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ON STRONG NÖRLUND SUMMABILITY FIELDS

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1. Introduction. Let p denote the sequence $\{p_n\}$ and set $p(z) = \Sigma p_n z^n$ wherever this series converges. (Where no limits are stated, sums are throughout to be taken from $n = 0$ to $n = \infty$.) We use a similar notation with other letters in place of p. Given two sequences p, q, the convolution $p*q$ is defined by

$$(p*q)_n = \sum_{\nu=0}^{n} p_{n-\nu}q_\nu ;$$

it is familiar, and easily verified, that the operation of convolution is commutative and associative. We write $P_n = (p*1)_n$ (where 1 denotes the sequence $\{1\}$), and take P_{-1} to mean 0. If, for all $n \geqq 0$, $P_n \neq 0$, then we define the Nörlund mean (N, p) of the sequence s as σ_n, where

$$\sigma_n = \frac{(p*s)_n}{P_n} \ (n \geqq 0)$$

and $\sigma_{-1} = 0$. If $\sigma_n \to \lambda$ as $n \to \infty$, then s is said to be limitable (N, p) to the number λ. We say s is absolutely limitable (N, p) or limitable $|N, p|$ if σ is of bounded variation, i.e.,

$$(1) \qquad \sum |\sigma_n - \sigma_{n-1}| < \infty$$

We write $x = a(y)$ to mean that $x = by$ for some sequence b of bounded variation; thus (1) can also be written $\sigma = a(1)$. We shall denote by $o(N, p)$ the set of all sequences limitable (N, p) to zero, and by $a(N, p)$ those which are limitable $|N, p|$.

It follows from Toeplitz's theorem [**2**, Theorem 2] that necessary and sufficient conditions for the regularity of (N, p) are that

$$(2) \qquad p_n = o(|P_n|)$$

and that

$$(3) \qquad \sum_{\nu=0}^{n} |p_\nu| = O(|P_n|).$$

Necessary and sufficient conditions in order that (N, p) should be absolutely regular (that is, that $s = a(1)$ should imply that $\sigma = a(1)$, and that $\{s_n\}$, $\{\sigma_n\}$ should have the same limit) were given by Mears [**5**]. The conditions are (2) together with

$$(4) \qquad \sum_{n=k}^{\infty} \left| \frac{P_{n-k}}{P_n} - \frac{P_{n-k-1}}{P_{n-1}} \right| \leqq M \qquad (k = 1, 2, \ldots .).$$

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We write throughout

$$s_n = \sum_{\nu=0}^{n} a_\nu.$$

Then, following Borwein and Cass [**1**], we describe the sequence s as strongly summable (N, p) with index $\lambda > 0$ to μ, and write $s_n \to \mu[N, p]_\lambda$, if

$$\sum_{\nu=0}^{n} |p_\lambda|^{1-\lambda}|w_\nu - \mu p_\nu|^\lambda = o(|P_n|),$$

where $w = p*a$; it is assumed that, for all n, $p_n \neq 0$. As was pointed out in [**3**], this definition is of use only when $|P_n| \to \infty$ as $n \to \infty$.

We recall that if (N, p) is regular, or absolutely regular, then $p(z)$ is regular in $|z| < 1$ (see, for example, [**2**, p. 65]). Generalising some earlier results of Peyerimhoff [**7**], Miesner [**6**] obtained relations between the sets $o(N, p)$ and $o(N, q)$, and between the sets $a(N, p)$ and $a(N, q)$, where $q(z) = p(z)r(z)$, and where $r(z)$ satisfies appropriate restrictions. Miesner's theorems are as follows.

THEOREM A. *Let $r(z) = \sum r_n z^n$ be absolutely convergent for $|z| \leqq 1$, with $r(z) \neq 0$ for $z = 0$ and $|z| = 1$, and having inside the unit circle the roots $\alpha_1, \alpha_2, \ldots, \alpha_k$ with multiplicities $\gamma_1, \gamma_2, \ldots, \gamma_k (\gamma_i > 0)$. Let $p(z) = \sum p_n z^n$ be a function regular for $|z| < 1$, and suppose that the root α_i* of *$r(z)$ is a root of $p(z)$ with multiplicity $\lambda_i \geqq 0 (i = 1, \ldots, k)$. Define $q(z) = r(z)p(z)$.*

(i) *If p satisfies conditions* (2) *and* (3), *then $s \in o(N, q)$ if and only if*

$$s_n = t_n + \sum_{i=1}^{k} \alpha_i^{-n} \sum_{j=\lambda_i}^{\lambda_i+\gamma_i-1} c_{ij} A_n^{\,j}, \tag{5}$$

with $t \in o(N, p)$, $A_n^j = \binom{n+j}{n}$ and c_{ij} constants.

(ii) *If p satisfies the conditions* (2), (3), *and* (4), *then $s \in a(N, q)$ if and only if* (5) *holds with $t \in a(N, p)$.*

THEOREM B. *Let*

$$r(z) = \rho(z)k(z) = \rho(z) \prod_{i=1}^{m} (z - \beta_i)^{\lambda_i}$$

with $|\beta_i| = 1$, $\beta_i \neq 1$, λ_i a positive integer, and where the Taylor series of $\rho(z)$ is absolutely convergent and different from zero for $|z| \leqq 1$. Let $p(z)$ be a function regular in $|z| < 1$, and put $q(z) = r(z)p(z)$.

(*i*) *If p satisfies conditions* (2) *and* (3), *then $s \in o(N, q)$ if and only if*

$$s = \left(\prod_{i=1}^{m} T_{\beta_i}^{\ \lambda_i}\right) t, \tag{6}$$

where $t \in o(N, p)$ and the operator T is defined by

$$(T_\beta t)_n = -\sum_{\nu=0}^{n} t_\nu \beta^{\nu-n-1}, \tag{7}$$

and $(T_{\beta_1} T_{\beta_2})t = T_{\beta_1}(T_{\beta_2} t) = T_{\beta_2}(T_{\beta_1} t)$.

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(ii) *If p satisfies the conditions* (2), (3), *and* (4), *then* $s \in a(N, q)$ *if and only if* (6) *holds with* $t \in a(N, p)$.

As Miesner remarks, the case in which $r(z)$ has some zeros on $|z| = 1$ and some zeros in $|z| < 1$ may be dealt with by combining Theorems A and B.

The main object of the present paper is to obtain analogues of Theorems A and B for strong Nörlund summability with index 1. However, we wish first to point out that Theorem B can be simplified and generalised, since it is the analogue of this modified version which will be given.

2. Statement of the theorems. We have stated Theorem B with (6) in the form given by Miesner, but using the notation for convolution previously introduced, (6) can be simplified. Let

$$l(z) = \sum l_n z^n = (k(z))^{-1}$$

and observe that the expression on the right in (7) is just the convolution of t with the sequence $\{-\beta^{-1-n}\}$. Since

$$(z - \beta)^{-1} = -\sum \beta^{-1-n} z^n \quad \text{for } |z| < 1,$$

it is clear that l is the convolution of the sequences $\{-\beta_i^{-1-n}\}_{i=1,\ldots,m}$ taken with the appropriate multiplicities. Thus we can write (6) more simply as

$$s = l * t. \tag{8}$$

In this form the result can be improved.

THEOREM 1. *Let* $r(z) = \rho(z)k(z)$, *where* $\sum |\rho_n| < \infty$, $\rho(z) \neq 0$ *for* $|z| \leqq 1$. *Suppose that* $\sum |k_n| < \infty$, *that* $k_0 \neq 0$ (*so that* $l(z) = (k(z))^{-1}$ *is regular in some neighbourhood of the origin*), *and that* $k(1) \neq 0$. *Let* $p(z)$ *be regular in* $|z| < 1$, *and put* $q(z) = p(z)r(z)$.

(i) *If p satisfies* (2) *and* (3), *then* $s \in o(N, q)$ *if and only if* (8) *holds with* $t \in o(N, p)$.

(ii) *If p satisfies* (2), (3) *and* (4), *then* $s \in a(N, q)$ *if and only if* (8) *holds with* $t \in a(N, p)$.

We remark that the hypotheses on $k(z)$ are satisfied in particular when $k(z)$ is a product of a finite number of terms of the form $(1 - \alpha_i z)^{\lambda_i}$ with $|\alpha_i| = 1$, $\alpha_i \neq 1$, $\mathscr{R}(\lambda_i) > 0$. Thus Theorem 1 includes the extension of Theorem B to the case in which the zeros of $k(z)$ on $|z| = 1$ may be of fractional order. Of course, since the regularity (or absolute regularity) of (N, p) implies that $p(z)$ is regular in $|z| < 1$, the case of zeros of fractional order inside $|z| < 1$ cannot arise and so Theorem A cannot be extended in this way. While the hypotheses of Theorem 1 do not exclude the possibility that $k(z)$ might have zeros in $|z| < 1$, this case is better dealt with by Theorem A.

THEOREM 2. *Let* $r(z) = \sum r_n z^n$ *be absolutely convergent for* $|z| \leqq 1$ *with* $r(z) \neq 0$ *for* $|z| = 1$ *or* $z = 0$, *and having inside the unit circle the roots* $\alpha_1, \alpha_2, \ldots, \alpha_k$ *with* *multiplicities* $\gamma_1, \gamma_2, \ldots, \gamma_k (> 0)$. *Let* (N, p) *be regular and suppose that the root* α_i *of* $r(z)$ *is a root of* $p(z)$ *with multiplicity* $\lambda_i \geqq 0$ $(i = 1, 2, \ldots, k)$ *and that*

$$(9) \qquad \sum_{\nu=0}^{n} |p_\nu - p_{\nu-1}| = o(|P_n|) \qquad (p_{-1} = 0).$$

Define $q(z) = p(z)r(z)$. *Then* $s_n \to 0\,[N, q]_1$ *if and only if* (5) *holds with* $t_n \to 0\,[N, p]_1$.

We remark that, if (9) is not assumed, the conclusion of the theorem may be false even in simple cases. As an example, take

$$p(z) = \frac{1}{1 - z^2}; \qquad r(z) = 1 + 2z.$$

Thus

$$q_n = \begin{cases} 1 & (n \text{ even}) \\ 2 & (n \text{ odd}). \end{cases}$$

Now $r(-\frac{1}{2}) = 0$, so that, if $s_n = (-2)^n$, then (5) holds with $t_n = 0$. But it is false that $\{s_n\}$ is summable $[N, q]_1$ to 0. For, writing $w(z) = a(z)q(z)$, we have

$$a(z) = (1 - z)s(z) = \frac{1 - z}{1 + 2z}; \qquad w(z) = \frac{1}{1 + z}.$$

Thus $w_n = (-1)^n$, so that

$$\sum_{\nu=0}^{n} |w_\nu| \neq o(|Q_n|).$$

THEOREM 3. *Let* $r(z)$ *satisfy the same conditions as in Theorem* 1. *Let* $p(z)$ *satisfy* (2) *and* (3), *and let* $q(z) = p(z)r(z)$. *Then* $s_n \to 0\,[N, q]_1$ *if and only if* (8) *holds with* $t_n \to 0\,[N, p]_1$.

We remark that the hypotheses of Theorem 3 does not imply that $|P_n| \to \infty$ as $n \to \infty$. However, as already indicated, it is only the case in which $|P_n| \to \infty$ which is of interest.

3. Proof of Theorem 1. If we write $m(z) = p(z)k(z)$, then $q(z) = m(z)\rho(z)$. By [**6**, Lemma 2], (N, p) regular (absolutely regular) implies that (N, m) is regular (absolutely regular). By a further application of this lemma we deduce that (N, q) is regular in Case (i) and absolutely regular in Case (ii). But, by a theorem of Wiener and Levy (see, e.g., [**8**, Volume **1**, p. 246]), quoted in [**6**] as Lemma 3, it follows that the Taylor series of $(\rho(z))^{-1}$ is also absolutely convergent in $|z| \leqq 1$. It therefore follows from [**6**, Corollary 1] that summability (N, q) is equivalent to summability (N, m) (in Case (i)) and (N, q) is absolutely equivalent to (N, m) in Case (ii).

Thus for (i) we have to show $s \in o(N, m)$ if and only if (8) holds with $t \in o(N, p)$. Now $s \in o(N, m)$ means by definition that $(m*s)_n = o(|M_n|)$, and similarly,

$t \in o(N, p)$ means $(p*t)_n = o(|P_n|)$. But translating equation (9) of [**6**] into our notation, $o(|M_n|)$ is the same as $o(|P_n|)$. Further,

$$m*s = (p*k)*s = p*t,$$

since the hypothesis $s = l*t$ is equivalent to $t = k*s$. Hence (i) is proved.

For (ii) we argue in a similar way, but use equation (13) of [**6**] in place of equation (9).

4. Proof of Theorem 2. We first prove some lemmas. The first of these is not, in fact, required for the proof of Theorem 2, but is given because it helps to indicate the scope of that theorem. It follows from it that, if $|P_n| \to \infty$ as $n \to \infty$ (the important case), then (9) is necessarily satisfied whenever (N, p) is regular and absolutely regular. For the assumption that $p = a(P)$ is *weaker* than (4), since it is equivalent to the assertion that (4) holds in the special case $k = 1$.

LEMMA 1. *Suppose that* (N, p) *is regular,* $|P_n| \to \infty$, *and that* $p = a(P)$. *Then* (9) *holds.*

Proof. Let $p_n = \theta_n P_n$, so that

$$\theta_n = a(1); \quad P_{n-1} = P_n - p_n = P_n(1 - \theta_n) \quad (n \geqq 1).$$

Hence

$$p_{n-1} = P_n \theta_{n-1}(1 - \theta_n) \quad (n > 0),$$

and

$$p_n - p_{n-1} = P_n\{\theta_n - \theta_{n-1}(1 - \theta_n)\} = P_n(\theta_n - \theta_{n-1}) + p_n\theta_{n-1}.$$

Since (N, p) is regular and $|P_n| \to \infty$, we clearly have

$$\sum_{\nu=1}^{n} |P_\nu(\theta_\nu - \theta_{\nu-1})| = o(|P_n|) \text{ and } \sum_{\nu=1}^{n} |p_\nu||\theta_{\nu-1}| = o(|P_n|),$$

and hence (9) holds.

LEMMA 2. *Suppose that* $\sum |r_n| < \infty$, $r(0) \neq 0$, $r(1) \neq 0$. *Suppose that* p *satisfies* (2) *and* (3), *and let* $q(z) = r(z)p(z)$. *Then* q *satisfies* (2) *and* (3), *and*

$$Q_n = P_n(r(1) + o(1)).$$

If, further, (9) *holds, then*

$$\sum_{\nu=0}^{n} |q_\nu - q_{\nu-1}| = o(|Q_n|). \tag{10}$$

Proof. The first part of the lemma is given by [**6**, Lemma 2(i)]. For the second part, since

$$q_\nu - q_{\nu-1} = \sum_{k=0}^{\nu} (p_{\nu-k} - p_{\nu-k-1})r_k,$$

we have

$$\sum_{\nu=0}^{n} |q_\nu - q_{\nu-1}| \leqq \sum_{k=0}^{n} |r_k| \sum_{\nu=k}^{n} |p_{\nu-k} - p_{\nu-k-1}|$$
$$= o(|P_n|) = o(|Q_n|).$$

LEMMA 3. *Suppose that* $p(z)$ *is regular in* $|z| < 1$. *Let* $|\alpha| < 1$, $\alpha \neq 0$, *and write*

$$q(z) = \left(1 - \frac{z}{\alpha}\right)p(z).$$

If q *satisfies* (10), *then* p *satisfies* (9).

Proof. Since $p(z)$ is regular in $|z| < 1$, we must have $q(\alpha) = 0$. Hence

$$P_n = \alpha^{-n} \sum_{\nu=0}^{n} Q_\nu \alpha^\nu = -\alpha^{-n} \sum_{\nu=n+1}^{\infty} Q_\nu \alpha^\nu,$$

and thus

$$\frac{P_n}{Q_n} = -\sum_{\mu=1}^{\infty} \frac{Q_{n+\mu}}{Q_n} \alpha^\mu.$$

Choose $\eta > 0$ so that $|\alpha|(1 + \eta) < 1$. Since (10) implies that $Q_{n+1} \sim Q_n$ we see that, provided n is sufficiently large,

$$|Q_{n+\mu}| \leqq (1 + \eta)^\mu |Q_n|, \tag{11}$$

for all $\mu \geqq 1$. Also, for fixed μ, $Q_{n+\mu}/Q_n \to 1$ as $n \to \infty$; and it therefore follows by dominated convergence that

$$\frac{P_n}{Q_n} \to -\sum_{\mu=1}^{\infty} \alpha^\mu = -\frac{\alpha}{1 - \alpha}$$

as $n \to \infty$. Hence $o(|P_n|)$ and $o(|Q_n|)$ are equivalent. Also, since $q(\alpha) = 0$,

$$\begin{aligned} p_n - p_{n-1} &= \alpha^{-n} \sum_{\nu=0}^{n} (q_\nu - q_{\nu-1})\alpha^\nu \\ &= -\alpha^{-n} \sum_{\nu=n+1}^{\infty} (q_\nu - q_{\nu-1})\alpha^\nu \\ &= -\sum_{\nu=1}^{\infty} (q_{n+\nu} - q_{n+\nu-1})\alpha^\nu. \end{aligned}$$

Thus

$$\sum_{\mu=0}^{n} |p_\mu - p_{\mu-1}| \leqq \sum_{\nu=1}^{\infty} |\alpha|^\nu \sum_{\mu=0}^{n} |q_{\mu+\nu} - q_{\mu+\nu-1}|. \tag{12}$$

Given $\epsilon > 0$, the inner sum in (12) does not exceed

$$\sum_{\mu=0}^{n+\nu} |q_\mu - q_{\mu-1}| < \epsilon |Q_{n+\nu}|,$$

for all sufficiently large $n + \nu$, and hence for all $\nu \geqq 0$ if n is sufficiently large. Again using (11), we see that for sufficiently large n, (12) does not exceed

$$\epsilon |Q_n| \sum_{\nu=1}^{\infty} |\alpha|^\nu (1 + \eta)^\nu. \tag{13}$$

Since the sum in (13) is constant, we see that

$$\sum_{\nu=0}^{n} |p_\nu - p_{\nu-1}| = o(|Q_n|)$$
$$= o(|P_n|),$$

and hence the result.

LEMMA 4. *Suppose that p satisfies* (2), (3), *and* (9), *and that $p(z)$ has a zero of order $\lambda \geqq 0$ for $z = \alpha$, where $|\alpha| < 1$. Let $q(z) = (1 - z/\alpha)p(z)$. Then $s_n \to 0$ $[N, q]_1$ if and only if, for some constant c,*

(14) $$s_n = t_n + cA_n{}^\lambda \alpha^{-n},$$

where $t_n \to 0$ $[N, p]_1$.

Proof. Write

$$p(z) = \left(1 - \frac{z}{\alpha}\right)^\lambda h(z),$$

so that $h(z)$ is regular $|z| < 1$, and $h(\alpha) \neq 0$. Suppose first that $s_n \to 0$ $[N, q]_1$. Define t_n by (14) (where c is to be chosen later), and write

$$t_n = \sum_{\nu=0}^{n} b_\nu.$$

Then

$$p(z)b(z) = p(z)\left(a(z) - \frac{c(1-z)}{(1-z/\alpha)^{\lambda+1}}\right)$$
$$= \frac{q(z)}{1-z/\alpha}\left(a(z) - \frac{c(1-z)}{(1-z/\alpha)^{\lambda+1}}\right)$$
$$= \frac{w(z) - c(1-z)h(z)}{1-z/\alpha},$$

where we write $w(z) = q(z)a(z)$. Thus we are given that

$$\sum_{\nu=0}^{n} |w_\nu| = o(|Q_n|).$$

Hence $w(z)$ is regular in $|z| < 1$ (even though $a(z)$ may not be). We now choose

$$c = \frac{w(\alpha)}{(1-\alpha)h(\alpha)}.$$

We also write

$$d(z) = \sum d_n z^n = w(z) - c(1-z)h(z).$$

Thus $d(\alpha) = 0$; also

(15) $$d_n = w_n - c(h_n - h_{n-1}).$$

Hence

$$\begin{aligned} p(z)b(z) &= \frac{d(z)}{1 - z/\alpha} \\ &= \sum_{n=0}^{\infty} \left(\frac{z}{\alpha}\right)^n \sum_{\nu=0}^{n} d_\nu \alpha^\nu \\ &= -\sum_{n=0}^{\infty} \left(\frac{z}{\alpha}\right)^n \sum_{\nu=n+1}^{\infty} d_\nu \alpha^\nu. \end{aligned}$$

Thus we have to show that

$$\sum_{\nu=0}^{n} \left| \sum_{\mu=\nu+1}^{\infty} d_\mu \alpha^{\mu-\nu} \right| = o(|P_n|). \tag{16}$$

Now, by repeated application of Lemmas 2 and 3 (9) is equivalent to

$$\sum_{\nu=0}^{n} |h_\nu - h_{\nu-1}| = o(|H_n|); \tag{17}$$

also, noticing that $o(|H_n|) = o(|P_n|) = o(|Q_n|)$, it follows from (15) that

$$\sum_{\nu=0}^{n} |d_\nu| = o(|P_n|).$$

We now apply the argument to obtain (13) in Lemma 3 to get

$$\begin{aligned} \sum_{\nu=0}^{n} \left| \sum_{\mu=1}^{\infty} d_{\nu+\mu} \alpha^\mu \right| &\leqq \sum_{\mu=1}^{\infty} |\alpha|^\mu \sum_{\nu=0}^{n} |d_{\nu+\mu}| \\ &= o(|P_n|), \end{aligned}$$

and hence (16).

Now suppose that $t_n \to 0$ $[N, p]_1$. We know (for example, by [**4**, Theorem 1]) that (N, p) implies (N, q); hence, by [**1**, Theorem 1], $[N, p]_1$ implies $[N, q]_1$; thus $t_n \to 0$ $[N, q]_1$. Thus it remains only to show that

$$A_n^\lambda \alpha^{-n} \to 0 \ [N, q]_1.$$

But

$$\frac{q(z)(1 - z)}{(1 - z/\alpha)^{\lambda+1}} = (1 - z)h(z),$$

so that this assertion is equivalent to

$$\sum_{\nu=0}^{n} |h_\nu - h_{\nu-1}| = o(|Q_n|).$$

But we have already seen that $o(|Q_n|)$ is equivalent to $o(|H_n|)$, and this is therefore given by (17).

LEMMA 5. *Suppose that p satisfies* (2) *and* (3). *Suppose that $\sum |r_n| < \infty$, and that $r(z) \neq 0$ for $|z| \leqq 1$. Let $q(z) = r(z)p(z)$. Then $[N, p]_1$, $[N, q]_1$ are equivalent.*

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This follows at once from the theorem of Wiener and Lévy already mentioned, together with [**1**, Proposition 2 and Corollary 1].

LEMMA 6. *Suppose that* $\sum |r_n| < \infty$, *and that* $r(z)$ *has zeros of order greater than or equal to* $\gamma_i(\gamma_i > 0)$ *at the points* $\alpha_i(i = 1, 2, \ldots k)$, *where* $0 < |\alpha_i| < 1$. *Let*

$$r(z) = \prod_{i=1}^{k} \left(1 - \frac{z}{\alpha_i}\right)^{\gamma_i} r_1(z).$$

Then the expansion of $r_1(z)$ *as a power series converges absolutely for* $|z| \leqq 1$.

This is [**6**, Lemma 4].

Proof of Theorem 2. Let

$$p^*(z) = p(z) \prod_{i=1}^{k} \left(1 - \frac{z}{\alpha_i}\right)^{\gamma_i}.$$

It follows by repeated applications of Lemma 4 (the regularity of the relevant methods being ensured by Lemma 2) that $s_n \to 0\ [N, p^*]_1$ if and only if (5) holds with $t_n \to 0\ [N, p]_1$. But we deduce from Lemmas 5 and 6 that $[N, p^*]_1$ and $[N, q]_1$ are equivalent, and the theorem follows.

5. Proof of theorem 3. We use the same notation as in the proof of Theorem 1. By Lemma 5, $[N, q]_1$ and $[N, m]_1$ are equivalent, and hence $s_n \to 0\ [N, q]_1$ is equivalent to the assertion that

$$\sum_{\nu=0}^{n} |(m*a)_\nu| = o(|M_n|). \tag{18}$$

But, defining t by (8), and writing

$$t_n = \sum_{\nu=0}^{n} b_\nu,$$

we have

$$m*a = (p*k)*a = p*b.$$

Also, by Lemma 2, $o(|M_n|) = o(|P_n|)$, so that (18) is equivalent to

$$\sum_{\nu=0}^{n} |(p*b)_\nu| = o(|P_n|).$$

But this is the definition of the assertion that $t_n \to 0\ [N, p]_1$. Hence the result.

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