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Abstract

In this article we give a geometric construction of a tilting perverse sheaf on Drinfeld's compactification, by applying the nearby cycles functor to a family of nondegenerate Whittaker sheaves. Its restrictions along the defect stratification are shown to be certain perverse sheaves attached to the nilpotent radical of the Langlands dual Lie algebra. We also describe the subquotients of the monodromy filtration using the Picard–Lefschetz oscillators introduced by Schieder. We give an argument that the subquotients are semisimple based on the action, constructed by Feigin, Finkelberg, Kuznetsov, and Mirković, of the Langlands dual Lie algebra on the global intersection cohomology of quasimaps into flag varieties.

1. Introduction

1.1 In this article we study the degeneration of a Whittaker sheaf on Drinfeld's compactification to an object of the principal series category (for us 'sheaf' will mean D-module, but our results carry over *mutatis mutandi* to ℓ-adic sheaves in characteristic $p > 0$ for $\ell \neq p$). In [AG15b] this degeneration is implemented by a gluing functor, meaning !-extension from the general locus followed by !-restriction to the special fiber. This operation produces a complex of sheaves, which the authors of [AG15b] link to the constant term of Poincaré series.

Nearby cycles provide a different, although closely related, method of degeneration. Notably, if we apply nearby cycles (more precisely, the geometric Jacquet functor of [ENV04]) to the perverse cohomological shift of a nondegenerate Whittaker sheaf, the result is still a perverse sheaf. It comes equipped with a nilpotent endomorphism, which gives rise to the so-called monodromy filtration. We describe the restrictions of this nearby cycles sheaf to the defect strata in terms of the Langlands dual Lie algebra, showing in particular that these restrictions are perverse, and hence that the nearby cycles sheaf is tilting. We also describe the associated graded sheaf of the monodromy filtration on nearby cycles along with its Lefschetz $\mathfrak{sl}_2$-action, in terms of the Picard–Lefschetz oscillators, which are certain factorizable perverse sheaves with $\mathfrak{sl}_2$-action introduced in [Sch18].

In [Cam17], the author introduced a similar construction in the situation of a finite-dimensional flag variety. Namely, we showed that the nearby cycles of a one-parameter family of nondegenerate Whittaker sheaves on a flag variety is the big projective sheaf, which is isomorphic to the tilting extension of the constant perverse sheaf on the big cell. Thus in both cases taking nearby cycles of Whittaker sheaves produces tilting sheaves.

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1.2 Fix an algebraically closed field k of characteristic zero and a smooth connected curve X over k. Let G be a connected reductive group over k. We assume for simplicity that the derived subgroup $[G,G]$ is simply connected. We write I for the set of vertices of the Dynkin diagram of G, and Z_G for the center. Fix a Borel subgroup B with unipotent radical N, and put $T := B/N$.

For $\mathscr{Y}$ an algebraic stack locally of finite type, we write $D(\mathscr{Y})$ for the unbounded derived category of D-modules on $\mathscr{Y}$ and denote by $D_{\mathrm{hol}}(\mathscr{Y})$ the full subcategory consisting of bounded complexes with holonomic cohomologies. The reader is free to view $D(\mathscr{Y})$ as a triangulated category for most of the paper, with the exception of § 3, where the differential graded (DG) enhancement of $D(\mathscr{Y})$ is used.

We denote by $\overline{\mathrm{Bun}}_{N^\omega}$ Drinfeld's compactification of the moduli stack Bun_{N^ω} of canonically twisted N-bundles. A choice of isomorphism $N/[N,N] \cong \mathbb{G}_a^I$ gives rise to a map

$$\mathrm{ev} : \mathrm{Bun}_{N^\omega} \longrightarrow \mathbb{G}_a,$$

constructed for example in [FGV01, § 4.1].

Let χ be a nontrivial exponential D-module on $\mathbb{G}_a$. As in [FGV01], one shows that $\mathrm{ev}^\Delta \chi$ extends cleanly to a perverse sheaf $\mathscr{W}_1$ on $\overline{\mathrm{Bun}}_{N^\omega}$, where ev^Δ denotes cohomologically normalized inverse image along the smooth morphism ev. The sheaf $\mathscr{W}_1$ has irregular singularities because χ does, so, to be clear, by 'perverse sheaf' we mean any holonomic D-module.

Choose a dominant regular cocharacter $\gamma : \mathbb{G}_m \to T$, which determines an action

$$a_\gamma : \mathbb{G}_m \times \overline{\mathrm{Bun}}_{N^\omega} \to \overline{\mathrm{Bun}}_{N^\omega}.$$

We will denote by $\mathscr{W}$ the perverse sheaf $a_\gamma^\Delta \mathscr{W}_1$ on $\mathbb{G}_m \times \overline{\mathrm{Bun}}_{N^\omega}$. Consider the embeddings

$$\overline{\mathrm{Bun}}_{N^\omega} \xrightarrow{\ i\ } \mathbb{A}^1 \times \overline{\mathrm{Bun}}_{N^\omega} \xleftarrow{\ j\ } \mathbb{G}_m \times \overline{\mathrm{Bun}}_{N^\omega}.$$

The $\mathbb{G}_m$-equivariant object $i^! j_! \mathscr{W}$ of $D(\overline{\mathrm{Bun}}_{N^\omega})$ is studied in [AG15b].

We will consider instead a closely related perverse sheaf on $\overline{\mathrm{Bun}}_{N^\omega}$, namely the nearby cycles $\Psi(\mathscr{W})$ with respect to the projection $\mathbb{A}^1 \times \overline{\mathrm{Bun}}_{N^\omega} \to \mathbb{A}^1$. This sheaf is unipotently $\mathbb{G}_m$-monodromic, and (the logarithm of) the monodromy endomorphism of nearby cycles agrees with the obstruction to $\mathbb{G}_m$-equivariance, a nilpotent endomorphism. The composition $\Psi \circ a_\gamma^\Delta$ for a $\mathbb{G}_m$-equivariant one-parameter family is studied in [ENV04] under the name *geometric Jacquet functor*.

Up to isomorphism $\Psi(\mathscr{W})$ does not depend on the choice of χ. In particular $\Psi(\mathscr{W})$ is Verdier self-dual, since Ψ commutes with Verdier duality and the Verdier dual of $\mathscr{W}$ is the clean extension of $\mathrm{ev}^\Delta \chi^{-1}$.

The object $i^! j_! \mathscr{W}$ can be recovered from $\Psi(\mathscr{W})$ as the derived invariants of monodromy, and conversely $\Psi(\mathscr{W})$ is the derived coinvariants of the natural $H^\bullet(\mathbb{G}_m)$-action on $i^! j_! \mathscr{W}$ (see Definition 3.1.1 and Proposition 3.2.2). One can summarize by saying that nearby cycles and the 'gluing functor' $i^! j_!$ are related by Koszul duality.

From the standpoint of the geometric Langlands program this Koszul duality motivates the study of $\Psi(\mathscr{W})$, since the gluing functors play a crucial role in the proof of the geometric Langlands equivalence sketched in [Gai15]. Namely, they are used to achieve a spectral description of the so-called extended Whittaker category. Nearby cycles has the advantage over the gluing functor of being t-exact, so that $\Psi(\mathscr{W})$, unlike $i^! j_! \mathscr{W}$, is a perverse sheaf.

1.3 The stack $\overline{\mathrm{Bun}}_{N^\omega}$ has a stratification by defect of the generalized N-bundle. Our first main theorem is a description of the restrictions of $\Psi(\mathscr{W})$ to the strata. In particular we will prove that this sheaf is tilting with respect to the defect stratification, meaning its !- and $*$-restrictions to the strata are perverse (although generally not lisse).

Write Λ for the lattice of cocharacters of T, and denote by $\Lambda^{\mathrm{pos}} \subset \Lambda$ the positive coweights with respect to B. The defect stratification of $\overline{\mathrm{Bun}}_{N^\omega}$ is indexed by Λ^{pos}, and for each $\mu \in \Lambda^{\mathrm{pos}}$ we denote the locally closed stratum embedding by

$$\mathfrak{j}_{=\mu} : \overline{\mathrm{Bun}}_{N^\omega,=\mu} \longrightarrow \overline{\mathrm{Bun}}_{N^\omega}.$$

Denote by $X^{(n)}$ the n-fold symmetrized power of X. If $\mu = \sum_{i\in I} n_i\alpha_i$ for some nonnegative integers n_i (here α_i is the simple coroot corresponding to i) then the corresponding configuration space of points in X is defined by

$$X^\mu := \prod_{i\in I} X^{(n_i)},$$

and there is a smooth surjection

$$\mathfrak{m}_\mu : \overline{\mathrm{Bun}}_{N^\omega,=\mu} \longrightarrow X^\mu.$$

In [BG08] the authors introduced certain factorizable perverse sheaves Ω^μ on the configuration spaces X^μ. The !-fiber of Ω^μ at a point $\sum_i \mu_i x_i$ is

$$\bigotimes_i C^\bullet(\check{\mathfrak{n}})^{\mu_i},$$

where $\check{\mathfrak{n}}$ is the unipotent radical of a Borel subalgebra in the Langlands dual Lie algebra $\check{\mathfrak{g}}$, and $C^\bullet(\check{\mathfrak{n}})$ is its $\check{T}$-graded cohomological Chevalley complex.

THEOREM 1.3.1. *For any $\mu \in \Lambda^{\mathrm{pos}}$ there is an isomorphism $\mathfrak{j}^!_{=\mu}\Psi(\mathscr{W}) \tilde{\to} \mathfrak{m}^\triangle_\mu \Omega^\mu$.*

In fact, Ω^μ is indecomposable and $\mathfrak{m}_\mu$ has contractible fibers, so the isomorphism in the theorem is automatically unique up to scaling. Since $\Psi(\mathscr{W})$ is Verdier self-dual, Theorem 1.3.1 implies that it is tilting with respect to the defect stratification.

1.4 Recall that the *monodromy filtration* on $\Psi(\mathscr{W})$ is the unique filtration by perverse sheaves

$$F_{-m} \subset \cdots \subset F_{m-1} \subset F_m = \Psi(\mathscr{W})$$

such that for all $1 \leqslant i \leqslant n$ the ith power of the monodromy endomorphism induces an isomorphism $F_i/F_{i-1} \tilde{\to} F_{-i}/F_{-i-1}$. The associated graded sheaf $\mathrm{gr}\,\Psi(\mathscr{W})$ has an action of the so-called Lefschetz $\mathfrak{sl}_2$ such that the lowering operator is induced by the monodromy endomorphism and F_i/F_{i-1} has weight i for the Cartan operator.

We now formulate a description of $\mathrm{gr}\,\Psi(\mathscr{W})$ in terms of certain factorizable perverse sheaves with $\mathfrak{sl}_2$-action, called the *Picard–Lefschetz oscillators* after [Sch18] (see also § 3.2 of [Sch17]).

First let us define the Picard–Lefschetz oscillators on $X^{(n)}$. Let std denote the standard two-dimensional representation of the Lefschetz $\mathfrak{sl}_2$, and write sgn for the sign character of the symmetric group Σ_n. The $\Sigma_n \times \mathfrak{sl}_2$-representation $\mathrm{sgn} \otimes \mathrm{std}^{\otimes n}$ (here Σ_n also permutes the std factors) gives rise to a local system with $\mathfrak{sl}_2$-action on the disjoint locus $X^{(n)}_{\mathrm{disj}}$, by applying the associated bundle construction to the canonical Σ_n-torsor $X^n_{\mathrm{disj}} \to X^{(n)}_{\mathrm{disj}}$. Then $\mathscr{P}_n$ is defined

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as the intermediate extension to $X^{(n)}$ of the (perverse cohomological shift of the) local system attached to $\mathrm{sgn} \otimes \mathrm{std}^{\otimes n}$. The perverse sheaf $\mathscr{P}_n$ carries an $\mathfrak{sl}_2$-action by functoriality, and is evidently semisimple.

Recall that a *Kostant partition* of $\mu \in \Lambda^{\mathrm{pos}}$ is an expression of the form $\mu = \sum_{\beta \in R^+} n_\beta \beta$, where the n_β are nonnegative integers and R^+ denotes the set of positive coroots. Denote by $\mathrm{Kost}(\mu)$ the set of Kostant partitions of μ. To any $\mathfrak{k} \in \mathrm{Kost}(\mu)$ given by $\mu = \sum_{\beta \in R^+} n_\beta \beta$ we attach a partially symmetrized power

$$X^{\mathfrak{k}} := \prod_{\beta \in R^+} X^{(n_\beta)},$$

which is equipped with a canonical finite map $\iota^{\mathfrak{k}} : X^{\mathfrak{k}} \to X^\mu$.

The *Picard–Lefschetz oscillator* on X^μ is defined by the formula

$$\mathscr{P}^\mu := \bigoplus_{\mathfrak{k} \in \mathrm{Kost}(\mu)} \iota^{\mathfrak{k}}_* \Big(\underset{\beta \in R^+}{\boxtimes} \mathscr{P}_{n_\beta} \Big).$$

In particular, we have $\mathscr{P}^{n\alpha} = \mathscr{P}_n$ for α a simple coroot. By construction, $\mathscr{P}^\mu$ is a semisimple perverse sheaf with $\mathfrak{sl}_2$-action.

THEOREM 1.4.1. *There is an $\mathfrak{sl}_2$-equivariant isomorphism*

$$\mathrm{gr}\, \Psi(\mathscr{W}) \xrightarrow{\sim} \bigoplus_{\mu \in \Lambda^{\mathrm{pos}}} \mathfrak{j}_{=\mu,!*} \mathfrak{m}_\mu^{\Delta} \mathscr{P}^\mu. \tag{1.4.1}$$

In particular $\mathrm{gr}\, \Psi(\mathscr{W})$ is semisimple. Using the theorem we can compute the kernel of the monodromy operator, whose simple subquotients are the lowest weight sheaves for the Lefschetz $\mathfrak{sl}_2$-action.

COROLLARY 1.4.1.1. *The canonical morphism $\mathfrak{j}_{=0,!} \mathrm{IC}_{\mathrm{Bun}_{N^\omega}} \to \Psi(\mathscr{W})$ is an isomorphism onto the kernel of the monodromy operator.*

1.5 A key step in the proof of Theorem 1.4.1 is establishing that $\mathrm{gr}\, \Psi(\mathscr{W})$ is semisimple. We give a standard weight-theoretic argument, justified by Mochizuki's theory of weights for holonomic D-modules developed in [Moc15]. We also present a more direct proof of Theorem 1.4.1 which uses the remarkable action, constructed in [FFKM99], of $\check{\mathfrak{g}}$ on the intersection cohomology of quasimap spaces. As we now explain, this action can be obtained via a deformation-theoretic argument from the global unramified categorical geometric Langlands conjecture, together with its expected compatibility with geometric Eisenstein series. This leads us to a natural generalization of the action from [FFKM99], which the author intends to return to in future work.

Write $\overline{\mathfrak{p}} : \overline{\mathrm{Bun}}_B \to \mathrm{Bun}_G$ and $\overline{\mathfrak{q}} : \overline{\mathrm{Bun}}_B \to \mathrm{Bun}_T$ for the canonical morphisms. The functor of *compactified Eisenstein series* $\mathrm{Eis}_{!*} : D(\mathrm{Bun}_T) \to D(\mathrm{Bun}_G)$, introduced in [BG02], is defined by

$$\mathrm{Eis}_{!*}(-) = \overline{\mathfrak{p}}_*(\mathrm{IC}_{\overline{\mathrm{Bun}}_B} \overset{!}{\otimes} \overline{\mathfrak{q}}^!(-)).$$

The rest of this section plays a motivational role only, and is otherwise not necessary to understand the contents of this paper.

For any algebraic group H, we denote by LS_H the DG algebraic stack of de Rham H-local systems on X. For any DG stack $\mathscr{Y}$ we denote by $\mathrm{QCoh}(\mathscr{Y})$ the unbounded derived category of quasicoherent sheaves on $\mathscr{Y}$.

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The form of the geometric Langlands conjecture which we now state is a coarse version of the one formulated in [AG15a], but at this level of precision it goes back to Beilinson and Drinfeld. It is expected to enjoy various compatibilities, but here we only describe the conjectural interaction with Eisenstein series. Let

$$\mathfrak{r}^{\mathrm{spec}} : \mathrm{LS}_{\check{T}} \longrightarrow \mathrm{LS}_{\check{G}}$$

be the map induced by the inclusion $\check{T} \to \check{G}$.

Conjecture 1.5.1 (Global unramified geometric Langlands). There is a fully faithful functor

$$\mathbb{L}_G : \mathrm{QCoh}(\mathrm{LS}_{\check{G}}) \longrightarrow D(\mathrm{Bun}_G)$$

which makes the following square commute up to natural isomorphism.

$$\begin{tikzcd} \mathrm{QCoh}(\mathrm{LS}_{\check{T}}) \arrow[r, "\rho(\omega)\circ\mathbb{L}_T"] \arrow[d, "\mathfrak{r}^{\mathrm{spec}}_*"] & D(\mathrm{Bun}_T) \arrow[d, "\mathrm{Eis}_{!*}"] \\ \mathrm{QCoh}(\mathrm{LS}_{\check{G}}) \arrow[r, "\mathbb{L}_G"] & D(\mathrm{Bun}_G) \end{tikzcd}$$

Here $\rho(\omega)$ denotes the automorphism of $D(\mathrm{Bun}_T)$ given by translation by the same-named T-bundle.

Kodaira–Spencer theory tells us that the tangent space to $\mathrm{LS}_{\check{G}}$ at a $\check{G}$-local system $E_{\check{G}}$ is identified with $H^\bullet_{\mathrm{dR}}(X, \check{\mathfrak{g}}_{E_{\check{G}}})[1]$, where $\check{\mathfrak{g}}_{E_{\check{G}}}$ denotes the local system attached to $E_{\check{G}}$ and the adjoint representation $\check{\mathfrak{g}}$. Moreover, the (derived) endomorphism algebra of the skyscraper sheaf $\delta_{E_{\check{G}}}$ in $\mathrm{QCoh}(\mathrm{LS}_{\check{G}})$ is the enveloping algebra of the DG Lie algebra $H^\bullet_{\mathrm{dR}}(X, \check{\mathfrak{g}}_{E_{\check{G}}})$, and in particular $H^\bullet_{\mathrm{dR}}(X, \check{\mathfrak{g}}_{E_{\check{G}}})$ acts on $\delta_{E_{\check{G}}}$.

Applying these principles in the case $E_{\check{G}} = \mathfrak{r}^{\mathrm{spec}}(E_{\check{T}})$ for a $\check{T}$-local system $E_{\check{T}}$, we derive the following concrete consequence of Conjecture 1.5.1. According to geometric class field theory $\mathbb{L}_T(\delta_{E_{\check{T}}})$ is a multiplicative flat line bundle, and in particular is invariant under translation by any T-bundle.

Conjecture 1.5.2. The DG Lie algebra $H^\bullet_{\mathrm{dR}}(X, \check{\mathfrak{g}}_{E_{\check{T}}})$ acts on $\mathrm{Eis}_{!*}(\mathbb{L}_T(\delta_{E_{\check{T}}}))$.

Moreover, we expect that the full endomorphism algebra of $\mathrm{Eis}_{!*}(\mathbb{L}_T(\delta_{E_{\check{T}}}))$ is the enveloping algebra of $H^\bullet_{\mathrm{dR}}(X, \check{\mathfrak{g}}_{E_{\check{T}}})$. We also remark that $\mathbb{L}_T(\delta_{E_{\check{T}}})$ has a simple and nonconjectural description: it is the multiplicative line bundle with flat connection on Bun_T corresponding to $E_{\check{T}}$ under geometric class field theory. This means that it is characterized (among multiplicative flat line bundles) by the property that for any coweight $\lambda : \check{T} \to \mathbb{G}_m$, its inverse image along the map $X \to \mathrm{Bun}_T$ sending $x \mapsto \mathscr{P}^{\mathrm{triv}}_T(\lambda \cdot x)$ is $\lambda(E_{\check{T}})$.

When $E_{\check{T}}$ is trivial, which is the only case we will use in this paper, Conjecture 1.5.2 says that $\check{\mathfrak{g}} \otimes H^\bullet_{\mathrm{dR}}(X)$ acts on $\bar{\mathfrak{p}}_* \, \mathrm{IC}_{\overline{\mathrm{Bun}}_B}$. In § 6.3 we specify the action of certain generators of $\check{\mathfrak{g}} \otimes H^\bullet_{\mathrm{dR}}(X)$, namely the factors in the triangular decomposition. In [FFKM99] the authors verify the necessary relations for $\check{\mathfrak{g}} \otimes H^0(X)$, which suffices for our application. This approach seems intractable when approaching Conjecture 1.5.2 in full generality: the derivedness of the Lie algebra $\check{\mathfrak{g}} \otimes H^\bullet_{\mathrm{dR}}(X)$, or more generally $H^\bullet_{\mathrm{dR}}(X, \check{\mathfrak{g}}_{E_{\check{T}}})$, makes the checking of relations difficult or impossible to do 'by hand.'

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2. Drinfeld compactifications and Zastava spaces

2.1 Let $2\rho : \mathbb{G}_m \to T$ denote the sum of the simple coroots and fix a square root $\omega_X^{\otimes 1/2}$ of ω_X. We define

$$\rho(\omega_X) := 2\rho(\omega_X^{\otimes 1/2}) \in \mathrm{Bun}_T(k).$$

By definition, $\overline{\mathrm{Bun}}_{N^\omega}$ is the fiber product

$$\begin{tikzcd} \overline{\mathrm{Bun}}_{N^\omega} \arrow[r] \arrow[d] & \overline{\mathrm{Bun}}_B \arrow[d] \\ \operatorname{Spec} k \arrow[r, "\rho(\omega_X)"] & \mathrm{Bun}_T \end{tikzcd}$$

(see [BG02, § 0.2.1] for the definition of Drinfeld's compactification $\overline{\mathrm{Bun}}_B$). The T-bundle $\rho(\omega_X)$ and the action of T on N give rise to a group scheme N^ω over X, and the open stratum $\mathrm{Bun}_{N^\omega} \to \overline{\mathrm{Bun}}_{N^\omega}$ is identified with the moduli stack of N^ω-bundles, as the notation suggests. By construction, T acts on $\overline{\mathrm{Bun}}_{N^\omega}$ in such a way that $\overline{\mathrm{Bun}}_{N^\omega} \to \overline{\mathrm{Bun}}_B$ factors through a closed embedding

$$\overline{\mathrm{Bun}}_{N^\omega}/T \longrightarrow \overline{\mathrm{Bun}}_B.$$

For each $\mu \in \Lambda^{\mathrm{pos}}$ the corresponding stratum $\overline{\mathrm{Bun}}_{N^\omega,=\mu}$ fits into a fiber square

$$\begin{tikzcd} \overline{\mathrm{Bun}}_{N^\omega,=\mu} \arrow[r] \arrow[d] & \mathrm{Bun}_B \arrow[d] \\ X^\mu \arrow[r] & \mathrm{Bun}_T \end{tikzcd}$$

where the lower horizontal morphism is the twisted Abel–Jacobi map $D \mapsto \rho(\omega_X)(D)$. We write

$$\overline{\mathrm{Bun}}_{N^\omega,\leqslant\mu} := \bigcup_{\mu' \leqslant \mu} \overline{\mathrm{Bun}}_{N^\omega,=\mu'}$$

and $\mathrm{j}_{\leqslant\mu}$ for the corresponding open embedding.

The embedding $\mathrm{j}_{=\mu}$ of the stratum, which is known to be affine, extends to a finite map

$$\mathrm{j}_{\geqslant\mu} : \overline{\mathrm{Bun}}_{N^\omega,\geqslant\mu} \longrightarrow \overline{\mathrm{Bun}}_{N^\omega},$$

where $\overline{\mathrm{Bun}}_{N^\omega,\geqslant\mu} := X^\mu \times_{\mathrm{Bun}_T} \overline{\mathrm{Bun}}_B$.

2.2 Now we introduce the Zastava spaces, which are factorizable local models for $\overline{\mathrm{Bun}}_{N^\omega}$. There are several versions of Zastava space, and notations vary significantly within the literature (ours is similar to [AG15b]).

Define the Zastava space Z to be the open locus in $\overline{\mathrm{Bun}}_{N^\omega} \times_{\mathrm{Bun}_G} \mathrm{Bun}_{B^-}$ where the generalized N-reduction and B^--reduction are transverse generically on X. It is well known that Z is a scheme, with connected components

$$Z^\lambda := Z \cap (\overline{\mathrm{Bun}}_{N^\omega} \times_{\mathrm{Bun}_G} \mathrm{Bun}_{B^-}^{\lambda+\deg\rho(\omega_X)})$$

indexed by $\lambda \in \Lambda^{\mathrm{pos}}$.

The map $\mathfrak{p}^- : \mathrm{Bun}_{B^-} \to \mathrm{Bun}_G$ gives rise to $'\mathfrak{p}^- : Z \to \overline{\mathrm{Bun}}_{N^\omega}$. It is shown in [BG02] that $\mathfrak{p}^{-,\lambda}$ and therefore $'\mathfrak{p}^{-,\lambda}$ are smooth for λ sufficiently dominant. Moreover, given a quasicompact open $U \subset \overline{\mathrm{Bun}}_{N^\omega}$, for λ sufficiently dominant the image of $Z^\lambda \to \overline{\mathrm{Bun}}_{N^\omega}$ contains U.

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For each $0 \leqslant \mu \leqslant \lambda$ we have the corresponding stratum $Z^\lambda_{=\mu} := Z^\lambda \times_{\overline{\mathrm{Bun}}_{N^\omega}} \overline{\mathrm{Bun}}_{N^\omega,=\mu}$ with locally closed embedding

$$'\mathrm{j}^\lambda_{=\mu} : Z^\lambda_{=\mu} \longrightarrow Z^\lambda.$$

Define $\mathring{Z}^\lambda := Z^\lambda_{=0}$. Similarly, we have $Z^\lambda_{\leqslant\mu} := Z^\lambda \times_{\overline{\mathrm{Bun}}_{N^\omega}} \overline{\mathrm{Bun}}_{N^\omega,\leqslant\mu}$ with the open embedding

$$'\mathrm{j}^\lambda_{\leqslant\mu} : Z^\lambda_{\leqslant\mu} \longrightarrow Z^\lambda.$$

Put $Z^\lambda_{\geqslant\mu} := Z \times_{\overline{\mathrm{Bun}}_{N^\omega}} \overline{\mathrm{Bun}}_{N^\omega,\geqslant\mu}$, so that $'\mathrm{j}^\lambda_{=\mu}$ extends to the finite map

$$'\mathrm{j}^\lambda_{\geqslant\mu} : Z^\lambda_{\geqslant\mu} \longrightarrow Z^\lambda.$$

2.3 Let us recall the factorization structure on Z. For every $\lambda \in \Lambda^{\mathrm{pos}}$ there is a canonical map $\pi^\lambda : Z^\lambda \to X^\lambda$, which is well known to be affine. We write $(X^{\lambda_1} \times X^{\lambda_2})_{\mathrm{disj}} \subset X^{\lambda_1} \times X^{\lambda_2}$ for the open locus where the two divisors are disjoint. Similarly, put

$$(Z^{\lambda_1} \times Z^{\lambda_2})_{\mathrm{disj}} := (Z^{\lambda_1} \times Z^{\lambda_2}) \times_{X^{\lambda_1}\times X^{\lambda_2}} (X^{\lambda_1} \times X^{\lambda_2})_{\mathrm{disj}}.$$

The factorization structure is a canonical morphism $(Z^{\lambda_1} \times Z^{\lambda_2})_{\mathrm{disj}} \to Z^{\lambda_1+\lambda_2}$ which fits into a fiber square, as follows.

$$\begin{tikzcd} (Z^{\lambda_1} \times Z^{\lambda_2})_{\mathrm{disj}} \arrow[r] \arrow[d] & Z^{\lambda_1+\lambda_2} \arrow[d] \\ (X^{\lambda_1} \times X^{\lambda_2})_{\mathrm{disj}} \arrow[r] & X^{\lambda_1+\lambda_2} \end{tikzcd}$$

The factorization structure is compatible with the defect stratification in the following sense. The factorization structure on the strata consists of, for each decomposition $\mu_1+\mu_2 = \mu$ satisfying $0 \leqslant \mu_1 \leqslant \lambda_1$ and $0 \leqslant \mu_2 \leqslant \lambda_2$, a morphism $(Z^{\lambda_1}_{=\mu_1} \times Z^{\lambda_2}_{=\mu_2})_{\mathrm{disj}} \to Z^{\lambda_1+\lambda_2}_{=\mu}$ which fits into a fiber square, as follows.

$$\begin{tikzcd} \coprod_{\mu_1+\mu_2=\mu}(Z^{\lambda_1}_{=\mu_1} \times Z^{\lambda_2}_{=\mu_2})_{\mathrm{disj}} \arrow[r] \arrow[d] & Z^{\lambda_1+\lambda_2}_{=\mu} \arrow[d] \\ (X^{\lambda_1} \times X^{\lambda_2})_{\mathrm{disj}} \arrow[r] & X^{\lambda_1+\lambda_2} \end{tikzcd}$$

One has similar factorization structures on $Z^\lambda_{\leqslant\mu}$ and $Z^\lambda_{\geqslant\mu}$. Moreover, these factorization structures are compatible with $'\mathrm{j}^\lambda_{=\mu}$, $'\mathrm{j}^\lambda_{\geqslant\mu}$, etc.

2.4 We will also need the compactified Zastava space $\overline{Z}$, which is the open locus in $\overline{\mathrm{Bun}}_{N^\omega} \times_{\mathrm{Bun}_G} \overline{\mathrm{Bun}}_{B^-}$ where the generalized N- and B^--reductions are generically transverse. In particular there is an open embedding $'\mathrm{j}^- : Z \to \overline{Z}$ obtained from $\mathrm{j}^- : \mathrm{Bun}_{B^-} \to \overline{\mathrm{Bun}}_{B^-}$ by base change.

For any $\nu \in \Lambda^{\mathrm{pos}}$ we put

$${}_{=\nu}\overline{Z} := \overline{Z} \times_{\overline{\mathrm{Bun}}_{B^-}} \overline{\mathrm{Bun}}_{B^-,=\nu},$$

and similarly for ${}_{\leqslant\nu}\overline{Z}$ and ${}_{\geqslant\nu}\overline{Z}$.

The projections π^λ extend to proper morphisms $\overline{\pi}^\lambda : \overline{Z}^\lambda \to X^\lambda$. The factorization structure on Z extends to $\overline{Z}$ in a way compatible with both defect stratifications on $\overline{Z}$.

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3. Nearby cycles and adelic invariance

3.1 We now reinterpret Beilinson's construction of nearby cycles from [Bei87], which will help to streamline our first proof of Theorem 1.3.1. In this section we use the language of DG categories in the sense of Lurie, meaning k-linear stable ∞-categories: see [GR17, ch. I.1] for a summary of the theory.

Let Y be a scheme of finite type equipped with an action of $\mathbb{G}_m$ and a $\mathbb{G}_m$-equivariant morphism $f : Y \to \mathbb{A}^1$. Write Y_0 for the fiber of Y over 0 and $\mathring{Y}$ for the preimage of $\mathbb{G}_m \subset \mathbb{A}^1$. We temporarily denote the embeddings by

$$Y_0 \xrightarrow{i} Y \xleftarrow{j} \mathring{Y}.$$

The projection $\mathring{Y} \to \mathbb{G}_m$ gives rise to an action of $D(\mathbb{G}_m)$, viewed as a symmetric monoidal DG category under !-tensor product, on the DG category $D(\mathring{Y})$. We claim that this naturally induces an action of the DG algebra $H^\bullet(\mathbb{G}_m)$ on the identity functor of $D(\mathring{Y})$. Namely, we view the action of $D(\mathbb{G}_m)$ on $D(\mathring{Y})$ as a monoidal functor

$$D(\mathbb{G}_m) \longrightarrow \mathrm{Fun}(D(\mathring{Y}), D(\mathring{Y})).$$

Since $H^\bullet(\mathbb{G}_m)$ is the algebra of endomorphisms of the monoidal unit in $D(\mathbb{G}_m)$, this monoidal functor induces the desired action.

In particular, for any holonomic D-module $\mathscr{F}$ on $\mathring{Y}$, we have an action of $H^\bullet(\mathbb{G}_m)$ on $i^! j_! \mathscr{F}$ by functoriality. The point $1 \in \mathbb{G}_m$ induces an augmentation $H^\bullet(\mathbb{G}_m) \to k$, and since $H^\bullet(\mathbb{G}_m)$ is generated by a single element in cohomological degree 1, the endomorphism algebra $\mathrm{End}_{H^\bullet(\mathbb{G}_m)\text{-mod}}(k)$ is canonically isomorphic to the polynomial ring $k[t]$. Here $H^\bullet(\mathbb{G}_m)$-mod is the DG category of modules over the DG algebra $H^\bullet(\mathbb{G}_m)$.

Definition 3.1.1. We define unipotent nearby cycles with respect to f to be the functor

$$\Psi : D_{\mathrm{hol}}(\mathring{Y}) \longrightarrow D(Y_0)$$

given on objects by the formula

$$\Psi(\mathscr{F}) = k \underset{H^\bullet(\mathbb{G}_m)}{\otimes} i^! j_! \mathscr{F}.$$

In particular $\mathrm{End}_{H^\bullet(\mathbb{G}_m)\text{-mod}}(k) = k[t]$ acts on Ψ, and we call the action of the generator the *monodromy endomorphism* of Ψ.

More precisely, the action of $H^\bullet(\mathbb{G}_m)$ on the identity functor of $D(\mathring{Y})$ defines a lift of the latter to a functor

$$D(\mathring{Y}) \longrightarrow H^\bullet(\mathbb{G}_m)\text{-mod}(D(\mathring{Y})),$$

and likewise on the holonomic subcategory. Then Ψ is by definition the composition

$$D_{\mathrm{hol}}(\mathring{Y}) \longrightarrow H^\bullet(\mathbb{G}_m)\text{-mod}(D_{\mathrm{hol}}(\mathring{Y})) \xrightarrow{i^! j_!} H^\bullet(\mathbb{G}_m)\text{-mod}(D(Y_0)) \longrightarrow D(Y_0),$$

where the last functor is tensor product with the augmentation $H^\bullet(\mathbb{G}_m)$-module. Since $H^\bullet(\mathbb{G}_m)$ is isomorphic to $\mathrm{Sym}(k[-1])$, an action of $H^\bullet(\mathbb{G}_m)$ is the same as an action of the abelian DG Lie algebra $k[-1]$. Tensor product with the augmentation $H^\bullet(\mathbb{G}_m)$-module corresponds to (homotopy) $k[-1]$-coinvariants.

3.2 The following lemma will be used to compare our construction of Ψ with Beilinson's. It says that the homotopy coinvariants of $H^1(\mathbb{G}_m)[-1]$ acting on the constant sheaf $k_{\mathbb{G}_m}$ is the 'infinite Jordan block.'

For any $a \geqslant 1$ let L_a be the shifted D-module on $\mathbb{G}_m$ corresponding to the local system whose monodromy is a unipotent Jordan block of rank a. There are canonical injections $L_a \to L_{a+1}$ (in the heart of the constructible t-structure), and we put

$$L_\infty := \operatorname*{colim}_{a \geqslant 1} L_a.$$

Observe that L_∞ has a canonical injective 'shift' endomorphism with cokernel $k_{\mathbb{G}_m}$.

Lemma 3.2.1. *There is a canonical isomorphism*

$$k \underset{H^\bullet(\mathbb{G}_m)}{\otimes} k_{\mathbb{G}_m} \xrightarrow{\sim} L_\infty$$

in $D(\mathbb{G}_m)$, which identifies the action of the generator in $k[t] = \operatorname{End}_{H^\bullet(\mathbb{G}_m)\text{-mod}}(k)$ with the canonical endomorphism of L_∞.

Proof. First, observe that since any self-extension of L_∞ splits, the action of $H^1(\mathbb{G}_m)[-1]$ on L_∞ is trivial. Thus the canonical inclusion $k_{\mathbb{G}_m} = L_1 \to L_\infty$ factors through a map

$$k \underset{H^\bullet(\mathbb{G}_m)}{\otimes} k_{\mathbb{G}_m} \longrightarrow L_\infty. \tag{3.2.1}$$

To prove that this map is an isomorphism, we use a Koszul-type resolution of the augmentation $H^\bullet(\mathbb{G}_m)$-module. Namely, put $M_1 := H^\bullet(\mathbb{G}_m)$, and let

$$M_2 := \operatorname{cofib}(H^\bullet(\mathbb{G}_m)[-1] \to H^\bullet(\mathbb{G}_m))$$

be the cofiber (i.e., mapping cone) of the action of a generator of $H^1(\mathbb{G}_m)$. Proceeding inductively, we define

$$M_{a+1} := \operatorname{cofib}(H^\bullet(\mathbb{G}_m)[-a] \to M_a),$$

so there are canonical maps $M_a \to M_{a+1}$. We define M_∞ to be the colimit of the M_a. Since the module M_a has cohomology $k \oplus k[-a]$, the canonical map $M_\infty \to k$ to the augmentation $H^\bullet(\mathbb{G}_m)$-module is an isomorphism. In particular $k[t]$ acts on M_∞, and it is not hard to see that the generator acts by the canonical 'shift' map $M_\infty \to M_\infty$ with cofiber M_1.

Thus we may replace the left-hand side of (3.2.1) with $M_\infty \otimes_{H^\bullet(\mathbb{G}_m)} k_{\mathbb{G}_m}$. A straightforward inductive argument shows that $M_a \otimes_{H^\bullet(\mathbb{G}_m)} k_{\mathbb{G}_m}$ maps isomorphically onto L_a, which proves that (3.2.1) is an isomorphism which moreover preserves the filtrations on both sides. The action of $k[t]$ on both sides agrees by inspection: both endomorphisms shift the filtration by 1. □

Proposition 3.2.2. *The functor Ψ has the following properties.*

(i) *It coincides with the construction in [Bei87] and in particular preserves holonomicity.*
(ii) *If $\mathscr{F}$ is $\mathbb{G}_m$-equivariant, then $\Psi(\mathscr{F})$ is unipotently $\mathbb{G}_m$-monodromic, and the monodromy endomorphism agrees with the obstruction to $\mathbb{G}_m$-equivariance.*

Proof. Let $\mathscr{F}$ be a holonomic D-module on $\overset{\circ}{Y}$. Then Beilinson's nearby cycles is given by the formula

$$\tilde{\Psi}(\mathscr{F}) := \operatorname*{colim}_{a \geqslant 1} i^! j_!(\mathscr{F} \overset{*}{\otimes} f^* L_a).$$

Moreover, we have $\tilde{\Psi}(\mathscr{F}) = H^0 i^! j_!(\mathscr{F} \overset{*}{\otimes} f^* L_a)$ for large a, which is evidently holonomic. The functor

$$D_{\mathrm{hol}}(\mathbb{G}_m) \longrightarrow D(Y_0)$$

given on objects by $\mathscr{M} \mapsto i^! j_!(\mathscr{F} \overset{*}{\otimes} f^* \mathscr{M})$ admits a unique colimit-preserving extension to the category of ind-holonomic sheaves on $\mathbb{G}_m$, so one can write

$$\tilde{\Psi}(\mathscr{F}) = i^! j_!(\mathscr{F} \overset{*}{\otimes} f^* L_\infty).$$

The functor $\tilde{\Psi}$ carries a canonical endomorphism induced by the shift endomorphism $L_\infty \to L_\infty$. Now it follows from Lemma 3.2.1 that $\Psi(\mathscr{F}) \tilde{\to} \tilde{\Psi}(\mathscr{F})$, preserving the monodromy endomorphisms on both sides.

For (ii), note that the $\mathbb{G}_m$-equivariance of $i^! j_! \mathscr{F}$ implies that $\Psi(\mathscr{F}) = H^0 i^! j_!(\mathscr{F} \overset{*}{\otimes} f^* L_a)$ is $\mathbb{G}_m$-monodromic (here a is large). By construction, the monodromy endomorphism is induced by the canonical endomorphism of L_a with one-dimensional kernel and cokernel. But the latter is precisely the obstruction to $\mathbb{G}_m$-equivariance for L_a, so the claim follows from the functoriality of this obstruction. $\square$

It follows from part (i) of Proposition 3.2.2 that Ψ enjoys the standard properties of the unipotent nearby cycles functor: it is t-exact, commutes with Verdier duality, and commutes with proper direct image and smooth inverse image.

3.3 Before proving Theorem 1.3.1, we will show that $\mathfrak{j}^!_{=\mu}\Psi(\mathscr{W})$ is pulled back from X^μ for any $\mu \in \Lambda^{\mathrm{pos}}$. This property is equivalent to invariance under the 'adelic N^ω,' as we now explain.

For any $x \in X$, we define the open substack $\overline{\mathrm{Bun}}^x_{N^\omega} \subset \overline{\mathrm{Bun}}_{N^\omega}$ to consist of those generalized N^ω-bundles whose defect is disjoint from x. A point of the ind-algebraic stack $\mathscr{H}^x_{N^\omega}$ consists of two points of $\overline{\mathrm{Bun}}^x_{N^\omega}$ together with an identification over $X \backslash \{x\}$. Note that $\mathscr{H}^x_{N^\omega}$ has the structure of a groupoid acting on $\overline{\mathrm{Bun}}^x_{N^\omega}$. The fibers of $\mathscr{H}^x_{N^\omega}$ over $\overline{\mathrm{Bun}}^x_{N^\omega} \times \overline{\mathrm{Bun}}^x_{N^\omega}$ are isomorphic to ind-affine space $\mathrm{colim}_n \mathbb{A}^n$, which implies that the functor which forgets $\mathscr{H}^x_{N^\omega}$-equivariance is fully faithful, i.e., $\mathscr{H}^x_{N^\omega}$-equivariance is a property.

We say that an object of $D(\overline{\mathrm{Bun}}_{N^\omega})$ is $N^\omega(\mathbb{A})$-*equivariant* if, for every $x \in X$, its restriction to $\overline{\mathrm{Bun}}^x_{N^\omega}$ is $\mathscr{H}^x_{N^\omega}$-equivariant.

Proposition 3.3.1. *An object $\mathscr{F}$ of $D(\overline{\mathrm{Bun}}_{N^\omega})$ is $N^\omega(\mathbb{A})$-equivariant if and only if, for every $\mu \in \Lambda^{\mathrm{pos}}$, the canonical morphism*

$$\mathfrak{m}^*_\mu \mathfrak{m}_{\mu,*} \mathfrak{j}^!_{=\mu} \mathscr{F} \longrightarrow \mathfrak{j}^!_{=\mu} \mathscr{F}$$

is an isomorphism.

For each $x \in X$, denote by $\mathscr{O}_x$ the completed local ring of X at x, with fraction field K_x. If R is a k-algebra, we denote by $R \hat{\otimes} \mathscr{O}_x$ and $R \hat{\otimes} K_x$ the respective completed tensor products.

The local Hecke stack $\mathscr{H}^{\mathrm{loc},x}_{N^\omega}$ is defined as follows: a $\operatorname{Spec} R$-point of $\mathscr{H}^{\mathrm{loc},x}_{N^\omega}$ consists of two N^ω-bundles over $\operatorname{Spec}(R \hat{\otimes} \mathscr{O}_x)$ equipped with an isomorphism over $\operatorname{Spec}(R \hat{\otimes} K_x)$. There is a natural restriction map $\mathrm{res}^x : \mathscr{H}^x_{N^\omega} \to \mathscr{H}^{\mathrm{loc},x}_{N^\omega}$.

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Our choice of isomorphism $N/[N,N] \cong \mathbb{G}_a^{\oplus I}$ induces a map $\mathscr{H}_{N^\omega}^{\mathrm{loc},x} \to \mathbb{G}_a$ in the following way. The projection $N \to [N,N]$ induces a morphism

$$\mathscr{H}_{N^\omega}^{\mathrm{loc},x} \longrightarrow \prod_I \mathscr{H}_{\mathbb{G}_a^\omega}^{\mathrm{loc},x},$$

where $\mathbb{G}_a^\omega$ is the additive group scheme over X attached to the canonical line bundle ω_X. Note that there is a canonical isomorphism of groupoids

$$\mathscr{H}_{\mathbb{G}_a^\omega}^{\mathrm{loc},x} \xrightarrow{\sim} \mathbb{G}_a^\omega(\mathscr{O}_x)\backslash \mathbb{G}_a^\omega(K_x)/\mathbb{G}_a^\omega(\mathscr{O}_x),$$

so taking residues defines a morphism $\mathrm{rsd}^x : \mathscr{H}_{\mathbb{G}_a^\omega}^{\mathrm{loc},x} \to \mathbb{G}_a$. The composition

$$\mathrm{rsd}_\psi^x : \mathscr{H}_{N^\omega}^x \xrightarrow{\mathrm{res}^x} \mathscr{H}_{N^\omega}^{\mathrm{loc},x} \longrightarrow \prod_I \mathscr{H}_{\mathbb{G}_a^\omega}^{\mathrm{loc},x} \xrightarrow{\prod \mathrm{rsd}^x} \prod_I \mathbb{G}_a \xrightarrow{\mathrm{add}} \mathbb{G}_a$$

is an additive character, meaning it is a morphism of groupoids.

It follows that $\widetilde{\chi}^x := \mathrm{rsd}_\psi^{x,!}\chi$ is a character sheaf on $\mathscr{H}_{N^\omega}^x$, i.e., it is multiplicative for the groupoid structure. In particular we can speak of $(\mathscr{H}_{N^\omega}^x, \widetilde{\chi}^x)$-equivariant sheaves on $\overline{\mathrm{Bun}}_{N^\omega}^x$, which form a full subcategory of $D(\overline{\mathrm{Bun}}_{N^\omega}^x)$. Likewise, if a sheaf on $\overline{\mathrm{Bun}}_{N^\omega}$ is $(\mathscr{H}_{N^\omega}^x, \widetilde{\chi}^x)$-equivariant for all $x \in X$ we say that it is $(N^\omega(\mathbb{A}), \widetilde{\chi})$-*equivariant*. Although we will not use this fact, the category of $(N^\omega(\mathbb{A}), \widetilde{\chi})$-equivariant sheaves on $\overline{\mathrm{Bun}}_{N^\omega}$ is equivalent to the derived category of vector spaces, being generated by the object $\mathscr{W}_1$ introduced above. Moreover $(\mathscr{H}_{N^\omega}^x, \widetilde{\chi}^x)$-equivariance for a single $x \in X$ implies $(N^\omega(\mathbb{A}), \widetilde{\chi})$-equivariance.

Observe that there is a natural T-action on $\mathscr{H}_{N^\omega}^{\mathrm{loc},x}$. Using the chosen dominant regular cocharacter $\gamma : \mathbb{G}_m \to T$, the resulting $\mathbb{G}_m$-action on $\mathscr{H}_{N^\omega}^{\mathrm{loc},x}$ is contracting. In particular, it extends to an action $\mathbb{A}^1 \times \mathscr{H}_{N^\omega}^{\mathrm{loc},x} \to \mathscr{H}_{N^\omega}^{\mathrm{loc},x}$ of the multiplicative monoid $\mathbb{A}^1$. The !-pullback of χ along the composition

$$\mathbb{A}^1 \times \mathscr{H}_{N^\omega}^x \xrightarrow{\mathrm{id}_{\mathbb{A}^1} \times \mathrm{res}^x} \mathbb{A}^1 \times \mathscr{H}_{N^\omega}^{\mathrm{loc},x} \longrightarrow \mathscr{H}_{N^\omega}^{\mathrm{loc},x} \longrightarrow \mathbb{G}_a$$

defines an $\mathbb{A}^1$-family $\widetilde{\chi}_{\mathrm{ext}}^x$ of character sheaves on $\mathscr{H}_{N^\omega}^x$. Its !-restriction to $\{1\} \times \mathscr{H}_{N^\omega}^x$ is $\widetilde{\chi}_x$, and it is trivial along $\{0\} \times \mathscr{H}_{N^\omega}^x$.

Lemma 3.3.2. *The sheaf $\Psi(\mathscr{W})$ is $N^\omega(\mathbb{A})$-equivariant.*

Proof. Fix $x \in X$; we omit restriction to $\overline{\mathrm{Bun}}_{N^\omega}^x$ from the notation in what follows. By construction $\mathscr{W}$ is $\widetilde{\chi}_{\mathrm{ext}}^x|_{\mathbb{G}_m \times \mathscr{H}_{N^\omega}^x}$-equivariant. Since $\widetilde{\chi}_{\mathrm{ext}}^x|^!_{\{0\} \times \mathscr{H}_{N^\omega}^x}$ is the trivial character sheaf, it follows from our construction of Ψ (or equivalently, Beilinson's) that $\Psi(\mathscr{W})$ is $\mathscr{H}_{N^\omega}^x$-equivariant as desired. $\square$

4. Restriction to the strata

4.1 Now we give the first proof of Theorem 1.3.1 by deducing it from [AG15b, Theorem 1.3.6], which describes the restrictions to the strata of $i_!j_!\mathscr{W}$ in terms of the perverse sheaf Ω. Since we work with a fixed dominant regular coweight γ rather than the entire torus T, it will be necessary to prove a slightly different formulation of the latter theorem.

As in [AG15b, § 7.2], the inclusion

$$D(\overline{\mathrm{Bun}}_{N^\omega})^{N^\omega(\mathbb{A})} \longrightarrow D(\overline{\mathrm{Bun}}_{N^\omega})$$

of $N^\omega(\mathbb{A})$-equivariant sheaves on $\overline{\mathrm{Bun}}_{N^\omega}$ admits a right adjoint, which we denote by

$$\mathrm{Av}_*^{N^\omega(\mathbb{A})} : D(\overline{\mathrm{Bun}}_{N^\omega}) \longrightarrow D(\overline{\mathrm{Bun}}_{N^\omega})^{N^\omega(\mathbb{A})}.$$

Let i and j be as in § 1.2, and write $p : \mathbb{A}^1 \times \overline{\mathrm{Bun}}_{N^\omega} \to \overline{\mathrm{Bun}}_{N^\omega}$ for the projection. We will also abusively denote $p \circ j$ by p.

Proposition 4.1.1. *There is a canonical isomorphism* $i^! j_! \mathscr{W} \tilde{\to} \mathrm{Av}_*^{N^\omega(\mathbb{A})} p_! \mathscr{W}$.

Proof. See [AG15b, § 10.3], where the claim is proved for the action of the entire torus T. The same proof applies *mutatis mutandi* to our claim, which involves only the $\mathbb{G}_m$-action. □

First proof of Theorem 1.3.1. Since $\mathfrak{j}^!_{=\mu}$ preserves (homotopy) colimits, we see that

$$\mathfrak{j}^!_{=\mu} \Psi(\mathscr{W}) \xrightarrow{\sim} k \underset{H^\bullet(\mathbb{G}_m)}{\otimes} \mathfrak{j}^!_{=\mu} i^! j_! \mathscr{W}.$$

By Proposition 4.1.1 we have

$$\mathfrak{j}^!_{=\mu} i^! j_! \mathscr{W} \xrightarrow{\sim} \mathfrak{j}^!_{=\mu} \mathrm{Av}_*^{N^\omega(\mathbb{A})} p_! \mathscr{W},$$

and Proposition 3.3.1 implies that

$$\mathfrak{j}^!_{=\mu} \mathrm{Av}_*^{N^\omega(\mathbb{A})} p_! \mathscr{W} \xrightarrow{\sim} \mathfrak{m}_\mu^* \mathfrak{m}_{\mu,*} \mathfrak{j}^!_{=\mu} p_! \mathscr{W}.$$

Now Theorem 1.3.6 of [AG15b] yields

$$\mathfrak{m}_\mu^* \mathfrak{m}_{\mu,*} \mathfrak{j}^!_{=\mu} p_! \mathscr{W} \xrightarrow{\sim} \mathfrak{m}_\mu^\Delta \Omega^\mu \otimes H_c^\bullet(\mathbb{G}_m)[1]. \tag{4.1.1}$$

It remains to show that under the composed isomorphism

$$\mathfrak{j}^!_{=\mu} i^! j_! \mathscr{W} \xrightarrow{\sim} \mathfrak{m}_\mu^\Delta \Omega^\mu \otimes H_c^\bullet(\mathbb{G}_m)[1],$$

the action of $H^\bullet(\mathbb{G}_m)$ on the left-hand side corresponds to the natural action on $H_c^\bullet(\mathbb{G}_m)$ on the right-hand side. Since $H_c^\bullet(\mathbb{G}_m) = H^\bullet(\mathbb{G}_m)[-1]$ as $H^\bullet(\mathbb{G}_m)$-modules, this will finish the proof.

It is clear that

$$\mathfrak{j}^!_{=\mu} i^! j_! \mathscr{W} \xrightarrow{\sim} \mathfrak{m}_\mu^* \mathfrak{m}_{\mu,*} \mathfrak{j}^!_{=\mu} p_! \mathscr{W}$$

intertwines the actions of $H^\bullet(\mathbb{G}_m)$, since it is obtained by evaluating a morphism of functors on $\mathscr{W}$. Tracing through the proof of Theorem 1.3.6 in [AG15b], we see that the isomorphism (4.1.1) is also obtained by evaluating a morphism of functors on $\mathscr{W}$, with the appearance of $H_c^\bullet(\mathbb{G}_m)$ accounted for by the isomorphism

$$a_{\gamma,!} \mathscr{W} \xrightarrow{\sim} \mathscr{W}_1 \otimes H_\bullet(\mathbb{G}_m)[-1] \xrightarrow{\sim} \mathscr{W}_1 \otimes H_c^\bullet(\mathbb{G}_m)[1].$$

The latter isomorphism intertwines the actions of $H^\bullet(\mathbb{G}_m)$ as needed. □

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4.2 The rest of this subsection is devoted to the second proof of Theorem 1.3.1. This proof applies [Ras16, Theorem 4.6.1], which says that Ω can be realized as the twisted cohomology of Zastava space. Accordingly, we must formulate the analogue of Theorem 1.3.1 on Zastava space. First, the Whittaker sheaf: we claim that

$$\mathscr{W}_{Z^\lambda} := (\mathrm{id}_{\mathbb{G}_m} \times {}'\mathfrak{p}^{-,\lambda})^! \mathscr{W}[\dim \overline{\mathrm{Bun}}_{N^\omega} - \dim Z^\lambda]$$

is perverse for any λ. For λ sufficiently dominant $\mathfrak{p}^{-,\lambda}$ is smooth, so that $\mathscr{W}_{Z^\lambda}$ is the cohomologically normalized inverse image of the perverse sheaf $\mathscr{W}$. If $\lambda' \leqslant \lambda$ then we can pull back $\mathscr{W}_{Z^\lambda}$ along $\mathrm{id}_{\mathbb{G}_m}$ times the factorization morphism

$$(Z^{\lambda'} \times Z^{\lambda-\lambda'})_{\mathrm{disj}} \longrightarrow Z^\lambda,$$

and it is not hard to see that we obtain the restriction of $\mathscr{W}_{Z^{\lambda'}} \boxtimes \mathscr{W}_{Z^{\lambda-\lambda'}}$. Since the factorization map is étale, this implies that $\mathscr{W}_{Z^{\lambda'}}$ is perverse as desired.

Since the map $\overline{\mathrm{Bun}}_{N^\omega} \to \mathrm{Bun}_G$ is T-equivariant for the trivial action of T on Bun_G, we obtain an action of T on Z which makes ${}'\mathfrak{p}^-$ a T-equivariant map. In particular our fixed dominant regular cocharacter $\gamma : \mathbb{G}_m \to T$ induces an action of $\mathbb{G}_m$ on Z, and since $\mathscr{W}$ was $\mathbb{G}_m$-equivariant, so is $\mathscr{W}_Z$.

Theorem 4.2.1. *For any $\mu \in \Lambda^{\mathrm{pos}}$ there is an isomorphism*

$$'\mathrm{j}^!_{=\mu} \Psi(\mathscr{W}_Z) \xrightarrow{\sim} {}'\mathfrak{m}^\Delta_\mu \Omega^\mu.$$

We will need to use the factorization structure on Z in the following way. First, observe that $\mathscr{W}_Z$ admits a natural factorization structure. Thus $\Psi(\mathscr{W}_Z)$ admits a factorization structure by the Künneth formula for nearby cycles. Although the Künneth formula holds for the total nearby cycles functor, in this case the total nearby cycles equals the unipotent nearby cycles because $\mathscr{W}_Z$ is $\mathbb{G}_m$-equivariant.

4.3 In the second proof, Theorems 1.3.1 and 4.2.1 will be proved simultaneously by an inductive argument. The argument uses the following key lemma.

Let $f : \mathscr{X} \to \mathscr{Y}$ be a morphism of Artin stacks with $\mathscr{Y}$ smooth, and suppose we are given a function $\mathscr{Y} \to \mathbb{A}^1$. Let $g : S \to \mathscr{Y}$ be a morphism where S is an affine scheme and consider the following cartesian square.

$$\begin{tikzcd} \mathscr{X} \times_{\mathscr{Y}} S \arrow[r, "'g"] \arrow[d, "'f"] & \mathscr{X} \arrow[d, "f"] \\ S \arrow[r, "g"] & \mathscr{Y} \end{tikzcd}$$

Write $i : S_0 \to S$ for the inclusion of the vanishing locus of the function $S \to \mathscr{Y} \to \mathbb{A}^1$.

Lemma 4.3.1. *For any $\mathscr{F} \in D(\mathscr{X})$ which is universally locally acyclic (ULA) over $\mathscr{Y}$ and any $\mathscr{G} \in D(S)$, there is a canonical isomorphism*

$$\Psi('g^!(\mathscr{F}) \overset{!}{\otimes} {}'f^!(\mathscr{G})) \xrightarrow{\sim} {}'g^!(\mathscr{F}) \overset{!}{\otimes} {}'f^!(i^! \Psi(\mathscr{G})).$$

Fix $\nu \in \Lambda^{\mathrm{pos}}$. We will apply the lemma in the case $\mathscr{X} = \mathbb{A}^1 \times \overline{\mathrm{Bun}}^\lambda_{B^-,\leqslant\nu}$, $\mathscr{F} = \mathrm{IC}_{\mathbb{A}^1} \boxtimes \mathrm{j}^-_!(\mathrm{IC}_{\mathrm{Bun}^\lambda_{B^-}})|_{\overline{\mathrm{Bun}}^\lambda_{B^-,\leqslant\nu}}$, and $\mathscr{Y} = \mathbb{A}^1 \times \mathrm{Bun}_G$. Let us check that the ULA property holds when λ is sufficiently dominant relative to ν.

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According to [BG08, Corollary 4.5], the perverse sheaf $\mathfrak{j}^-_!(\mathrm{IC}_{\mathrm{Bun}_{B^-}})|_{\overline{\mathrm{Bun}}^{\lambda}_{B^-,\leqslant\nu}}$ has a filtration by $\eta \leqslant \nu$, with its subquotients being the perverse sheaves $\mathfrak{j}^-_{\geqslant\eta,!}(\Omega^\eta \boxtimes \mathrm{IC}_{\overline{\mathrm{Bun}}^{\lambda-\eta}_{B^-,\leqslant\nu-\eta}})$. Here Ω^η is attached to $\check{\mathfrak{n}}^-$ rather than $\check{\mathfrak{n}}$. Since the ULA property is stable under extensions and exterior products, it suffices to show that these subquotients are ULA over Bun_G.

Observe that the diagram

$$\begin{tikzcd} X^\eta \times \overline{\mathrm{Bun}}^{\lambda-\eta}_{B^-} \arrow[r, "\mathfrak{j}^-_{\geqslant\eta}"] \arrow[d] & \overline{\mathrm{Bun}}^{\lambda}_{B^-} \arrow[d] \\ \overline{\mathrm{Bun}}^{\lambda-\eta}_{B^-} \arrow[r] & \mathrm{Bun}_G \end{tikzcd}$$

commutes, where the left vertical arrow is projection onto the second factor. Since $\mathfrak{j}^-_{\geqslant\eta}$ is proper, the functor $\mathfrak{j}^-_{\geqslant\eta,!}$ preserves the ULA property. Thus it suffices to prove that, for λ sufficiently dominant, $\mathrm{IC}_{\overline{\mathrm{Bun}}^{\lambda-\eta}_{B^-,\leqslant\nu-\eta}}$ is ULA over Bun_G for all $\eta \leqslant \nu$. This follows immediately from [Cam16, Corollary 4.1.1.1].

Second proof of Theorems 1.3.1 and 4.2.1. Observe that Theorem 1.3.1 is trivial on the open stratum, since

$$\Psi(\mathscr{W})|_{\mathrm{Bun}_{N^\omega}} = \Psi(\mathscr{W}|_{\mathbb{G}_m \times \mathrm{Bun}_{N^\omega}}) = \Psi(\mathrm{IC}_{\mathbb{G}_m} \boxtimes \mathrm{IC}_{\mathrm{Bun}_{N^\omega}}) = \mathrm{IC}_{\mathrm{Bun}_{N^\omega}},$$

and similarly for Theorem 4.2.1 on $\mathring{Z}$.

We begin by proving Theorem 4.2.1 for the deepest strata, i.e., the closed embeddings

$${}'\mathfrak{j}^\mu_{=\mu} : X^\mu \longrightarrow Z^\mu.$$

Recall that $\Psi(\mathscr{W}_Z)$ is $\mathbb{G}_m$-monodromic by construction, so the contraction principle says that

$${}'\mathfrak{j}^{\mu,!}_{=\mu}\Psi(\mathscr{W}_Z) = \pi^\mu_!\Psi(\mathscr{W}_Z).$$

Write $\mathring{\pi}^\mu := \pi^\mu \circ {}'\mathfrak{j}^\mu_{=0}$. Theorem 4.6.1 in [Ras16] implies that there is an isomorphism

$$(\mathrm{id}_{\mathbb{G}_m} \times \mathring{\pi})_!(\mathscr{W}_Z|_{\mathbb{G}_m \times \mathring{Z}}) \xrightarrow{\sim} \mathrm{IC}_{\mathbb{G}_m} \boxtimes \Omega,$$

compatible with the factorization structures. Since $\mathscr{W}_Z$ is !-extended from $\mathbb{G}_m \times \mathring{Z}$, we obtain

$$(\mathrm{id}_{\mathbb{G}_m} \times \pi)_!\mathscr{W}_Z \xrightarrow{\sim} \mathrm{IC}_{\mathbb{G}_m} \boxtimes \Omega.$$

Since $\pi = \overline{\pi} \circ {}'\mathfrak{j}^-$ and Ψ commutes with proper pushforwards, we have

$$\overline{\pi}_!\Psi((\mathrm{id}_{\mathbb{G}_m} \times {}'\mathfrak{j}^-)_!\mathscr{W}_Z) \xrightarrow{\sim} \Omega.$$

Therefore it suffices to prove that the canonical morphism

$${}'\mathfrak{j}^-_!\Psi(\mathscr{W}_Z) \longrightarrow \Psi((\mathrm{id}_{\mathbb{G}_m} \times {}'\mathfrak{j}^-)_!\mathscr{W}_Z)$$

is an isomorphism. Since Ψ commutes with Verdier duality we can replace the !-pushforwards with $*$-pushforwards.

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Fix $S \to \overline{\mathrm{Bun}}_{N^\omega}$ with S an affine scheme and apply Lemma 4.3.1 with $f = \mathrm{id}_{\mathbb{A}^1} \times \overline{\mathfrak{p}}^-$,

$$g : \mathbb{A}^1 \times S \longrightarrow \mathbb{A}^1 \times \overline{\mathrm{Bun}}_{N^\omega} \longrightarrow \mathbb{A}^1 \times \mathrm{Bun}_G,$$

$\mathscr{F} = \mathrm{IC}_{\mathbb{A}^1} \boxtimes \mathfrak{j}^-_*(\omega_{\mathrm{Bun}_{B^-}})|_{\overline{\mathrm{Bun}}^\lambda_{B^-,\leqslant\nu}}$, and $\mathscr{G} = \mathscr{W}|^!_{\mathbb{A}^1\times S}$. Then the lemma yields an isomorphism

$$\Psi((\mathrm{id}_{\mathbb{G}_m} \times {}'\mathfrak{j}^-)_*\mathscr{W}_Z)|_{\leqslant\nu\overline{Z}^\lambda} \xrightarrow{\sim} {}'\mathfrak{j}^-_*\Psi(\mathscr{W}_Z)|_{\leqslant\nu\overline{Z}^\lambda}$$

for λ sufficiently dominant. Changing λ if necessary so that $\lambda \geqslant \mu$, we can restrict this isomorphism along the map

$$({}_{\leqslant\nu}\overline{Z}^\mu \times \mathring{Z}^{\lambda-\mu})_{\mathrm{disj}} \longrightarrow {}_{\leqslant\nu}\overline{Z}^\lambda.$$

By factorizability we obtain the desired isomorphism on ${}_{\leqslant\nu}\overline{Z}^\mu$. Since ν was arbitrary, Theorem 4.2.1 is proved for the deepest strata.

Now we prove Theorem 1.3.1. Fix $\mu \in \Lambda^{\mathrm{pos}}$ and choose $\lambda \geqslant \mu$ dominant enough that $Z^\lambda_{=\mu}$ surjects smoothly onto $X^\mu \times_{\mathrm{Bun}_T} \mathrm{Bun}_B$. Note that $(X^\mu \times \mathring{Z}^{\lambda-\mu})_{\mathrm{disj}}$ is one of the connected components of the fiber product

$$(X^\mu \times X^{\lambda-\mu})_{\mathrm{disj}} \times_{X^\lambda} Z^\lambda_{=\mu},$$

and that the former surjects onto $\overline{\mathrm{Bun}}_{N^\omega,=\mu}$. Theorem 4.2.1 for the deepest and open strata implies that the cohomologically normalized pullback of $\mathfrak{j}^!_{=\mu}\Psi(\mathscr{W})$ to $(X^\mu \times \mathring{Z}^{\lambda-\mu})_{\mathrm{disj}}$ is $\Omega^\mu \boxtimes \mathrm{IC}_{\mathring{Z}^{\lambda-\mu}}$. Theorem 1.3.1 follows once we observe that the composition

$$(X^\mu \times \mathring{Z}^{\lambda-\mu})_{\mathrm{disj}} \longrightarrow \overline{\mathrm{Bun}}_{N^\omega,=\mu} \xrightarrow{\mathfrak{m}_\mu} X^\mu$$

is the projection onto the first factor and apply Lemma 3.3.2.

The previous paragraph implies Theorem 4.2.1 holds on the stratum $Z^\lambda_{=\mu}$. Let $\lambda' \geqslant \mu$ and change λ if necessary so that $\lambda \geqslant \lambda'$. By restricting along the morphism

$$(Z^{\lambda'}_{=\mu} \times \mathring{Z}^{\lambda-\lambda'})_{\mathrm{disj}} \longrightarrow Z^\lambda_{=\mu}$$

and invoking factorization, we obtain Theorem 4.2.1.

The remainder of Theorems 1.3.1 and 4.2.1 follows as in the first proof. □

5. First proof of Theorem 1.4.1

5.1 Like Theorem 1.3.1, we formulate the analogue of Theorem 1.4.1 on Zastava space.

THEOREM 5.1.1. *For any $\lambda \in \Lambda^{\mathrm{pos}}$, there is an $\mathfrak{sl}_2$-equivariant isomorphism of factorizable sheaves*

$$\mathrm{gr}\,\Psi(\mathscr{W}_{Z^\lambda}) \xrightarrow{\sim} \bigoplus_{0\leqslant\mu\leqslant\lambda} {}'\mathfrak{j}^\lambda_{=\mu,!*}\,{}'\mathfrak{m}^{\lambda,\Delta}_\mu \mathscr{P}^\mu. \tag{5.1.1}$$

Now we work out three of the simplest cases of Theorem 5.1.1. For brevity, we will write $\Psi := \Psi(\mathscr{W}_{Z^\lambda})$.

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Example 5.1.2. Let α be a simple coroot. There is an isomorphism $Z^\alpha \cong X \times \mathbb{A}^1$ under which $\mathring{Z}^\alpha \cong X \times \mathring{\mathbb{A}}^1$, where $\mathring{\mathbb{A}}^1 := \mathbb{A}^1 \backslash \{0\}$. The canonical map $\mathring{Z}^\alpha \to \mathbb{A}^1$ is given in these terms by $(x,t) \mapsto 1/t$. It follows from [Cam17, Example 4.3] that Ψ is the cohomologically normalized pullback of the unique indecomposable tilting sheaf on $\mathbb{A}^1$ which extends $\mathrm{IC}_{\mathring{\mathbb{A}}^1}$. Moreover, the monodromy filtration

$$F_{-1} \subset F_0 \subset F_1 = \Psi$$

satisfies $F_{-1} \cong \mathrm{IC}_{Z^\alpha_{=\alpha}}$, $F_0/F_{-1} \cong \mathrm{IC}_{Z^\alpha}$, and $F_1/F_0 \cong \mathrm{IC}_{Z^\alpha_{=\alpha}}$, where $Z^\alpha_{=\alpha} = X \times \{0\} \subset X \times \mathbb{A}^1 = Z^\alpha$. This confirms Theorem 5.1.1 in the case $\lambda = \alpha$.

Example 5.1.3. Now consider the case $\lambda = 2\alpha$. By Theorem 4.2.1 for $\mu = 2\alpha$, we have a short exact sequence

$$\Omega^{2\alpha} \longrightarrow \Psi \longrightarrow {}'\mathrm{j}^{2\alpha}_{\leqslant\alpha,*}\,{}'\mathrm{j}^{2\alpha,*}_{\leqslant\alpha}\Psi$$

(recall that $\Omega^{2\alpha}$ is the clean extension of the sign local system on $X^{(2)}_{\mathrm{disj}}$). Similarly, applying Theorem 4.2.1 for $\mu = \alpha$ and 0 we obtain an exact triangle

$${}'\mathrm{j}^{2\alpha}_{=\alpha,*}\,\mathrm{IC}_{Z^{2\alpha}_{=\alpha}} \longrightarrow {}'\mathrm{j}^{2\alpha}_{\leqslant\alpha,*}\,{}'\mathrm{j}^{2\alpha,*}_{\leqslant\alpha}\Psi \longrightarrow {}'\mathrm{j}^{2\alpha}_{=0,*}\,\mathrm{IC}_{\mathring{Z}^{2\alpha}},$$

where we used the fact that $\Omega^\alpha \cong \mathrm{IC}_X$. Applying Verdier duality to the equation in [BG08, Corollary 4.5] (or rather the analogous equation on Zastava space), we have

$$[{}'\mathrm{j}^{2\alpha}_{=0,*}\,\mathrm{IC}_{\mathring{Z}^{2\alpha}}] = [\mathrm{IC}_{Z^{2\alpha}}] + [\mathrm{IC}_{\overline{Z^{2\alpha}_{=\alpha}}}] + [\Omega^{2\alpha}],$$

where we identified $\Omega^{2\alpha}$ with its Verdier dual. Finally, one computes the simple constituents of ${}'\mathrm{j}^{2\alpha}_{=\alpha,*}\,\mathrm{IC}_{Z^{2\alpha}_{=\alpha}}$ as follows. First, consider the short exact sequence

$$\mathrm{IC}_{\overline{Z^{2\alpha}_{=\alpha}}} \longrightarrow {}'\mathrm{j}^{2\alpha}_{=\alpha,*}\,\mathrm{IC}_{Z^{2\alpha}_{=\alpha}} \longrightarrow {}'\mathrm{j}^{2\alpha,!}_{=2\alpha}\,\mathrm{IC}_{\overline{Z^{2\alpha}_{=\alpha}}}[1].$$

To compute the third term, observe that there is a Cartesian square

$$\begin{tikzcd}
X^2 \arrow[r] \arrow[d] & Z^{2\alpha}_{\geqslant\alpha} \arrow[d, "{}'\mathrm{j}^{2\alpha}_{\geqslant\alpha}"] \\
X^{(2)} \arrow[r, "{}'\mathrm{j}^{2\alpha}_{=2\alpha}"] & Z^{2\alpha}
\end{tikzcd}$$

and that the !-restriction of $\mathrm{IC}_{Z^{2\alpha}_{\geqslant\alpha}}$ along the top horizontal morphism is $\mathrm{IC}_{X^2}[-1]$. Since ${}'\mathrm{j}^{2\alpha}_{\geqslant\alpha}$ is finite and birational onto its image, we can use base change to compute

$${}'\mathrm{j}^{2\alpha,!}_{=2\alpha}\,\mathrm{IC}_{\overline{Z^{2\alpha}_{=\alpha}}}[1] = {}'\mathrm{j}^{2\alpha,!}_{=2\alpha}\,{}'\mathrm{j}^{2\alpha}_{\geqslant\alpha,*}\,\mathrm{IC}_{Z^{2\alpha}_{\geqslant\alpha}}[1] = \mathrm{IC}_{Z^{2\alpha}_{=2\alpha}} \oplus \Omega^{2\alpha}.$$

Summarizing, we have

$$[\Psi] = [\mathrm{IC}_{Z^{2\alpha}}] + 2[\mathrm{IC}_{\overline{Z^{2\alpha}_{=\alpha}}}] + [\mathrm{IC}_{Z^{2\alpha}_{=2\alpha}}] + 3[\Omega^{2\alpha}].$$

Now we will determine which graded component of $\mathrm{gr}\,\Psi$ each simple subquotient lies in. In what follows, 'weight' refers to an eigenvalue of the Lefschetz Cartan operator. Since the monodromy filtration is compatible with the factorization structure, when we pull back $\mathrm{gr}\,\Psi$ along the factorization map

$$(Z^\alpha \times Z^\alpha)_{\mathrm{disj}} \longrightarrow Z^{2\alpha}$$

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we get $(\operatorname{gr}\Psi(\mathscr{W}_{Z^\alpha}))^{\boxtimes 2}$ restricted to $(Z^\alpha\times Z^\alpha)_{\mathrm{disj}}$. By the previous example, the latter sheaf with $\mathfrak{sl}_2$-action is isomorphic to

$$(\mathrm{std}^{\otimes 2}\otimes \mathrm{IC}_{X^2})\oplus(\mathrm{std}\otimes\mathrm{IC}_{X\times Z^\alpha})\oplus(\mathrm{std}\otimes\mathrm{IC}_{Z^\alpha\times X})\oplus(\mathrm{triv}\otimes\mathrm{IC}_{Z^\alpha\times Z^\alpha}). \tag{5.1.2}$$

It follows immediately that $\mathrm{IC}_{Z^{2\alpha}}$ has weight 0 and that the two copies of $\mathrm{IC}_{\overline{Z^{2\alpha}_{=\alpha}}}$ have weights ± 1. Since $\mathrm{std}^{\otimes 2}\cong V_2\oplus\mathrm{triv}$, the three copies of $\Omega^{2\alpha}$ have weights -2, 0, and 2, and $\mathrm{IC}_{Z^{2\alpha}_{=2\alpha}}$ has weight 0.

In terms of the monodromy filtration

$$F_{-2}\subset F_{-1}\subset F_0\subset F_1\subset F_2=\Psi,$$

we have $F_{-2}\cong\Omega^{2\alpha}\cong F_2/F_0$, $F_{-1}/F_{-2}\cong\mathrm{IC}_{\overline{Z^{2\alpha}_{=\alpha}}}\cong F_1/F_0$, and F_0/F_{-1} has simple constituents $\mathrm{IC}_{Z^{2\alpha}}$, $\mathrm{IC}_{Z^{2\alpha}_{=2\alpha}}$, and $\Omega^{2\alpha}$. So in order to prove Theorem 5.1.1 in the case $\lambda=2\alpha$, it remains to show that F_0/F_{-1} is semisimple. Its pullback along the factorization map is semisimple, and since semisimplicity is étale local the restriction of F_0/F_{-1} to $Z^{2\alpha}\backslash\pi^{-1}(X)$ is semisimple. But Ψ has no simple subquotients supported on $\pi^{-1}(X)$, so F_0/F_{-1} is the intermediate extension of its restriction to $Z^{2\alpha}\backslash\pi^{-1}(X)$ and therefore semisimple.

Example 5.1.4. Suppose $\lambda=\alpha+\beta$ is a coroot, where α and β are distinct simple coroots. Applying Theorem 4.2.1 for $\mu=\alpha+\beta$, we obtain the short exact sequence

$$\Omega^\lambda\longrightarrow\Psi\longrightarrow {}'\mathrm{j}^\lambda_{<\lambda,*}\,{}'\mathrm{j}^{\lambda,*}_{<\lambda}\Psi.$$

Similar considerations yield the short exact sequence

$${}'\mathrm{j}^\lambda_{=\alpha,*}\,\mathrm{IC}_{Z^\lambda_{=\alpha}}\oplus{}'\mathrm{j}^\lambda_{=\beta,*}\,\mathrm{IC}_{Z^\lambda_{=\beta}}\longrightarrow{}'\mathrm{j}^\lambda_{<\lambda,*}\,{}'\mathrm{j}^{\lambda,*}_{<\lambda}\Psi\longrightarrow{}'\mathrm{j}^\lambda_{=0,*}\,\mathrm{IC}_{\overset{\circ}{Z}{}^\lambda}.$$

Applying [BG08, Corollary 4.5], we have

$$[{}'\mathrm{j}^\lambda_{=0,*}\,\mathrm{IC}_{\overset{\circ}{Z}{}^\lambda}]=[\mathrm{IC}_{Z^\lambda}]+[\mathrm{IC}_{\overline{Z^\lambda_{=\alpha}}}]+[\mathrm{IC}_{\overline{Z^\lambda_{=\beta}}}]+[\Upsilon^\lambda],$$

where Υ^λ is the Verdier dual of Ω^λ. According to [AG15b, § 1.3.2], in this case Ω^λ is the $*$-extension of $\mathrm{IC}_{X^2_{\mathrm{disj}}}$ to $X^\lambda=X^2$, whence Υ^λ is the !-extension. In particular $[\Omega^\lambda]=[\Upsilon^\lambda]=[\mathrm{IC}_{X^2}]+[\mathrm{IC}_X]$. As for the remaining simple constituents, consider the short exact sequence

$$\mathrm{IC}_{\overline{Z^\lambda_{=\alpha}}}\longrightarrow{}'\mathrm{j}^\lambda_{=\alpha,*}\,\mathrm{IC}_{Z^\lambda_{=\alpha}}\longrightarrow{}'\mathrm{j}^{\lambda,!}_{=\alpha}\,\mathrm{IC}_{\overline{Z^\lambda_{=\alpha}}}[1],$$

and similarly for β. The third term is IC_{X^2}, so finally we see that

$$[\Psi]=[\mathrm{IC}_{Z^\lambda}]+2[\mathrm{IC}_{\overline{Z^\lambda_{=\alpha}}}]+2[\mathrm{IC}_{\overline{Z^\lambda_{=\beta}}}]+4[\mathrm{IC}_{X^2}]+2[\mathrm{IC}_X].$$

Now we compute the weights of the simple subquotients of Ψ. We have the factorization morphism

$$(Z^\alpha\times Z^\beta)_{\mathrm{disj}}\longrightarrow Z^{\alpha+\beta},$$

and after pulling back $\operatorname{gr}\Psi$ the result is (5.1.2), up to relabeling β as α. As in Example 5.1.3, it follows that $\mathrm{IC}_{Z^{\alpha+\beta}}$ has weight 0, the two copies of $\mathrm{IC}_{\overline{Z^\lambda_{=\alpha}}}$ have weights ±1 and likewise for $\mathrm{IC}_{\overline{Z^\lambda_{=\beta}}}$, and the four copies of IC_{X^2} have weights -2, 0, 0, and 2. We will see below that in any case where λ is a coroot, there are two simple subquotients of Ψ isomorphic to IC_X, with weights ±1.

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In terms of the monodromy filtration, we have $F_{-2} \cong \mathrm{IC}_{X^2} \cong F_2/F_1$, F_{-1}/F_{-2} and F_1/F_0 each have simple subquotients $\mathrm{IC}_{\overline{Z^\lambda_{=\alpha}}}$, $\mathrm{IC}_{\overline{Z^\lambda_{=\beta}}}$, and IC_X, and F_0/F_{-1} has simple subquotients IC_{Z^λ} and IC_{X^2}, the latter with multiplicity two. As in Example 5.1.3, one uses factorization to show that F_0/F_{-1} is semisimple. To prove the semisimplicity of F_{-1}/F_{-2} and F_1/F_0, it is enough to show that there are no extensions between IC_X and $\mathrm{IC}_{\overline{Z^\lambda_{=\alpha}}}$. By Verdier duality, it suffices to prove that

$$\mathrm{Ext}^1_{D(Z^\lambda)}(\mathrm{IC}_X, \mathrm{IC}_{\overline{Z^\lambda_{=\alpha}}}) = 0.$$

We have $'\mathfrak{j}^{\lambda,!}_{=\lambda}\,\mathrm{IC}_{\overline{Z^\lambda_{=\alpha}}} = \mathrm{IC}_{X^2}[-1]$, whence $\Delta^!\,\mathrm{IC}_{\overline{Z^\lambda_{=\alpha}}} = \mathrm{IC}_X[-2]$. Thus

$$\mathrm{RHom}_{D(Z^\lambda)}(\mathrm{IC}_X, \mathrm{IC}_{\overline{Z^\lambda_{=\alpha}}}) = H^\bullet(X)[-2],$$

and in particular Ext^1 vanishes.

5.2 The following lemma will be used in both proofs of Theorems 1.4.1 and 5.1.1. For any $\mu \in \Lambda^{\mathrm{pos}}$ write

$$\Delta^\mu : X \times_{\mathrm{Bun}_T} \overline{\mathrm{Bun}}_B \longrightarrow \overline{\mathrm{Bun}}_{N^\omega}$$

for the finite birational map defined as the composition of $\mathfrak{j}_{\geqslant\mu}$ and the embedding

$$X \times_{\mathrm{Bun}_T} \overline{\mathrm{Bun}}_B = X \times_{X^\mu} \overline{\mathrm{Bun}}_{N^\omega,\geqslant\mu} \longrightarrow \overline{\mathrm{Bun}}_{N^\omega,\geqslant\mu}$$

induced by the diagonal map $X \to X^\mu$.

Lemma 5.2.1. *If μ is a coroot, then there is an indecomposable subquotient $\mathscr{M}$ of $\Psi(\mathscr{W})$ with a filtration*

$$\mathscr{M}_{-1} \subset \mathscr{M}_0 \subset \mathscr{M}_1 = \mathscr{M}$$

such that $\mathscr{M}_{-1} \cong \Delta^\mu_*\,\mathrm{IC}_{X\times_{\mathrm{Bun}_T}\overline{\mathrm{Bun}}_B}$, $\mathscr{M}_0/\mathscr{M}_{-1} \cong \mathrm{IC}_{\overline{\mathrm{Bun}}_{N^\omega}}$, and $\mathscr{M}/\mathscr{M}_0 \cong \Delta^\mu_*\,\mathrm{IC}_{X\times_{\mathrm{Bun}_T}\overline{\mathrm{Bun}}_B}$.

Proof. First, we claim the subsheaf $\mathfrak{j}_{=0,!}\,\mathrm{IC}_{\mathrm{Bun}_{N^\omega}}$ of $\Psi(\mathscr{W})$ has a quotient $\mathscr{M}_0$ of the form described above. Recall that $\mathfrak{j}_{=0,!}\,\mathrm{IC}_{\mathrm{Bun}_{N^\omega}}$ has a descending filtration with subquotients $\mathfrak{j}_{=\nu,!*}\mathfrak{r}^\Delta_\nu\Omega^\nu$, and in particular has $\mathfrak{j}_{=\mu,!*}\Omega^\mu$ as a subquotient and $\mathrm{IC}_{\overline{\mathrm{Bun}}_{N^\omega}}$ as a quotient. The former sheaf has $\Delta^\mu_*\,\mathrm{IC}_{X\times_{\mathrm{Bun}_T}\overline{\mathrm{Bun}}_B}$ as a quotient because μ is a coroot, so it suffices to show that for any $0 < \nu < \mu$ we have

$$\mathrm{Ext}^1(\mathfrak{j}_{=\nu,!*}\mathfrak{r}^\Delta_\nu\Omega^\nu, \Delta^\mu_*\,\mathrm{IC}_{X\times_{\mathrm{Bun}_T}\overline{\mathrm{Bun}}_B}) = 0,$$

or dually

$$\mathrm{Ext}^1(\Delta^\mu_*\,\mathrm{IC}_{X\times_{\mathrm{Bun}_T}\overline{\mathrm{Bun}}_B}, \mathfrak{j}_{=\nu,!*}\mathfrak{r}^\Delta_\nu\Upsilon^\nu) = 0,$$

where Υ^ν is the Verdier dual of Ω^ν. One computes using base change that

$$\Delta^{\mu,!}\mathfrak{j}_{=\nu,!*}\mathfrak{r}^\Delta_\nu\Upsilon^\nu = \Delta^!\Upsilon^\nu \overset{!}{\otimes} \Delta^{\mu,!}\,\mathrm{IC}_{\overline{\mathrm{Bun}}_{N^\omega}}.$$

Since $\Delta^{\mu,!}\,\mathrm{IC}_{\overline{\mathrm{Bun}}_{N^\omega}}$ is concentrated in cohomological degrees greater than or equal to 1, and both $\Delta^!\Upsilon^\nu$ and $\Delta^{\mu,!}\,\mathrm{IC}_{\overline{\mathrm{Bun}}_{N^\omega}}$ have lisse (actually constant) cohomology sheaves, their !-tensor product is concentrated in cohomological degrees greater than or equal to 2. It follows that the Ext^1 above vanishes.

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We have shown that $\mathfrak{j}_{=0,!}\,\mathrm{IC}_{\mathrm{Bun}_{N^\omega}}$ has a quotient $\mathscr{M}_0$ which fits into a short exact sequence

$$\Delta^\mu_*\,\mathrm{IC}_{X\times_{\mathrm{Bun}_T}\overline{\mathrm{Bun}}_B} \longrightarrow \mathscr{M}_0 \longrightarrow \mathrm{IC}_{\overline{\mathrm{Bun}}_{N^\omega}}.$$

This sequence does not split because $\mathscr{M}_0$ is the quotient of the indecomposable sheaf $\mathfrak{j}_{=0,!}\,\mathrm{IC}_{\mathrm{Bun}_{N^\omega}}$ with simple cosocle $\mathrm{IC}_{\overline{\mathrm{Bun}}_{N^\omega}}$. Dually, we obtain a subsheaf $\mathscr{M}/\mathscr{M}_{-1}$ of $\mathfrak{j}_{=0,*}\,\mathrm{IC}_{\mathrm{Bun}_{N^\omega}}$ of the desired form, from which follows the existence of $\mathscr{M}$. $\square$

We will first give a proof of Theorem 1.4.1 under the assumption that $\mathrm{gr}\,\Psi(\mathscr{W})$ is semisimple. The semisimplicity can be proved via Mochizuki's theory of weights for holonomic D-modules, since $\mathscr{W}$ is pure and (up to shift) the monodromy filtration on nearby cycles of a pure sheaf coincides with the weight filtration (see [Moc15, Corollary 9.1.10]).

First proof of Theorems 1.4.1 and 5.1.1. Both sides of the isomorphism (1.4.1) restrict to $\mathrm{IC}_{\mathrm{Bun}_{N^\omega}}$. Suppose that we have constructed the isomorphism over $\overline{\mathrm{Bun}}_{N^\omega,<\mu}$. Then for $\lambda \geqslant \mu$ sufficiently dominant, pulling back along $'\mathfrak{p}^-$ yields the isomorphism (5.1.1) over $Z^\lambda_{<\mu}$. One obtains (5.1.1) on $Z^\mu_{<\mu}$ by pullback along the factorization map

$$(Z^\mu_{<\mu}\times \mathring{Z}^{\lambda-\mu})_{\mathrm{disj}} \longrightarrow Z^\lambda_{<\mu},$$

since the inverse images of both sides of the isomorphism factorize and are constant along the second component. The same argument yields (5.1.1) on $Z^{\mu'}_{<\mu} = Z^{\mu'}$ for $\mu' < \mu$.

On the other hand, one can use factorization to obtain (5.1.1) on $Z^\mu\backslash\pi^{-1}(X)$. Namely, for $\mu_1+\mu_2=\mu$, $\mu_1,\mu_2<\mu$, the pullback of both sides of the isomorphism along

$$(Z^{\mu_1}\times Z^{\mu_2})_{\mathrm{disj}} \longrightarrow Z^\mu$$

are identified. Since this factorization map is étale but not necessarily an embedding, we must argue that the isomorphism descends to its image. This immediately reduces to the case that $\mu = n\cdot\alpha$ for some $\alpha\in\Delta$. In this case, both sides of (5.1.1) are the intermediate extension of their restriction to $\pi^{-1}(X^\mu_{\mathrm{disj}})$, so it suffices to show that the isomorphism over $(Z^\alpha)^n_{\mathrm{disj}}$ descends to $\pi^{-1}(X^\mu_{\mathrm{disj}})\subset Z^\mu$. Since $(Z^\alpha)^n_{\mathrm{disj}}$ is a Σ_n-torsor over $\pi^{-1}(X^\mu_{\mathrm{disj}})$ and Σ_n is generated by transpositions, the claim reduces to the case $n=2$. But this was already done in Example 5.1.3.

Note that $Z^\mu_{<\mu}\cup(Z^\mu\backslash\pi^{-1}(X)) = Z^\mu\backslash\Delta(X)$. The isomorphisms of the previous two paragraphs clearly agree on $Z^\mu_{<\mu}\cap(Z^\mu\backslash\pi^{-1}(X))$, hence glue to an isomorphism away from the main diagonal.

If μ is not a coroot, then we claim that $\Psi(\mathscr{W}_{Z^\mu})$ has no simple subquotients supported on the main diagonal, whence $\mathrm{gr}\,\Psi(\mathscr{W}_{Z^\mu})$ is the intermediate extension of its restriction to $Z^\mu\backslash\Delta(X)$. This is true for the right-hand side of (5.1.1) by construction, so the claim implies that the isomorphism extends to Z^μ in this case. By Theorem 4.2.1 there is a filtration of $\Psi(\mathscr{W}_{Z^\mu})$ by the sheaves $'\mathfrak{j}^\mu_{=\nu,*}{}'\mathfrak{m}^{\mu,\Delta}_\nu\Omega^\nu$ for $0\leqslant\nu\leqslant\mu$. Using [BG08, Corollary 4.5], one can show that $'\mathfrak{j}^\mu_{=\nu,*}{}'\mathfrak{m}^{\mu,\Delta}_\nu\Omega^\nu$ surjects onto $'\mathfrak{j}^\mu_{=\mu,*}\,\mathrm{add}_*(\Omega^\nu\boxtimes\Upsilon^{\mu-\nu})$, and that no subquotient of the kernel is supported on X^μ. Now the claim follows, because out of the latter sheaves only Ω^μ and Υ^μ could have subquotients supported on the diagonal, and by § 3.3 of [BG08] this occurs if and only if μ is a coroot.

Suppose that μ is a coroot. Then $\Delta_*\,\mathrm{IC}_X$ appears as a subquotient of Ω^μ and of Υ^μ with multiplicity one. By the analysis in the previous paragraph $\Delta_*\,\mathrm{IC}_X$ appears as a summand of $\mathrm{gr}\,\Psi(\mathscr{W}_{Z^\mu})$ with multiplicity two, and there are no other subquotients supported on the main diagonal. Thus the isomorphism (5.1.1) extends to Z^μ, and it remains to show that $\mathfrak{sl}_2$ acts on the summand $\mathrm{IC}_X^{\oplus 2}$ of $\mathrm{gr}\,\Psi(\mathscr{W}_{Z^\mu})$ as the standard representation.

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The only other possibility is that $\mathfrak{sl}_2$ acts on $\mathrm{IC}_X^{\oplus 2}$ trivially, which would imply that the subquotient $\mathscr{M}$ from Lemma 5.2.1 is a subquotient of F_0/F_{-1}. But $\mathscr{M}$ is indecomposable and F_0/F_{-1} is semisimple, so this is impossible.

Having constructed the isomorphism of Theorem 5.1.1 over Z^μ, we can complete the inductive step of Theorem 1.4.1 by extending the isomorphism from $\overline{\mathrm{Bun}}_{N^\omega,<\mu}$ to $\overline{\mathrm{Bun}}_{N^\omega,\leqslant\mu}$. Choose $\lambda \geqslant \mu$ dominant enough that $Z^\lambda_{\leqslant\mu}$ surjects smoothly onto $\overline{\mathrm{Bun}}_{N^\omega,\leqslant\mu}$. As in the proof of Theorem 1.3.1, note that $(Z^\mu \times \overset{\circ}{Z}{}^{\lambda-\mu})_{\mathrm{disj}}$ is one of the connected components of the fiber product

$$(X^\mu \times X^{\lambda-\mu})_{\mathrm{disj}} \times_{X^\lambda} Z^\lambda_{\leqslant\mu},$$

and that the former surjects onto $\overline{\mathrm{Bun}}_{N^\omega,\leqslant\mu}$. By factorization, the cohomologically normalized pullback of $\mathrm{gr}\,\Psi(\mathscr{W})$ to $(Z^\mu \times \overset{\circ}{Z}{}^{\lambda-\mu})_{\mathrm{disj}}$ is $(\mathrm{gr}\,\Psi(\mathscr{W}_{Z^\mu})) \boxtimes \mathrm{IC}_{\overset{\circ}{Z}{}^{\lambda-\mu}}$.

A factorization argument as in the proof of Theorem 4.2.1 allows us to construct the isomorphism (5.1.1) over $Z^\lambda_{\leqslant\mu}$ for arbitrary $\lambda \geqslant \mu$, which completes the proof of Theorem 5.1.1. □

Proof of Corollary 1.4.1.1. It suffices to prove the corresponding claim on Z^λ for any $\lambda \in \Lambda^{\mathrm{pos}}$. The morphism

$$'\mathfrak{j}^\lambda_{=0,!}\,\mathrm{IC}_{\mathrm{Bun}_{N^\omega}} \longrightarrow \Psi(\mathscr{W})$$

is injective because $\Psi(\mathscr{W})$ is tilting, so it suffices to show that $'\mathfrak{j}^\lambda_{=0,!}\,\mathrm{IC}_{\mathrm{Bun}_{N^\omega}}$ and the kernel of monodromy have the same class in the Grothendieck group. By factorization and induction this holds away from the main diagonal, and [BG08, Corollary 4.5] implies that the only subquotient of $'\mathfrak{j}^\lambda_{=0,!}\,\mathrm{IC}_{\mathrm{Bun}_{N^\omega}}$ supported on the main diagonal is $\Delta_*\,\mathrm{IC}_X$ with multiplicity one. Theorem 5.1.1 implies that the same is true for the kernel of the monodromy operator on $\Psi(\mathscr{W})$. □

6. Second proof of Theorem 1.4.1

6.1 In this section we will give a proof of Theorems 1.4.1 and 5.1.1 which does not use weights for irregular holonomic D-modules to prove the semisimplicity of $\mathrm{gr}\,\Psi(\mathscr{W})$, but instead depends on Conjecture 1.5.2 (but only the part proved in [FFKM99]). First we make the statement of the conjecture more precise in the case of a trivial $\check{T}$-local system by specifying the action of generators of $\check{\mathfrak{g}} \otimes H^\bullet(X)$ on $\overline{\mathfrak{p}}_*\,\mathrm{IC}_{\overline{\mathrm{Bun}}_B}$.

We construct the action of $\check{\mathfrak{h}} \otimes H^\bullet(X)$ as follows. Pullback along the evaluation map

$$X \times \mathrm{Bun}_T \longrightarrow \mathrm{pt}/T$$

defines a homomorphism

$$\mathrm{Sym}(\mathfrak{h}^*[-2]) = H^\bullet(\mathrm{pt}/T) \longrightarrow H^\bullet(X) \otimes H^\bullet(\mathrm{Bun}_T).$$

By adjunction we obtain a morphism $\mathfrak{h}^* \otimes H_\bullet(X)[-2] \longrightarrow H^\bullet(\mathrm{Bun}_T)$. Identifying $\mathfrak{h}^* \cong \check{\mathfrak{h}}$ and $H_\bullet(X)[-2] \cong H^\bullet(X)$, the latter using Poincaré duality, we obtain a morphism

$$\check{\mathfrak{h}} \otimes H^\bullet(X) \longrightarrow H^\bullet(\mathrm{Bun}_T).$$

Then the action of $H^\bullet(\mathrm{Bun}_T)$ on ω_{Bun_T} induces by functoriality the desired action of $\check{\mathfrak{h}} \otimes H^\bullet(X)$ on $\mathrm{Eis}_{!*}\,\omega_{\mathrm{Bun}_T} = \overline{\mathfrak{p}}_*\,\mathrm{IC}_{\overline{\mathrm{Bun}}_B}$.

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Next we construct the action of $\check{\mathfrak{n}} \otimes H^\bullet(X)$. Denote by $\mathscr{U}(\check{\mathfrak{n}})$ the factorization algebra whose fiber at $\sum_i \mu_i x_i \in X^\mu$ is

$$\bigotimes_i U(\check{\mathfrak{n}})^{\mu_i},$$

where the superscript μ_i indicates the corresponding $\check{T}$-graded component. The following is part of [BG08, Theorem 5.6], which itself is a restatement of results from [FFKM99]: for any $\mu \in \Lambda^{\mathrm{pos}}$ there is a canonical morphism

$$\mathrm{j}_{\geqslant\mu,!}(\mathscr{U}(\check{\mathfrak{n}})^\mu \boxtimes \mathrm{IC}_{\overline{\mathrm{Bun}}_B}) \longrightarrow \mathrm{IC}_{\overline{\mathrm{Bun}}_B},$$

which induces an isomorphism

$$\mathscr{U}(\check{\mathfrak{n}})^\mu \boxtimes \mathrm{IC}_{\mathrm{Bun}_B} \xrightarrow{\sim} \mathrm{j}^!_{=\mu}\, \mathrm{IC}_{\overline{\mathrm{Bun}}_B}.$$

Here we abuse notation slightly by denoting the maps $\mathrm{j}_{\geqslant\mu} : X^\mu \times \overline{\mathrm{Bun}}_B \to \overline{\mathrm{Bun}}_B$ and $\mathrm{j}_{=\mu} : X^\mu \times \mathrm{Bun}_B \to \overline{\mathrm{Bun}}_B$ by the same symbols we used in the case of $\overline{\mathrm{Bun}}_{N^\omega}$. The latter are obtained from the former by base change along $\overline{\mathrm{Bun}}_{N^\omega} \to \overline{\mathrm{Bun}}_B$.

Pushing forward to Bun_G, we obtain a morphism

$$H^\bullet(X^\mu, \mathscr{U}(\check{\mathfrak{n}})^\mu) \otimes \overline{\mathfrak{p}}_* \,\mathrm{IC}_{\overline{\mathrm{Bun}}_B} \longrightarrow \overline{\mathfrak{p}}_*\, \mathrm{IC}_{\overline{\mathrm{Bun}}_B}.$$

The object $\mathscr{U}(\check{\mathfrak{n}})^\mu$ is concentrated in (perverse) cohomological degrees greater than or equal to 1, and if μ is a coroot then we have $H^1(\mathscr{U}(\check{\mathfrak{n}})^\mu) = \mathrm{IC}_X$. The resulting morphism $k_X \to \mathscr{U}(\check{\mathfrak{n}})^\mu$ induces

$$H^\bullet(X) \otimes \overline{\mathfrak{p}}_*\, \mathrm{IC}_{\overline{\mathrm{Bun}}_B} \longrightarrow H^\bullet(X^\mu, \mathscr{U}(\check{\mathfrak{n}})^\mu) \otimes \overline{\mathfrak{p}}_*\, \mathrm{IC}_{\overline{\mathrm{Bun}}_B} \longrightarrow \overline{\mathfrak{p}}_*\, \mathrm{IC}_{\overline{\mathrm{Bun}}_B},$$

which defines the action of $\check{\mathfrak{n}}_\mu \otimes H^\bullet(X) \cong H^\bullet(X)$ on $\overline{\mathfrak{p}}_*\, \mathrm{IC}_{\overline{\mathrm{Bun}}_B}$.

Dually, for any $\mu \in \Lambda^{\mathrm{pos}}$ there is a canonical morphism

$$\mathrm{IC}_{\overline{\mathrm{Bun}}_B} \longrightarrow \mathrm{j}_{\geqslant\mu,*}(\mathscr{U}^\vee(\check{\mathfrak{n}}^-)^\mu \boxtimes \mathrm{IC}_{\overline{\mathrm{Bun}}_B}),$$

which induces an isomorphism

$$\mathrm{j}^*_{=\mu}\, \mathrm{IC}_{\overline{\mathrm{Bun}}_B} \xrightarrow{\sim} \mathscr{U}^\vee(\check{\mathfrak{n}}^-)^\mu \boxtimes \mathrm{IC}_{\mathrm{Bun}_B}.$$

Here $\mathscr{U}^\vee(\check{\mathfrak{n}}^-)$ is by definition the Verdier dual of $\mathscr{U}(\check{\mathfrak{n}}^-)$.

Thus we obtain a morphism

$$\overline{\mathfrak{p}}_*\, \mathrm{IC}_{\overline{\mathrm{Bun}}_B} \longrightarrow H^\bullet(X^\mu, \mathscr{U}(\check{\mathfrak{n}}^-)^\mu)^\vee \otimes \overline{\mathfrak{p}}_*\, \mathrm{IC}_{\overline{\mathrm{Bun}}_B},$$

or by adjunction

$$H^\bullet(X^\mu, \mathscr{U}(\check{\mathfrak{n}}^-)^\mu) \otimes \overline{\mathfrak{p}}_*\, \mathrm{IC}_{\overline{\mathrm{Bun}}_B} \longrightarrow \overline{\mathfrak{p}}_*\, \mathrm{IC}_{\overline{\mathrm{Bun}}_B}.$$

If μ is a coroot, then as before we have a morphism $H^\bullet(X) \to H^\bullet(X^\mu, \mathscr{U}(\check{\mathfrak{n}}^-)^\mu)$, which defines the action of $\check{\mathfrak{n}}^-_{-\mu} \otimes H^\bullet(X) \cong H^\bullet(X)$ on $\overline{\mathfrak{p}}_*\, \mathrm{IC}_{\overline{\mathrm{Bun}}_B}$.

6.2 Fix a coroot μ. Recall the subquotient $\mathscr{M}$ of $\Psi(\mathscr{W})$ from Lemma 5.2.1. The action of $\check{\mathfrak{g}}$ on $\overline{\mathfrak{p}}_* \mathrm{IC}_{\overline{\mathrm{Bun}}_B}$ relates to our problem through the following key lemma, whose proof will occupy this subsection.

Lemma 6.2.1. *The sheaf $\mathscr{M}$ does not descend to $\overline{\mathrm{Bun}}_{N^\omega}/\mathbb{G}_m$.*

First, observe that $\mathscr{M}_0$ and $\mathscr{M}/\mathscr{M}_{-1}$ descend to $\overline{\mathrm{Bun}}_{N^\omega}/T$ and hence to $\overline{\mathrm{Bun}}_{N^\omega}/\mathbb{G}_m$, being subquotients of $\mathfrak{j}_{=0,!}\,\mathrm{IC}_{\mathrm{Bun}_{N^\omega}}$ and $\mathfrak{j}_{=0,*}\,\mathrm{IC}_{\mathrm{Bun}_{N^\omega}}$ respectively. The obstruction to descent of $\mathscr{M}$ to $\overline{\mathrm{Bun}}_{N^\omega}/\mathbb{G}_m$ is the resulting composition

$$\Delta^\mu_* \mathrm{IC}_{(X\times_{\mathrm{Bun}_T}\overline{\mathrm{Bun}}_B)/\mathbb{G}_m} \longrightarrow \mathrm{IC}_{\overline{\mathrm{Bun}}_{N^\omega}/\mathbb{G}_m}[1] \longrightarrow \Delta^\mu_* \mathrm{IC}_{(X\times_{\mathrm{Bun}_T}\overline{\mathrm{Bun}}_B)/\mathbb{G}_m}[2]. \tag{6.2.1}$$

Similarly, the obstruction to its descent to $\overline{\mathrm{Bun}}_{N^\omega}/T$ is the composition

$$\Delta^\mu_* \mathrm{IC}_{(X\times_{\mathrm{Bun}_T}\overline{\mathrm{Bun}}_B)/T} \longrightarrow \mathrm{IC}_{\overline{\mathrm{Bun}}_{N^\omega}/T}[1] \longrightarrow \Delta^\mu_* \mathrm{IC}_{(X\times_{\mathrm{Bun}_T}\overline{\mathrm{Bun}}_B)/T}[2]. \tag{6.2.2}$$

Denote by $\mathring{\Delta}^\mu : X \times_{\mathrm{Bun}_T} \mathrm{Bun}_B \to \overline{\mathrm{Bun}}_{N^\omega}$ the locally closed embedding given by composing Δ^μ with the open embedding $X \times_{\mathrm{Bun}_T} \mathrm{Bun}_B \to X \times_{\mathrm{Bun}_T} \overline{\mathrm{Bun}}_B$. Composition with the canonical morphisms

$$\mathring{\Delta}^\mu_! \mathrm{IC}_{(X\times_{\mathrm{Bun}_T}\mathrm{Bun}_B)/T} \to \Delta^\mu_* \mathrm{IC}_{(X\times_{\mathrm{Bun}_T}\overline{\mathrm{Bun}}_B)/T}$$

and

$$\Delta^\mu_* \mathrm{IC}_{(X\times_{\mathrm{Bun}_T}\overline{\mathrm{Bun}}_B)/T} \to \mathring{\Delta}^\mu_* \mathrm{IC}_{(X\times_{\mathrm{Bun}_T}\mathrm{Bun}_B)/T}$$

gives

$$\mathrm{End}(\Delta^\mu_* \mathrm{IC}_{(X\times_{\mathrm{Bun}_T}\overline{\mathrm{Bun}}_B)/T}) \longrightarrow H^\bullet(X \times \mathrm{pt}/T), \tag{6.2.3}$$

since the map $(X \times_{\mathrm{Bun}_T} \mathrm{Bun}_B)/T \to X \times \mathrm{pt}/T$ induces an isomorphism on cohomology.

Lemma 6.2.2. *The image of the endomorphism (6.2.2) under (6.2.3) is*

$$-1 \otimes h_\mu \in H^0(X) \otimes \mathfrak{h}^* \subset H^2(X \times \mathrm{pt}/T).$$

Proof. Theorem 5.1.5 in [BG02] says that $\mathrm{IC}_{\overline{\mathrm{Bun}}_B}$ is ULA over Bun_T, which implies that the !-restriction of $\mathrm{IC}_{\overline{\mathrm{Bun}}_B}$ to $\overline{\mathrm{Bun}}_{N^\omega}/T$ is $\mathrm{IC}_{\overline{\mathrm{Bun}}_{N^\omega}/T}[\dim T - \dim \mathrm{Bun}_T]$. It follows that the !-restriction of $\Delta^\mu_* \mathrm{IC}_{X\times\overline{\mathrm{Bun}}_B}$ to $\overline{\mathrm{Bun}}_{N^\omega}/T$ is a shift of $\Delta^\mu_* \mathrm{IC}_{(X\times_{\mathrm{Bun}_T}\overline{\mathrm{Bun}}_B)/T}$, where we abusively write $\Delta^\mu : X \times \overline{\mathrm{Bun}}_B \to \overline{\mathrm{Bun}}_B$ for the similarly defined finite map. This gives rise to a commutative square

$$\begin{tikzcd}
\mathrm{End}(\Delta^\mu_* \mathrm{IC}_{X\times\overline{\mathrm{Bun}}_B}) \arrow[r] \arrow[d] & H^\bullet(X \times \mathrm{Bun}_T) \arrow[d] \\
\mathrm{End}(\Delta^\mu_* \mathrm{IC}_{(X\times_{\mathrm{Bun}_T}\overline{\mathrm{Bun}}_B)/T}) \arrow[r] & H^\bullet(X \times \mathrm{pt}/T)
\end{tikzcd}$$

where the upper horizontal arrow is defined similarly to (6.2.3) and the right vertical arrow is id_X times restriction along $\rho(\omega) : \mathrm{pt}/T \to \mathrm{Bun}_T$. The previous subsection implies that $\mathscr{M}_0$ and $\mathscr{M}/\mathscr{M}_{-1}$ extend to $\overline{\mathrm{Bun}}_B$, giving rise to a morphism

$$\Delta^\mu_* \mathrm{IC}_{X\times\overline{\mathrm{Bun}}_B} \longrightarrow \mathrm{IC}_{\overline{\mathrm{Bun}}_B}[1] \longrightarrow \Delta^\mu_* \mathrm{IC}_{X\times\overline{\mathrm{Bun}}_B}[2] \tag{6.2.4}$$

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which restricts to (6.2.2) on $\overline{\mathrm{Bun}}_{N^\omega}/T$. Thus it suffices to show that the image of (6.2.4) in $H^2(X \times \mathrm{Bun}_T)$ restricts to $-1 \otimes h_\mu$ along $\mathrm{id}_X \times \rho(\omega)$.

Observe that

$$\mathrm{End}(\Delta^\mu_* \mathrm{IC}_{X\times\overline{\mathrm{Bun}}_B}) = \mathrm{Hom}(\Delta^\mu_*(k_X \boxtimes \mathrm{IC}_{\overline{\mathrm{Bun}}_B}), \Delta^\mu_*(\omega_X \boxtimes \mathrm{IC}_{\overline{\mathrm{Bun}}_B}))[-2],$$

so $\overline{\mathfrak{p}}_*$ induces a morphism

$$\mathrm{End}(\Delta^\mu_* \mathrm{IC}_{X\times\overline{\mathrm{Bun}}_B}) \longrightarrow \mathrm{Hom}(H^\bullet(X) \otimes \overline{\mathfrak{p}}_* \mathrm{IC}_{\overline{\mathrm{Bun}}_B}, H_\bullet(X) \otimes \overline{\mathfrak{p}}_* \mathrm{IC}_{\overline{\mathrm{Bun}}_B})[-2].$$

Composing with the canonical map $k \to H^\bullet(X)$ and its dual $H_\bullet(X) \to k$, we obtain

$$\mathrm{End}(\Delta^\mu_* \mathrm{IC}_{X\times\overline{\mathrm{Bun}}_B}) \longrightarrow \mathrm{End}(\overline{\mathfrak{p}}_* \mathrm{IC}_{\overline{\mathrm{Bun}}_B})[-2]. \tag{6.2.5}$$

By construction, the image of (6.2.2) under (6.2.5) coincides with the action of $f_\mu e_\mu \in U(\check{\mathfrak{g}})$. By composing with the morphisms $\mathfrak{p}_! \mathrm{IC}_{\mathrm{Bun}_B} \to \overline{\mathfrak{p}}_* \mathrm{IC}_{\overline{\mathrm{Bun}}_B}$ and $\overline{\mathfrak{p}}_* \mathrm{IC}_{\overline{\mathrm{Bun}}_B} \to \mathfrak{p}_* \mathrm{IC}_{\mathrm{Bun}_B}$, we obtain

$$\mathrm{End}(\overline{\mathfrak{p}}_* \mathrm{IC}_{\overline{\mathrm{Bun}}_B}) \longrightarrow \mathrm{Hom}(\mathfrak{p}_! \mathrm{IC}_{\mathrm{Bun}_B}, \mathfrak{p}_* \mathrm{IC}_{\mathrm{Bun}_B}). \tag{6.2.6}$$

Note that (6.2.6) annihilates the endomorphism of $\overline{\mathfrak{p}}_* \mathrm{IC}_{\overline{\mathrm{Bun}}_B}$ given by the action of $e_\mu f_\mu$, since it factors through a sheaf supported on the boundary. Proposition 4.9 in [FFKM99] says that the relation $[e_\mu, f_\mu] = h_\mu$ holds in $\mathrm{End}(\overline{\mathfrak{p}}_* \mathrm{IC}_{\overline{\mathrm{Bun}}_B})$, which implies that the images of $f_\mu e_\mu$ and $-h_\mu$ under (6.2.6) coincide.

Now consider the commutative square

$$\begin{tikzcd}
\mathrm{End}(\Delta^\mu_* \mathrm{IC}_{X\times\overline{\mathrm{Bun}}_B}) \arrow[r, "(6.2.3)"] \arrow[d, "(6.2.5)"'] & H^\bullet(X \times \mathrm{Bun}_T) \arrow[d] \\
\mathrm{End}(\overline{\mathfrak{p}}_* \mathrm{IC}_{\overline{\mathrm{Bun}}_B})[-2] \arrow[r, "(6.2.6)"] & \mathrm{Hom}(\mathfrak{p}_! \mathrm{IC}_{\mathrm{Bun}_B}, \mathfrak{p}_* \mathrm{IC}_{\mathrm{Bun}_B})[-2]
\end{tikzcd}$$

where the right vertical morphism is the composition

$$\begin{aligned}
H^\bullet(X \times \mathrm{Bun}_T) \;&=\; \mathrm{Hom}(\mathring{\Delta}^\mu_! \mathrm{IC}_{X\times\mathrm{Bun}_B}, \mathring{\Delta}^\mu_* \mathrm{IC}_{X\times\mathrm{Bun}_B}) \\
&\longrightarrow \mathrm{Hom}(H^\bullet(X) \otimes \mathfrak{p}_! \mathrm{IC}_{\mathrm{Bun}_B}, H_\bullet(X) \otimes \mathfrak{p}_* \mathrm{IC}_{\mathrm{Bun}_B})[-2] \\
&\longrightarrow \mathrm{Hom}(\mathfrak{p}_! \mathrm{IC}_{\mathrm{Bun}_B}, \mathfrak{p}_* \mathrm{IC}_{\mathrm{Bun}_B})[-2].
\end{aligned}$$

Note that $\mathrm{id}_X \times \rho(\omega) : X \times \mathrm{pt}/T \to X \times \mathrm{Bun}_T$ admits a canonical retraction, given by the projection $X \times \mathrm{Bun}_T \to X$ and the evaluation map $X \times \mathrm{Bun}_T \to \mathrm{pt}/T$. So far we have shown that the image of (6.2.2) under the resulting composition

$$\begin{aligned}
R^2 \mathrm{End}(\Delta^\mu_* \mathrm{IC}_{(X\times_{\mathrm{Bun}_T}\overline{\mathrm{Bun}}_B)/T}) &\longrightarrow H^2(X \times \mathrm{pt}/T) \longrightarrow H^2(X \times \mathrm{Bun}_T) \\
&\longrightarrow R^0 \mathrm{Hom}(\mathfrak{p}_! \mathrm{IC}_{\mathrm{Bun}_B}, \mathfrak{p}_* \mathrm{IC}_{\mathrm{Bun}_B})
\end{aligned}$$

agrees with the image of $-1 \otimes h_\mu$ under

$$H^2(X \times \mathrm{pt}/T) \longrightarrow H^2(X \times \mathrm{Bun}_T) \longrightarrow R^0 \mathrm{Hom}(\mathfrak{p}_! \mathrm{IC}_{\mathrm{Bun}_B}, \mathfrak{p}_* \mathrm{IC}_{\mathrm{Bun}_B}),$$

so it suffices to show that the latter composition is injective.

We have $H^2(X \times \mathrm{pt}/T) = \check{\mathfrak{h}} \oplus H^2(X)$, and we have already used the fact that for each $\lambda \in \Lambda$, an element $h \in \check{\mathfrak{h}}$ maps to $\langle h, \lambda\rangle$ times the canonical morphism $\mathfrak{p}_! \mathrm{IC}_{\mathrm{Bun}^\lambda_B} \to \mathfrak{p}_* \mathrm{IC}_{\mathrm{Bun}^\lambda_B}$. One checks that the canonical generator of $H^2(X)$ maps to canonical map $\mathfrak{p}_! \mathrm{IC}_{\mathrm{Bun}_B} \to \mathfrak{p}_* \mathrm{IC}_{\mathrm{Bun}_B}$ itself, which proves the desired injectivity. □

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Note that Lemma 6.2.2 already implies that $\mathscr{M}$ does not descend to $\overline{\mathrm{Bun}}_{N^\omega}/T$, since $h_\mu \neq 0$.

Proof of Lemma 6.2.1. The morphism (6.2.1) induces an element of $H^2(X \times \mathrm{pt}/\mathbb{G}_m)$ in the same way that (6.2.2) gives rise to $-1 \otimes h_\mu \in H^2(X \times \mathrm{pt}/T)$. Moreover, these constructions fit into a commutative square

$$\begin{tikzcd} \mathrm{End}(\Delta^\mu_* \mathrm{IC}_{(X \times_{\mathrm{Bun}_T} \overline{\mathrm{Bun}}_B)/T}) \arrow[r] \arrow[d] & H^\bullet(X \times \mathrm{pt}/T) \arrow[d] \\ \mathrm{End}(\Delta^\mu_* \mathrm{IC}_{(X \times_{\mathrm{Bun}_T} \overline{\mathrm{Bun}}_B)/\mathbb{G}_m}) \arrow[r] & H^\bullet(X \times \mathrm{pt}/\mathbb{G}_m) \end{tikzcd}$$

where the vertical morphisms are induced by γ, and in particular (6.2.2) maps to (6.2.1) along the left vertical morphism. The image of h_μ under $H^2(\mathrm{pt}/T) \to H^2(\mathrm{pt}/\mathbb{G}_m) = k$ is the positive integer $\langle h_\mu, \gamma \rangle$, so the lemma follows. $\square$

6.3 We need another, more elementary lemma. Fix $\mu \in \Lambda^{\mathrm{pos}}$ and $\mathfrak{k} \in \mathrm{Kost}(\mu)$ given by $\mu = \sum n_\beta \beta$.

Lemma 6.3.1. *If $\mathscr{L}$ is a nonconstant simple summand of $\boxtimes_{\beta \in R^+} \mathscr{P}_{n_\beta}$, then*

$$\Delta^! \iota^{\mathfrak{k}}_* \mathscr{L} = 0 = \Delta^* \iota^{\mathfrak{k}}_* \mathscr{L}.$$

Proof. This follows from the fact that the local system on $X^{(n)}_{\mathrm{disj}}$ associated to a nontrivial irreducible Σ_n-representation extends cleanly over the main diagonal. $\square$

Second proof of Theorems 1.4.1 and 5.1.1. We proceed as in the first proof, until we have constructed the isomorphism (5.1.1) over $Z^\mu \backslash \Delta(X)$ and reduced to the case that μ is a coroot. We showed that in this case $\Psi(\mathscr{W}_{Z^\mu})$ contains $\Delta_* \mathrm{IC}_X$ with multiplicity two but has no other subquotients supported on the main diagonal (in particular, the two sides of (5.1.1) agree in the Grothendieck group). Either $\mathfrak{sl}_2$ acts trivially on both copies of $\Delta_* \mathrm{IC}_X$ or they have weights 1 and -1. In order to show that (5.1.1) extends to Z^μ we need to rule out the first case, and then show that $\mathrm{gr}\, \Psi(\mathscr{W}_{Z^\mu})$ is semisimple.

We must rule out the possibility that $\mathfrak{sl}_2$ acts trivially on the subquotient $\mathscr{M}$ from Lemma 5.2.1. Since the monodromy endomorphism of $\Psi(\mathscr{W}_{Z^\mu})$ is the obstruction to $\mathbb{G}_m$-equivariance, this would imply that $\mathscr{M}$ is $\mathbb{G}_m$-equivariant, contradicting Lemma 5.2.1.

Now we finish the proof that $\mathrm{gr}\, \Psi(\mathscr{W}_{Z^\mu})$ is semisimple. Using the $\mathfrak{sl}_2$-action, it decomposes into the direct sum of its isotypic components, indexed by the irreducible $\mathfrak{sl}_2$-representations. The previous paragraph says that the two copies of $\Delta_* \mathrm{IC}_X$ are subquotients of the std-isotypic component. Thus the other isotypic components have no subquotients supported on the main diagonal, so they are the same as the corresponding isotypic components on the right-hand side of (5.1.1) and in particular are semisimple. We will show that for any simple subquotient $\mathscr{L} \neq \Delta_* \mathrm{IC}_X$ of the std-isotypic component we have

$$\mathrm{Ext}^1(\Delta_* \mathrm{IC}_X, \mathscr{L}) = 0 = \mathrm{Ext}^1(\mathscr{L}, \Delta_* \mathrm{IC}_X),$$

from which it follows that $\mathrm{IC}_X^{\oplus 2}$ is a direct summand of the std-isotypic component. Since the other summand has no subquotients supported on the main diagonal, it is semisimple by the induction hypothesis.

We will show that $H^i(\Delta^! \mathscr{L}) = 0$ for $i \leqslant 1$, which implies that $\mathrm{Ext}^1(\Delta_* \mathrm{IC}_X, \mathscr{L}) = 0$. The other vanishing follows by applying Verdier duality: since $\mathrm{gr}\, \Psi(\mathscr{W}_{Z^\mu})$ is Verdier self-dual, so is its std-isotypic component. Observe that $\mathscr{L}$ has the form $'\mathfrak{j}^{\mu}_{=\nu,!*}{}'\mathfrak{m}^{\mu,\Delta}_{\nu}\mathscr{F}$ for some $0 < \nu \leqslant \mu$ and a simple summand $\mathscr{F}$ of $\mathscr{P}^\nu$. The case $\nu = 0$ is excluded because then $\mathscr{L} = \mathrm{IC}_{Z^\mu}$ has weight 0.

First suppose that $\nu < \mu$. Then we have

$$\Delta^{!\prime}\mathfrak{j}^{\mu}_{=\nu,!*}{}'\mathfrak{m}^{\mu,\Delta}_{\nu}\mathscr{F} \xrightarrow{\sim} \Delta^! \mathscr{F} \otimes^! \Delta^! \mathrm{IC}_{Z^{\mu-\nu}}.$$

Since $\Delta^! \mathrm{IC}_{Z^{\mu-\nu}}$ is concentrated in cohomological degrees greater than or equal to 1 (see [BG08]) and $\Delta^! \mathscr{F}$ is concentrated in degrees greater than or equal to 0, and both complexes have lisse cohomology sheaves, their !-tensor product is concentrated in degrees greater than or equal to 2 as desired.

Finally, we address the case $\nu = \mu$, where $\mathscr{F} = \mathscr{L}$. By Lemma 6.3.1, we can assume $\mathscr{L}$ is a summand of $\mathrm{add}_* \mathrm{IC}_{\prod X^{(n_\beta)}}$ for some Kostant partition $\mu = \sum n_\beta \beta$. If $\sum n_\beta \geqslant 3$ then the claim follows by base change. By assumption $\mathscr{L} \neq \Delta_* \mathrm{IC}_X$, so $\sum n_\beta > 1$. This leaves only the case $\sum n_\beta = 2$, and since μ is a coroot the only possibility is that $\mu = \beta_1 + \beta_2$ is a sum of two distinct coroots. As shown in Lemma 5.1.4, in this case the std-isotypic component of $\mathrm{gr}\, \Psi(\mathscr{W}_{Z^\mu})$ is just $\mathrm{IC}_X^{\oplus 2}$. □

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