# ON THE CHARACTERISATION OF ALTERNATING GROUPS BY CODEGREES

## MALLORY DOLORFINO<sup>®</sup>, LUKE MARTIN<sup>®</sup>, ZACHARY SLONIM<sup>®</sup>, YUXUAN SUN<sup>®</sup> and YONG YANG<sup>®</sup>

(Received 16 October 2023; accepted 18 November 2023; first published online 26 January 2024)

#### Abstract

Let *G* be a finite group and Irr(*G*) the set of all irreducible complex characters of *G*. Define the codegree of  $\chi \in Irr(G)$  as  $cod(\chi) := |G : ker(\chi)|/\chi(1)$  and let  $cod(G) := \{cod(\chi) | \chi \in Irr(G)\}$  be the codegree set of *G*. Let A<sub>n</sub> be an alternating group of degree  $n \ge 5$ . We show that A<sub>n</sub> is determined up to isomorphism by  $cod(A_n)$ .

2020 Mathematics subject classification: primary 20D06; secondary 20C15.

Keywords and phrases: codegree, alternating group.

## 1. Introduction

Let *G* be a finite group and Irr(*G*) the set of all irreducible complex characters of *G*. For any  $\chi \in \text{Irr}(G)$ , define the codegree of  $\chi$  as  $\text{cod}(\chi) := |G : \text{ker}(\chi)|/\chi(1)$  and the codegree set of *G* as  $\text{cod}(G) := \{\text{cod}(\chi) | \chi \in \text{Irr}(G)\}$ . We refer the reader to the authors' previous paper [8] for the current literature on codegrees.

The following conjecture appears in the *Kourovka Notebook of Unsolved Problems in Group Theory* [12, Question 20.79].

CODEGREE VERSION OF HUPPERT'S CONJECTURE. Let *H* be a finite nonabelian simple group and *G* a finite group such that cod(H) = cod(G). Then  $G \cong H$ .

In [8], the authors verified the conjecture for all sporadic simple groups. In this paper, we provide a general proof verifying this conjecture for all alternating groups of degree greater than or equal to 5.

THEOREM 1.1. Let  $A_n$  be an alternating group of degree  $n \ge 5$  and G a finite group. If  $cod(G) = cod(A_n)$ , then  $G \cong A_n$ .



This research was conducted under NSF-REU grant DMS-1757233, DMS-2150205 and NSA grant H98230-21-1-0333, H98230-22-1-0022 by Dolorfino, Martin, Slonim and Sun during the Summer of 2022 under the supervision of Yang. Yang was also partially supported by a grant from the Simons Foundation (#918096).

<sup>©</sup> The Author(s), 2024. Published by Cambridge University Press on behalf of Australian Mathematical Publishing Association Inc.

Throughout the paper, we follow the notation used in Isaacs' book [10] and the ATLAS of Finite Groups [6].

#### 2. Proof of Theorem 1.1

First, we note that the cases n = 5, 6 and 7 have already been proven in [1, 2], so in the following, we always assume that n > 7. Now, let *G* be a minimal counterexample and *N* be a maximal normal subgroup of *G*. So  $cod(G) = cod(A_n)$  and G/N is simple. By [8, Lemma 2.5],  $cod(G/N) \subseteq cod(A_n)$ . Then, by [9, Theorem B],  $G/N \cong A_n$  so  $N \neq 1$  since  $G \not\cong A_n$ .

## Step 1: N is a minimal normal subgroup of G.

Suppose *L* is a nontrivial normal subgroup of *G* with L < N. Then by [8, Lemma 2.6],  $cod(G/N) \subseteq cod(G/L) \subseteq cod(G)$ . However,  $cod(G/N) = cod(A_n) = cod(G)$ , so equality must be attained in each inclusion. Thus,  $cod(G/L) = cod(A_n)$  which implies that  $G/L \cong A_n$  since *G* is a minimal counterexample. This is a contradiction since we also have  $G/N \cong A_n$ , but L < N.

## Step 2: N is the only nontrivial, proper normal subgroup of G.

Otherwise, we assume *M* is another proper nontrivial normal subgroup of *G*. If *N* is included in *M*, then M = N or M = G since G/N is simple, which is a contradiction. Then  $N \cap M = 1$  and  $G = N \times M$ . Since *M* is also a maximal normal subgroup of *G*, we have  $N \cong M \cong A_n$ . Choose  $\psi_1 \in \operatorname{Irr}(N)$  and  $\psi_2 \in \operatorname{Irr}(M)$  such that  $\operatorname{cod}(\psi_1) = \operatorname{cod}(\psi_2) = \max(\operatorname{cod}(A_n))$ . Set  $\chi = \psi_1 \cdot \psi_2 \in \operatorname{Irr}(G)$ . Then  $\operatorname{cod}(\chi) = (\max(\operatorname{cod}(A_n)))^2 \notin \operatorname{cod}(G)$ , which is a contradiction.

Step 3:  $\chi$  is faithful, for each nontrivial  $\chi \in Irr(G|N) := Irr(G) - Irr(G/N)$ . From the proof of [8, Lemma 2.5],

$$\operatorname{Irr}(G/N) = \{\hat{\chi}(gN) = \chi(g) \mid \chi \in \operatorname{Irr}(G) \text{ and } N \leq \ker(\chi)\}.$$

By the definition of Irr(G|N), it follows that if  $\chi \in Irr(G|N)$ , then  $N \not\leq ker(\chi)$ . Thus, since N is the unique nontrivial, proper, normal subgroup of G,  $ker(\chi) = G$  or  $ker(\chi) = 1$ . Therefore,  $ker(\chi) = 1$  for all nontrivial  $\chi \in Irr(G|N)$ .

### *Step 4: N is an elementary abelian group.*

Suppose that *N* is not abelian. Since *N* is a minimal normal subgroup, by [7, Theorem 4.3A(iii)],  $N = S^n$ , where *S* is a nonabelian simple group and  $n \in \mathbb{Z}^+$ . By [14, Lemma 4.2] and [11, Theorem 4.3.34], there is a nontrivial character  $\chi \in Irr(N)$  which extends to some  $\psi \in Irr(G)$ . Now,  $\ker(\psi) = 1$  by Step 3, so  $\operatorname{cod}(\psi) = |G|/\psi(1) = |G/N| \cdot |N|/\chi(1)$ . However, by assumption,  $\operatorname{cod}(G) = \operatorname{cod}(G/N)$ . Thus,  $\operatorname{cod}(\psi) \in \operatorname{cod}(G) = \operatorname{cod}(G/N)$ , so  $\operatorname{cod}(\psi) = |G/N|/\phi(1)$  for some  $\phi \in Irr(G/N)$ . Hence, |G/N| is divisible by  $\operatorname{cod}(\psi)$  which contradicts the fact that  $\operatorname{cod}(\psi) = |G/N| \cdot |N|/\chi(1)$ , as  $\chi(1) \neq |N|$ . Thus, *N* must be abelian.

Now to show that N is elementary abelian, let a prime p divide |N|. Then N has a p-Sylow subgroup K, and K is the unique p-Sylow subgroup of N since N is abelian,

so *K* is characteristic in *N*. Thus, *K* is a normal subgroup of *G*, so K = N as *N* is minimal. Thus,  $|N| = p^n$ . Now, take the subgroup  $N^p = \{n^p \mid n \in N\}$  of *N*, which is proper by Cauchy's theorem. Since  $N^p$  is characteristic in *N*, it must be normal in *G*, so  $N^p$  is trivial by the uniqueness of *N*. Thus, every element of *N* has order *p* and *N* is elementary abelian.

*Step 5:*  $C_G(N) = N$ .

First note that since N is normal,  $C_G(N) \leq G$ . Additionally, since N is abelian by Step 4,  $N \leq C_G(N)$ . By the maximality of N, we must have  $C_G(N) = N$  or  $C_G(N) = G$ . If  $C_G(N) = N$ , we are done.

If not, then  $C_G(N) = G$ , so N must be in the centre of G. Then since N is the unique minimal normal subgroup of G by Step 2, |N| must be prime. If not, there always exists a proper nontrivial subgroup K of N, and K is normal since it is contained in Z(G), contradicting the minimality of N. Hence, we have  $N \leq Z(G)$  which implies that  $Z(G) \cong N$ . This is because N is a maximal normal subgroup of G so if not, we would have Z(G) = G, implying G is abelian which is a contradiction. Thus, N is isomorphic to a subgroup of the Schur multiplier of G/N by [10, Corollary 11.20].

Now, we note that it is well known that for n > 7, the Schur multiplier of  $A_n$  is  $\mathbb{Z}_2$ , so  $G \cong 2.A_n$  [17]. From [3, Theorem 4.3], 2. $A_n$  always has a faithful irreducible character  $\chi$  of degree  $2^{\lfloor (n-2)/2 \rfloor}$ . Recall that by Step 2, there is only one nontrivial proper normal subgroup of  $G \cong 2.A_n$ . In particular,  $N \cong \mathbb{Z}_2$  is the only nontrivial proper normal subgroup of G. Thus,  $|\ker(\chi)| = 1$  or 2. Then  $\operatorname{cod}(\chi) = |2.A_n : \ker(\chi)|/\chi(1)$ . If  $|\ker(\chi)| = 1$ , then  $\operatorname{cod}(\chi) = n!/2^{\lfloor (n-2)/2 \rfloor}$ , and if  $|\ker(\chi)| = 2$ , then  $\operatorname{cod}(\chi) = (n!/2)/2^{\lfloor (n-2)/2 \rfloor} = n!/2^{\lfloor n/2 \rfloor}$ . In either case, for any prime  $p \neq 2$ ,  $|\operatorname{cod}(\chi)|_p = |n!|_p = |A_n|_p$ . However,  $\operatorname{cod}(\chi) \in \operatorname{cod}(A_n)$  since  $\operatorname{cod}(G) = \operatorname{cod}(A_n)$ . Therefore, there is a character degree of  $A_n$  which is a power of 2.

However, from [13], for n > 7,  $A_n$  only has a character degree equal to a power of 2 when  $n = 2^d + 1$  for some positive integer *d*. In this case,  $2^d = n - 1 \in cd(A_n)$  so we need  $|A_n|/n - 1 = |2.A_n|/2^{\lfloor (n-2)/2 \rfloor}$  or  $|2.A_n|/2^{\lfloor n/2 \rfloor}$ . Hence,

$$\frac{1}{n-1} = \frac{2}{2^{\lfloor (n-2)/2 \rfloor}} = \frac{1}{2^{\lfloor (n-2)/2 \rfloor - 1}} \quad \text{or} \quad \frac{1}{2^{\lfloor n/2 \rfloor - 1}}$$

so  $n - 1 = 2^{\lfloor (n-2)/2 \rfloor - 1}$  or  $2^{\lfloor n/2 \rfloor - 1}$ . However, the only integer solution to either of these equations occurs when n = 9 and  $9 - 1 = 8 = 2^3 = 2^{\lfloor 9/2 \rfloor - 1}$ . In this case, we check the ATLAS [6] to find that the codegree sets of  $A_9$  and  $2.A_9$  do not have the same order. This is a contradiction, so  $\mathbf{C}_G(N) = N$ .

Step 6. Let  $\lambda$  be a nontrivial character in  $\operatorname{Irr}(N)$  and  $\vartheta \in \operatorname{Irr}(I_G(\lambda)|\lambda)$ , the set of irreducible constituents of  $\lambda^{I_G(\lambda)}$ , where  $I_G(\lambda)$  is the inertia group of  $\lambda$  in G. Then  $|I_G(\lambda)|/\vartheta(1) \in \operatorname{cod}(G)$ . Also,  $\vartheta(1)$  divides  $|I_G(\lambda)/N|$  and |N| divides |G/N|. Lastly,  $I_G(\lambda) < G$ , that is,  $\lambda$  is not G-invariant.

Let  $\lambda$  be a nontrivial character in Irr(N) and  $\vartheta \in \text{Irr}(I_G(\lambda)|\lambda)$ . Let  $\chi$  be an irreducible constituent of  $\vartheta^G$ . By [10, Corollary 5.4],  $\chi \in \text{Irr}(G)$ , and by [10, Definition 5.1], we have  $\chi(1) = (|G|/|I_G(\lambda)|) \cdot \vartheta(1)$ . Moreover,  $\ker(\chi) = 1$  by Step 2, and thus

 $\operatorname{cod}(\chi) = |G|/\chi(1) = |I_G(\lambda)|/\vartheta(1)$ , so  $|I_G(\lambda)|/\vartheta(1) \in \operatorname{cod}(G)$ . Now, since *N* is abelian,  $\lambda(1) = 1$ , so we have  $\vartheta(1) = \vartheta(1)/\lambda(1)$  which divides  $|I_G(\lambda)|/|N|$ , so |N| divides  $|I_G(\lambda)|/\vartheta(1)$ . Moreover,  $\operatorname{cod}(G) = \operatorname{cod}(G/N)$ , and all elements in  $\operatorname{cod}(G/N)$  divide |G/N|, so |N| divides |G/N|.

Next, we want to show  $I_G(\lambda)$  is a proper subgroup of G. To reach a contradiction, assume  $I_G(\lambda) = G$ . Then ker $(\lambda) \leq G$ . From Step 2, ker $(\lambda) = 1$ , and from Step 4, N is a cyclic group of prime order. Thus, by the Normaliser–Centraliser theorem,  $G/N = \mathbf{N}_G(N)/\mathbf{C}_G(N) \leq \operatorname{Aut}(N)$  so G/N is abelian, which is a contradiction.

### Step 7: Final contradiction.

From Step 4, *N* is an elementary abelian group of order  $p^m$  for some prime *p* and integer  $m \ge 1$ . By the Normaliser–Centraliser theorem,  $A_n \cong G/N = N_G(N)/C_G(N) \le Aut(N)$  and m > 1. Note that in general,  $Aut(N) \cong GL(m, p)$ . By Step 6, |N| divides |G/N|, so  $|N| = p^m$  divides  $|A_n|$  and  $G/N \cong A_n \le GL(m, p)$ . We prove by contradiction that this cannot occur.

First, we claim that if  $p^m$  divides  $|A_n|$  and  $A_n \leq (GL(m, p))$ , then p must equal 2. To show this, we note that for p > 2, by [4], if  $p^m$  divides  $|A_n|$ , then m < n/2. However, by [16, Theorem 1.1], if n > 6, the minimal faithful degree of a modular representation of  $A_n$  over a field of characteristic p is at least n - 2. Since embedding  $A_n$  as a subgroup of GL(m, p) is equivalent to giving a faithful representation of degree m over a field of characteristic  $p, we have <math>m \ge n - 2$ . This is a contradiction since n/2 > n - 2 implies n < 4. Therefore, p = 2.

Now, let p = 2. As above, from [4], we obtain  $|n!|_2 \le 2^{n-1}$ . Thus, if  $2^m$  divides  $|A_n|$ , then  $2^m \le |A_n|_2 \le 2^{n-2}$  so  $m \le n-2$ . We will deal first with n > 8 and then treat the case n = 8 later. For n > 8, [15, Theorem 1.1] shows that the minimal faithful degree of a modular representation of  $A_n$  over a field of characteristic 2 is at least n - 2. Therefore, we must have  $m \ge n - 2$ , so we have equality, m = n - 2.

Let  $\lambda \in \operatorname{Irr}(N)$ ,  $\vartheta \in \operatorname{Irr}(I_G(\lambda)|\lambda)$  and  $T := I_G(\lambda)$ . Then  $1 < |G:T| < |N| = 2^{n-2}$  for |G:T| is the number of all conjugates of  $\lambda$ . By Step 5,  $|T|/\vartheta(1) \in \operatorname{cod}(G)$  and moreover |N| divides  $|T|/\vartheta(1)$ . Since  $|N| = |N|_2 = |A_n|_2$  and  $\operatorname{cod}(G) = \operatorname{cod}(A_n)$ , it follows that  $||T|/\vartheta(1)|_2 = |N|$ . Thus,  $||T/N|/\vartheta(1)|_2 = 1$  so the 2-parts of |T/N| and  $\vartheta(1)$  are equal. Thus, for every  $\vartheta \in \operatorname{Irr}(T \mid \lambda)$ , we have  $|\vartheta(1)|_2 = |T/N|_2$ . However,  $|T/N| = \sum_{\vartheta \in \operatorname{Irr}(T|\lambda)} \vartheta(1)^2$ . Hence, if  $|\vartheta(1)|_2 = 2^k \ge 2$  for every  $\vartheta \in \operatorname{Irr}(T \mid \lambda)$ , we would have  $|T/N|_2 = 2^{2k}$ , which contradicts the fact that  $|\vartheta(1)|_2 = |T/N|_2$ . Therefore,  $|T/N|_2 = 1$ . Thus, since  $|G/N|_2 \ge |N| = 2^{n-2}$ , we have  $|G:T|_2 = |G/N: T/N|_2 \ge 2^{n-2}$ , so  $|G:T| \ge 2^{n-2} = |N|$ , which is a contradiction.

Now we turn to the case n = 8. We have p = 2 and m = 4, 5 or 6. In this case,  $A_8 \cong GL(4, 2)$  and  $2^6$  divides  $|A_8|$ . We look at each possibility for *m* in turn. If m = 6, then  $|N|_2 = |A_8|_2$ . For this case, the same argument as above holds since 6 = 8 - 2, and we reach a contradiction.

Second, let m = 5. As above,  $|G:T| < |N| = 2^5$  and  $|T|/\vartheta(1) \in \operatorname{cod}(G)$  such that  $2^5$  divides  $|T|/\vartheta(1)$ . Further,  $||T/N|/\vartheta(1)|_2 \le 2$  so  $|T/N|_2 \le 4$  and  $|G/N:T/N|_2 \ge 16$ . Thus, 16 divides |G/N:T/N| and |G/N:T/N| < 32. However, we may check the index

#### Alternating groups

of all subgroups of  $G/N \cong A_8$  using [6] and find that none of them satisfy these two properties.

Third, let m = 4. Then  $G/N \cong A_8 \cong GL(4, 2)$  and  $N = (\mathbb{Z}_2)^4$  so G is an extension of GL(4, 2) by N. We may computationally calculate the codegree set for any such group using MAGMA [5]. There are only four such nonisomorphic extensions and we find that none of them have the same codegree set as  $A_8$ . (The MAGMA code is available at https://github.com/zachslonim/Characterizing-Alternating-Groups-by-Their-Codegrees.) In every case,  $|N| = p^m$  produces a contradiction, so N = 1 and  $G \cong A_n$ .

## Acknowledgements

The authors gratefully acknowledge the financial support of NSF and NSA, and also thank Texas State University for providing a great working environment and support. The authors would also like to thank Professor Richard Stanley for his help.

#### References

- [1] N. Ahanjideh, 'Nondivisibility among irreducible character co-degrees', *Bull. Aust. Math. Soc.* **105** (2022), 68–74.
- [2] A. Bahri, Z. Akhlaghi and B. Khosravi, 'An analogue of Huppert's conjecture for character codegrees', *Bull. Aust. Math. Soc.* **104**(2) (2021), 278–286.
- [3] C. Bessenrodt and J. B. Olsson, 'Prime power degree representations of the double covers of the symmetric and alternating groups', J. London Math. Soc. 66(2) (2002), 313–324.
- [4] C. Bessenrodt, H. P. Tong-Viet and J. Zhang, 'Huppert's conjecture for alternating groups', J. Algebra 470 (2017), 353–378.
- [5] J. C. W. Bosma and C. Playoust, 'The Magma algebra system I: the user language', J. Symbolic Comput. 24 (1997), 235–265.
- [6] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker and R. A. Wilson, *Atlas of Finite Groups* (Clarendon Press, Oxford, 1985).
- [7] J. D. Dixon and B. Mortimer, *Permutation Groups*, Graduate Texts in Mathematics, 163 (Springer, New York, 1996).
- [8] M. Dolorfino, L. Martin, Z. Slonim, Y. Sun and Y. Yang, 'On the characterisation of sporadic simple groups by codegrees', *Bull. Aust. Math. Soc.*, to appear. Published online (27 March 2023); doi:10.1017/S0004972723000187.
- [9] N. N. Hung and A. Moreto, 'The codegree isomorphism problem for finite simple groups', Preprint, 2023, arXiv:2301.00446.
- [10] I. M. Isaacs, Character Theory of Finite Groups (Academic Press, New York, 1976).
- [11] G. James and A. Kerber, *The Representation Theory of the Symmetric Group* (Addison-Wesley, Reading, MA, 1981).
- [12] E. I. Khukrho and V. D. Mazurov, *Unsolved Problems in Group Theory*, The Kourovka Notebook, 20 (Russian Academy of Sciences, Novosibirsk, 2022).
- [13] G. Malle and A. E. Zalesskii, 'Prime power degree representations of quasi-simple groups', Arch. Math. (Basel) 77 (2001), 461–468.
- [14] A. Moretó, 'Complex group algebra of finite groups: Brauer's problem 1', Adv. Math. 208 (2007), 236–248.
- [15] A. Wagner, 'The faithful linear representations of least degree of  $S_n$  and  $A_n$  over a field of characteristic 2', *Math. Z.* **151** (1976), 127–138.
- [16] A. Wagner, 'The faithful linear representations of least degree of  $S_n$  and  $A_n$  over a field of odd characteristic', *Math. Z.* **154** (1977), 104–113.
- [17] R. A. Wilson, The Finite Simple Groups (Springer, London, 2009).

MALLORY DOLORFINO, Department of Mathematics, Kalamazoo College, Kalamazoo, Michigan, USA e-mail: mallory.dolorfino19@kzoo.edu

LUKE MARTIN, Department of Mathematics, Gonzaga University, Spokane, Washington, USA e-mail: lwmartin2019@gmail.com

ZACHARY SLONIM, Department of Mathematics, University of California, Berkeley, Berkeley, California, USA e-mail: zachslonim@berkeley.edu

YUXUAN SUN, Department of Mathematics and Statistics, Haverford College, Haverford, Pennsylvania, USA e-mail: ysun1@haverford.edu

YONG YANG, Department of Mathematics, Texas State University, San Marcos, Texas, USA e-mail: yang@txstate.edu

120