcal{M}_{p,r} = \{a \in A_{\mathbb{R}} : \square_p^r(a) = 0\}$ . This is the topic of the current section, where we describe the structure of  $\mathcal{M}_{p,r}$  as a representation of the orthogonal Lie algebra  $\mathfrak{so}(p+1, q+1)$  and show that its complex analogue is simply the category- $\mathcal{O}$  dual of  $N(\lambda) = R_r$  (see Corollaries 10.12 and 10.13 for the precise statements).

We now make this precise. We continue with the notation from §§ 2 and 4 and start with the formal definitions. Since we will be able to prove results for both the real and the complex cases, we fix  $1 \leq p \leq n$  and keep the notation  $\tilde{F}_p \in A_{\mathbb{R}}$  as well as  $\mathfrak{s} = \mathfrak{s}(p, r) \subset \mathcal{D}(A_{\mathbb{R}})$  and  $\square_p \in \mathcal{D}(A_{\mathbb{R}})$  from the beginning of § 9. The spaces that interest us are defined as follows.

DEFINITION 10.1. Let  $r \in \mathbb{N}^*$ . The elements of the subspace

$$\mathcal{M}_{p,r} = \{a \in A_{\mathbb{R}} : \square_p^r(a) = 0\}$$

will be called *higher harmonics* or *harmonics of level  $r$  in  $A_{\mathbb{R}}$* . We clearly also have the analogous complex space

$$M_{p,r} = \{a \in A : \square_p^r(a) = 0\}.$$

Observe that  $A = A_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$  and that  $\square_p^r$  has real coefficients. Hence, if  $a = u + iv \in A$  with  $u, v \in A_{\mathbb{R}}$ , we have that  $\square_p^r(a) = \square_p^r(u) + i\square_p^r(v) = 0 \iff \square_p^r(u) = \square_p^r(v) = 0$ . Therefore  $M_{p,r} = \mathcal{M}_{p,r} \otimes_{\mathbb{R}} \mathbb{C}$ .

In the notation of Theorem 9.8, set  $\tau = \mathcal{F} \circ \varkappa \circ \tilde{\psi}_{\mathbb{R}}$ . Thus  $\tau : \mathfrak{s} \rightarrow \mathcal{S}(\square_p^r)$  is a Lie algebra homomorphism that induces an isomorphism  $\tau : U(\mathfrak{s})/\mathcal{J}_r \cong \mathcal{S}(\square_p^r)$  for the appropriate ideal  $\mathcal{J}_r$ . By Proposition 9.4, upon tensoring with the complex numbers, we obtain an isomorphism  $\tau \otimes_{\mathbb{R}} \mathbb{C} : U(\mathfrak{g})/J_r \cong \mathcal{S}(\Delta_1^r)$ .

LEMMA 10.2. (1) *The map  $\tau$  defines a representation of  $\mathfrak{s}$  in the space  $\mathcal{M}_{p,r}$  of higher harmonics.*  
 (2) *Similarly,  $\tau \otimes_{\mathbb{R}} \mathbb{C}$  defines a representation of  $\mathfrak{g}$  in the space  $M_{p,r}$ .*

*Proof.* It suffices to prove (1). Let  $f \in \mathcal{M}_{p,r}$  and  $Y \in \mathfrak{s}$ . Then  $\mathcal{Y} = \tau(Y)$  is a symmetry of the operator  $\square_p^r$ , whence  $\square_p^r(\mathcal{Y}(f)) = (\square_p^r \cdot \mathcal{Y})(f) \in \mathcal{D}(A_{\mathbb{R}}) \cdot \square_p^r(f) = 0$ . Therefore  $\mathcal{Y}(f) \in \mathcal{M}_{p,r}$  and so  $\tau$  takes values in  $\text{End}_{\mathbb{R}}(\mathcal{M}_{p,r})$ , as required.  $\square$

Remark 10.3. In fact, the  $\mathfrak{g}$ -modules  $M_{p,r}$  are twists of each other (see the proof of Corollary 10.13 for the details), so it will suffice to prove results for just one of them. We will use the module

$$M_r = M_{n,r} = \{a \in A : \Delta_1^r(a) = 0\},$$

as was also defined in the introduction. For most of this section we will study this module.

We are interested in the following pairing.

DEFINITION 10.4. The pairing  $\langle | \rangle$  is defined by

$$\langle | \rangle : A \times A \longrightarrow \mathbb{C}, \quad \langle a | f \rangle = \mathcal{F}(a)(f)|_0 = \mathcal{F}(a)(f)(0).$$

The *orthogonal* of a subspace  $W \subseteq A$  is written  $W^\perp = \{f \in A : \langle W | f \rangle = 0\}$ .

Recall from §2 that the Lie subalgebra  $\mathfrak{k} \subset \mathfrak{g}$  generated by the  $E_{jk}$ ,  $2 \leq j, k \leq \ell$ , is isomorphic to  $\mathfrak{so}(n, \mathbb{C})$ . The next result is an easy variant on classical results, but since we could not find an appropriate reference, we include a proof in the Appendix.

LEMMA 10.5. Set  $\mathcal{I}_r = \tilde{\mathbb{F}}_p^r A$ .

- (1) *The bilinear form  $\langle | \rangle$  is symmetric and nondegenerate. It is also  $\mathfrak{k}$ -invariant in the sense that  $\langle Y.a | f \rangle + \langle a | Y.f \rangle = 0$  for all  $a, f \in A$  and  $Y \in \mathfrak{k}$ .*
- (2)  *$M_r^\perp = \mathcal{I}_r$  and  $\mathcal{I}_r^\perp = M_r$ .*
- (3) *The form  $\langle | \rangle$  induces a nondegenerate  $\mathfrak{k}$ -invariant symmetric pairing  $\langle | \rangle : R_r \times M_r \longrightarrow \mathbb{C}$ .*

By Lemma 10.2,  $M_r$  has a  $\mathfrak{g}$ -module structure given by  $a \mapsto \tau(Y).a = \mathcal{F}(\tilde{\psi}(\vartheta(Y)))(a)$  for all  $Y \in \mathfrak{g}$  and  $a \in M_r$ . On the other hand,  $R_r = N(\lambda)$  is a  $\mathfrak{g}$ -module through the map  $\psi_r : U(\mathfrak{g}) \rightarrow \mathcal{D}(R)$  induced by  $\tilde{\psi}$ . We set  $Y.p = \psi_r(Y)(p)$  for all  $Y \in \mathfrak{g}$  and  $p \in R_r$ . These structures are related as follows.

THEOREM 10.6. *The pairing  $\langle | \rangle : R_r \times M_r \longrightarrow \mathbb{C}$  is  $\mathfrak{g}$ -invariant in the sense that*

$$\langle Y.p | g \rangle = \langle p | \tau(\vartheta(Y)).g \rangle = \langle p | \mathcal{F}(\tilde{\psi}(Y))(g) \rangle$$

for all  $Y \in \mathfrak{g}$ ,  $p \in R_r$  and  $g \in M_r$ .

*Proof.* We first claim that the result is equivalent to proving that

$$\mathcal{F}[\tilde{\psi}(Y)(p)](g)|_0 = \mathcal{F}(p)[\mathcal{F}(\tilde{\psi}(Y))(g)]|_0. \tag{10.7}$$

To see this, note that  $\langle Y.p | g \rangle = \langle \tilde{\psi}(Y)(p) | g \rangle = \mathcal{F}[\tilde{\psi}(Y)(p)](g)(0)$  and  $\tau(\vartheta(Y)).g = \mathcal{F}(\tilde{\psi}(Y))(g)$ . Hence  $\langle p | \tau(\vartheta(Y)).g \rangle = \mathcal{F}(p)[\mathcal{F}(\tilde{\psi}(Y))(g)](0)$ , as claimed.

Write  $V_j = \frac{1}{2}U_j\Delta_1 - I(d)\partial_{U_{-j}}$  as in (4.2). The following formulæ, which are easily checked, will be needed in the proof:

$$\mathcal{F}(E_{ab}) = E_{ba} \quad \text{and} \quad Q_b = \mathcal{F}(V_b) = \mathbf{F}\partial_{U_b} - U_{-b}I(d) \quad \text{for } a, b \in \{0, \pm 2, \dots, \pm \ell\}. \tag{10.8}$$

Recall from § 2 that  $\mathfrak{g} = \mathfrak{r}^- \oplus \mathfrak{m} \oplus \mathfrak{r}^+$  with  $\mathfrak{m} = \mathbb{C}E_{11} \oplus \mathfrak{k}$ . Observe that if (10.7) is true for  $Y, Z \in \mathfrak{g}$ , then it is true for  $[Y, Z]$ . Thus, we can reduce the verification to the case where  $Y$  is a (scalar multiple of a) root vector  $y_\alpha$  of the Chevalley basis given in Proposition 2.4. We may therefore prove (10.7) by considering the cases  $Y \in \mathfrak{k}$ ,  $Y \in \mathfrak{r}^+$  and  $Y \in \mathfrak{r}^-$  separately.

Suppose first that  $Y = y_\alpha \in \mathfrak{k}$  for  $\alpha \in \Phi_1$ . From Proposition 2.4 we may assume that  $Y = E_{ab}$  with  $a, b \in \{0, \pm 2, \dots, \pm \ell\}$ . Then  $\tilde{\psi}(Y) = Y = E_{ab}$  and, since  $Y$  is a derivation,  $Y(p) = [Y, p]$ . Hence  $\mathcal{F}(Y(p)) = [\mathcal{F}(p), \mathcal{F}(Y)]$  and we get  $\mathcal{F}(Y(p))(g) = \mathcal{F}(p)(\mathcal{F}(Y)(g)) - \mathcal{F}(Y)(\mathcal{F}(p)(g))$ . But  $\mathcal{F}(Y) = E_{ba} = U_b\partial_{U_a} - U_{-a}\partial_{U_{-b}}$  (see (10.8)), and so  $\mathcal{F}(Y)(g)(0) = 0$  for all  $g \in A$  (in particular for  $g = \mathcal{F}(p)(g)$ ). We then get  $\mathcal{F}(Y(p))(g)|_0 = \mathcal{F}(p)(\mathcal{F}(Y)(g))|_0$ , as desired.

Suppose next that  $Y = y_\alpha \in \mathfrak{r}^-$ , where either  $\alpha = -(\varepsilon_1 \pm \varepsilon_b)$  or  $\alpha = -\varepsilon_1$ . Here, since  $\tilde{\psi}(y_\alpha) = x_\alpha$ , we may assume that  $\tilde{\psi}(Y) = U_b$  for some  $b \in \{0, \pm 2, \dots, \pm \ell\}$ ; see Lemma 4.3. Therefore we have

$$\mathcal{F}[\tilde{\psi}(Y)(p)](g) = \mathcal{F}(U_b p)(g) = [\mathcal{F}(p)\mathcal{F}(U_b)](g) = \mathcal{F}(p)[\mathcal{F}(\tilde{\psi}(Y))(g)]$$

and, in particular,  $\mathcal{F}[\tilde{\psi}(Y)(p)](g)|_0 = \mathcal{F}(p)[\mathcal{F}(\tilde{\psi}(Y))(g)]|_0$ .

Finally, suppose that  $Y = y_\alpha \in \mathfrak{r}^+$ , where either  $\alpha = \varepsilon_1 \pm \varepsilon_b$  or  $\alpha = \varepsilon_1$ . From Lemma 4.3 we may assume that  $\tilde{\psi}(Y) = V_b$  for some  $b \in \{0, \pm 2, \dots, \pm \ell\}$ . Recall from (10.8) that  $\mathcal{F}(V_b) = Q_b = \mathbf{F}\partial_{U_b} - U_{-b}I(d)$ . We will need an auxiliary calculation: for  $p, f \in A$ , we claim that

$$\begin{aligned} \mathcal{F}(\Delta_1 p - \Delta_1(p))(f)|_0 &= \left[ \mathcal{F}(p\Delta_1)(f) + 2 \left\{ U_0\mathcal{F}(\partial_{U_0}(p)) + \sum_{\pm j=2}^{\ell} U_{-j}\mathcal{F}(\partial_{U_j}(p)) \right\} (f) \right] (0) \\ &= \mathcal{F}(p\Delta_1)(f)|_0 = 2\mathbf{F}\mathcal{F}(p)(f)|_0 = 0. \end{aligned} \tag{10.9}$$

To see this, notice that  $\mathcal{F}(\Delta_1 p - \Delta_1(p)) = \mathcal{F}(p\Delta_1) + \mathcal{F}(2\{\partial_{U_0}(p)\partial_{U_0} + \sum_{\pm j=2}^{\ell} \partial_{U_j}(p)\partial_{U_{-j}}\})$ . Then (10.9) follows by applying this equality to  $f$  and then evaluating at 0.

We now return to the proof in the case where  $\tilde{\psi}(Y) = V_b$ . We have

$$\mathcal{F}[\tilde{\psi}(Y)(p)] = \mathcal{F}(V_b(p)) = \mathcal{F}(\frac{1}{2}U_b\Delta_1(p) - I(d)(\partial_{U_{-b}}(p))) = \frac{1}{2}\mathcal{F}(\Delta_1(p))\partial_{U_b} - \mathcal{F}[I(d)(\partial_{U_{-b}}(p))].$$

On the other hand, using

$$\mathcal{F}(I(d)\partial_{U_{-b}}p)(g) = [\mathcal{F}(p)U_{-b}\mathcal{F}(I(d))](g) = [\mathcal{F}(p)U_{-b}I(d)](g) = \mathcal{F}(p)(U_{-b}I(d)(g)),$$



we get

$$\begin{aligned} \mathcal{F}(p)[\mathcal{F}(\tilde{\psi}(Y))(g)] &= \mathcal{F}(p)(Q_b(g)) = \mathcal{F}(p)(\mathbb{F}\partial_{U_b}(g) - U_{-b}I(d)(g)) \\ &= \frac{1}{2}\mathcal{F}(\Delta_1 p)(\partial_{U_b}(g)) - \mathcal{F}(I(d)\partial_{U_{-b}}p)(g). \end{aligned}$$

It follows that

$$\begin{aligned} \mathcal{F}(p)[\mathcal{F}(\tilde{\psi}(Y))(g)] - \mathcal{F}[\tilde{\psi}(Y)(p)](g) &= \frac{1}{2}\{\mathcal{F}(\Delta_1 p) - \mathcal{F}(\Delta_1(p))\}(\partial_{U_b}(g)) \\ &\quad - \{\mathcal{F}(I(d)\partial_{U_{-b}}p) - \mathcal{F}[I(d)(\partial_{U_{-b}}(p))]\}(g). \end{aligned} \tag{10.10}$$

It is easily checked that  $I(d)\partial_{U_{-b}}p - I(d)(\partial_{U_{-b}}(p)) = I(d)p\partial_{U_{-b}} + \partial_{U_{-b}}(p)\mathbb{E}_1$ . Therefore

$$\mathcal{F}[I(d)\partial_{U_{-b}}p - I(d)(\partial_{U_{-b}}(p))](g) = U_{-b}\mathcal{F}(p)(I(d)(g)) + \mathbb{E}_1(\mathcal{F}(\partial_{U_{-b}}(p))(g)),$$

and this polynomial vanishes at 0 since  $(U_{-b}q)|_0 = \mathbb{E}_1(q)|_0 = 0$  for all  $q \in A$ . Equation (10.10) then gives  $\{\mathcal{F}(p)[\mathcal{F}(\tilde{\psi}(Y))(g)] - \mathcal{F}[\tilde{\psi}(Y)(p)](g)\}(0) = \frac{1}{2}\mathcal{F}(\Delta_1 p - \Delta_1(p))(\partial_{U_b}(g))(0)$ . Now applying (10.9) to  $f = \partial_{U_b}(g)$  yields  $\{\mathcal{F}(p)[\mathcal{F}(\tilde{\psi}(Y))(g)] - \mathcal{F}[\tilde{\psi}(Y)(p)](g)\}(0) = 2\mathbb{F}\mathcal{F}(p)(\partial_{U_b}(g))|_0 = 0$ , as required.  $\square$

As in [Jan83, §2.1], the dual  $M^*$  of a  $\mathfrak{g}$ -module  $M$  is endowed with a  $\mathfrak{g}$ -module structure through

$$(Y.f)(x) = f(\vartheta(Y).x) \quad \text{for all } Y \in \mathfrak{g}, f \in \mathfrak{g}^* \text{ and } x \in M. \tag{10.11}$$

Let  $\mathcal{O}$  denote the category of highest-weight modules, as in §2. If  $M \in \text{Ob } \mathcal{O}$ , then  $M = \bigoplus_{\mu \in \mathfrak{h}^*} M^\mu$  where  $M^\mu$  is the  $\mu$ -weight space of  $\mathfrak{h}$  in  $M$ . As in [Jan83, 4.10], the  $\mathcal{O}$ -dual  $M^\vee$  of  $M$  is then defined by

$$M^\vee = \bigoplus_{\mu \in \mathfrak{h}^*} (M^\vee)^\mu, \quad (M^\vee)^\mu \cong (M^\mu)^*.$$

By [Jan83, 4.10(2,3)], the contravariant functor  $M \rightarrow M^\vee$  is exact and satisfies  $L(\omega)^\vee \cong L(\omega)$  for all  $\omega \in \mathfrak{h}^*$ .

For the next result, recall from Theorem 4.13 that if  $R_r = N(\lambda)$  is not a simple  $\mathfrak{g}$ -module, then it has a simple socle  $Z_r \cong L(\mu)$ , for an appropriate weight  $\mu$  and finite-dimensional factor  $\overline{R} = R_r/Z_r \cong L(\lambda)$ .

**COROLLARY 10.12.** *Under the  $\mathfrak{g}$ -module structure of  $M_r$  given by Lemma 10.2, there is a  $\mathfrak{g}$ -module isomorphism  $N(\lambda)^\vee \cong M_r$ . Furthermore:*

- (1) *if  $n$  is even with  $r < n/2$  or if  $n$  is odd, then  $M_r \cong L(\lambda)$ ;*
- (2) *if  $n$  is even with  $r \geq n/2$ , then  $M_r$  has a simple finite-dimensional socle  $\overline{R}^\vee \cong L(\lambda)$ , with quotient  $M_r/\overline{R}^\vee = Z_r^\vee \cong L(\mu)$ . The formula for the weight  $\mu$  is given in (4.14).*

*Proof.* By Lemma 4.8,  $N(\lambda)$  is an object of category  $\mathcal{O}$ . By Theorem 10.6, the pairing  $\langle | \rangle$  enables us to define a linear injection

$$\xi : R_r \hookrightarrow M_r^* \quad \text{given by} \quad \xi(p)(a) = \langle p | a \rangle \quad \text{for all } p \in R_r \text{ and } a \in M_r.$$

Moreover, the  $\mathfrak{g}$ -invariance of  $\langle | \rangle$  means that  $\xi(Y.p) = Y.\xi(p)$  under the  $\mathfrak{g}$ -module structure on  $M_r^*$  defined by (10.11); thus,  $\xi$  is  $\mathfrak{g}$ -linear. To deduce that  $\xi$  restricts to an isomorphism  $\xi : R_r \xrightarrow{\sim} M_r^\vee$ , it suffices to check that  $\xi : R_r^\nu \xrightarrow{\sim} (M_r^\nu)^*$  for all  $\nu \in \mathfrak{h}^*$ . Let  $Y \in \mathfrak{h}$ ,  $p \in R_r^\nu$  and  $f \in M_r^{\nu'}$ . From  $\vartheta(Y) = Y$  one gets that  $\nu(Y)\langle p | f \rangle = \nu'(Y)\langle p | f \rangle$ . Therefore  $\langle R_r^\nu | M_r^{\nu'} \rangle = 0$  for  $\nu \neq \nu'$ , and it follows that  $\langle | \rangle$  gives a nondegenerate pairing  $R_r^\nu \times M_r^\nu \rightarrow \mathbb{C}$ , as desired. Statements (1) and (2) are then consequences of the  $\mathcal{O}$ -duality and the structure of the  $\mathfrak{g}$ -module  $N(\lambda) = R_r$  obtained in Theorem 4.13.  $\square$

We now return to the real case as discussed at the beginning of the section and describe the natural analogues of Corollary 10.12.

**COROLLARY 10.13.** *Fix  $r \geq 1$  and  $1 \leq p \leq n$  and consider the  $\mathfrak{so}(p, r)$ -module  $\mathcal{M}_{p,r}$  from Definition 10.1 and Lemma 10.2. Then:*

- (1) *if  $n$  is even with  $r < n/2$  or if  $n$  is odd, then  $\mathcal{M}_{p,r}$  is simple;*
- (2) *if  $n$  is even with  $r \geq n/2$ , then  $\mathcal{M}_{p,r}$  has a simple finite-dimensional socle  $S'$ , with an infinite-dimensional simple quotient  $\mathcal{M}_{p,r}/S'$ .*

*Proof.* We first want to obtain an analogue of Corollary 10.12 for any  $M_{p,r}$ . To do this, recall that the automorphism  $\phi$  from (9.1) satisfies  $\phi(\Delta_1) = \square_p$ . Hence, if we define the twisted  $\mathcal{D}(A)$ -module structure  ${}^\phi A$  on  $A$  by  $\theta \circ a = \phi(\theta)a$  for  $\theta \in \mathcal{D}(A)$  and  $a \in A$ , then a simple computation shows that  ${}^\phi M_r = M_{p,r}$ . Since twisting preserves submodule lattices and dimensions, it follows from Corollary 10.12 that the statement of the present corollary holds for the  $\mathfrak{g}$ -modules  $M_{p,r}$ .

As was observed in (10.1),  $\mathcal{M}_{p,r} \otimes_{\mathbb{R}} \mathbb{C} = M_{p,r}$  for each  $p$  and  $r$ . Thus, just as in the proof of Corollary 9.10, faithful flatness then implies that the desired results for the  $\mathcal{M}_{p,r}$  follow from the corresponding results for the  $M_{p,r}$ . □

### 11. Conformal densities and the ambient construction

The  $\mathfrak{g}$ -module  $M_1 = H$  of harmonic polynomials from Definition 10.1 is an ‘incarnation’ of the scalar singleton module introduced by Dirac through the ambient method; cf. [Dir35, Dir36, EG91, Bek11]. In this section, we will briefly recall this construction and its generalisation to the higher harmonics, and then show that these objects have a natural algebraic analogue that gives another incarnation of the module  $M_r \cong N(\lambda)^\vee$ . This will prove Theorem 1.8 from the introduction.

The ambient method allows one to work with the standard representation of  $\mathfrak{so}(p+1, q+1)$  by linear vector fields on  $\mathbb{R}^{n+2}$  and leads to a conformal model of the representation of  $\mathfrak{g}$  on  $M_r$ . To see this, one first constructs the following conformal compactification of  $\mathbb{R}^{p,q}$ . Let  $\mathbb{Q} = U_1 U_{-1} + F_p$  be the quadratic form of signature  $(p+1, q+1)$  on  $\mathbb{R}^{n+2}$  with coordinate functions  $U_1, U_{-1}, X_1, \dots, X_n$ . and write  $\mathcal{C} = \{z \in \mathbb{R}^{n+2} : \mathbb{Q}(z) = 0\}$  for the corresponding light cone. Then the associated projective quadric  $\mathcal{Q} = \{\mathbb{Q} = 0\} \subset \mathbb{RP}^{n+1}$  is a conformal compactification of  $\mathbb{R}^{p,q}$ , under the identification of  $\mathbb{R}^{p,q}$  with the open cell  $\mathcal{Q} \cap \{U_1 = 1\}$ . (The space  $\mathcal{Q}$  is sometimes known as the boundary of the anti-de Sitter space when  $(p, q) = (n-1, 1)$ .) In this setting, a conformal density of weight  $w$  on  $\mathcal{Q}$  can be viewed as a homogeneous function of weight  $w$  on the light cone  $\mathcal{C}$ . The conformal Laplacian is the operator induced by the Laplacian  $\Delta = \partial_{U_1} \partial_{U_{-1}} + \square_p$ . It acts on densities of weight  $-n/2 + 1$ , as shown for example in [EG91, Propositions 4.4 and 4.7] and [GJMS92, Proposition 2.1], and the corresponding space of harmonic densities of weight  $-n/2 + 1$  is then an  $\mathfrak{so}(p+1, q+1)$ -module. For Minkowski space–time, when  $(p, q) = (n-1, 1)$ , this  $\mathfrak{so}(n, 2)$ -module corresponds to the scalar singleton module  $D(n/2 - 1, 0)$  of [Bek11].

Analogous questions arise for homogeneous functions on the ‘generalised light cone’  $\{\mathbb{Q}^r = 0\}$ , equivalently for densities  $\varphi$  of weight  $w$  such that  $\Delta \cdot \varphi = 0 \pmod{\mathbb{Q}^r}$ . Again, by [EG91] and [GJMS92], the above construction works for densities of weight  $-n/2 + r$ . This produces an  $\mathfrak{so}(p+1, q+1)$ -module, which in the case of the Minkowski space–time corresponds to the higher-order singleton as defined in [BG13].

In this section we aim to give an algebraic version of this construction and relate it to the  $\mathfrak{g}$ -modules  $R_r$  and  $M_r$  obtained in the previous sections. Roughly speaking, in our setting (and in the notation of § 2), the generalised light cone  $\{\mathbb{Q}^r = 0\}$  is replaced by a factor  $B/\mathbb{Q}^r B$  of a

polynomial ring in  $n + 2$  variables, while the ‘conformal Laplacian’ is replaced by the operator  $\Delta \in \mathcal{D}(B)$  and the ‘densities’ by homogeneous polynomials on (a finite extension of)  $B/\mathbb{Q}^r B$ .

We now formalise this approach. We adopt the notation of §§ 2 and 3; in particular, we use the presentation of  $\mathfrak{g}$  provided by Proposition 2.3. Following (4.1), write  $B = A[U_{-1}, U_1]$  for  $A = \mathbb{C}[U_0, U_{\pm 2}, \dots, U_{\pm \ell}]$  and let  $B[U_1^{-1}] = A[U_{-1}, U_1^{\pm 1}]$  be the localization of  $B$  at the powers of  $U_1$ . In the notation of (2.1),  $U_{-1} = (\mathbb{Q} - \mathbb{F})U_1^{-1}$  for  $\mathbb{Q} = \sum_{j=1}^{\ell} U_j U_{-j} + \frac{1}{2}U_0^2$ , and so  $B[U_1^{-1}] = A[\mathbb{Q}, U_1^{\pm 1}]$ . We also need to adjoin a square root of  $U_1$  to  $B[U_1^{-1}]$ ; thus, formally, set

$$S = B[U_1^{-1}, T^{\pm 1}]/(T^2 - U_1)$$

for an indeterminate  $T$ . The class of  $T$  in  $S$  is denoted by  $t$ . Notice that

$$S = A[U_{-1}, t^{\pm 1}] = A[\mathbb{Q}, t^{\pm 1}] = \mathbb{C}[t^{\pm 1}, U_{\pm 2}, \dots, U_{\pm \ell}, U_0, \mathbb{Q}]. \tag{11.1}$$

Clearly,  $\mathbb{Q}$  and  $t$  are indeterminates over  $A$ , and  $S = B[U_1^{-1}] \oplus B[U_1^{-1}]t$  is a free  $B[U_1^{-1}]$ -module of rank 2.

*Remark 11.2.* A derivation  $D \in \text{Der}_{\mathbb{C}}(B)$  extends to  $B[U_1^{-1}, T^{\pm 1}]$  by setting  $D(T) = (1/2T)D(U_1)$ . Then  $D((T^2 - U_1)f) = (T^2 - U_1)D(f)$  for any  $f \in B[U_1^{-1}, T^{\pm 1}]$ , and so  $D$  also defines a derivation on  $S$ . In particular, the derivations  $E_{ab} \in \mathfrak{g}$  as defined in (2.2) act on  $S$ , and this endows  $S$  with a natural  $\mathfrak{g}$ -module structure. Furthermore, the derivation  $\partial_{U_1}$  extends to  $S$  by  $\partial_{U_1}(t) = 1/2t$ , so we can write  $\partial_{U_1} = (1/2t)\partial_t \in \text{Der}_{\mathbb{C}}(S)$ . With this notation and that from (2.1) and (2.2), we can therefore write

$$\mathbf{E} = \frac{t}{2}\partial_t + \sum_{j \neq 1} U_j \partial_{U_j}, \quad E_{j1} = \frac{U_j}{2t}\partial_t - U_{-1}\partial_{U_{-j}} = \frac{U_j}{2t}\partial_t - \frac{1}{t^2}(\mathbb{Q} - \mathbb{F})\partial_{U_{-j}}$$

as elements of  $\text{Der}_{\mathbb{C}}(S)$ .

We now fix an integer  $r \in \mathbb{N}^*$  and set

$$\bar{S} = \bar{S}_r = S/\mathbb{Q}^r S.$$

Let  $D \in \text{Der}_{\mathbb{C}}(S)$  be a derivation such that  $D(\mathbb{Q}) = c\mathbb{Q}$  for some  $c \in S$ , for example  $D = \mathbf{E}$  or  $E_{ab}$ . Then  $D$  induces a derivation, again denoted by  $D$ , in  $\text{Der}_{\mathbb{C}}(\bar{S})$ . In particular, the algebra  $\bar{S}$  also has a  $\mathfrak{g}$ -module structure. As such,  $S$  and  $\bar{S}$  are graded by the weights of the Euler operator  $\mathbf{E}$  and we can make the following definition.

**DEFINITION 11.3.** Let  $p \in \mathbb{Q}$ . An element  $u \in C = S$  or  $\bar{S}$  is said to be *homogeneous of weight  $p$*  if  $\mathbf{E}(u) = pu$ . The space of homogeneous elements of weight  $p$  is denoted by  $C(p)$ . Set  $\text{wt}(0) = 0$  and  $\text{wt}(u) = p$  if  $0 \neq u \in C(p)$ ; thus  $\mathbf{E}(u) = \text{wt}(u)u$ . If  $V$  is a subspace of  $C$ , set  $V(p) = C \cap C(p)$ .

Notice that  $U_j \in S(1)$  for all  $j$  and that  $t \in S(\frac{1}{2})$ . Hence,  $S = \bigoplus_{\beta \in (1/2)\mathbb{Z}} S(\beta)$  and, since  $\mathbb{Q}^r$  is homogeneous of degree  $2r$ , we have:

$$\bar{S} = \bigoplus_{\beta \in (1/2)\mathbb{Z}} \bar{S}(\beta), \quad \text{where } \bar{S}(\beta) = S(\beta)/\mathbb{Q}^r S(\beta - 2r).$$

We can think of the elements of  $\bar{S}(\beta)$  as ‘half-densities’ on the subscheme  $\{\mathbb{Q}^r = 0\} \subset \mathbb{C}^{n+2}$ .

LEMMA 11.4. Let  $\bar{q}$  denote the class of  $q$  in  $\bar{S}$  and pick  $\beta \in \frac{1}{2}\mathbb{Z}$ . Then:

- (1)  $\bar{S}(\beta)$  is a  $\mathfrak{g}$ -submodule of the  $\mathfrak{g}$ -module  $\bar{S}$ ;
- (2)  $\bar{S} = \bigoplus_{j=0}^{r-1} A[t^{\pm 1}]\bar{q}^j$  and  $\bar{S}(\beta) = \bigoplus_{j=0}^{r-1} A[t^{\pm 1}](\beta - 2j)\bar{q}^j$  as  $A[t^{\pm 1}]$ -modules.

*Proof.* This follows from (11.1) combined with the fact that  $[E, Y] = 0$  for all  $Y \in \mathfrak{g}$ . □

By Remark 11.2, the Laplacian  $\Delta = 2 \sum_{j=1}^{\ell} \partial_{U_j} \partial_{U_{-j}} + \partial_{U_0}^2$  extends trivially to  $B[U_1^{-1}, T]$  and then restricts to  $S$ . This differential operator on  $S$ , still denoted by  $\Delta$ , can also be written as

$$\Delta = \frac{1}{t} \partial_t \partial_{U_{-1}} + \Delta_1 \quad \text{for } \Delta_1 = 2 \sum_{j \geq 2} \partial_{U_j} \partial_{U_{-j}} + \partial_{U_0}^2.$$

As usual, the set of *harmonic elements in  $S$*  is defined to be  $\mathcal{H}(S) = \{f \in S : \Delta(f) = 0\}$ . Since  $[E, \Delta] = -2\Delta$ , this is a graded subspace of  $S$  and  $\mathcal{H}(S) = \bigoplus_{\beta \in (1/2)\mathbb{Z}} \mathcal{H}(\beta)$ . Clearly,  $\Delta(S(\beta)) \subseteq S(\beta - 2)$  for all  $\beta \in \frac{1}{2}\mathbb{Z}$ .

We now want to find  $\beta$  such that  $\Delta$  induces an operator from  $\bar{S}(\beta)$  to  $\bar{S}(\beta - 2)$ . The next result can be compared with [EG91, Propositions 4.4 and 4.7], [GJMS92, Proposition 2.1] and [BZ91, II Proposition].

PROPOSITION 11.5. Let  $\delta = -d = -n/2 + r$ . Then the Laplacian  $\Delta$  induces an operator

$$\bar{\Delta}_\delta = \bar{\Delta} : \bar{S}(\delta) = \frac{S(\delta)}{q^r S(\delta - 2r)} \longrightarrow \bar{S}(\delta - 2) = \frac{S(\delta - 2)}{q^r S(\delta - 2(r + 1))}$$

by the formula  $\bar{\Delta}(\bar{f}) = \overline{\Delta(f)}$  for all  $f \in S(\delta)$ .

*Proof.* As in (3.13), one shows that for all  $k \in \mathbb{N}^*$  and homogeneous elements  $u \in S$ ,

$$\Delta(uq^k) = \Delta(u)q^k + k[n + 2 + 2wt(u) + 2(k - 1)]uq^{k-1}. \tag{11.6}$$

We are looking for  $\delta \in \frac{1}{2}\mathbb{Z}$  such that  $\Delta(uq^r) \in q^r S(\delta - 2(r + 1))$  for all  $u \in S(\delta - 2r)$ . However, by (11.6), if  $\delta = -n/2 + r$  then

$$\Delta(uq^r) = \Delta(u)q^r + r[n + 2 + 2\delta - 2(r + 1)]uq^{r-1} = \Delta(u)q^r$$

for all  $u \in S(\delta - 2r)$ . This proves the proposition. □

Thanks to Proposition 11.5, we can define the following subspace of ‘harmonic elements of degree  $\delta$ ’, or ‘harmonic conformal densities of weight  $\delta$ ’. It gives a (complex) version of the ‘scalar singleton module’ defined by Dirac; see, for example, [Bek11, § 3.5].

DEFINITION 11.7. Let  $r \in \mathbb{N}^*$  and  $\delta = -n/2 + r$  and set

$$N_\lambda = \text{Ker } \bar{\Delta}_\delta = \{\bar{f} \in \bar{S}(\delta) : \bar{\Delta}_\delta(\bar{f}) = 0\}.$$

Since  $[Y, \Delta] = 0$  for all  $Y \in \mathfrak{g}$ , it is easy to see that  $N_\lambda$  is a  $\mathfrak{g}$ -submodule of  $\bar{S}(\delta)$ .

It follows from Lemma 11.4(2) that any element  $\bar{f} \in \bar{S}(\beta)$ , for  $f \in S(\beta)$ , can be uniquely written as

$$\bar{f} = \sum_{j=0}^{r-1} f_j \bar{q}^j \quad \text{with } f = \sum_{j=0}^{r-1} f_j q^j \text{ and } f_j \in A[t^{\pm 1}](\beta - 2j). \tag{11.8}$$

Let  $a \in A$  and  $\nu \in \mathbb{Z}$ . Since  $\partial_{U_{-1}}(at^\nu) = 0$  and  $\Delta(a) = \Delta_1(a)$ , it follows that  $\Delta(at^\nu) = \Delta_1(a)t^\nu$ . Applying this observation to the  $f_j$  shows that  $\Delta(f_j) = \Delta_1(f_j)$  for  $0 \leq j \leq r - 1$ .

PROPOSITION 11.9. (1) Let  $f = \sum_{j=0}^s f_j \mathbb{Q}^j \in S(\delta)$  with  $0 \leq s \leq r - 1$  and  $f_j \in A[t^{\pm 1}](\delta - 2j)$  as in (11.8). Then

$$\overline{\Delta}(f) = 0 \iff \Delta(f) = 0 \iff \begin{cases} \Delta_1(f_j) = -2(j + 1)(r - (j + 1))f_{j+1} & \text{if } 0 \leq j \leq s - 1, \\ \Delta_1(f_s) = 0. \end{cases}$$

Consequently,

$$N_\lambda = \{\bar{f} \in \overline{S}(\delta) : \Delta(f) = 0\} = \overline{\mathcal{H}(S)(\delta)} \subset \overline{S}(\delta).$$

(2) There exists an isomorphism of  $\mathfrak{m}$ -modules

$$\text{ev}_0 : N_\lambda \xrightarrow{\sim} \mathcal{S}_\lambda = \{a \in A[t^{\pm 1}](\delta) : \Delta_1^r(a) = 0\}$$

given by  $\text{ev}_0(\bar{f}) = f_0$ , when  $f = \sum_{j=0}^{r-1} f_j \mathbb{Q}^j \in S(\delta)$  with  $f_j \in A[t^{\pm 1}](\delta - 2j)$ .

Proof. (1) By (11.6) we have

$$\Delta(f) = \sum_{j=0}^s \Delta(f_j \mathbb{Q}^j) = \Delta_1(f_j) \mathbb{Q}^s + \sum_{j=0}^{s-1} \{\Delta_1(f_j) + (j + 1)[2\text{wt}(f_{j+1}) + N + 2j]f_{j+1}\} \mathbb{Q}^j.$$

It follows from  $\Delta_1(f_j) \in A[t^{\pm 1}]$  that  $\Delta(f) \in \bigoplus_{j=0}^s A[t^{\pm 1}] \mathbb{Q}^j$ . Since  $\text{wt}(f_{j+1}) = \delta - 2(j + 1)$  if  $f_{j+1} \neq 0$ , one deduces that  $\Delta(f) = 0$  is equivalent to the following system of equations in the ring  $A[t, t^{-1}]$ :

$$\begin{cases} \Delta_1(f_s) = 0, \\ \Delta_1(f_j) = -2(j + 1)(r - (j + 1))f_{j+1} & \text{if } 0 \leq j \leq s - 1. \end{cases}$$

By definition,  $\overline{\Delta}(\bar{f}) = \overline{\Delta}(f)$  and so  $\overline{\Delta}(\bar{f}) = 0$  is equivalent to  $\Delta(f) \in S(\delta - 2(r + 1))\mathbb{Q}^r$ . Recalling that  $\Delta(f) \in \bigoplus_{j=0}^{r-1} A[t^{\pm 1}] \mathbb{Q}^j$  and using  $S = \bigoplus_{j \in \mathbb{N}} A[t^{\pm 1}] \mathbb{Q}^j$ , one deduces that  $\Delta(f) = 0$  if and only if  $\overline{\Delta}(\bar{f}) = 0$ .

Finally, if  $\bar{x} \in N_\lambda$  we can find  $f$  as in (11.8) such that  $\bar{x} = \bar{f}$ . By definition,  $\overline{\Delta}(\bar{f}) = 0$  and hence  $\Delta(f) = 0$ . The equality  $N_\lambda = \overline{\mathcal{H}(S)(\delta)}$  follows.

(2) By induction, the equalities  $\Delta_1(f_j) = -2(j + 1)(r - (j + 1))f_{j+1}$  for  $0 \leq j \leq r - 2$  yield  $\Delta_1^p(f_0) = c_p f_p$ , where  $c_p = (-2)^p p!(r - 1) \cdots (r - p)$  for  $0 \leq p \leq r - 1$ . It follows from (1) that the map  $\bar{f} \mapsto f_0$  with  $f \in \mathcal{H}(S)(\delta)$  takes values in  $\mathcal{S}_\lambda$  and is injective. This also implies that if  $a \in \mathcal{S}_\lambda$ , one can obtain  $f = \sum_{j=0}^{r-1} f_j \mathbb{Q}^j \in \mathcal{H}(S)(\delta)$  by setting  $f_0 = a$  and  $f_p = (1/c_p) \Delta_1^p(a)$  for  $1 \leq p \leq r - 1$ . This yields a vector space isomorphism  $N_\lambda \xrightarrow{\sim} \mathcal{S}_\lambda$ .

Recall from § 2 that  $\mathfrak{m} = \mathbb{C}E_{11} \oplus \mathfrak{k}$  where  $E_{11} = (t/2)\partial_t - U_{-1}\partial_{U_{-1}}$  and  $\mathfrak{k} = \sum_{i,j \notin \{\pm 1\}} \mathbb{C}E_{ij}$ . Clearly, therefore,  $A[t^{\pm 1}]$  is  $\mathfrak{m}$ -stable. From  $[E_{11}, \Delta_1] = 0 = [\mathfrak{k}, \Delta_1]$  it follows that  $\mathcal{S}_\lambda$  is an  $\mathfrak{m}$ -module. Let  $Y \in \mathfrak{m}$  and  $\bar{f} \in N_\lambda$  with  $f = \sum_{j=0}^{r-1} f_j \mathbb{Q}^j$  as above. Since  $Y \cdot \mathbb{Q} = Y(\mathbb{Q}) = 0$ , one has  $Y \cdot \sum_{j=0}^{r-1} f_j \mathbb{Q}^j = \sum_{j=0}^{r-1} (Y \cdot f_j) \mathbb{Q}^j$  with  $Y \cdot f_j \in A[t^{\pm 1}](\delta - 2j)$ . Hence  $\text{ev}_0(Y \cdot f) = (Y \cdot f)_0 = Y \cdot f_0 = Y \cdot \text{ev}_0(f)$  and so the linear isomorphism  $\text{ev}_0 : N_\lambda \xrightarrow{\sim} \mathcal{S}_\lambda$  is indeed an isomorphism of  $\mathfrak{m}$ -modules.  $\square$

As we show next, this theorem implies that, as  $\mathfrak{k}$ -modules,  $N_\lambda$  is isomorphic to the module  $M_r = M_{n,r} = \{a \in A : \Delta_1^r(a) = 0\}$  from Remark 10.3. Before proving this, note that as  $\Delta_1^r(A(m)) \subset A(m - 2r)$  for all  $m \in \mathbb{N}$ , the space  $M_r$  is a graded subspace of  $A$ . Thus any  $a \in M_r$  can be written as  $a = \sum_{m \in \mathbb{N}} a(m)$  with  $a(m) \in M_r(m)$ . On the other hand, by Proposition 11.9(2) and the fact that  $\text{wt}(t) = \frac{1}{2}$ , any  $g \in \mathcal{S}_\lambda$  may be written as  $g = \sum_{m \in \mathbb{N}} g(m)t^{2(\delta - m)}$  with  $g(m) \in A(m)$ .

COROLLARY 11.10. *The  $\mathfrak{k}$ -modules  $M_r \subset A$  and  $\mathcal{S}_\lambda \subset A[t^{\pm 1}]$  are isomorphic via the map*

$$\sigma : \mathcal{S}_\lambda \xrightarrow{\sim} M_r, \quad \sum_{m \in \mathbb{N}} a(m)t^{2(\delta-m)} \mapsto \sum_{m \in \mathbb{N}} a(m).$$

*In particular, the map  $\sigma \circ \text{ev}_0 : N_\lambda \xrightarrow{\sim} M_r$  is an isomorphism of  $\mathfrak{k}$ -modules.*

*Proof.* Let  $g = \sum_{m \in \mathbb{N}} a(m)t^{2(\delta-m)} \in \mathcal{S}_\lambda$  and recall that  $\Delta_1(at^\beta) = \Delta_1(a)t^\beta$  for all  $a \in A$  and  $\beta \in \mathbb{Z}$ . Then  $\Delta_1^r(g) = \sum_m \Delta_1(a(m))t^{2(\delta-m)} = 0$  forces  $\Delta_1^r(a(m)) = 0$  for all  $m$ , and hence  $\sum_m a(m) \in M_r$ . Conversely, the same computation shows that if  $\sum_m a(m) \in M_r$ , then  $\sum_{m \in \mathbb{N}} a(m)t^{2(\delta-m)} \in \mathcal{S}_\lambda$ . Thus  $\sigma$  is a linear bijection. The  $\mathfrak{k}$ -linearity of  $\sigma$  is a consequence of  $Y.t^\beta = 0$  for all  $\beta$  and  $Y \in \mathfrak{k}$ . The second assertion then follows from Proposition 11.9(2).  $\square$

By Lemma 10.2,  $M_r$  is a  $\mathfrak{g}$ -module, so it is natural to ask whether  $\sigma \circ \text{ev}_0$  is actually  $\mathfrak{g}$ -linear in this corollary. This is true and forms the main result of the section.

THEOREM 11.11. *The map  $\sigma \circ \text{ev}_0 : N_\lambda \rightarrow M_r$  is an isomorphism of  $\mathfrak{g}$ -modules. Equivalently,*

$$\sigma(\text{ev}_0(Y.\bar{f})) = \tau(Y).g \quad \text{for all } Y \in \mathfrak{g} \text{ and } g = \sigma(\text{ev}_0(\bar{f})) \in M_r \text{ with } \bar{f} \in N_\lambda.$$

*Proof.* Throughout the proof we write  $\bar{f} = \sum_{j=0}^{r-1} f_j \bar{Q}^j$  with  $f = \sum_{j=0}^{r-1} f_j Q^j$  as in (11.8); thus  $\text{ev}_0(\bar{f}) = f_0 \in \mathcal{S}_\lambda$  by Proposition 11.9. Let  $Y \in \mathfrak{g}$  and recall that the derivation  $Y$  satisfies  $Y(Q) = 0$ . Therefore  $Y.f = \sum_j (Y.f_j)Q^j$ , and it follows from the definition of  $\text{ev}_0$  that  $\text{ev}_0(Y.\bar{f}) = \text{ev}_0(Y.f_0)$ . Using this, the theorem is then obviously equivalent to

$$\sigma(\text{ev}_0(Y.f_0)) = \tau(Y).\sigma(f_0) \quad \text{for all } Y \in \mathfrak{g} \text{ and } f_0 \in \mathcal{S}_\lambda. \tag{11.12}$$

By Corollary 11.10,  $f_0 = \sum_{m \in \mathbb{N}} a(m)t^{2(\delta-m)}$  with  $a(m) \in A(m)$  for all  $m$ , and  $\sigma(f_0) = \sum_m a(m)$ . For simplicity we set  $\gamma(m) = 2(\delta - m)$ . Thus

$$Y.f_0 = \sum_m (Y.a(m))t^{\gamma(m)} + \sum_m a(m)(Y.t^{\gamma(m)}) = \sum_m (Y.a(m))t^{\gamma(m)} + \sum_m \gamma(m)a(m)(Y.t)t^{\gamma(m)-1}$$

while  $\tau(Y).\sigma(f_0) = \sum_m \tau(Y).a(m)$ . As usual, in order to prove that (11.12) holds, it suffices to prove the statement when  $Y$  is a root vector in  $\mathfrak{g}$  and hence (a scalar multiple of) some  $y_\alpha$  from the Chevalley system given in Proposition 2.4. As in the proof of Theorem 10.6, we will consider the three cases  $Y \in \mathfrak{k}$ ,  $Y \in \mathfrak{r}^-$  and  $Y \in \mathfrak{r}^+$  separately.

Suppose first that  $Y = y_\alpha \in \mathfrak{k}$  for  $\alpha \in \Phi_1$ . We may assume that  $Y = E_{ab}$  with  $a, b \in \{0, \pm 2, \dots, \pm \ell\}$ . Then  $E_{ab}.t = 0$  and  $\tau(Y) = \mathcal{F}(\tilde{\psi}(\vartheta(Y))) = \mathcal{F}(\tilde{\psi}(E_{ba})) = \mathcal{F}(E_{ba}) = E_{ab} = Y$  by (10.8). Thus (11.12) is equivalent to the  $\mathfrak{k}$ -linearity of  $\sigma$ , which follows from Corollary 11.10.

Suppose next that  $Y = y_\alpha \in \mathfrak{r}^-$ , where either  $\alpha = -(\varepsilon_1 \pm \varepsilon_b)$  or  $\alpha = -\varepsilon_1$ . Hence, by Proposition 2.4,  $Y = y_\alpha$  is either equal to  $E_{\mp b, 1}$  when  $\alpha = -(\varepsilon_1 \pm \varepsilon_b)$  and  $b \in \{2, \dots, \ell\}$ , or equal to  $\sqrt{2}E_{0,1}$  when  $\alpha = -\varepsilon_1$ . By the definition of  $\vartheta$  and  $\tilde{\psi}$  we have  $\tilde{\psi}(\vartheta(Y)) = x_{-\alpha} \in \mathfrak{r}^+$ . Using Lemma 4.3, this root vector is either  $V_{-b}$  when  $\alpha = -(\varepsilon_1 - \varepsilon_b)$ , or  $V_b$  when  $\alpha = -(\varepsilon_1 + \varepsilon_b)$ , or  $\sqrt{2}V_0$  when  $\alpha = -\varepsilon_1$ . Recall that  $\mathcal{F}(V_a) = Q_a = \mathbf{F}\partial_{U_a} - U_{-a}I(d)$ . Therefore, up to the scalar  $\sqrt{2}$ , we may assume that  $Y = E_{j1}$  and  $\tau(Y) = Q_{-j}$  with  $j \in \{0, \pm 2, \dots, \pm \ell\}$ . Since  $E_{j1} = (U_j/2t)\partial_t - U_{-1}\partial_{U_{-j}}$  and  $a(m) \in A = \mathbb{C}[U_0, U_{\pm 2}, \dots, U_{\pm \ell}]$ , we obtain

$$E_{j1}.f_0 = -\sum_m U_{-1}\partial_{U_{-j}}(a(m))t^{\gamma(m)} + \sum_m \frac{\gamma(m)}{2}U_j a(m)t^{\gamma(m)-2}.$$



Recalling from (2.1) that  $U_{-1} = (1/U_1)(\mathbb{Q} - \mathbb{F}) = t^{-2}(\mathbb{Q} - \mathbb{F})$ , it follows that

$$\begin{aligned} E_{j1}.f_0 &= -\sum_m (\mathbb{Q} - \mathbb{F})\partial_{U_{-j}}(a(m))t^{\gamma(m)-2} + \sum_m \frac{\gamma(m)}{2}U_j a(m)t^{\gamma(m)-2} \\ &= \sum_m \left( \frac{\gamma(m)}{2}U_j a(m) + \mathbb{F}\partial_{U_{-j}}(a(m)) \right) t^{\gamma(m)-2} - \mathbb{Q} \sum_m \partial_{U_{-j}}(a(m))t^{\gamma(m)-2}. \end{aligned}$$

Thus, as  $\gamma(m) = 2(\delta - m) = -2(d + m)$ , we have

$$\text{ev}_0(E_{j1}.f_0) = \sum_m (-(d + m)U_j + \mathbb{F}\partial_{U_{-j}})a(m)t^{\gamma(m)-2}$$

and  $\sigma(\text{ev}_0(E_{j1}.f_0)) = \sum_m (-(d + m)U_j + \mathbb{F}\partial_{U_{-j}})(a(m))$ . Since  $I(d)(a(m)) = (m + d)a(m)$ , observe that

$$Q_{-j}.\sigma(f_0) = Q_{-j}.\sum_m a(m) = \sum_m Q_{-j}.a(m) = \sum_m (\mathbb{F}\partial_{U_{-j}}(a(m)) - (d + m)U_j a(m)),$$

and we obtain  $\sigma(\text{ev}_0(E_{j1}.f_0)) = Q_{-j}.\sigma(f_0)$  as required.

Finally, suppose that  $Y = y_\alpha \in \mathfrak{r}^+$ , where either  $\alpha = \varepsilon_1 \pm \varepsilon_b$  or  $\alpha = \varepsilon_1$ . From Lemma 4.3 we see that we may take  $Y = E_{1b}$ ; using Lemma 4.3 and the definition of  $\mathcal{F}$ , we may assume that  $Y = E_{1j}$ , whence  $\tau(Y) = \partial_{U_j}$  with  $j \in \{0, \pm 2, \dots, \pm \ell\}$ . Recall that  $E_{1j} = U_1\partial_{U_j} - U_{-j}\partial_{U_{-1}} = t^2\partial_{U_j} - U_{-j}\partial_{U_{-1}}$ ; from  $E_{1j}(t) = 0$  and  $\partial_{U_{-1}}(a(m)) = 0$  we get that  $E_{1j}.f_0 = \sum_m \partial_{U_j}(a(m))t^{\gamma(m)+2}$ . On the other hand we have  $\partial_{U_j}.\sigma(f_0) = \sum_m \partial_{U_j}(a(m))$ , and it follows that  $\sigma(\text{ev}_0(E_{1j}.f_0)) = \sum_m \partial_{U_j}(a(m)) = \partial_{U_j}.\sigma(f_0)$  as desired.  $\square$

In summary, by combining Corollary 10.12 with Theorem 11.11, we have the following relationship between the three main  $\mathfrak{g}$ -modules appearing in this paper: the module  $N(\lambda) = R_r$  of regular functions on the scheme  $\{\mathbb{F}^r = 0\} \subset \mathbb{C}^n$ , the module  $M_r$  of higher harmonic polynomials on  $\mathbb{C}^n$ , and the module  $N_\lambda$  of harmonic conformal densities of weight  $-n/2 + r$  on  $\mathbb{C}^{n+2}$ .

COROLLARY 11.13. *There exist isomorphisms of  $\mathfrak{g}$ -modules*

$$N(\lambda) \cong N_\lambda^\vee \quad \text{or, equivalently,} \quad R_r \cong M_r^\vee.$$

Moreover:

- (1) if  $n$  is odd or if  $n$  is even with  $r < n/2$ , then  $R_r \cong L(\lambda) \cong M_r$ ;
- (2) if  $n$  is even with  $r \geq n/2$ , then  $R_r$  has a simple socle  $Z_r$  isomorphic to  $L(\mu)$ , where  $\mu$  is given by (4.14), and the quotient  $Q_r = R_r/Z_r$  is isomorphic to the simple finite-dimensional module  $L(\lambda)$ ;
- (3) if  $n$  is even with  $r \geq n/2$ , then  $M_r$  has a simple (finite-dimensional) socle  $Q_r^\vee \cong L(\lambda)$  and the quotient  $Z_r^\vee = M_r/Q_r^\vee \cong L(\mu)$ .  $\square$

We conclude this section with remarks about the module  $N_\lambda$  and its annihilator.

Remark 11.14. (1) Recall that  $\delta = -d = -n/2 + r$ . Since  $t^{2\delta} \in \mathcal{H}(S)(\delta)$ , the class  $e_\lambda = [t^{2\delta}] \in \overline{S}(\delta)$  belongs to  $N_\lambda$ . It is easy to see that  $e_\lambda$  is a highest-weight vector in  $N_\lambda$ , with weight  $\lambda = -d\varepsilon_1 = \delta\varepsilon_1$ . In more detail:

- (i)  $E_{ab}.t^{2\delta} = 0 = E_{c,0}.t^{2\delta}$  for  $1 \leq a < b \leq \ell$  and  $1 \leq c \leq \ell$  and so, by Proposition 2.4,  $\mathfrak{n}^+.e_\lambda = 0$ ;
- (ii)  $E_{11}.t^{2\delta} = (U_1/2t)\partial_t(t^{2\delta}) = \delta t^{2\delta}$  and  $E_{jj}.t^{2\delta} = 0$  for  $2 \leq j \leq \ell$ , showing that  $e_\lambda$  has weight  $\lambda = \delta\varepsilon_1$ .

Therefore, if  $n$  is even with  $r < n/2$  or if  $n$  is odd, one gets  $N_\lambda = U(\mathfrak{g}).e_\lambda$ .

Assume now that  $n$  is even with  $r \geq n/2$ . Then  $F^\vee = U(\mathfrak{g}).e_\lambda \cong L(\lambda)$  is finite-dimensional. The quotient  $N_\lambda/F^\vee$  is isomorphic to  $L(\mu)$ , and by similar computations it is easily seen that the class  $e_\mu \in N_\lambda/F^\vee$  of the element  $U_2^{\delta+1}t^{-2} = U_2^{\delta+1}U_1^{-1} \in \mathcal{H}(S)(\delta)$  is a highest-weight vector of weight  $\mu$ . Thus,  $N_\lambda/F^\vee = U(\mathfrak{g}).e_\mu \cong L(\mu)$ . Through the isomorphism  $N_\lambda \xrightarrow{\sim} M_r$ , the elements  $e_\lambda$  and  $e_\mu$  correspond, respectively, to 1 and  $U_2^{\delta+1}$  (which, in turn, gives the element  $\xi^m$  in the proof of Theorem 4.13).

(2) By Corollaries 11.13 and 4.17, the annihilator of  $N_\lambda$  is equal to the primitive ideal  $J_r$ . Let  $\mathcal{U}$  denote the subalgebra of  $\mathcal{D}(B)$  generated by the elements  $E_{ij} \in \mathfrak{g}$ . Then  $\mathcal{U}$  is a factor of the enveloping algebra  $U(\mathfrak{g})$  and  $\overline{\mathcal{U}} = U(\mathfrak{g})/J_r$  is a primitive quotient.

When  $r = 1$ ,  $\mathcal{U}$  and  $\overline{\mathcal{U}}$  are, respectively, complexifications of the ‘off-shell higher-spin algebra’ and ‘on-shell higher-spin algebra’ as defined in [Vas03, §3], [Bek09, §3.1.1] and [Bek12, §4.1]. In particular, on the Minkowski space–time (i.e. in the real case  $\mathbb{R}^{p,q} = \mathbb{R}^{n-1,1}$  as in §9), the algebra of symmetries  $\mathcal{S}(\square_{n-1})$  is isomorphic to the ‘on-shell higher-spin algebra’ of [Bek09, §3.1.3, Corollary 3].

### Appendix

Here we provide a proof of Lemma 10.5 and repeat (a minor variant of) the statement of the result for the reader’s convenience.

LEMMA A.1. (1) Set  $\mathcal{I}_r = F^r A$ . The bilinear form  $\langle | \rangle$  from Definition 10.4 satisfies the following properties.

- (i)  $\langle A(p) | A(q) \rangle = 0$  for  $p \neq q$  in  $\mathbb{N}$ , and  $\langle a | \phi \rangle = \mathcal{F}(a)(\phi) = \mathcal{F}(\phi)(a)$  for all  $a, \phi \in A(p)$ .
- (ii)  $\langle | \rangle$  is symmetric, nondegenerate and  $\mathfrak{k}$ -invariant.
- (iii)  $L^{\perp\perp} = L$  for any graded subspace  $L \subset A$ .
- (iv)  $M_r^\perp = \mathcal{I}_r$  and  $\mathcal{I}_r^\perp = M_r$ .

(2) The pairing  $\langle | \rangle$  induces a nondegenerate  $\mathfrak{k}$ -invariant symmetric pairing  $\langle | \rangle : R_r \times M_r \rightarrow \mathbb{C}$ .

*Proof.* (1) (i, ii) With the standard notation, let  $U^{\mathbf{j}}$  and  $U^{\mathbf{k}}$  be monomials in the  $U_j$  (where  $\mathbf{j}$  and  $\mathbf{k}$  are multi-indices). Then  $\mathcal{F}(U^{\mathbf{j}}) = \partial_U^{\mathbf{j}}$  and  $\langle U^{\mathbf{j}} | U^{\mathbf{k}} \rangle = \partial_U^{\mathbf{j}}(U^{\mathbf{k}}) = c_{\mathbf{j},\mathbf{k}} \delta_{\mathbf{j},\mathbf{k}}$  for some  $c_{\mathbf{j}} \in \mathbb{N}^*$ . This proves (i) and implies that the pairing is symmetric and nondegenerate.

Let  $K \cong \text{SO}(\mathbb{F}) \subset \text{Aut}(A)$  be the algebraic group such that  $\text{Lie}(K) = \mathfrak{k}$ . It is well known (and easy to see) that if  $g \in K$ , then  $\mathcal{F}(g.a) = g.\mathcal{F}(a)$  for all  $a \in A$ . Hence  $\langle g.a | g.\phi \rangle = (g.\mathcal{F}(a))(g.\phi)(0) = [g.\mathcal{F}(a)(\phi)](0) = \mathcal{F}(a)(\phi)(0) = \langle a | \phi \rangle$ . Thus,  $\langle | \rangle$  is  $K$ -invariant and therefore  $\mathfrak{k}$ -invariant.

(iii) By (i) we have perfect pairings  $\langle | \rangle_m : A(m) \times A(m) \rightarrow \mathbb{C}$  for all  $m \in \mathbb{N}$ . For a subspace  $P \subset A(m)$ , denote by  $P^\circ$  its orthogonal with respect to  $\langle | \rangle_m$ ; then  $P^\circ = P^\perp \cap A(m)$ . As  $A(m)$  is finite-dimensional, one has  $P^{\circ\circ} = P$ . It is easily seen that  $L^\perp$  is graded and that  $L^\perp(m) = L(m)^\circ$  for all  $m$ . Let  $a \in L^{\perp\perp}$  and write  $a = \sum_m a(m)$  with  $a(m) \in A(m)$ . We have  $\langle a | p \rangle = \langle a(k) | p \rangle = 0$  for all  $p \in L(k)^\circ$ . Hence  $a(k) \in L(k)^{\circ\circ} = L(k)$  and we obtain that  $a \in L$ .

(iv) Since  $\mathcal{F}(\mathbf{F}^r) = 2^{-r} \Delta_1^r$ , it is clear that  $M_r \subset \mathcal{I}_r^\perp$ . Conversely, if  $\phi \in \mathcal{I}_r^\perp$  and  $U^j \in A$ , one has  $\partial_U^j[\Delta_1^r(\phi)](0) = 2^r \langle \mathbf{F}^r U^j \mid \phi \rangle = 0$ . It then follows from (i) that  $\Delta_1^r(\phi) = 0$  and  $\phi \in M_r$ . Thus  $\mathcal{I}_r^\perp = M_r$ .

(2) Observe that  $\mathcal{I}_r$  and  $M_r$  are  $\mathfrak{k}$ -invariant subspaces of  $A$  (see Corollary 11.10). Since  $R_r = A/\mathcal{I}_r$ , the claim therefore follows from (1)(ii) and (1)(iv).  $\square$

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