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ANY SPINE OF THE CUBE IS 2-COLLAPSIBLE

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1. Introduction. M. Cohen [**1**] defined a polyhedron K to be n-collapsible if $K \times I^n$ PL collapses. He proved that any spine of the cube B^3 is 3-collapsible. This was a step directed toward the Zeeman Conjecture [**4**], which asserts that every compact contractible 2-polyhedron is 1-collapsible. In this paper we improve the result of Cohen by one dimension (Theorem 3): Any spine of the cube is 2-collapsible. The central question of 1-collapsibility remains unanswered.

Gillman and Rolfsen [**3**] have shown that any standard spine of the cube is 1-collapsible. Conjecture: If K is any spine of the cube, then $K \times I$ collapses to a standard spine of the cube. This would imply our main theorem. Lacking a proof of this conjecture, we must resort to an argument independent of [**3**].

THEOREM 1. *Let $A_1, A_2, \ldots, A_n$ be a finite collection of pairwise disjoint contractible PL subsets of the cube. Then the decomposition obtained by shrinking each A_i to a point is* 1*-collapsible.*

THEOREM 2. *Let G be a finite graph embedded in a* 2*-manifold M, and suppose that no component of G is homeomorphic to a circle. Then $G \times I$ collapses to a subpolyhedron PL homeomorphic to a regular neighborhood of G in M.*

Note. The condition on G in Theorem 2 is necessary to rule out the situation in which G could be the centerline of a möbius band. If M is orientable, it is unnecessary.

THEOREM 3. *Any spine of the cube is* 2*-collapsible.*

Finally, we pose a question which could improve the proof of Theorem 3, to yield 1-collapsibility.

2. Proof of theorem 1. We generalize the idea of a spanning arc in a natural way:

Definition. A k-od is homeomorphic to the cone over k points. These k points are called the *end points* of the k-od. A k-od spans B^3 if its end points lie in ∂B^3, and the rest of the k-od lies in Int B^3.

LEMMA 1.1. *A decomposition of B^3 whose only nondegenerate element is a spanning k-*od *yields a quotient space that is* 1*-collapsible.*

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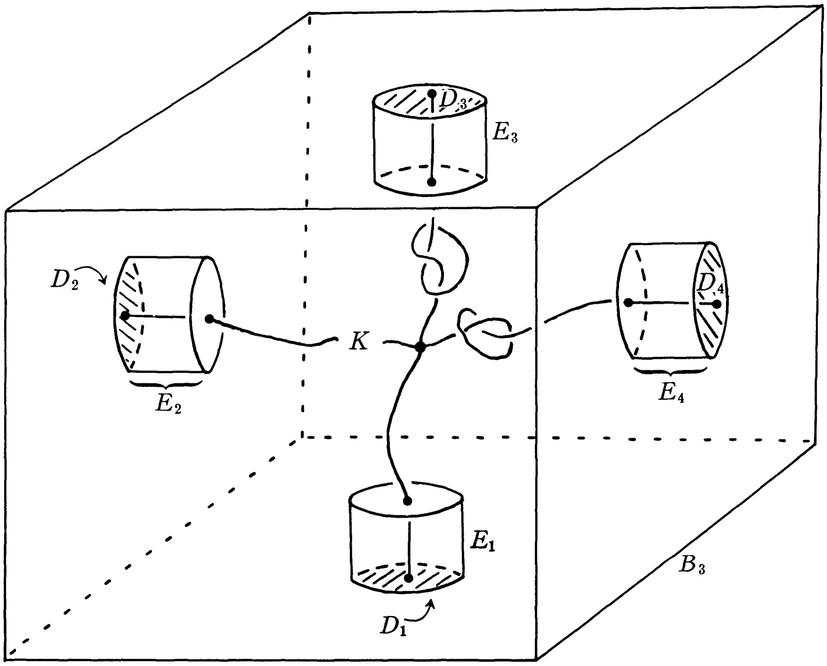

FIGURE I

Proof. Let K denote the spanning k-od, and B^* denote the decomposition space. We select disjoint disks $D_1, \ldots, D_k$ in ∂B^3 with each D_i containing an end point of K at its center, as shown in Figure I. We may collapse $B^* \times I$ to

$$\hat{B} = B^* \times 0 \cup \bigcup_{i=1}^{k} D_i^* \times [0, 1],$$

where D_i^* is the image of D_i in B^*. The set $\hat{B}$ may be regarded as the decomposition of B^3 in which the k legs of K are identified into a single leg. The nondegenerate elements of $\hat{B}$, viewed in this manner, are sets of k points identified together. Whereas the decomposition B^* was not in general collapsible, the decomposition $\hat{B}$ is collapsible. This is shown by the following "Unknotting Procedure" (see Figure I):

The disks D_i are fattened slightly in B^3, yielding disjoint 3-cells E_i, for $i = 1, \ldots, k$. Each E_i intersects K in an unknotted spanning arc of E_i. We collapse $E_i - K$ by pushing inwards on $D_i - K$, for $i = 1, 2, \ldots, k - 1$. More precisely, let $\xi_i = D_i \cap K$, a point. Regarding E_i as $D_i \times [0, 1]$ with D_i identified with $D_i \times 0$ and $E_i \cap K = \xi_i \times [0, 1]$, we collapse $D_i \times [0, 1]$ to

$$D_i \times 1 \cup (\partial D_i \cup \xi_i) \times [0, 1]$$

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in standard fashion. Note that E_k remains uncollapsed. This will enable us to "untie the knot" in the leg of K which intersects E_k, and to "unlink" it from other legs.

The uncollapsed arcs $\xi_i \times [0\ 1]$, for $i = 1, \ldots, k - 1$ seem to remain dangling from our 3-ball after these collapses, but we may simply remove them, in that they are identified with an arc in E_k; they are therefore superfluous in the description of the decomposition space at this stage. Let K_* be the sub k-od of K gotten by throwing away $\xi_i \times [0, 1)$, $i = 1, \ldots, k$. At this point, the subset of $\hat{B}$ we are left with can be regarded as a 3-ball containing a non-spanning k-od K_* in which the k legs have been identified to a single leg. Although $k - 1$ legs of K_* reach the boundary, the kth leg does not. The kth leg may thus be untied, that is, viewed as a straight radius of the 3-ball which does not quite reach the boundary. We now collapse the 3-ball inward from the boundary towards this straight radius until this leg of the k-od does reach the boundary like the others. This completes one stage of the Unknotting Procedure.

By repetition of this procedure, all legs can be straightened, after which the set is easily collapsed. This completes the proof of Lemma 1.1.

LEMMA 1.2. *If X 1-collapses, Y 1-collapses, and $X \cap Y$ is a single point, then $X \cup Y$ 1-collapses.*

Proof of Lemma 1.2. The set $(X \cup Y) \times I$ collapses to

$$X \times [0, 1/2] \cup Y \times [1/2, 1].$$

Since X 1-collapses,

$$X \times [0, 1/2] \searrow (X \cap Y) \times 1/2,$$

and similarly for $Y \times [1/2, 1]$. (Recall that if a polyhedron collapses, it collapses to any point.)

Proof of Theorem 1. Let $A_1, \ldots, A_n$ be a finite collection of pairwise disjoint contractible PL subsets of the 3-ball B^3. Let R_i be a regular neighborhood of A_i in B^3. Then R_i is a 3-cell, and the decomposition obtained by shrinking the A_i's is the same as that obtained by shrinking the R_i's. By Lemma 1.2, we may assume that $B^3 - R_i$ is connected. Thus, $R_i \cap \partial B^3$ consists of 2-cells, say k_i of them. We select a spanning k_i-od K_i of R_i such that R_i is a regular neighborhood of K_i in B^3. Then the decomposition of B^3 obtained by shrinking the R_i's is the same as that obtained by shrinking the K_i's. By applying the technique of Lemma 1.1 to each K_i, $i = 1, 2, \ldots, n$, we unknot each K_i in turn, and unlink it from K_j for $j \neq i$. This completes the proof.

3. Proof of theorem 2. By neglecting vertices of order 2, we may assume without loss of generality that all vertices of G are of order $\geqq 3$.

Except for a neighborhood of the vertices, $G \times I$ and a regular neighborhood of G in M are naturally homeomorphic in a fashion which takes point $\times I$ to a fiber of the regular neighborhood. Let h be such a homeomorphism. We now show how to deal with a neighborhood of a vertex v of G.

Let D denote a disk neighborhood of v in M. Then $G \cap D$ is a k-od, $k \geqq 3$, with edges $E_1, E_2, \ldots, E_k$, labeled in the cyclic order of their appearance in D. The corresponding collapse of $G \times I$ can be described by a "First Approximation," which yields the correct cyclic order, then a "Final Modification," which deals with possible undesired "twists" that may occur.

First Approximation. Let E_i have end points $e_i \in \partial D$ and $v \in \operatorname{Int} D$. The set $E_1 \times I$ is not collapsed at all. For $2 \leqq i \leqq k$, we select disjoint closed subintervals of the unit interval I in the natural order which we call $I_2, \ldots, I_k$. The set $E_i \times I$ is collapsed so that it linearly tapers from $e_i \times I$ to $v \times I_i$. See Figure II.

Final Modification. It is possible that $G \times I$, collapsed by the First Approximation, will not properly extend the homeomorphism h, because some of the E_i's will need a "twist" in order to match with h along $e_i \times I$. For each such E_i, a modification to the collapse is made as follows: Select any integer j with $2 \leqq j \leqq k, j \neq i$. Such a j exists since v has order $\geqq 3$ in G. A small square in $E_j \times I$ is not collapsed, and a small square in

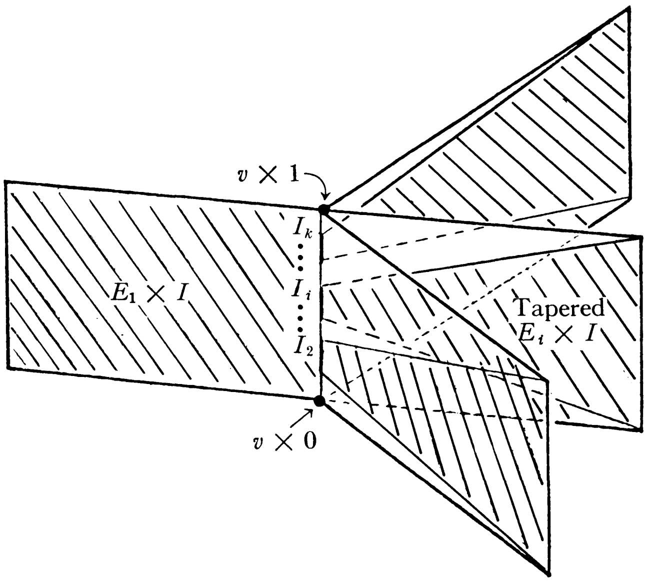

FIGURE II

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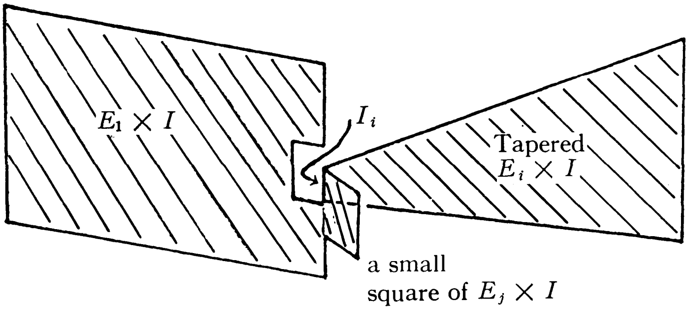

FIGURE III

$E_1 \times I$ is collapsed, so that the neighborhood of E_i "twists" through $E_j \times I$ before attaching to $E_1 \times I$ along $v \times I$. See Figure III. This completes the proof.

4. Proof of theorem 3. Let K be any spine of the cube B^3. We may assume K is at most 2-dimensional, since K must itself have a 2-dimensional spine. We denote by K^2 the intrinsic open 2-skeleton of K, that is, local 2-manifold points. We denote by K^1 the intrinsic 1-skeleton, that is, points that locally look like an n-od $\times I$ with $n \geqq 3$. The remaining points comprise the intrinsic 0-skeleton K^0. By Lemma 1.2, it will suffice to prove 1-collapsibility for each component of the closure of K^2; thus, we may assume that K^2 is dense in K. Let $r : B^3 \rightarrow K$ be a natural retraction, that is,

$$r^{-1}(p) = \begin{cases} \text{an arc, for } p \in K^2, \\ \text{an } n\text{-od with } n \geqq 3, \text{ for } p \in K^1, \\ \text{a contractible polyhedron, for } p \in K^0. \end{cases}$$

One way to define such a retraction r is as a composition $r = r_2 r_1 r_0$, where $r_i : N_{i-1} \rightarrow N_i$ is a PL retraction, and where N_i a regular neighborhood of K mod K^i (so N_{-1} can be taken as B^3, and $N_2 = K$).

$$B^3 = N_{-1} \xrightarrow{r_0} N_0 \xrightarrow{r_1} N_1 \xrightarrow{r_2} N_2 = K.$$

Each r_i shares the corresponding property of r displayed above over $K^i - K^{i-1}$, and is a collar-collapse over $N_i - K^i$. Details for constructing such an r are provided in [**2**] (see especially Theorem 5.1 and Addendum 5.2).

Let B^* denote the "pinched cube" obtained from B^3 by identifying $r^{-1}(p)$ to a point for each $p \in K^0$. We will show that $K \times I \searrow B^*$, so that Theorem 3 will follow from Theorem 1.

Let G denote the dual 1-skeleton of K under some fixed triangulation T.

LEMMA 3.1. *$G \times I$ collapses to a subpolyhedron PL homeomorphic to* $r^{-1}(G)$.

This lemma is proved with the same technique as that of Theorem 2.

Let $U_1, U_2, \ldots, U_n$ be the components of K^2. Each U_i must be orientable. Let $C_i \subset U_i$ be a thin open neighborhood of [closure U_i] $-$ U_i. Setting $G_i = [U_i - C_i] \cap G$ we may identify $r^{-1}(G_i)$ with $G_i \times I$ for each i. The vertices of $G - \cup G_i$ all lie in K^1; they are all of order $\geqq 3$ in G. Thus, the argument of Theorem 2 yields the collapse of $G \times I$ to a subpolyhedron PL homeomorphic to $r^{-1}(G)$ near each vertex of $G - \cup G_i$, proving Lemma 3.1.

Let v be a vertex of the triangulation T of K. Then $\text{link}(v) \subset G$.

LEMMA 3.2. *The PL collapse of Lemma* 3.1, *restricted to* link(v), *is also a PL collapse, yielding a collapse of* link$(v) \times I$ *to a subpolyhedron PL homeomorphic to* $r^{-1}(\text{link}(v))$.

Proof. The collapse of Lemma 3.1 takes place locally in small neighborhoods of vertices v_j of $G - \cup G_i$. If such a v_j lies in link(v), then its neighborhood in G coincides with its neighborhood in link(v). This completes the proof of Lemma 3.2.

Lemma 3.2 enables us to extend the collapse of Lemma 3.1 linearly towards v, pinching $r^{-1}(v)$ to a point and proving Theorem 3.

We are left with a natural question.

Question. Does the collapse of Lemma 3.2 extend to a collapse of the pair [star$(v) \times I$, link$(v) \times I$] to the pair $(r^{-1}(\text{star}(v)), r^{-1}(\text{link}(v))]$? An affirmative answer would imply that any spine of the cube is 1-collapsible.

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