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RESEARCH ARTICLE

Groups of symplectic involutions on symplectic varieties of Kummer type and their fixed loci

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Abstract

We describe the Galois action on the middle ℓ-adic cohomology of smooth, projective fourfolds $K_A(v)$ that occur as a fiber of the Albanese morphism on moduli spaces of sheaves on an abelian surface A with Mukai vector v. We show this action is determined by the action on $H^2_{\text{ét}}(A_{\bar{k}}, \mathbb{Q}_\ell(1))$ and on a subgroup $G_A(v) \leqslant (A \times \hat{A})[3]$, which depends on v. This generalizes the analysis carried out by Hassett and Tschinkel over $\mathbb{C}$ [21]. As a consequence, over number fields, we give a condition under which $K_2(A)$ and $K_2(\hat{A})$ are not derived equivalent.

The points of $G_A(v)$ correspond to involutions of $K_A(v)$. Over $\mathbb{C}$, they are known to be symplectic and contained in the kernel of the map $\operatorname{Aut}(K_A(v)) \to \mathrm{O}(H^2(K_A(v), \mathbb{Z}))$. We describe this kernel for all varieties $K_A(v)$ of dimension at least 4.

When $K_A(v)$ is a fourfold over a field of characteristic 0, the fixed-point loci of the involutions contain K3 surfaces whose cycle classes span a large portion of the middle cohomology. We examine the fixed-point locus on fourfolds $K_A(0, l, s)$ over $\mathbb{C}$ where A is $(1, 3)$-polarized, finding the K3 surface to be elliptically fibered under a Lagrangian fibration of $K_A(0, l, s)$.

Contents

1. Introduction

Given a polarized abelian surface (A, H) defined over an arbitrary field k, we may study moduli spaces of geometrically H-stable sheaves on A with a fixed Mukai vector $v = (r, l, s)$; that is, fixed rank, Néron-Severi class of the determinental line bundle and Euler characteristic. Under mild conditions on the Mukai vector, the moduli spaces $M_A(v)$ are smooth and projective. Their Albanese varieties are $A \times \hat{A}$, and we denote a fiber of the Albanese morphism by $K_A(v)$.

If defined over $\mathbb{C}$, the variety $K_A(v)$ is a hyperkähler variety of dimension $v^2 - 2$ and is deformation equivalent to the *generalized Kummer variety* $K_n(A) \cong K_A(1, 0, -n-1)$, where $n := \frac{v^2}{2} - 1$, which is given by the fiber over 0 of the summation map acting on the Hilbert scheme of length-$(n+1)$ points on A. Following Fu and Li [17], who study these varieties over other fields, we call the $K_A(v)$ symplectic varieties (see Proposition 2.5). There are four known deformation types of hyperkähler varieties: $\mathrm{K3}^{[n]}$-type, Kummer type (or Kum_n-type) and the two sporadic examples of O'Grady [49, 50]. The varieties $K_A(v)$ are of Kummer n-type. It has been shown [41, Prop. 2.4] that under a lattice-theoretic condition, if $n + 1$ is a prime power, any hyperkähler of Kummer n-type is the fiber of the Albanese map of a moduli space of stable objects on an abelian surface A. So varieties $K_A(v)$ do not exhaust the class but are, at this point, the best understood.

In [21], Hassett and Tschinkel analyze the cohomology of complex generalized Kummer fourfolds $K_2(A)$. They show that $H^4(K_2(A), \mathbb{Q})$ is generated by $H^2(K_2(A), \mathbb{Q})$ and an 81-dimensional vector space spanned by the cycle classes of 81 distinct K3 surfaces in $K_2(A)$. These surfaces are each contained in the fixed locus of a symplectic involution of the form $t_x^*\iota^*$, where ι is multiplication by -1 on A, and t_x is translation by a point of the three-torsion $A[3]$ of A. Hassett and Tschinkel use deformation theory to show that the middle cohomology for any hyperkähler variety X of Kum_2-type has a similar decomposition. The cohomology of Kummer-type hyperkähler varieties is also studied in [19].

In this paper, we extend these results by characterizing the Galois action on the ℓ-adic étale cohomology of fourfolds $K_A(v)$ over nonclosed fields. As one might expect from the results of Hassett–Tschinkel, there is an 81-dimensional subspace of $H^4_{\text{ét}}(K_2(A)_{\bar{k}}, \mathbb{Q}_\ell(2))$ whose Galois action is determined by the structure of $A[3]$. However, deformation-theoretic tools are too coarse to keep track of how the Galois action changes for other fourfolds $K_A(v)$, which we find depends on v:

Theorem 1.1 (Theorem 4.4, Proposition 4.6). *Suppose $K_A(v)$ is a smooth, projective variety over an arbitrary field k. Then, there is a subgroup $G_{A_{\bar{k}}}(v) \leqslant (A_{\bar{k}} \times \hat{A}_{\bar{k}})[3]$ and a Galois equivariant isomorphism*

$$H^4_{\acute{e}t}(K_A(v)_{\bar{k}}, \mathbb{Q}_\ell(2)) \cong \mathrm{Sym}^2 H^2_{\acute{e}t}(K_A(v)_{\bar{k}}, \mathbb{Q}_\ell(1)) \oplus V,$$

where V is the 80-dimensional subrepresentation of the permutation representation $\mathbb{Q}_\ell[G_{A_{\bar{k}}}(v)]$ such that

$$\mathbb{Q}_\ell[G_{A_{\bar{k}}}(v)] \cong V \oplus \mathbb{Q}_\ell,$$

and the trivial representation $\mathbb{Q}_\ell$ is the span of $(0, 0) \in G_{A_{\bar{k}}}(v)$. The Galois action on the group ring $\mathbb{Q}_\ell[G_{A_{\bar{k}}}(v)]$ is induced by the action on $G_{A_{\bar{k}}}(v)$.

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By a generalization of the work of Yoshioka [56], this means the Galois action on the middle cohomology is determined by the action on $H^2_{\text{ét}}(A_{\bar{k}}, \mathbb{Q}_\ell(1))$ and the action on the subgroup $G_{A_{\bar{k}}}(v)$, which is the kernel of the isogeny $\phi\colon A \times \hat{A} \to A \times \hat{A}$ given by $(x, y) \mapsto (\phi_M(y) - sx, \phi_L(x) + ry)$ (See Section 3.1). This stands in surprising contrast to the case of moduli spaces of sheaves on K3 surfaces – symplectic varieties of K3$^{[n]}$-type – where the cohomology representations depend only on that of the K3 surface [15, Thm. 2].

As a consequence, by studying the even cohomology of $K_2(A)$ for A defined over a number field, we are able to show the following result on derived equivalence:

Corollary 1.2 (Corollary 5.8). *Let A be an abelian surface over a number field k for which the permutation representations associated to $A_{\bar{k}}[3]$ and $\hat{A}_{\bar{k}}[3]$ are not isomorphic. Then, $K_2(A)$ and $K_2(\hat{A})$ are not derived equivalent over k.*

In forthcoming work [16] on Galois actions on torsion subgroups of abelian surfaces, examples of such abelian surfaces are constructed. Intriguingly, this corollary shows that if $K_2(A)$ and $K_2(\hat{A})$ are derived equivalent after base change to $\mathbb{C}$, then the kernel of the Fourier–Mukai transform cannot be given by naturally associated bundles that would descend to the field of definition for A. Corollary 1.2 complements the recent work of Magni [37], which provides a sufficient condition for the existence of such equivalences over algebraically closed fields of characteristic zero.

The cohomology group V in Theorem 1.1 is generated by K3 surfaces contained in the fixed-point loci of symplectic involutions on $K_A(v)$. We give a case-by-case explicit description of $G_A(v)$, and hence, an explicit description of these symplectic involutions, which dictate the Galois action on V.

By work of Boissière–Nieper-Wisskirchen–Sarti in [8], Hassett–Tschinkel in [21] and Kapfer–Menet in [31], for any hyperkähler variety X over $\mathbb{C}$ of Kum$_{n-1}$-type, the kernel

$$\ker(\mathrm{Aut}(X) \to \mathrm{O}(H^2(X, \mathbb{Z}))) \cong \mathbb{Z}/2\mathbb{Z} \ltimes (\mathbb{Z}/n\mathbb{Z})^4$$

consists of symplectic automorphisms of X; when $\dim X = 4$, the kernel contains all of the symplectic involutions of X. We give an explicit description of this kernel for hyperkähler varieties $K_A(v)$ of any dimension at least 4 over $\mathbb{C}$.

Theorem 1.3 (Theorem 3.15). *Suppose $K_A(v)$ is a smooth, projective variety over $k = \mathbb{C}$. Then,*

$$\ker(\mathrm{Aut}(K_A(v)) \to \mathrm{O}(H^2(K_A(v), \mathbb{Z})))$$

consists of automorphisms of the following two forms:

$$L_y \otimes t_x^* \quad \text{and} \quad \kappa_{(x,y)} := L_y \otimes t_x^* \kappa,$$

where $\kappa = \iota^*$ if $K_A(v)$ is an Albanese fiber over symmetric line bundles, and otherwise κ is a composition of ι^* with a translation. The $\kappa_{(x,y)}$ are symplectic involutions of $K_A(v)$, and when $\dim K_A(v) = 4$, *these are all of the symplectic involutions.*

In the complex case, the group $G_A(v)$ also appears in [38] as Γ_v. Markman defines Γ_v as the kernel of the map ϕ above as well as in terms of Clifford algebras (§10.1, Remark 4.3 *op.cit.*). The result [38, Lemma 10.1] and its proof show Γ_v embeds into the monodromy group of $K_A(v)$, acts trivially on $H^2(K_A(v), \mathbb{Z})$ and $H^3(K_A(v), \mathbb{Z})$, and that $M_A(v)$ is isomorphic to a quotient of $A \times \hat{A} \times K_A(v)$ by an action of Γ_v. Thus, the fact that the automorphisms $L_y \otimes t_x^*$ are symplectic is not new, but we provide a proof to make our study of this family self-contained.

Beyond their analysis of the middle cohomology for $K_2(A)$, Hassett and Tschinkel explicitly describe the fixed-point loci of the symplectic involutions. They show that the locus fixed by the standard involution contains the Kummer K3 surface

$$\overline{\{(a_1, a_2, a_3) \mid a_1 = 0, a_2 = -a_3, a_2 \neq 0\}},$$

as well as a unique isolated point supported at the identity element 0. Tarí in [54] finishes the description by showing there are 35 more isolated points, which are tuples of two-torsion points of A. The deformation invariance of the symplectic involutions implies that the fixed locus of any $\iota_{(x,y)}$ in $K_A(v)$ also consists of a K3 surface and 36 isolated points [31, Thm. 7.5].

Motivated by these results, we seek a similar description of the fixed-point loci in fourfolds $K_A(0, l, s)$, whose general member is a degree $s + 3$ line bundle on a genus 4 curve in the linear system $|L|$ with $c_1(L) = l$. These moduli spaces admit a Lagrangian fibration, which aids in our study. We give the following description:

Theorem 1.4 (Theorem 7.2). *The K3 surface in the fixed-point locus of ι^* acting on $K_A(0, l, s)$ is elliptically fibered with four singular fibers of type I_1 and* 10 *singular fibers of type I_2.*

The singular fibers in this elliptic fibration agree with a natural elliptic fibration on the Kummer K3 surface $K_1(A)$ when A is $(1, 3)$-polarized – a necessary condition for $K_A(0, l, s)$ to be a fourfold. The K3 surface appears to be closely connected to the relative Jacobian of $K_1(A) \to \mathbb{P}^1$.

We also describe the isolated points in the fixed-point locus using the Abel map for the curves in $|L|$.

Outline

In Section 2, we provide a brief introduction to moduli spaces of sheaves, and Kummer-type varieties arising from them, over arbitrary fields. In Section 3, we identify which automorphisms of $M_A(v)$ given by translation and tensoring by a degree 0 line bundle restrict to automorphisms of $K_A(v)$, and then show how these give rise to the description of the symplectic automorphisms discussed in Theorem 1.3. We also begin the analysis of the fixed-point loci for the symplectic involutions. In Section 4, we study the middle cohomology of fourfolds $K_A(v)$, proving Theorem 1.1. In Section 5, we compare our results to questions about derived equivalences between abelian surfaces and their generalized Kummer fourfolds. Namely, we give criteria in Section 5.1 for when a derived equivalence between abelian surfaces A and B induces an isomorphism between $G_A(v)$ and $G_B(v)$, and we prove Corollary 1.2 in Section 5.2.

The second half of the paper is dedicated to studying the fixed-point locus of ι^* for fourfolds $K_A(0, l, s)$ over $\mathbb{C}$, including the proof of Theorem 1.4. In Section 6, we study the general geometry of $K_A(0, l, s)$ and the fixed-point locus, and then focus on the elliptic fibers of the K3 surface in Section 7. In Section 8, we describe the isolated points in the fixed-point locus.

Notation

We write the standard involution on an abelian surface A, the morphism multiplying by -1 in the group law of A, as $\iota\colon A \to A$. We write $K_n(A)$ for the generalized Kummer variety of dimension $2n$. In particular, we write $K_1(A)$ for the Kummer K3 surface of A.

For a smooth projective variety X over a field k, let $X_{\bar{k}} := X \times_k \bar{k}$. We denote by $\widetilde{H}(X_{\bar{k}}, \mathbb{Z}_\ell)$ the ℓ-adic Mukai lattice of X, which is the direct sum of the even cohomology twisted into weight zero:

$$\widetilde{H}(X_{\bar{k}}, \mathbb{Z}_\ell) := \bigoplus_{i=0}^{\dim X} H^{2i}_{\text{ét}}(X_{\bar{k}}, \mathbb{Z}_\ell(i)).$$

This lattice is given the usual Mukai pairing. For $X = A$ an abelian surface, $(\alpha, \beta) = -\alpha_0\beta_4 + \alpha_2\beta_2 - \alpha_4\beta_0$. We will always assume that our Mukai vectors v satisfy the conditions given in Setting 2.4 unless indicated otherwise.

Throughout, $D(X)$ denotes the bounded derived category of coherent sheaves on X.

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2. Moduli spaces over arbitrary fields

Let A be an abelian surface defined over an arbitrary field k.

Definition 2.1. Let $\omega \in H^4_{\text{ét}}(A_{\bar{k}}, \mathbb{Z}_\ell(2))$ be the numerical equivalence class of a point on $A_{\bar{k}}$. A *Mukai vector on* A is an element of

$$N(A) := \mathbb{Z} \oplus \mathrm{NS}(A) \oplus \mathbb{Z}\omega,$$

where $N(A)$ is a subgroup of $\widetilde{H}(A_{\bar{k}}, \mathbb{Z}_\ell)$ under the natural inclusion.

Given a coherent sheaf $\mathcal{F}$ on A, we assign to it a Mukai vector $v(\mathcal{F}) \in N(A)$ given by its rank, the Néron-Severi class of its determinantal line bundle and its Euler characteristic. We will write this as $v(\mathcal{F}) = (r, l, s)$.

By fixing a Mukai vector v and a polarization H on A, we can construct the moduli space $M_{A,H}(v)$ parametrizing H-semistable sheaves on A. We use the more compact notation $M_A(v)$. We ask that the Mukai vector satisfies the following conditions in order to ensure that the moduli space is nicely behaved (i.e., is a nonempty, smooth, projective variety over k).

Definition 2.2.

(a) A Mukai vector $v \in N(A)$ is *geometrically primitive* if its image under $N(A) \to N(A_{\bar{k}})$ is primitive in the lattice.

(b) A Mukai vector (r, l, s) is *positive* if one of the following is satisfied:
   (i) $r > 0$
   (ii) $r = 0$, l is effective and $s \neq 0$
   (iii) $r = 0$, $l = 0$ and $s < 0$.

(c) A polarization $H \in \mathrm{Pic}(A)$ is *v-generic* if every H-semistable sheaf with Mukai vector v defined over $\bar{k}$ is H-stable.

A polarization is often v-generic if it is not contained in a locally finite union of certain hyperplanes in $\mathrm{NS}(A_{\bar{k}})_{\mathbb{R}}$ defined in [28, Def. 4.C.1], but this is not always enough to ensure genericity (see, for example, [15, Ex. 1.7]).

When $v^2 = 0$ and H is v-generic, Mukai showed that $M_H(v)$ is an abelian surface [46, Rmk. 5.13]. We focus on the higher-dimensional case.

Proposition 2.3. *Let $v \in N(A)$ be a geometrically primitive and positive Mukai vector with $v^2 \geq 2$, and let H be a v-generic polarization on A. Then, $M_A(v)$ is a nonempty, smooth, projective, geometrically irreducible variety of dimension $v^2 + 2$ over k.*

Proof. The projectivity and smoothness are shown in [17, Prop. 6.9], which relies on classic results in [44] as well as [34] for the construction of moduli spaces of semistable sheaves over arbitrary fields. Geometric irreducibility of $M_A(v)$ follows from [29, Thm. 4.1] (note that the authors work over $\mathbb{C}$, but their proof holds for any algebraically closed field). Finally, the dimension claim follows from [44, Cor. 0.2] once we know $M_A(v)$ is nonempty; nonemptiness is a consequence of [56, Thm. 0.1] along with a lifting argument as in [17, Prop. 6.9] when the field has positive characteristic. □

Let $v := (r, l, s)$ be a Mukai vector as in Proposition 2.3 and let

$$\Phi_P \colon D(A) \to D(\hat{A})$$

denote *the Fourier–Mukai transform* on A, which has kernel the Poincaré bundle P on $A \times \hat{A}$. In [56, Thm. 4.1], Yoshioka proves over $\mathbb{C}$ that the Albanese variety of $M_H(v)$ is $A \times \hat{A}$ and fixing any $\mathcal{F}_0 \in M_H(v)$, we define the Albanese morphism as follows:

$$\begin{aligned} M_A(v) &\to \hat{A} \times A \\ \mathcal{F} &\mapsto (\det(\mathcal{F}) \otimes \det(\mathcal{F}_0)^{-1}, \det(\Phi_P(\mathcal{F})) \otimes \det(\Phi_P(\mathcal{F}_0))^{-1}) \end{aligned} \tag{2.1}$$

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This construction also shows that over an arbitrary field k, the following map gives the Albanese torsor of $M_H(v)$:

$$\begin{aligned} \mathrm{alb}\colon M_A(v) &\to \mathrm{Pic}^l_A \times \mathrm{Pic}^m_{\hat{A}} \\ \mathcal{F} &\mapsto (\det(\mathcal{F}), \det(\Phi_P(\mathcal{F}))), \end{aligned} \tag{2.2}$$

where m is the Néron-Severi class in the Mukai vector $\Phi_P(v) := (s, m, r)$, which is the negative of the Poincaré dual of l by [45, Prop. 1.17].

Setting 2.4. Let A be an abelian surface defined over a field k. Let $v := (r, l, s) \in N(A)$ be a geometrically primitive and positive Mukai vector with $v^2 \geq 6$ and char $k \nmid \frac{v^2}{2}$. Let H be a v-generic polarization on A. Fix (L, M) a pair of line bundles in $\mathrm{Pic}^l(A) \times \mathrm{Pic}^m(\hat{A})$. Let $K_A(v)$ be the fiber of alb over (L, M).

Over $\mathbb{C}$, [56, Thm. 0.2] shows that $K_A(v)$ is a hyperhähler variety, and the following result generalizes this to other fields.

Proposition 2.5 [56, Thm. 0.2], [17, Prop. 6.9]. *Suppose we have data as in Setting 2.4. Then, $K_A(v)$ is a smooth, projective symplectic variety of dimension $v^2 - 2$ and is deformation equivalent to the generalized Kummer variety $K_{(v^2-2)/2}(A)$.*

For $K_A(v)$ over a field of characteristic zero, which we may assume is a subfield of $\mathbb{C}$, $K_A(v)_{\mathbb{C}}$ is a hyperkähler variety. In positive characteristic, Fu and Li [17, Def. 3.1] define a symplectic variety X to be a smooth connected variety, where $\pi_1^{\text{ét}}(X) = 0$ and X admits a nowhere degenerate closed algebraic 2-form.

We are interested in symplectic involutions on $K_A(v)$. We will show in Theorem 3.15 that these all involve the induced action of the standard involution ι on A. Pullback ι^* sends degree 0 line bundles on A to their inverses. For any line bundle $\mathcal{L} \in \mathrm{Pic}(A)$, the multiplication by n map has the property that $[n]^*\mathcal{L} \cong \mathcal{L}^{n^2} \otimes M$ for some $M \in \mathrm{Pic}^0(A)$. Thus, $\mathcal{L}$ and $\iota^*\mathcal{L}$ differ by a degree 0 line bundle, so are always in the same Néron-Severi class.

In order for ι^* to give a well-defined morphism on $K_A(v)$, $K_A(v)$ must be a fiber of the Albanese morphism over a pair of symmetric line bundles L and M, which we prefer to do when possible for notational simplicity. In the case of generalized Kummer varieties $K_{n-1}(A)$ or varieties $K_A(v)$ whose Mukai vector has trivial Néron-Severi class, it is always possible to choose the fiber over the structure sheaves of A and $\hat{A}$. For other choices of Mukai vector, we show in Lemma 2.6 below that over an algebraically closed field, we may always choose such a pair of symmetric line bundles.

Lemma 2.6. *Let A be an abelian variety over an algebraically closed field k. Then, any class in* $\mathrm{NS}(A)$ *has a symmetric representative. Moreover, there is a short exact sequence of the following form, where* $\mathrm{Pic}^{\mathrm{sym}}(A)$ *is the subgroup of all symmetric line bundles:*

$$0 \to \mathrm{Pic}^0(A)[2] \to \mathrm{Pic}^{\mathrm{sym}}(A) \to \mathrm{NS}(A) \to 0.$$

Proof. The action of ι^* on the following short exact sequence

$$0 \to \mathrm{Pic}^0(A) \to \mathrm{Pic}(A) \to \mathrm{NS}(A) \to 0$$

gives rise to the long exact sequence

$$0 \to \mathrm{Pic}^0(A)[2] \to \mathrm{Pic}(A)^{\mathrm{sym}} \to \mathrm{NS}(A) \to H^1(\mathbb{Z}/2\mathbb{Z}, \mathrm{Pic}^0(A)) \to \cdots,$$

where $\mathrm{NS}^{\mathrm{sym}}(A) = \mathrm{NS}(A)$ since, for any line bundle $\mathcal{L}$, $\iota^*\mathcal{L}$ is in the same Néron-Severi class as $\mathcal{L}$. The group $H^1(\mathbb{Z}/2\mathbb{Z}, \mathrm{Pic}^0(A))$ is trivial since crossed homomorphisms correspond to elements in $\mathrm{Pic}^0(A)$ and principal crossed homomorphisms correspond to choices of element in $\mathrm{Pic}^0(A)$ that have a square root, which is all of them, since we are working over an algebraically closed field. □

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The proof above requires the field k to be algebraically closed, but we will often work over a nonclosed field. In that case, the existence of a symmetric line bundle in a given Néron-Severi class is not guaranteed. Rather than working over a finite extension of the ground field in order to acquire a symmetric bundle, we will simply alter ι^* by a correction factor to get an associated involution on $K_A(v)$ (see Construction 3.10).

3. Symplectic involutions on $K_A(v)$

In [8, Cor. 5(2)], the authors show that, for $X = K_{n-1}(A)$ over $\mathbb{C}$, the kernel of

$$\nu\colon \operatorname{Aut} X \to \mathrm{O}(H^2(X,\mathbb{Z}))$$

is isomorphic to $\mathbb{Z}/2\mathbb{Z} \ltimes (\mathbb{Z}/n\mathbb{Z})^4$, generated by ι and translation by elements of $A[n]$. In fact, this group of automorphisms is isomorphic to $\mathbb{Z}/2\mathbb{Z} \ltimes (\mathbb{Z}/n\mathbb{Z})^4$ for any hyperkähler variety X of Kum_{n-1}-type, since it is a deformation invariant [21, Thm. 2.1]. Moreover, when $\dim X = 4$, $\ker \nu$ contains all of the symplectic involutions [31, Thm. 7.5(i)]. Markman identifies a subgroup $\Gamma_v \cong (\mathbb{Z}/n\mathbb{Z})^4$ of $\ker \nu$ when $X = K_A(v)$ as coming from the kernel of ϕ defined below [38, §10.1]. In this section, we give an explicit description of $\ker \nu$ for $K_A(v)_{\bar{k}}$ when we are in the more general Setting 2.4 and k is arbitrary; this will allow us to understand the action of the Galois group on the fixed-point loci of the involutions in $\ker \nu$.

In Section 3.1, we identify which automorphisms of $M_A(v)$ given by translation and tensoring by a degree 0 line bundle restrict to automorphisms of $K_A(v)$ and show they form a group isomorphic to $(\mathbb{Z}/n\mathbb{Z})^4$. We also identify the group of such automorphisms when v is not primitive. The other automorphism needed to generate $\ker \nu$ is ι^* when $K_A(v)$ is the Albanese fiber over symmetric line bundles; in Section 3.2, we produce an involution κ to replace ι^* in the more general setting. We then study the fixed loci of the compositions of κ with the automorphisms produced in Section 3.1. In Section 3.3, we show that these compositions are symplectic and act trivially on $H^2(K_A(v),\mathbb{Z})$.

3.1. Automorphisms from translation and tensor

In this section, we work with data as in Setting 2.4, with the additional assumption that k is an algebraically closed field, and we define $n := \frac{v^2}{2}$. Because $k = \bar{k}$ and $\operatorname{char} k \nmid n$, we have $A[n] \cong (\mathbb{Z}/n\mathbb{Z})^4$.

We recall that given a line bundle $\mathcal{L} \in \operatorname{Pic}(A)$, $\phi_{\mathcal{L}}\colon A \to \hat{A}$ is defined by $\phi_{\mathcal{L}}(x) := t_x^*\mathcal{L} \otimes \mathcal{L}^{-1}$, where $t_x\colon A \to A$ is translation by a point $x \in A$. We denote by $L_y \in \operatorname{Pic}^0(A)$ the line bundle corresponding to a point $y \in \hat{A}$. Note that $\phi_{\mathcal{L}}$ is dependent only on the Néron-Severi class of $\mathcal{L}$, so we will use the notation $\phi_{[\mathcal{L}]}$.

Pullback by the translation map and tensoring by degree 0 line bundles give automorphisms of $M_A(v)$, and we are interested in when these automorphisms respect the Albanese morphism. That is, we identify in Theorem 3.1 below which of the $L_y \otimes t_x^* \in \operatorname{Aut} M_A(v)$ restrict to automorphisms of $K_A(v)$.

Theorem 3.1. *Let v be a Mukai vector as in Setting 2.4. There are exactly n^4 elements $(x, y) \in A \times \hat{A}$ for which the automorphism $L_y \otimes t_x^*$ on $M_A(v)$ restricts to an automorphism on $K_A(v)$. These elements form a subgroup*

$$G_A(v) \leqslant (A \times \hat{A})[n],$$

whose set of k-points is isomorphic to $(\mathbb{Z}/n\mathbb{Z})^4$.

The elements of $G_A(v)$ are the solutions to the following equations on $\hat{A}$ and A, where l and m are the Néron-Severi classes of L and M:

$$\phi_l(x) = -ry \quad \text{and} \quad \phi_m(y) = sx. \tag{3.1}$$

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Equivalently, $G_A(v)$ is the kernel of the following isogeny:

$$\begin{aligned}\phi\colon A \times \hat{A} &\to A \times \hat{A} \\ (x, y) &\mapsto (\phi_m(y) - sx, \phi_l(x) + ry).\end{aligned} \tag{3.2}$$

The proof of Theorem 3.1 requires analysis of ϕ_l and ϕ_m. We will crucially need the following lemma.

Lemma 3.2 (Yoshioka [56, Lem. 4.2]).

$$\phi_m \circ \phi_l = -\chi \cdot 1_A \quad \textit{and} \quad \phi_l \circ \phi_m = -\chi \cdot 1_{\hat{A}},$$

where $\chi := \chi(L) = \chi(M) = \frac{l^2}{2} = n + rs$.

Additionally, we recall that for any $\mathcal{F} \in D(A)$,

$$\Phi_P(t_x^*\mathcal{F}) = L_{-x} \otimes \Phi_P(\mathcal{F}) \quad \text{and} \quad \Phi_P(\mathcal{F} \otimes L_y) = t_y^*\Phi_P(\mathcal{F}).$$

This follows from [43, (3.1)]. Though the statement is not quite identical to the one we give here, it immediately follows from biduality of the Poincaré bundle [26, 9.12].

Proof of Theorem 3.1. The main issue in this proof is that maps of the form $L_y \otimes t_x^*$ are not, in general, well-defined as automorphisms on $K_A(v)$. Given $\mathcal{F} \in K_A(v)$, $L_y \otimes t_x^*\mathcal{F}$ has the same Mukai vector as $\mathcal{F}$ but may not have the same image under the Albanese morphism. For instance, pullback by t_x^*, in general, preserves Néron–Severi classes of line bundles and acts trivially on the structure sheaf, but it does not act trivially on all line bundles.

We therefore seek the $(x, y) \in A \times \hat{A}$ that satisfy the following conditions:

$$\begin{aligned} L &= \det(\mathcal{F}) = \det(L_y \otimes t_x^*\mathcal{F}) = L_y^{\otimes r} \otimes t_x^* \det(\mathcal{F}) = L_y^{\otimes r} \otimes t_x^* L \\ M &= \det(\Phi_P(\mathcal{F})) = \det(\Phi_P(L_y \otimes t_x^*(\mathcal{F}))) = \det(t_y^*(L_{-x} \otimes \Phi_P(\mathcal{F}))) \\ &\qquad\qquad = t_y^*(L_{-x}^{\otimes s} \otimes \det(\Phi_P(\mathcal{F}))) = t_y^*(L_{-x}^{\otimes s} \otimes M) = L_{-x}^{\otimes s} \otimes t_y^* M. \end{aligned}$$

We may rewrite these conditions as the equations (3.1). Equivalently, these (x, y) are the kernel of the map ϕ in (3.2).

Precomposing the map ϕ with $\psi\colon A \times \hat{A} \to A \times \hat{A}$, where $\psi(x, y) = (\phi_m(y) - rx, \phi_l(x) + sy)$, and applying Lemma 3.2, we have

$$\begin{aligned} \phi \circ \psi(x, y) &= \phi \circ (\phi_m(y) - rx, \phi_l(x) + sy) \\ &= (\phi_m(\phi_l(x) + sy) - s(\phi_m(y) - rx), \phi_l(\phi_m(y) - rx) + r(\phi_l(x) + sy)) \\ &= (-\chi \cdot x + rsx, -\chi \cdot y + rsy) = -n(x, y). \end{aligned}$$

Thus, $\phi \circ \psi = [-n]$, so ϕ is surjective and is hence an isogeny. Similarly, $\psi \circ \phi = [-n]$ and $G_A(v) \leqslant (A \times \hat{A})[n]$.

We show $G_A(v) \cong (\mathbb{Z}/n\mathbb{Z})^4$ in Lemma 3.6. This will require an understanding of preimages of elements under ϕ_l and ϕ_m, which we study in Claims 3.4 and 3.5. □

Remark 3.3. Since the maps ϕ_l and ϕ_m are determined by the Néron–Severi classes of L and M, the proof of Theorem 3.1 shows that the automorphisms of $M_A(v)$ given by elements of $G_A(v)$ will restrict to automorphisms of not just one, but any fiber of the Albanese morphism on $M_A(v)$.

Furthermore, for any $(x, y) \in (A \times \hat{A})[n]$, the automorphism $L_y \otimes t_x^*$ induces a permutation of the Albanese fibers and if $(x, y) \notin G_A(v)$, this permutation does not have any fixed fibers.

If we extend the domain of $\det \times \det \Phi_P$ to elements of $D(A)$ with Mukai vector v (by mapping to the Grothendieck group before taking determinants), $L_y \otimes t_x^*$ acts on the fibers of this map as well.

Before proving Lemma 3.6, we need results on the kernels of ϕ_l and ϕ_m:

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Claim 3.4. *Let $p \neq \operatorname{char} k$ be a prime and $\chi \neq 0$. Suppose p^q is the highest power of p dividing χ. Then, the group of p-power torsion points in $\ker \phi_l \cong \ker \phi_m$ is*

$$(\mathbb{Z}/p^{n_1}\mathbb{Z})^2 \times (\mathbb{Z}/p^{n_2}\mathbb{Z})^2,$$

where $0 \leq n_1 \leq n_2$ and $n_1 + n_2 = q$. If $n_1 > 0$, then L and M are p^{n_1}-st powers of other line bundles.

If L and M are separable, we may define their *polarization type* to be the termwise product of pairs (p^{n_1}, p^{n_2}) as p varies over primes dividing χ (cf. [6, §2]).

Proof. Since $\ker \phi_l \cong \ker \phi_{-l}$ and $\chi(L) \neq 0$, we may assume that L is ample. The proof of Riemann–Roch for abelian varieties in [47, §16] implies that the degree of ϕ_l is χ^2. The structure of $\ker \phi_l \cap A[p^q]$ is then determined by Lemma 3.2 and the fact that the Weil pairing e^L on the p-torsion is skew-symmetric [47, §20, Thm. 1]. Since ϕ_m is the negative of the dual of ϕ_l [6, §2], the group structure of p-power torsion points in $\ker \phi_m$ is isomorphic to that in $\ker \phi_l$. The last statement is a consequence of [47, §23, Thm. 3]. □

The images of any two elements of the same order under the compositions $\phi_l \circ \phi_m$ or $\phi_m \circ \phi_l$ will have the same order. However, ϕ_l and ϕ_m do not respect orders in this way.

Claim 3.5. *Let $p \neq \operatorname{char} k$ be a prime dividing χ, and assume that l and m are not p-th multiples of other classes, so $n_1 = 0$. Suppose $p^d \mid \chi$ for some $d \in \mathbb{N}$.*

(a) *Suppose $u \in A[p^d] \cap \ker \phi_l$. Then, the preimage of u in $\hat{A}[p^d]$ under ϕ_m is of the form $b + (\mathbb{Z}/p^d\mathbb{Z})^2$ for some $b \in \hat{A}[p^d]$.*
(b) *Suppose $v \in \hat{A}[p^d] \cap \ker \phi_m$. Then, the preimage of v in $\hat{A}[p^d]$ vnder ϕ_l is of the form $a + (\mathbb{Z}/p^d\mathbb{Z})^2$ for some $a \in A[p^d]$.*

Now, suppose $p^q = \chi$.

(c) *Suppose $u \in A$ and $\phi_l(u)$ has order p^c. Then, the preimage of u in $\hat{A}$ under ϕ_m is of the form $b + (\mathbb{Z}/p^q\mathbb{Z})^2$ for some $b \in \hat{A}[p^{c+q}]$.*
(d) *Suppose $v \in \hat{A}$ and $\phi_m(v)$ has order p^c. Then, the preimage of v in A under ϕ_l is of the form $a + (\mathbb{Z}/p^q\mathbb{Z})^2$ for some $b \in \hat{A}[p^{c+q}]$.*

Proof. (a) By Lemma 3.2, the composition $\phi_l \circ \phi_m$ is given by multiplication by $-\chi$. Thus, $\phi_m \circ \phi_l$ acts on $A[p^d]$ as the zero map, and hence,

$$\operatorname{im} \phi_m|_{\hat{A}[p^d]} \subseteq A[p^d] \cap \ker \phi_l.$$

By Claim 3.4, $A[p^d] \cap \ker \phi_l$ has p^{2d} elements and ϕ_m acting on $\hat{A}[p^d]$ is a p^{2d}-to-1 map. It follows by counting that $\operatorname{im} \phi_m|_{\hat{A}[p^d]} = A[p^d] \cap \ker \phi_l$. By Claim 3.4, the preimage of u is as stated.

Part (b) follows analogously.

(c) By Lemma 3.2, the preimage of $\phi_l(u)$ under $\phi_l \circ \phi_m$ consists of elements of order p^{c+q}. By Lemma 3.4, the result follows.

Part (d) follows analogously. □

The following result is proved using a case-by-case argument. The explicit argument given has the advantage of aiding in the analysis of examples. See [38, Lemma 10.1] for an approach using deformations over $\mathbb{C}$.

Lemma 3.6. *The solutions to the equations (3.1) form a group isomorphic to $(\mathbb{Z}/n\mathbb{Z})^4 \leqslant (A \times \hat{A})[n]$.*

Proof. Case 1: $\chi = 0$.

Both L and M must have degree 0, so ϕ_l and ϕ_m are both the 0-morphism. The equations (3.1) simplify to

$$0 = -ry \quad \text{and} \quad 0 = sx.$$

Furthermore, $n = -rs$. Since $v = (r, l, s)$ is positive and $v^2 \geq 4$, we must have $r > 0$ and $s < 0$. The solutions consist of all products of $|s|$-torsion points on A and r-torsion points on $\hat{A}$.

The group of solutions is isomorphic to $(\mathbb{Z}/r\mathbb{Z})^4 \times (\mathbb{Z}/|s|\mathbb{Z})^4$, hence $(\mathbb{Z}/n\mathbb{Z})^4$, since in this case, primitivity of the Mukai vector implies r and s are relatively prime.

Now, let p be a prime divisor of n and p^q be the highest power of p dividing n. We treat the remaining cases by analyzing solutions in $(A \times \hat{A})[p^q]$. We may then conclude by using the Sun Zi Remainder Theorem.

Case 2: $\chi \neq 0$ and at least one of r or s is relatively prime with p.

Suppose r is relatively prime with p. Fix an arbitrary $x \in A[p^q]$. The equation $\phi_l(x) = -ry$ then has exactly one solution y because multiplication by $-r$ acts bijectively on $\hat{A}[p^q]$.

Now we check that (x, y) is a solution to (3.1): Applying ϕ_m, we have $\phi_m \circ \phi_l(x) = -r\phi_m(y)$. Using Lemma 3.2, we then have $-rsx = -r\phi_m(y)$. Since x and y are p^q-torsion, multiplication by $-r$ acts bijectively, implying $sx = \phi_m(y)$.

Thus, for each $x \in A[p^q]$, there is one $y \in \hat{A}[p^q]$ so that (x, y) is a solution to (3.1). The projection map $(x, y) \mapsto x$ gives an isomorphism from solutions to (3.1) to $A[p^q] \cong (\mathbb{Z}/p^q\mathbb{Z})^4$.

If s is relatively prime with p, an analogous argument shows there is exactly one solution (x, y) to (3.1) for each $y \in \hat{A}[p^q]$ and that, again, the group of all solutions is isomorphic to $(\mathbb{Z}/p^q\mathbb{Z})^4$.

Cases 1 and 2 have covered all cases where r and s are not both divisible by p. Going forward, we assume $p \mid r$ and $p \mid s$. If $\mathrm{char}(k) \neq 0$, our assumption in Setting 2.4 that $\mathrm{char}(k) \nmid n$ implies in the following cases that $\mathrm{char}(k) \neq p$, and so we may apply Claim 3.4. By the primitivity of the Mukai vector, $n_1 = 0$ and n_2 is equal to the highest power of p dividing χ.

Let j be the highest power of p dividing r and k be the highest power of p dividing s. If r or s is 0, we choose the convention that j or k is ∞.

In each of Cases 3, 4, 5, we handle in stages the situations where q becomes higher and higher relative to j and k. From now on, we assume $j \geq k$. If $k > j$, the argument is analogous.

Case 3: $\chi \neq 0$, $0 < k \leq j$ and $q \leq j$. We observe that p^q is the highest power of p that divides χ.

Solutions $(x, y) \in (A \times \hat{A})[p^q]$ to the first equation in (3.1) are precisely those where $\phi_l(x) = 0$. By Claim 3.4, the group of such x is isomorphic to $(\mathbb{Z}/p^q\mathbb{Z})^2$.

Fix such an x. We observe that $sx \in A[p^q]$ and $\phi_l(sx) = 0$. By Claim 3.5(a), the preimage of sx under ϕ_m in $\hat{A}[p^q]$ is of the form $b + (\mathbb{Z}/p^q\mathbb{Z})^2$ for some $b \in \hat{A}[p^q]$. Thus, there are p^{4q} total solutions.

The projection $(x, y) \mapsto x$ gives a surjective group homomorphism $G_A(v) \twoheadrightarrow (\mathbb{Z}/p^q\mathbb{Z})^2$. The kernel of this map consists of all solutions where $x = 0$, which by Claim 3.4 is isomorphic to $(\mathbb{Z}/p^q\mathbb{Z})^2$. Since $G_A(v) \leqslant (A \times \hat{A})[p^q] \cong (\mathbb{Z}/p^q\mathbb{Z})^8$, this short exact sequence shows it must be isomorphic to $(\mathbb{Z}/p^q\mathbb{Z})^4$.

In Cases 4 and 5, we make a reduction argument. We observe that for any $(x, y) \in G_A(v)$, $(sx, sy) \in G_A(v) \cap (A \times \hat{A})[p^{q-k}]$. In each of Cases 4 and 5, we will show that the map

$$G_A(v) \xrightarrow{\cdot s} G_A(v) \cap (A \times \hat{A})[p^{q-k}] \tag{3.3}$$

given by multiplication by s is surjective and p^{4k}-to-1. This argument may be repeated to reduce each case to previous cases.

Case 4: $\chi \neq 0$, $0 < k \leq j < q$ and $q \leq j + k$.

We note that $p^q \mid \chi$. Since $q - k \leq j$, the argument in Case 3 shows that

$$G_A(v) \cap (A \times \hat{A})[p^{q-k}] \cong (\mathbb{Z}/p^{q-k}\mathbb{Z})^4. \tag{3.4}$$

Let $(u, v) \in G_A(v) \cap (A \times \hat{A})[p^{q-k}]$. We seek $(x, y) \in G_A(v) < (A \times \hat{A})[p^q]$, where $(sx, sy) = (u, v)$. First, we search for elements y, where $\phi_m(y) = u$ and $sy = v$. Thus, we look at the preimage of u under ϕ_m and analyze which of those elements give v when multiplied by s.

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Note that $\phi_l(u) = -rv = 0$. Since u is p^{q-k}-torsion, it is also p^q-torsion. So by Claim 3.5(a), the preimage of u under ϕ_m in $\hat{A}[p^q]$ is of the form $b + (\mathbb{Z}/p^q\mathbb{Z})^2$, where $b \in \hat{A}[p^q]$. Multiplying by s gives a p^{2k}-to-1 map on the following cosets:

$$b + (\mathbb{Z}/p^q\mathbb{Z})^2 \xrightarrow{\cdot s} sb + (\mathbb{Z}/p^{q-k}\mathbb{Z})^2.$$

We will now show that v is in the image of this map: the preimage of su under ϕ_m in $\hat{A}[p^q]$ is of the form $v + (\mathbb{Z}/p^q\mathbb{Z})^2$. The preimage of su under ϕ_m that is p^{q-k}-torsion is thus of the form $v + (\mathbb{Z}/p^{q-k}\mathbb{Z})^2$ and has exactly $p^{2(q-k)}$ elements. Now, the elements of $sb + (\mathbb{Z}/p^{q-k}\mathbb{Z})^2$ are p^{q-k}-torsion, there are $p^{2(q-k)}$ of them and their image under ϕ_m is su. Thus, these sets are equal, implying $v \in sb + (\mathbb{Z}/p^{q-k}\mathbb{Z})^2$. Thus, there are p^{2k} elements $y \in \hat{A}[p^q]$ with the desired properties.

Now we search for elements x, where $\phi_l(x) = -ry = -\frac{r}{s}v$ and $sx = u$. Note that since $j \geq k$, $-\frac{r}{s} = \frac{cp^e}{d}$ for some c, d relatively prime with p. We may define multiplying by $\frac{1}{d}$ on p-power torsion points by taking the preimage under multiplication by d since it is a bijection on such points. We examine the preimage of $-\frac{r}{s}v$ under ϕ_l and analyze which of those elements give u when multiplied by s.

Note $\phi_m(-\frac{r}{s}v) = -ru = 0$. So by Claim 3.5(b), the preimage of $-\frac{r}{s}v$ under ϕ_l is of the form $a + (\mathbb{Z}/p^q\mathbb{Z})^2$, where $a \in A[p^q]$.

Multiplying by s gives a p^{2k}-to-1 map on the following cosets:

$$a + (\mathbb{Z}/p^q\mathbb{Z})^2 \xrightarrow{\cdot s} sa + (\mathbb{Z}/p^{q-k}\mathbb{Z})^2.$$

We will now show that u is in the image of this map: The preimage of $-rv$ under ϕ_l in $A[p^q]$ is of the form $u + (\mathbb{Z}/p^q\mathbb{Z})^2$. The preimage of $-rv$ under ϕ_l that is p^{q-k}-torsion is thus of the form $u + (\mathbb{Z}/p^{q-k}\mathbb{Z})^2$. The elements of $sa + (\mathbb{Z}/p^{q-k}\mathbb{Z})^2$ are p^{q-k}-torsion and their image under ϕ_l is $-rv$. Thus, these sets are equal, implying $u \in sa + (\mathbb{Z}/p^{q-k}\mathbb{Z})^2$. In summary, there are p^{2k} elements $x \in A[p^q]$, where $\phi_l(x) = -\frac{r}{s}v = -ry$ and $sx = u$.

This shows that (3.3) is a surjective p^{4k}-to-1 map. Since multiplication by s decreases the order of the p-power torsion of an element by exactly p^k, by (3.4) we may conclude that $G_A(v) \cong (\mathbb{Z}/p^q\mathbb{Z})^4$.

Case 5: $\chi \neq 0$, $0 < k \leq j < q$ and $j + k < q$. In this case, p^{j+k} divides χ and no higher powers of p may divide χ.

By the argument in Case 4, we have

$$G_A(v) \cap (A \times \hat{A})[p^{j+k}] \cong (\mathbb{Z}/p^{j+k}\mathbb{Z})^4. \tag{3.5}$$

We will first extend our result for solutions of order up to p^{j+2k}. For convenience, define $t := \min\{q, j + 2k\}$.

Let

$$(u, v) \in G_A(v) \cap ((A \times \hat{A})[p^{t-k}] \setminus (A \times \hat{A})[p^j]).$$

We seek

$$(x, y) \in G_A(v) \cap (A \times \hat{A})[p^t]$$

so that $(sx, sy) = (u, v)$. First, we search for elements y, where $\phi_m(y) = u$ and $sy = v$; thus, we look at the preimage of u under ϕ_m and analyze which of those elements give v when multiplied by s.

If $\phi_l(u) = 0$, then the argument from Case 4 shows that there are p^{2k} elements $y \in \hat{A}[p^t]$, where $\phi_m(y) = u$ and $sy = v$.

If $\phi_l(u) \in \hat{A}[p^k] \setminus \{0\}$, then by Claim 3.5(c), the preimage of u under ϕ_m in $\hat{A}[p^t]$ is of the form $b + (\mathbb{Z}/p^{j+k}\mathbb{Z})^2$, where $b \in \hat{A}[p^t] \setminus \hat{A}[p^{j+k}]$.

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Note that $\phi_l(\phi_m(b)) = -rv$ and by Lemma 3.2, $-rsb = -(n+rs)b = -rv$. Multiplication by s gives a p^{2k}-to-1 map on the following cosets:

$$b + (\mathbb{Z}/p^{j+k}\mathbb{Z})^2 \xrightarrow{\cdot s} sb + (\mathbb{Z}/p^j\mathbb{Z})^2.$$

We will now show that v is in the image of this map.

The preimage of su under ϕ_m in $\hat{A}[p^{j+k}]$ is of the form $v + (\mathbb{Z}/p^{j+k}\mathbb{Z})^2$. The part of $v + (\mathbb{Z}/p^{j+k}\mathbb{Z})^2$ whose image under multiplication by $-r$ is $-rv$ is of the form $v+(\mathbb{Z}/p^j\mathbb{Z})^2$. Since ϕ_m maps $sb+(\mathbb{Z}/p^j\mathbb{Z})^2$ to su and multiplying this coset by $-r$ gives $-rv$, by counting elements, $sb + (\mathbb{Z}/p^j\mathbb{Z})^2 = v + (\mathbb{Z}/p^j\mathbb{Z})^2$. Hence, $v \in sb + (\mathbb{Z}/p^j\mathbb{Z})^2$. Thus, there are p^{2k} elements $y \in b + (\mathbb{Z}/p^{j+k}\mathbb{Z})^2$, where $\phi_m(y) = u$ and $sy = v$.

Now we search for elements x where $\phi_l(x) = -ry = -\frac{r}{s}v$ and $sx = u$. We examine the preimage of $-\frac{r}{s}v$ under ϕ_l and analyze which of those elements give u when multiplied by s.

If $\phi_m(-\frac{r}{s}v) = -ru = 0$, we may conclude using the arguments in Case 4. Otherwise, by Claim 3.5(d), the preimage of $-\frac{r}{s}v$ under ϕ_l in $A[p^t]$ is of the form $a + (\mathbb{Z}/p^{j+k}\mathbb{Z})^2$ where $a \in A[p^t] \setminus A[p^{j+k}]$. Note that $\phi_m(\phi_l(a)) = -ru$ and by Lemma 3.2, $-rsa = -ru$.

Multiplying by s gives a p^{2k}-to-1 map on cosets:

$$a + (\mathbb{Z}/p^{j+k}\mathbb{Z})^2 \xrightarrow{\cdot s} sa + (\mathbb{Z}/p^j\mathbb{Z})^2.$$

We will now show that u is in the image of this map. The preimage of $-rv$ under ϕ_l is of the form $u + (\mathbb{Z}/p^{j+k}\mathbb{Z})^2$. The part of $u + (\mathbb{Z}/p^{j+k}\mathbb{Z})^2$ whose image under multiplication by $-r$ is $-ru$ is of the form $u + (\mathbb{Z}/p^j\mathbb{Z})^2$. We have shown that ϕ_l maps $sa + (\mathbb{Z}/p^j\mathbb{Z})^2$ to $-rv$ and multiplying this coset by $-r$ gives $-ru$. By counting elements, we have the set equality $sa + (\mathbb{Z}/p^j\mathbb{Z})^2 = u + (\mathbb{Z}/p^j\mathbb{Z})^2$. Hence, $u \in sa + (\mathbb{Z}/p^j\mathbb{Z})^2$. Thus, there are p^{2k} elements $x \in a + (\mathbb{Z}/p^{j+k}\mathbb{Z})^2$, where $\phi_l(x) = -\frac{r}{s}v$ and $sx = u$. Thus, the following map is surjective and p^{4k}-to-1:

$$G_A(v) \cap (A \times \hat{A})[p^t] \xrightarrow{\cdot s} G_A(v) \cap (A \times \hat{A})[p^{t-k}]. \tag{3.6}$$

If $q \leq j + 2k$, we may now conclude, in combination with (3.5), that $G_A(v) \cong (\mathbb{Z}/p^q\mathbb{Z})^4$.

If $q > j + 2k$, (3.6) shows that $G_A(v) \cap (A \times \hat{A})[p^{j+2k}] \cong (\mathbb{Z}/p^{j+2k}\mathbb{Z})^4$. The above argument may be repeated for solutions of orders up to p^{j+3k} and then upward inductively to conclude that $G_A(v) \cong (\mathbb{Z}/p^q\mathbb{Z})^4$. □

Example 3.7. (a) For $K_2(A) \cong K_A(1, 0, -3)$, l and m are the trivial Néron-Severi classes (these are treated in general by Case 1 of the proof of Lemma 3.6), so $\phi_l(x) = 0$ and $\phi_m(y) = 0$. The equations (3.1) simplify to $0 = -y$ and $0 = -3x$, which recovers the fact that the group of symplectic automorphisms for $K_2(A)$ is generated by ι and translation by elements of $A[3]$ [8, Cor. 5(2)].

(b) In Sections 6–8, we consider fourfolds $K_A(v)$, where $v = (0, l, s)$ for l primitive and $\chi = 3$. If $s \equiv s' \bmod 3$, then $G_A(0, l, s) = G_A(0, l, s')$, leaving only three possible distinct groups of this form, which are described by a combination of Cases 2 and 3 of Lemma 3.6. Case 2 shows that $G_A(0, l, 1)$ and $G_A(0, l, 2)$, though in general distinct, each have one element $(x, y) \in (A \times \hat{A})[3]$ for every $y \in \hat{A}[3]$: for any $y \in \hat{A}[3]$, there is one $x \in \ker \phi_l$ so that $t_y^* M \simeq L_x \otimes M$. However, we see from Case 3 that $G_A(0, l, 0)$ is the product of $\ker \phi_l$ and $\ker \phi_m$.

The assumption in Theorem 3.1 that v is primitive is necessary for $G_A(v)$ to be isomorphic to $(\mathbb{Z}/n\mathbb{Z})^4$. In the case where $v = 2v_0$ for v_0 a primitive Mukai vector with $v_0^2 = 2$, which is used to construct O'Grady sixfolds, the solutions to the equations (3.1) are precisely of the form $(A \times \hat{A})[2] \cong (\mathbb{Z}/2\mathbb{Z})^8$, as is shown in [42, Lem. 5.1]. We generalize this result by extending Theorem 3.1 to find all solutions to (3.1) for any Mukai vector.

In the following result, we alter our hypotheses by naming the primitive vector of Setting 2.4 v_0 and considering a multiple of v_0.

Corollary 3.8. *Let $v = (r, l, s)$ be a Mukai vector on an abelian surface A so that $v = dv_0$, where $v_0 = (r_0, l_0, s_0)$ is primitive and $n := \frac{v_0^2}{2}$.*

Then, the group $G_A(v)$ of solutions $(x, y) \in A \times \hat{A}$ to the following equations

$$\phi_l(x) = -ry \quad \textit{and} \quad \phi_m(y) = sx \tag{3.7}$$

is isomorphic to $(\mathbb{Z}/dn\mathbb{Z})^4 \times (\mathbb{Z}/d\mathbb{Z})^4$.

Proof. Let m and m_0 be the respective Néron-Severi classes determined by Φ_P. We see from the definition of ϕ_l that $\phi_l = d \cdot \phi_{l_0}$ and likewise, $\phi_m = d \cdot \phi_{m_0}$: if we choose $(L_0, M_0) \in \mathrm{Pic}^{l_0}(A) \times \mathrm{Pic}^{m_0}(\hat{A})$ and $L := L_0^{\otimes d}$, $M := M_0^{\otimes d}$, then for any $x \in A$, we have

$$\phi_l(x) := t_x^* L \otimes L^{-1} = t_x^* L_0^{\otimes d} \otimes (L_0^{\otimes d})^{-1} = (t_x^* L_0 \otimes L_0^{-1})^{\otimes d} = d \cdot \phi_{l_0}(x).$$

Since ϕ_{l_0} and ϕ_{m_0} are group homomorphisms, we have, for any $(x, y) \in A \times \hat{A}$,

$$\phi_l(x) = \phi_{l_0}(dx) \quad \text{and} \quad \phi_m(y) = \phi_{m_0}(dy).$$

Thus, a pair $(x, y) \in A \times \hat{A}$ is a solution to (3.7) if and only if

$$\phi_{l_0}(dx) = -r_0 dy \quad \text{and} \quad \phi_{m_0}(dy) = s_0 dx,$$

or equivalently, (dx, dy) solves the equations (3.1) given by v_0. By Theorem 3.1, the set of solutions to the equations (3.1) given by v_0 is isomorphic to $(\mathbb{Z}/n\mathbb{Z})^4 \cong G_A(v_0)$. We may conclude by observing that the set of solutions to (3.7) is given by exactly the elements of $A \times \hat{A}$ that are in $G_A(v_0)$ after being multiplied by d. □

3.2. *Involutions and fixed loci*

Let A be an abelian surface over an arbitrary field k. If $K_A(v)$ is a fiber over symmetric line bundles, then ι^* gives an involution of $K_A(v)$. However, if symmetric bundles do not exist in the appropriate Néron-Severi classes over k, we show here how to define an involution κ to replace ι^*. For the remainder of the section, we fix a set of data as in Setting 2.4 and hence, fix a variety $K_A(v)$ over k.

We first give a lemma that will allow us to construct the involution κ.

Lemma 3.9. *Suppose we have an additional choice of line bundles $L' \in \mathrm{Pic}^l(A)$, $M' \in \mathrm{Pic}^m(\hat{A})$ over k. Let $K_A(v)' := \mathrm{alb}^{-1}(L', M')$. Then, there is an element $(a, b) \in (A \times \hat{A})(k)$ so that $L_b \otimes t_a^* \colon K_A(v) \to K_A(v)'$ is an isomorphism over k. It is unique up to composition with elements in $G_A(v)(k)$.*

Proof. Recall that for any $(x, y) \in A \times \hat{A}$, applying $L_y \otimes t_x^*$ to an element $\mathcal{F} \in K_A(v)$, we have

$$\det(L_y \otimes t_x^* \mathcal{F}) = L_y^{\otimes r} \otimes t_x^* L, \quad \text{and} \quad \det(\Phi_P(L_y \otimes t_x^* \mathcal{F})) = L_{-x}^{\otimes s} \otimes t_y^* M.$$

We also recall that the morphism $\phi\colon A \times \hat{A} \to A \times \hat{A}$ from (3.2) is an isogeny defined over k and sends sends (x, y) to

$$(t_y^* M \otimes M^{-1} \otimes L_{-x}^{\otimes s}, t_x^* L \otimes L^{-1} \otimes L_y^{\otimes r}).$$

The element (a, b) desired is precisely a preimage of $(L' \otimes L^{-1}, M' \otimes M^{-1}) \in (A \times \hat{A})(k)$ under ϕ. Finally, $L_b \otimes t_a^* : K_A(v) \to K_A(v)'$ is an isomorphism since it has an inverse $L_{-b} \otimes t_{-a}^*$. □

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Construction 3.10. Applying ι^* gives an isomorphism from $K_A(v)$ to $\mathrm{alb}^{-1}(\iota^*L, \iota^*M)$. By Lemma 3.9, there is an $(a, b) \in (A \times \hat{A})(k)$ such that $L_b \otimes t_a^*$ maps isomorphically from $\mathrm{alb}^{-1}(\iota^*L, \iota^*M)$ back to $K_A(v)$. So we have the following automorphism defined over k:

$$\begin{aligned} \kappa \colon K_A(v) &\to K_A(v) \\ \mathcal{F} &\mapsto L_b \otimes t_a^* \iota^* \mathcal{F}. \end{aligned}$$

Remark 3.11. We note that κ is an involution. More generally, for any $(c, d) \in A \times \hat{A}$, the morphism $L_d \otimes t_c^* \iota^*$ (which in general is an automorphism of $M(v)$ but perhaps not of $K(v)$) is an involution on $M_A(v)$. Indeed, $(\iota \circ t_c)^2 = \mathrm{id}$ on A and L_d is degree 0 and hence, fixed under pullback by translation; thus, for any $\mathcal{F} \in M_A(v)$, we have:

$$\begin{aligned} (L_d \otimes t_c^* \iota^*) \circ (L_d \otimes t_c^* \iota^*)(\mathcal{F}) &= L_d \otimes t_c^* \iota^* L_d \otimes t_c^* \iota^* t_c^* \iota^* (\mathcal{F}) \\ &= L_d \otimes t_c^* L_d^{-1} \otimes \mathcal{F} = L_d \otimes L_d^{-1} \otimes \mathcal{F} = \mathcal{F}. \end{aligned}$$

The following are thus involutions of $K_A(v)$, where $(x, y) \in G_A(v)(k)$:

$$\kappa_{(x,y)} := L_y \otimes t_x^* \kappa.$$

Under the simplifying assumption that L and M are symmetric, we may instead denote these involutions as

$$\iota_{(x,y)} := L_y \otimes t_x^* \iota^*.$$

Lemma 3.12. *Let $n := \frac{v^2}{2}$ be odd. Assume $k = \bar{k}$ and that L and M are symmetric, so ι^* is an involution on $K_A(v)$. Then, $\mathrm{Fix}(\iota_{(x,y)})$ is a translation of $\mathrm{Fix}(\iota_{(0,0)})$; that is, there exists $(u, w) \in G_A(v)$ so that*

$$\mathrm{Fix}(\iota_{(x,y)}) = L_w \otimes t_u^* (\mathrm{Fix}(\iota_{(0,0)})).$$

More generally, without the assumption that L and M are symmetric, there exists $(u, w) \in G_A(v)$ so that

$$\mathrm{Fix}(\kappa_{(x,y)}) = L_w \otimes t_u^* (\mathrm{Fix}(\kappa_{(0,0)})).$$

Proof. Let $\mathcal{F} \in K_A(v)$ be in the fixed locus of ι^*. Pick $(u, w) \in G_A(v)$ so that $2w = y$ and $2u = x$, which is possible since n is odd. For instance, when $\frac{v^2}{2} = 3$, $K_A(v)$ is a fourfold and the elements of $G_A(v)$ are all three-torsion, so we may choose $(-x, -y)$.

Then, $L_w \otimes t_u^* \mathcal{F}$ must be fixed by the involution

$$L_w \otimes t_u^* \iota^* (L_{-w} \otimes t_{-u}^*) = L_{2w} \otimes t_{2u}^* \iota^*.$$

The other direction of the containment is similar, as is the case with ι^* replaced by κ. □

Proposition 3.13. *Let $n := \frac{v^2}{2}$ be odd, assume $k = \bar{k}$ and let $L' \in \mathrm{Pic}^l(A)$ and $M' \in \mathrm{Pic}^m(\hat{A})$ be symmetric line bundles (cf. Lemma 2.6). Fix an involution κ as in Construction 3.10 on $K_A(v)$. Then, the fixed locus of κ in $K_A(v)$ is isomorphic to the fixed locus of $\iota^* = \iota_{(0,0)}$ in $K_A(v)'$.*

Proof. By Lemma 3.9, there is an $(x, y) \in A \times \hat{A}$ so that $L_y \otimes t_x^*$ gives an isomorphism from $K_A(v)$ to $K_A(v)'$. The composition

$$(L_{-y} \otimes t_{-x}^*) \circ \iota^* \circ (L_y \otimes t_x^*)$$

may be rearranged to $L_{-2y} \otimes t_{-2x}^* \iota^*$, where $L_{-2y} \otimes t_{-2x}^*$ gives an isomorphism from $\mathrm{alb}^{-1}(\iota^*L, \iota^*M)$ to $K_A(v)$. Thus, by the uniqueness statement of Lemma 3.9, there is an element $(u, w) \in G_A(v)$ for which

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$L_w \otimes t_u^* \circ L_{-2y} \otimes t_{-2x}^*$ is equal to the map $L_b \otimes t_a^*$ in the definition of κ. Then, $L_{-y} \otimes t_{-x}^* \operatorname{Fix}(\iota^*)$ is equal to $\operatorname{Fix}(\kappa_{(-u,-w)})$, which is isomorphic to $\operatorname{Fix}(\kappa)$ by Lemma 3.12. □

Finally, we give the following general result on the action of the Galois group on the geometric fixed loci.

Proposition 3.14. *Let k be an arbitrary field. For $(x, y) \in G_{A_{\bar{k}}}(v)$, the action of $\sigma \in \operatorname{Gal}(\bar{k}/k)$ sends the fixed locus of $\kappa_{(x,y)}$ in $K_A(v)_{\bar{k}}$ to the fixed locus of $\kappa_{(\sigma^{-1}x,\sigma^{-1}y)}$.*

Proof. Suppose $\mathcal{F}$ is fixed by $\kappa_{(x,y)}$. We use the equality $t_x \circ \sigma = \sigma \circ t_{\sigma^{-1}x}$ and the observation that σ commutes with ι and, moreover, κ, since κ is defined over the ground field k, to simplify the following equation:

$$\sigma^*\mathcal{F} \simeq \sigma^*(L_y \otimes t_x^*\kappa\mathcal{F}) \simeq \sigma^*L_y \otimes \sigma^*t_x^*\kappa\mathcal{F} \simeq \sigma^*L_y \otimes t_{\sigma^{-1}x}^*\kappa(\sigma^*\mathcal{F}).$$

Then, we have $\sigma^*L_y \simeq L_{\sigma^{-1}y}$, which we may verify using $\Phi_P\colon D(A) \to D(\hat{A})$:

$$\begin{aligned}\Phi_P(L_{-\sigma^{-1}y} \otimes \sigma^*L_y) &\simeq t_{-\sigma^{-1}y}^*\sigma^*k(-y)[-g] \simeq \sigma^*t_{-y}^*k(-y)[-g] \\ &\simeq \sigma^*k(0_{\hat{A}})[-g] \simeq k(0_{\hat{A}})[-g].\end{aligned}$$ □

3.3. Symplectic automorphisms and involutions

Let A be an abelian surface over $\mathbb{C}$. In the following lemma, we give a generalization of [8, Cor. 5(2)] to hyperkähler varieties $K_A(v)$ over $\mathbb{C}$.

Theorem 3.15. *Suppose we are in Setting 2.4 and we fix an involution κ as in Construction 3.10. Then, the kernel of*

$$\nu\colon \operatorname{Aut}(K_A(v)) \to \mathrm{O}(H^2(K_A(v), \mathbb{Z})) \tag{3.8}$$

consists of automorphisms of the form $L_y \otimes t_x^*$ and of the form $\kappa_{(x,y)} := L_y \otimes t_x^*\kappa$ for $(x, y) \in G_A(v)$. Thus, for any $(x, y) \in G_A(v)$, the automorphism $L_y \otimes t_x^*$ is symplectic. The $\kappa_{(x,y)}$ are symplectic involutions of $K_A(v)$, and when $\dim K_A(v) = 4$, these are all of the symplectic involutions.

Remark 3.16. While κ is not unique, by Lemma 3.9, the collection of elements in $\ker \nu$ is independent of the choice made in Construction 3.10.

Proof. Elements of $(x, y) \in A \times \hat{A}$ act on $M_A(v)$ via $L_y \otimes t_x^*$. Abelian varieties are path-connected, so the action of any element in $A \times \hat{A}$ is homotopic to the identity, which implies the induced action on $H^2(M_A(v), \mathbb{Z})$ is trivial. If $(x, y) \in G_A(v)$, then Theorem 3.1 shows that the action of $L_y \otimes t_x^*$ restricts to $K_A(v)$. By [56, Thm. 0.2(2)], the restriction map $H^2(M_A(v), \mathbb{Z}) \to H^2(K_A(v), \mathbb{Z})$ is a surjection. Therefore, $L_y \otimes t_x^*$ acts trivially on $H^2(K_A(v), \mathbb{Z})$ as well.

By [56, Thm. 0.2(2)], there is an isomorphism

$$H^2(K_A(v), \mathbb{Z}) \cong v^\perp,$$

where $v^\perp \subset H^{even}(A, \mathbb{Z})$ is the orthogonal complement to v under the Mukai pairing. Since ι^* acts by -1 on $H^1(A, \mathbb{Z})$, it acts trivially on $H^{even}(A, \mathbb{Z})$.

If we assume L and M are symmetric, ι^* is an automorphism of $K_A(v)$ and therefore must act trivially on $H^2(K_A(v), \mathbb{Z})$. If L and M are not both symmetric, since we are working over an algebraically closed field, we observe that κ is a composition of translation to an Albanese fiber over symmetric bundles, application of ι^* on that fiber and translation back (cf. proof of Proposition 3.13). Thus, κ will act trivially on $H^2(K_A(v), \mathbb{Z})$ as well.

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By the discussion above, $\ker \nu$ contains $2n^4$ elements, so by Theorem 3.1, we have identified all of them. The automorphisms in this kernel are clearly symplectic as the symplectic form generates part of $H^2(K_A(v), \mathbb{C})$.

For any nontrivial choice of $(x, y) \in G_A(v)$, $L_y \otimes t_x^*$ is not an involution, but by Section 3.2, $\kappa_{(x,y)}$ is an involution on $K_A(v)$.

Finally, suppose $\dim K_A(v) = 4$. By [31, Thm. 7.5(i)], all of the symplectic involutions of $K_A(v)$ act trivially on $H^2(K_A(v), \mathbb{Z})$. □

4. The middle cohomology of fourfolds $K_A(v)$

In this section, we work with data as in Setting 2.4 with the additional assumption that $v^2 = 6$, so $K_A(v)$ is a fourfold. We will prove results characterizing the middle cohomology of $K_A(v)$ when k has characteristic 0 in Section 4.1. We use these results to characterize the cohomology similarly when k has positive characteristic in Section 4.2 via a brief lifting argument.

4.1. Results in characteristic zero

Assume $K_A(v)$ is defined over an arbitrary field k of characteristic zero, so we may assume without loss of generality that $\bar{k} \hookrightarrow \mathbb{C}$. In this case, we can identify the Galois representations which make up the middle cohomology of $K_A(v)$. This will depend on understanding the fixed loci of $\kappa_{(x,y)}$ for $(x, y) \in G_{A_{\bar{k}}}(v)$.

Proposition 4.1. *Suppose $k = \bar{k}$. The fixed locus of any involution $\kappa_{(x,y)}$ for $(x, y) \in G_A(v)$ on a fourfold $K_A(v)$ consists of a K3 surface and 36 isolated points.*

Proof. First, suppose $k = \mathbb{C}$. Work of Hassett and Tschinkel [21] and Tarí [54] shows that the statement is true for $K_2(A)$. A discussion of the isolated fixed points in this case is given in Section 6.1.

Every hyperkähler fourfold $K_A(v)$ is deformation equivalent to $K_2(A)$ and by [21, Thm. 2.1], its group of symplectic involutions is also a deformation invariant. Thus, as in Kapfer and Menet [31, Thm. 7.5], the fixed loci are related by deformation as well, so the statement holds for $K_A(v)$.

Now let k be any algebraically closed field of characteristic zero. Since A is defined over k, we can assume without loss of generality that $k \hookrightarrow \mathbb{C}$. Let $K_A(v)_{\mathbb{C}} := K_A(v) \times_k \mathbb{C}$ and consider the Cartesian square

$$\begin{tikzcd} K_A(v)_{\mathbb{C}} \arrow[r, "\widetilde{\kappa}_{(x,y)}"] \arrow[d] & K_A(v)_{\mathbb{C}} \arrow[d] \\ K_A(v) \arrow[r, "\kappa_{(x,y)}"] & K_A(v), \end{tikzcd}$$

where $\widetilde{\kappa}_{(x,y)}$ is formed by replacing κ with its extension to $\mathbb{C}$, which we call $\tilde{\kappa}$. By Theorem 3.15, $\widetilde{\kappa}_{(x,y)}$ is a symplectic involution, and by the argument above, $\mathrm{Fix}(\widetilde{\kappa}_{(x,y)})$ is a K3 surface $Z := Z_{(x,y)} \subset K_A(v)_{\mathbb{C}}$ plus 36 isolated points.

By [14, Rmk. 3 following Thm 2.3],

$$\mathrm{Fix}(\widetilde{\kappa}_{(x,y)}) = \mathrm{Fix}(\kappa_{(x,y)}) \times_k \mathbb{C}.$$

This descent of the fixed-point locus means that $\mathrm{Fix}(\kappa_{(x,y)})$ consists of a surface $S := S_{(x,y)} \subset K_A(v)$ along with 36 k-points. We claim that S is a K3 surface. Indeed, we see via the valuative criterion of properness, using the fact that $\mathrm{Fix}(\kappa_{(x,y)})$ is a closed subscheme of $K_A(v)$, that $S \to \operatorname{Spec} k$ is proper. By flat base change, we have that $H^1(S, \mathcal{O}_S) \otimes \mathbb{C} \cong H^1(Z, \mathcal{O}_Z) = 0$, and $H^0(S, \omega_S) \otimes \mathbb{C} \cong H^0(Z, \omega_Z) = \mathbb{C}$,

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so ω_S has a nonvanishing global section and hence, is trivial. Finally, S is smooth by [11, Lem. 4.1], which completes the proof. □

See [30] for further discussion of these fixed-point loci in hyperkählers of Kummer type.

Let k now be arbitrary. Let $S_{(x,y)} \subset K_A(v)_{\bar{k}}$ be the K3 surface in $\mathrm{Fix}(\kappa_{(x,y)})$ and $s_{(x,y)} \in H^4_{\text{ét}}(K_A(v)_{\bar{k}}, \mathbb{Q}_\ell(2))$ the image of $[S_{(x,y)}] \in \mathrm{CH}^2 K_A(v)_{\bar{k}}$ under the cycle class map $\mathrm{CH}^2 K_A(v)_{\bar{k}} \to H^4_{\text{ét}}(K_A(v)_{\bar{k}}, \mathbb{Q}_\ell(2))$.

Lemma 4.2. *For $\sigma \in \mathrm{Gal}(\bar{k}/k)$, the induced action on the cycle classes $s_{(x,y)}$ for $(x,y) \in G_{A_{\bar{k}}}(v)$ is given by*

$$\sigma^* s_{(x,y)} = s_{(\sigma x, \sigma y)} \in H^4_{\acute{e}t}(K_A(v)_{\bar{k}}, \mathbb{Q}_\ell(2)).$$

Proof. By [40, Prop. 9.2], the cycle class map is Galois equivariant, so $\sigma^* s_{(x,y)}$ is the cycle class of $[\sigma^* S_{(x,y)}] \in \mathrm{CH}^2 K_A(v)_{\bar{k}}$. As in the proof of [40, Prop. 9.2], we have that $[\sigma^* S_{(x,y)}]$ is the preimage of $S_{(x,y)}$ under $\sigma^* \colon K_A(v)_{\bar{k}} \to K_A(v)_{\bar{k}}$. By Proposition 3.14, $(\sigma^*)^{-1}(S_{(x,y)}) = S_{(\sigma x, \sigma y)}$. Thus, we conclude that $\sigma^* s_{(x,y)} = s_{(\sigma x, \sigma y)}$, as desired. □

Definition 4.3. For a finite Galois module G, let $\mathbb{Q}_\ell[G]$ be the $\mathbb{Q}_\ell$-vector space with basis given by G, where the action of the Galois group on $\mathbb{Q}_\ell[G]$ is determined by the action on G: for $\sigma \in \mathrm{Gal}(\bar{k}/k)$ and $\sum_{g_i \in G} a_i g_i \in \mathbb{Q}_\ell[G]$,

$$\sigma \cdot \sum_{g_i \in G} a_i g_i = \sum_{g_i \in G} a_i (\sigma \cdot g_i).$$

We call $\mathbb{Q}_\ell[G]$ the *permutation representation*.

Recall that when k is not algebraically closed, the group $G_{A_{\bar{k}}}(v)$ naturally has the structure of a finite $\mathrm{Gal}(\bar{k}/k)$-module.

Theorem 4.4. *There is an isomorphism of Galois representations*

$$H^4_{\acute{e}t}(K_A(v)_{\bar{k}}, \mathbb{Q}_\ell(2)) \cong \mathrm{Sym}^2 H^2_{\acute{e}t}(K_A(v)_{\bar{k}}, \mathbb{Q}_\ell(1)) \oplus V,$$

where V is the 80-dimensional subrepresentation of $\mathbb{Q}_\ell[G_{A_{\bar{k}}}(v)]$ such that

$$\mathbb{Q}_\ell[G_{A_{\bar{k}}}(v)] \cong V \oplus \mathbb{Q}_\ell,$$

and the trivial representation $\mathbb{Q}_\ell$ is the span of $(0,0) \in G_{A_{\bar{k}}}(v)$.

Remark 4.5. As will be shown in Lemma 5.7, the action of the Galois group on $H^2_{\text{ét}}(K_A(v)_{\bar{k}}, \mathbb{Q}_\ell(1))$, and hence $\mathrm{Sym}^2 H^2_{\text{ét}}(K_A(v)_{\bar{k}}, \mathbb{Q}_\ell(1))$, is determined by the action on $H^2_{\text{ét}}(A_{\bar{k}}, \mathbb{Q}_\ell(1))$.

Proof. By Theorem 3.1, we have $3^4 = 81$ involutions

$$\begin{aligned} \kappa_{(x,y)} \colon K_A(v)_{\bar{k}} &\to K_A(v)_{\bar{k}} \\ \mathcal{F} &\mapsto L_y \otimes t_x^* \kappa \mathcal{F}, \end{aligned}$$

where $(x,y) \in G_{A_{\bar{k}}}(v)$.

As in the proof of Proposition 4.1, let $K_A(v)_{\mathbb{C}} := K_A(v) \times_k \mathbb{C}$ and $\widetilde{\kappa}_{(x,y)} \colon K_A(v)_{\mathbb{C}} \to K_A(v)_{\mathbb{C}}$ the base change of $\kappa_{(x,y)}$. By Proposition 4.1, $\mathrm{Fix}(\widetilde{\kappa}_{(x,y)})$ contains a K3 surface $Z_{(x,y)} \subset K_A(v)_{\mathbb{C}}$. This gives 81 distinct K3 surfaces in $K_A(v)_{\mathbb{C}}$, where the distinctness follows from [21, Thm. 2.1]. Via the cycle class map, these 81 surfaces give corresponding classes $z_{(x,y)} \in H^4(K_A(v)_{\mathbb{C}}, \mathbb{Q})$.

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Similarly, there are K3 surfaces $S_{(x,y)} \subset K_A(v)_{\bar{k}}$ and corresponding cohomology classes $s_{(x,y)} \in H^4_{\text{ét}}(K_A(v)_{\bar{k}}, \mathbb{Q}_\ell(2))$ such that $S_{(x,y)} \times_{\bar{k}} \mathbb{C} = Z_{(x,y)} \subset K_A(v)_{\mathbb{C}}$. Under the comparison and smooth base change isomorphisms

$$H^4(K_A(v)_{\mathbb{C}}, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell(2) \cong H^4_{\text{ét}}(K_A(v)_{\bar{k}}, \mathbb{Q}_\ell(2)),$$

the classes $z_{(x,y)}$ correspond to the classes $s_{(x,y)}$.

By [31, Thm. 7.5(ii)], the pair $(K_A(v)_{\mathbb{C}}, \widetilde{\kappa}_{(x,y)})$ is deformation equivalent to the pair $(K_2(A_{\mathbb{C}}), t_\tau \circ [-\mathrm{Id}]^{[[3]]})$ for some $\tau \in A_{\mathbb{C}}[3]$. In particular, these complex manifolds are diffeomorphic and so they have isomorphic cohomology rings. By [21, Prop. 4.3] (see also the discussion in [31, §6.4]), the $\mathbb{Q}_\ell$-span of $\{z_{(x,y)} - z_{(0,0)}\}_{(x,y)\in G_{A_{\mathbb{C}}}(v)}$ is an 80-dimensional vector space of $H^4(K_A(v)_{\mathbb{C}}, \mathbb{Q}_\ell(2))$ which is a direct sum complement to the subspace $\mathrm{Sym}^2 H^2(K_A(v)_{\mathbb{C}}, \mathbb{Q}_\ell(1))$.

Since the $s_{(x,y)}$ in $H^4(K_A(v)_{\bar{k}}, \mathbb{Q}_\ell(2))$ correspond to the $z_{(x,y)}$, it follows that

$$V := \mathrm{Span}_{\mathbb{Q}_\ell}\{s_{(x,y)} - s_{(0,0)}\}_{(x,y)\in G_{A_{\bar{k}}}(v)}$$

is an 80-dimensional subspace of $H^4_{\text{ét}}(K_A(v)_{\bar{k}}, \mathbb{Q}_\ell(2))$ which is a direct sum complement to $\mathrm{Sym}^2 H^2_{\text{ét}}(K_A(v)_{\bar{k}}, \mathbb{Q}_\ell(1))$.

By Lemma 4.2, we know that for $\sigma \in \mathrm{Gal}(\bar{k}/k)$,

$$\sigma^*(s_{(x,y)}) = s_{(\sigma x, \sigma y)}.$$

Thus, V is a Galois-invariant subspace of $H^4_{\text{ét}}(K_A(v)_{\bar{k}}, \mathbb{Q}_\ell(2))$. Noting that $\mathbb{Q}_\ell[G_{A_{\bar{k}}}(v)]$ is semisimple by Maschke's Theorem – the Galois action factors through a finite group representation determined by the finite extension of k over which $G_{A_{\bar{k}}}(v)$ is defined – and that $\sigma^*(s_{(0,0)}) = s_{(0,0)}$, this shows that V is the 80-dimensional subrepresentation of $\mathbb{Q}_\ell[G_{A_{\bar{k}}}(v)]$ such that

$$\mathbb{Q}_\ell[G_{A_{\bar{k}}}(v)] \cong V \oplus \mathbb{Q}_\ell,$$

where the trivial representation corresponds to $(0,0) \in G_{A_{\bar{k}}}(v)$. Hence, $H^4_{\text{ét}}(K_A(v)_{\bar{k}}, \mathbb{Q}_\ell(2))$ has the decomposition as stated. □

4.2. *Results in positive characteristic via lifting*

In this section we observe that, because Kummer varieties $K_A(v)$ defined over a field of positive characteristic lift to characteristic 0 [17], we may use Theorem 4.4 to give a similar description of the middle cohomology.

Proposition 4.6. *Suppose we have data as in Setting 2.4, where the base field k has characteristic $p > 0$. Then,*

$$H^4_{\acute{e}t}(K_A(v)_{\bar{k}}, \mathbb{Q}_\ell(2)) \cong \mathrm{Sym}^2 H^2_{\acute{e}t}(K_A(v)_{\bar{k}}, \mathbb{Q}_\ell(1)) \oplus V',$$

where V' is the 80-dimensional subrepresentation of $\mathbb{Q}_\ell[G_{A_{\bar{k}}}(v)]$ such that

$$\mathbb{Q}_\ell[G_{A_{\bar{k}}}(v)] \cong V' \oplus \mathbb{Q}_\ell,$$

and the trivial representation $\mathbb{Q}_\ell$ is the span of $(0,0) \in G_{A_{\bar{k}}}(v)$.

Proof. As explained in the proof of [17, Prop. 6.9], it is possible to lift $K_A(v)$ to characteristic 0 by lifting its defining data. That is, the data (A, v, H, L, M) defined over k has a lift $(\mathcal{A}, v_W, \mathcal{H}, \mathcal{L}, \mathcal{M})$ to a complete discrete valuation ring W of characteristic zero with residue field k and field of fractions $F := \mathrm{Frac}\, W$. Note that all of this lifting data can be recovered from a lift of (A, H, L). Indeed, lifting

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A automatically gives us a lift of $\hat{A}$, and lifting line bundles on $\hat{A}$ amounts to lifting their Néron–Severi class; a lift of the Néron–Severi class of M is given by the Néron–Severi class of $\det(\Phi_{\mathcal{P}}(\mathcal{L}))$. Call the specialization of v_W to the generic fiber v_F.

There is a surjection of Galois groups

$$\mathrm{Gal}(\bar{F}/F) \twoheadrightarrow \mathrm{Gal}(\bar{k}/k) \tag{4.1}$$

which is given by restricting automorphisms to the ring of integers of $\bar{F}$ and then passing to the quotient $\bar{k}$. By the smooth base change theorem [1, Exp. XVI, Corollaire 2.2], for $\ell \neq p$, there are isomorphisms

$$\begin{aligned} H^2_{\text{ét}}(K_{A_{\bar{F}}}(v_F), \mathbb{Q}_\ell(1)) &\cong H^2_{\text{ét}}(K_A(v)_{\bar{k}}, \mathbb{Q}_\ell(1)), \text{ and} \\ H^4_{\text{ét}}(K_{A_{\bar{F}}}(v_F), \mathbb{Q}_\ell(2)) &\cong H^4_{\text{ét}}(K_A(v)_{\bar{k}}, \mathbb{Q}_\ell(2)), \end{aligned} \tag{4.2}$$

which are equivariant with respect to the action of $\mathrm{Gal}(\bar{F}/F)$ on the left and $\mathrm{Gal}(\bar{k}/k)$ on the right, compatible with (4.1).

The isomorphisms of (4.2) are compatible with the ring structure on cohomology, so the isomorphism $H^4_{\text{ét}}(K_{A_{\bar{F}}}(v_F), \mathbb{Q}_\ell(2)) \cong H^4_{\text{ét}}(K_A(v)_{\bar{k}}, \mathbb{Q}_\ell(2))$ restricts to an isomorphism

$$\mathrm{Sym}^2 H^2_{\text{ét}}(K_{A_{\bar{F}}}(v), \mathbb{Q}_\ell(1)) \cong \mathrm{Sym}^2 H^2_{\text{ét}}(K_A(v)_{\bar{k}}, \mathbb{Q}_\ell(1)),$$

which is again compatible with the respective Galois group actions.

Let the following be the decomposition given by Theorem 4.4:

$$H^4_{\text{ét}}(K_{A_{\bar{F}}}(v_F), \mathbb{Q}_\ell(2)) \cong \mathrm{Sym}^2 H^2_{\text{ét}}(K_{A_{\bar{F}}}(v_F), \mathbb{Q}_\ell(1)) \oplus V,$$

and let $V' \subset H^4_{\text{ét}}(K_A(v)_{\bar{k}}, \mathbb{Q}_\ell(2))$ be the vector space complement to $\mathrm{Sym}^2 H^2_{\text{ét}}(K_A(v)_{\bar{k}}, \mathbb{Q}_\ell(1))$. Using the fact that V is a $\mathrm{Gal}(\bar{F}/F)$ subrepresentation of $H^4_{\text{ét}}(K_{A_{\bar{F}}}(v_F), \mathbb{Q}_\ell(2))$, we conclude that

$$H^4_{\text{ét}}(K_A(v)_{\bar{k}}, \mathbb{Q}_\ell(2)) \cong \mathrm{Sym}^2 H^2_{\text{ét}}(K_A(v)_{\bar{k}}, \mathbb{Q}_\ell(1)) \oplus V'$$

as $\mathrm{Gal}(\bar{k}/k)$ representations. In particular, there is an isomorphism $V \cong V'$ which is equivariant with respect to the action of $\mathrm{Gal}(\bar{F}/F)$ on the left and $\mathrm{Gal}(\bar{k}/k)$ on the right, again compatible with (4.1).

The subgroup $G_{A_{\bar{F}}}(v_F) \leqslant (A_{\bar{F}} \times \hat{A}_{\bar{F}})[3]$ is given by equations (3.1) determined by v_F, which is part of our lifted data. Thus, since the action of $\mathrm{Gal}(\bar{F}/F)$ on V is given by $G_{A_{\bar{F}}}(v)$, the action of $\mathrm{Gal}(\bar{k}/k)$ on V' must be the one determined analogously by $G_{A_{\bar{k}}}(v)$. □

5. Relation to derived equivalences

There are a number of results related to derived equivalences of smooth, projective symplectic varieties. For example, if X and Y are derived equivalent smooth complex projective surfaces, then $D(\mathrm{Hilb}^n X) \cong D(\mathrm{Hilb}^n Y)$ [51, Prop. 8]. If X and Y are K3 surfaces, then the converse holds, and if two moduli spaces of stable sheaves $M_X(v)$ and $M_Y(v')$ are derived equivalent, then X and Y are also derived equivalent [5, Cor. 9.7]. If X and Y are derived equivalent K3 surfaces over any field k, then the ℓ-adic étale cohomologies of any moduli $M_X(v)$ and $M_Y(v')$ of equal dimension are isomorphic as $\mathrm{Gal}(\bar{k}/k)$ representations [15, Thm. 2]. However, it is still an open question when such moduli are derived equivalent.

In the direction of symplectic varieties of Kummer type, complex abelian surfaces A and B are derived equivalent if and only if there is an isomorphism $K_1(A) \cong K_1(B)$ between their associated Kummer K3 surfaces [24, 53]. This result has also been proved for abelian surfaces over fields of odd characteristic [36]; the relation between Kummer surfaces and twisted derived equivalence of abelian surfaces has been examined in [35, Thm. 6.5.2]. While A and $\hat{A}$ are always derived equivalent over their field of definition, it is not known exactly when there is a derived equivalence between the generalized

Kummer fourfolds $K_2(A)$ and $K_2(\hat{A})$. Recently, it was shown that, over an algebraically closed field of characteristic zero, they are derived equivalent when A has a polarization of exponent coprime to 3 [37, Theorem 1].

Given these results, we ask the following two questions, which we examine in Sections 5.1 and 5.2, respectively.

Question 1. Suppose we have a derived equivalence of abelian surfaces $D^b(A) \cong D^b(B)$. How do the groups $G_A(v)$ introduced in Section 3 interact with the Rouquier isomorphism $A \times \hat{A} \simeq B \times \hat{B}$?

Question 2. Under what conditions are irreducible symplectic fourfolds of Kummer type derived equivalent?

Throughout this section, we will assume we are working with data as in Setting 2.4 and that all varieties $K_A(v)$ are an Albanese fiber over symmetric line bundles.

5.1. Compatibility with the Rouquier isomorphism

Proposition 5.1 (Rouquier, cf. [26, Prop. 9.45]). *Let A and B be abelian varieties and $F\colon D(A) \to D(B)$ a derived equivalence. There is an isomorphism $f\colon A \times \hat{A} \to B \times \hat{B}$, called the Rouquier isomorphism, which maps $(a, \alpha) \in A \times \hat{A}$ to the unique element $(b, \beta) \in B \times \hat{B}$ so that the following diagram commutes:*

$$\begin{tikzcd} D(A) \arrow[r, "F"] \arrow[d, "L_\alpha \otimes t_a^*"'] & D(B) \arrow[d, "L_\beta \otimes t_b^*"] \\ D(A) \arrow[r, "F"] & D(B). \end{tikzcd} \tag{5.1}$$

The following proposition gives some results addressing Question 1.

Proposition 5.2. *Let A and B be abelian surfaces over a field k, and let $v = (r, l, s) \in N(A)$ and $v' = (r', l', s') \in N(B)$.*

Let $F\colon D(A) \to D(B)$ be a derived equivalence such that $F(v) = v'$. Then, the base change of the Rouquier isomorphism to the algebraic closure $\bar{k}$ restricts to a group scheme isomorphism

$$f_{\bar{k}}\colon G_{A_{\bar{k}}}(v) \xrightarrow{\sim} G_{B_{\bar{k}}}(v') \tag{5.2}$$

under any of the following conditions:

(a) *For any elements $\mathcal{F}, \mathcal{G} \in M_A(v)$ such that* $\mathrm{alb}(\mathcal{F}) = \mathrm{alb}(\mathcal{G})$, *we have* $\det(F(\mathcal{F})) = \det(F(\mathcal{G}))$ *and* $\det(\Phi_P \circ F(\mathcal{F})) = \det(\Phi_P \circ F(\mathcal{G}))$;
(b) *F is a stability-preserving Fourier–Mukai transform; that is, if $E \in M_A(v)$, then $F(E)$ is in $M_B(v')$; or*
(c) *$k = \mathbb{C}$ and $\frac{v^2}{2} = 3$ (i.e., $K_A(v)$ is a fourfold).*

We note that the isomorphism (5.2) implies that the actions of $\mathrm{Gal}(\bar{k}/k)$ on $G_{A_{\bar{k}}}(v)$ and $G_{B_{\bar{k}}}(v')$ are isomorphic.

Proof. Let $(a, \alpha) \in G_{A_{\bar{k}}}(v)$. By Remark 3.3, to prove that $(b, \beta) := f_{\bar{k}}(a, \alpha) \in G_{B_{\bar{k}}}(v')$, it suffices to produce an element $\mathcal{H} \in D(B)$, where $v(\mathcal{H}) = v'$, $\det(\mathcal{H}) = \det(L_\beta \otimes t_b^*\mathcal{H})$, and $\det(\Phi_P(\mathcal{H})) = \det(\Phi_P(L_\beta \otimes t_b^*\mathcal{H}))$.

Under condition (a), for any $\mathcal{F} \in M_A(v)$, we may take $\mathcal{H} := F(\mathcal{F})$. In this case, we have $L_\beta \otimes t_b^*\mathcal{H} = F(L_\alpha \otimes t_a^*\mathcal{F})$. Since

$$\det(\mathcal{F}) = \det(L_\alpha \otimes t_a^*\mathcal{F}) \quad \text{and} \quad \det(\Phi_P(\mathcal{F})) = \det(\Phi_P(L_\alpha \otimes t_a^*\mathcal{F})),$$

condition (a) allows us to conclude that $\mathcal{H}$ has the needed property.

Under condition (b), F restricts to an isomorphism $M_A(v) \to M_B(v')$ and by the universal property of the Albanese morphism, there is a commutative diagram as follows:

$$\begin{tikzcd} M_A(v) \arrow[r, "F"] \arrow[d, "\mathrm{alb}"] & M_B(v') \arrow[d, "\mathrm{alb}"] \\ \mathrm{Pic}^l(A) \times \mathrm{Pic}^\lambda(\hat{A}) \arrow[r, dashed, "\exists"] & \mathrm{Pic}^{l'}(A) \times \mathrm{Pic}^{\lambda'}(\hat{A}) \end{tikzcd}$$

Thus, F satisfies condition (a).

By [21, Prop. 4.3] if $K_A(v)$ is a fourfold, the intersection of the fixed loci of κ and $(L_\alpha \otimes t_a^*)\kappa$ acting on $K_A(r,l,s)$ is nonempty. For instance, in $K_2(A)$, the intersection of $\mathrm{Fix}(\kappa)$ and $\mathrm{Fix}(\kappa_{(\tau,0)})$ where $\tau \in A[3]$ (cf. Lemma 3.12) contains $(0,\tau,-\tau)$.

Let $\mathcal{G}$ be an element in this intersection. It is thus fixed by $L_\alpha \otimes t_a^*$. Following the diagram above, we see that $\mathcal{H} := F(\mathcal{G})$ is fixed by $L_\beta \otimes t_b^*$ and thus, F satisfies the needed condition. □

Remark 5.3. The barrier to a proof of Proposition 5.2 under more general conditions is that it is not known that a general Fourier–Mukai equivalence will respect the Albanese morphism acting on $M_A(v)$.

The proof of Proposition 5.2 under condition (c) hinges on the selection of an element fixed by the automorphisms from Theorem 3.1. We anticipate that analogous results are available for higher-dimensional varieties of Kummer type. For instance, in $K_{n-1}(A)$, the intersection between $\mathrm{Fix}(\iota_{(0,0)})$ and $\mathrm{Fix}(\iota_{(\tau,0)})$ where $\tau \in A[n]$ contains $(0,\tau,2\tau,\ldots,(n-1)\tau)$.

Example 5.4. (a) For any abelian surface A we have the Fourier–Mukai equivalence $\Phi_P : D(A) \to D(\hat{A})$. For any Mukai vector v on A, condition (a) of Proposition 5.2 is satisfied for $F = \Phi_P$ since $\Phi_P \circ \Phi_P = \iota^* \circ [-2]$. If $v := (r,l,s)$, then $v' := F(v) = (s,m,r)$ [56, Lemma 3.1], and $G_{A_{\bar{k}}}(v)$ and $G_{\hat{A}_{\bar{k}}}(v')$ are very closely related via the canonical identification between an abelian surface and the dual of its dual. By Theorem 3.1, the elements in $G_{A_{\bar{k}}}(v)$ satisfy the equations shown in (3.1) and the elements of $G_{\hat{A}_{\bar{k}}}(v')$ satisfy the equations

$$\phi_m(y) = -sx, \quad \phi_l(x) = ry \quad \text{for } (y,x) \in \hat{A} \times \hat{\hat{A}}$$

Thus, $(x,y) \in G_{A_{\bar{k}}}(v)$ if and only if $(-y,x) \in G_{\hat{A}_{\bar{k}}}(v')$.

(b) Let A be an abelian surface defined over a field k of characteristic 0 with $\mathrm{NS}(A) = \mathbb{Z}l$ and $l^2 = 2n$. By [20, Lem. 3.6], the Fourier–Mukai equivalence $L \otimes^{\mathbb{L}} (-) : D(A) \to D(A)$ satisfies condition (a) of Proposition 5.2; in fact, $M_A(1,0,-n) \cong M_A(1,l,0)$. Moreover, by [56, Prop. 3.5], applying the Fourier–Mukai transform Φ_P followed by a shift $[-1]$ gives an isomorphism $M_A(1,l,0) \cong M_{\hat{A}}(0,-\hat{l},-1)$, where $\hat{l}$ is the Néron-Severi class of $\Phi_P(1,l,0)$. If l is an ample generator of $\mathrm{NS}(A)$, then $-\hat{l}$ is an ample generator of $\mathrm{NS}(\hat{A})$.

The shift functor $[1]$ acts on Mukai vectors by multiplication by -1, and in general, $G_A(v) = G_A(-v)$. Thus, there are isomorphisms of group schemes

$$G_A(1,0,-n) \cong G_A(1,l,0) \cong G_{\hat{A}}(0,-\hat{l},-1),$$

though as discussed in Remark 3.7, the groups $G_A(1,0,-n)$ and $G_A(1,l,0)$ are distinct subgroups of $(A \times \hat{A})[n]$.

5.2. *Derived equivalence of fourfolds of Kummer type*

The following result provides some information on Question 2 and allows us to produce an example where two such varieties over a number field k are not derived equivalent over k.

Proposition 5.5. *Let A and B be isogenous abelian surfaces over a finitely generated field k of characteristic 0. Let v and v' be Mukai vectors with $v^2 = v'^2 = 6$, so that $K_A(v)$ and $K_B(v')$ are fourfolds. If $K_A(v)$ and $K_B(v')$ are derived equivalent over k, then $\mathbb{Q}_\ell[G_{A_{\bar{k}}}(v)]$ and $\mathbb{Q}_\ell[G_{B_{\bar{k}}}(v')]$ are isomorphic as $\mathrm{Gal}(\bar{k}/k)$-representations.*

We begin with a lemma about the orthogonal complement to v in the Mukai lattice.

Lemma 5.6. *Let A be an abelian surface over a field k and v a Mukai vector with $v^2 \geq 2$. Let $v^\perp \subset \tilde{H}(A_{\bar{k}}, \mathbb{Q}_\ell)$ be the orthogonal complement to v under the Mukai pairing. Then, there is a Galois equivariant isomorphism $v^\perp \cong H^2_{ét}(A_{\bar{k}}, \mathbb{Q}_\ell(1)) \oplus \mathbb{Q}_\ell$.*

Proof. Let $w := (1, 0, -n)$ for $n := \frac{v^2}{2} \geq 1$, and note that

$$w^\perp = H^2_{\text{ét}}(A_{\bar{k}}, \mathbb{Q}_\ell(1)) \oplus \mathbb{Q}_\ell\langle(1, 0, n)\rangle.$$

We will show that $v^\perp \cong w^\perp$. For any $y \in \tilde{H}(A_{\bar{k}}, \mathbb{Q}_\ell)$ with $y^2 \neq 0$, let reflection through y be given by

$$x \mapsto x - \frac{2\langle x, y\rangle}{y^2} y.$$

Observe that $(v - w)^2 \neq 0$ or $(v + w)^2 \neq 0$, and so reflection through $v - w$ or $v + w$ gives an isometry $\tilde{H}(A_{\bar{k}}, \mathbb{Q}_\ell) \xrightarrow{\sim} \tilde{H}(A_{\bar{k}}, \mathbb{Q}_\ell)$ which sends v to $\pm w$. Thus, the isometry restricts to a Galois equivariant isomorphism $v^\perp \xrightarrow{\sim} w^\perp$. □

Lemma 5.7. *Let A be an abelian surface over a field k and v a Mukai vector with $v^2 \geq 6$. Then, there is a Galois equivariant isomorphism*

$$H^2_{ét}(K_A(v)_{\bar{k}}, \mathbb{Q}_\ell(1)) \cong H^2_{ét}(A_{\bar{k}}, \mathbb{Q}_\ell(1)) \oplus \mathbb{Q}_\ell.$$

Proof. By [56, Thm. 0.2(2)], along with the comparison theorem for singular and étale cohomology and the smooth base change theorem, we have a Galois equivariant isomorphism $H^2_{\text{ét}}(K_A(v)_{\bar{k}}, \mathbb{Q}_\ell(1)) \cong v^\perp$ (In fact, this isomorphism exists over $\mathbb{Z}_\ell$, while the isomorphism of Lemma 5.6 may only exist over $\mathbb{Q}_\ell$). This combined with Lemma 5.6 gives the result. □

Proof of Proposition 5.5. Suppose that $K_A(v)$ and $K_B(v')$ are derived equivalent, so they have isomorphic sums of even cohomologies after Tate twists [22, Lem. 3.1]: $\widetilde{H}(K_A(v)_{\bar{k}}, \mathbb{Q}_\ell) \cong \widetilde{H}(K_B(v')_{\bar{k}}, \mathbb{Q}_\ell)$. We know that the zeroth and top cohomologies of $K_A(v)$ and $K_B(v')$ are trivial Galois representations, and Lemma 5.7 gives that

$$H^2_{\text{ét}}(K_A(v)_{\bar{k}}, \mathbb{Q}_\ell(1)) \cong H^2_{\text{ét}}(A_{\bar{k}}, \mathbb{Q}_\ell(1)) \oplus \mathbb{Q}_\ell.$$

By Theorem 4.4 and Poincáre duality (cf. [23]), it follows that there is an isomorphism of Galois modules

$$\widetilde{H}(K_A(v), \mathbb{Q}_\ell) \cong \mathbb{Q}_\ell^{\oplus 4} \oplus H^2_{\text{ét}}(A_{\bar{k}}, \mathbb{Q}_\ell(1))^{\oplus 2} \oplus \mathrm{Sym}^2 H^2_{\text{ét}}(K_A(v)_{\bar{k}}, \mathbb{Q}_\ell(1)) \oplus V_A,$$

where $V_A := V$ from Theorem 4.4. There is a similar isomorphism for $\widetilde{H}(K_B(v'), \mathbb{Q}_\ell)$ involving V_B. We will check that these representations are semisimple, so that we can reduce to a comparison of V_A and V_B.

By [13, Thm. 3] and its extension to finitely generated fields of characteristic 0 in [57, Thm. 4.3], $H^2_{\text{ét}}(A_{\bar{k}}, \mathbb{Q}_\ell)$ is a semisimple representation, and thus so is $\mathrm{Sym}^2 H^2_{\text{ét}}(K_A(v)_{\bar{k}}, \mathbb{Q}_\ell(1))$. The $\mathrm{Gal}(\bar{k}/k)$-representation $\mathbb{Q}_\ell[G_{A_{\bar{k}}}(v)]$ factors through a finite group representation, determined by the finite extension of k over which $G_{A_{\bar{k}}}(v)$ is defined, and so by Maschke's theorem, it is also semisimple. Thus, the representation $\widetilde{H}(K_A(v), \mathbb{Q}_\ell)$ is semisimple. The same also holds for $\widetilde{H}(K_B(v'), \mathbb{Q}_\ell)$, so applying Schur's Lemma, this allows us to cancel isomorphic representations in the direct sums for $\widetilde{H}(K_A(v), \mathbb{Q}_\ell)$ and for $\widetilde{H}(K_B(v'), \mathbb{Q}_\ell)$. Since A and B are isogenous, there is an isomorphism $H^2_{\text{ét}}(A_{\bar{k}}, \mathbb{Q}_\ell) \cong H^2_{\text{ét}}(B_{\bar{k}}, \mathbb{Q}_\ell)$, so along with the observations above, we are reduced to an isomorphism $V_A \cong V_B$. This extends to an isomorphism $\mathbb{Q}_\ell[G_{A_{\bar{k}}}(v)] \cong \mathbb{Q}_\ell[G_{B_{\bar{k}}}(v')]$, as desired. □

We use this result to give a negative answer to Question 2 in the case of generalized Kummer varieties $K_2(A)$ and $K_2(\hat{A})$.

Corollary 5.8. *For an abelian surface A defined over a number field k for which* $\mathbb{Q}_\ell[A[3]]$ *and* $\mathbb{Q}_\ell[\hat{A}[3]]$ *are not isomorphic as Galois modules over k,* $K_2(A) \cong K_A(1,0,-3)$ *and* $K_2(\hat{A}) \cong K_A(3,0,-1)$ *are not derived equivalent over k.*

Proof. We have $G_{A_{\bar{k}}}(1,0,-3) = A[3]$ and by the discussion in Example 5.4(a), $G_{A_{\bar{k}}}(3,0,-1) = G_{\hat{A}_{\bar{k}}}(1,0,-3) = \hat{A}[3]$. The result then follows by Proposition 5.5. □

In [16] the authors exhibit an abelian surface A defined over a number field k where $\mathbb{Q}_\ell[A[3]]$ and $\mathbb{Q}_\ell[\hat{A}[3]]$ are not isomorphic as Galois modules over k.

Remark 5.9. If A is an abelian surface as in the proof of Corollary 5.8, any derived equivalence between $K_2(A)$ and $K_2(\hat{A})$ would have to be defined over a field larger than k. Moreover, the kernel of such a derived equivalence could not be constructed out of only universal bundles, since such bundles would naturally be defined over k, and the derived equivalence would descend.

Remark 5.10. The argument in Corollary 5.8 cannot be used to rule out derived equivalences between $K_2(A)$ and $K_2(\hat{A})$ in many contexts; for instance, it does not work when A is principally polarized, since such a polarization would give an isomorphism between $A[3]$ and $\hat{A}[3]$.

Proposition 5.5 also holds for Kummer varieties over fields of positive characteristic that satisfy the hypotheses of Proposition 4.6; Tate's theorem gives the needed semisimplicity result [55]. However, over a finite field in general, Tate's isogeny theorem implies there is an isomorphism between the Tate modules $T_\ell A$ and $T_\ell \hat{A}$. Thus, it would not be possible to use the approach of Corollary 5.8 to rule out a derived equivalence between $K_2(A)$ and $K_2(\hat{A})$ if A was defined over a finite field.

6. A $(1,3)$-polarized example: Lagrangian fibrations

In this and the following sections, we consider an extended example where we work over $\mathbb{C}$.

Let (A, L) be a polarized abelian surface where L is symmetric, $\mathrm{NS}(A) = \mathbb{Z}l$ for $l := c_1(L)$ and $l^2 = 6$, so L is a $(1,3)$-polarization (see Claim 3.4). Let $K_A(0,l,s)$ be as in Setting 2.4 and assume $M \in \mathrm{Pic}^m(\hat{A})$ is also symmetric. We will see below that the spaces $K_A(0,l,s)$ are fibered over $\mathbb{P}^2$ in Jacobians of irreducible genus 4 curves, and while they can be identified fiberwise as s varies, their global geometry differs: the discriminant of the Beauville–Bogomolov–Fujiki form on $\mathrm{Pic}(K_A(0,l,s))$ changes, so these moduli spaces are not, in general, birational.

We consider the fixed locus of $K_A(0,l,s)$ under the action of ι^*, which we refer to as $\mathrm{Fix}(\iota^*)$. By Lemma 3.12, the fixed locus of any symplectic involution on $K_A(0,l,s)$ is a translation of $\mathrm{Fix}(\iota^*)$. The moduli space $K_A(0,l,s)$ parametrizes rank 1 stable sheaves, or equivalently, rank 1 torsion-free sheaves, supported on irreducible curves in A. When the supporting curves are smooth, these sheaves are line bundles on the curves, but we also encounter curves with nodal singularities, in which case the space of rank 1 torsion-free sheaves naturally compactifies the space of line bundles.

In this section, we give necessary background and show that there is a natural fibration of $K_A(0,l,s)$ in abelian surfaces such that $\mathrm{Fix}(\iota^*) \subset K_A(0,l,s)$ contains an elliptically fibered K3 surface. In Section 7, we will analyze the singular fibers of this K3 surface, and in Section 8 we will analyze the isolated points of the fixed locus.

For comparison, we first give a description of $\mathrm{Fix}(\iota^*)$ in $K_A(1,0,-3)$ here.

6.1. $\mathrm{Fix}(\iota^*)$ *for* $K_2(A)$

The points in $K_2(A) \cong K_A(1,0,-3)$ consist of 0-dimensional length 3 subschemes of A for which the support sums to 0. It was shown in [21, Thm. 4.4] that $\mathrm{Fix}(\iota^*)$ contains the Kummer K3 surface

$$\overline{\{(a_1,a_2,a_3) \mid a_1 = 0, a_2 = -a_3, a_2 \neq 0\}} \tag{6.1}$$

as well as a unique isolated point supported at the identity element 0.

Any length 3 subscheme in $\mathrm{Fix}(\iota^*)$ containing a point $a \in A$ in its support that is not fixed by ι^* must be of the form $(0,a,-a)$, which is in the Kummer K3 surface described above. Thus, the remaining isolated points in $\mathrm{Fix}(\iota^*)$ found by Tarí [54] must consist of triples of three distinct points of $A[2] \cong (\mathbb{Z}/2\mathbb{Z})^4$ that sum to 0. The identity element cannot be contained in such a triple. Once we have chosen two of the points, the third is forced, and length 3 subschemes are unordered, so we have

$$\tfrac{1}{3}\tbinom{15}{2} = 35$$

such isolated points.

6.2. Stable sheaves and compactifications of the Jacobian

Let $\mathrm{Pic}^d(C)$ be the set of degree d line bundles on any curve C. We write Pic^d_C for the Picard scheme of degree d on a curve C, and we use $\overline{\mathrm{Pic}}^d_C$ to denote the moduli scheme parametrizing rank 1 degree d torsion-free sheaves on the mildly singular curves C that arise in this paper, which are all Gorenstein and, moreover, have planar singularities.

If C is elliptic, $\mathrm{Pic}^d_C \cong C$ for any d, and this fact has some generalizations to compactified Jacobians of singular genus 1 curves that we will find useful.

Proposition 6.1 [33, §3, p. 14],[12, Ex. 39]. *Let C be a genus 1 reduced curve that is irreducible and nodal. Then, $\overline{\mathrm{Pic}}^d_C \cong C$ for any d.*

The Abel map [33, Def. 1.0.5] and a generalization of it for compactified Jacobians of Gorenstein curves is useful to our arguments. We use the development of this map given by Kass in [33], though we do not need the full power of Kass's theory.

Generalized divisors on C are nonzero subsheaves of the sheaf of the total quotient ring of C, $I_D \subset \mathcal{K}$, that are coherent $\mathcal{O}_C$-modules. These divisors generalize Cartier divisors, which they coincide with when I_D is a line bundle. An *effective generalized divisor* on C is a 0-dimensional closed subscheme $Z \subset C$, meaning the following generalization of the Abel map continues to have the intuitive quality of sending points to corresponding elements in $\overline{\mathrm{Pic}}^{-d}_C$ [33, Def. 5.0.7], [2, Thm. 8.5]:

$$\begin{aligned} \alpha\colon \mathrm{Hilb}^d_C &\to \overline{\mathrm{Pic}}^{-d}_C \\ [D] &\mapsto I_D \end{aligned} \tag{6.2}$$

When the degree d is greater than or equal to the arithmetic genus g, this map is surjective and generically has fibers isomorphic to $\mathbb{P}^{d-g}$. If D is an effective generalized divisor, $\alpha^{-1}([I_D])$ is the complete linear system $|D|$. If $g = d$, the map is generically injective. The locus where α is non-injective in this case is *the exceptional locus* C^1_d, which consists of divisors D whose image under the canonical map lies on a hyperplane. Such divisors D, D' are linearly equivalent if there are canonical divisors K, K' such that $K - D = K' - D'$.

Related to Proposition 6.1, this generalized Abel map is an isomorphism when C is a nodal genus-1 curve.

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6.3. The Lagrangian fibration of $K_A(0,l,s)$

Since $l^2 = 6$, $K_A(0,l,s)$ is 4-dimensional, and since $\mathrm{NS}(A) = \mathbb{Z}l$ for $l := c_1(L)$, the curves $C \in |L|$ are irreducible. Hence, all rank 1 torsion-free sheaves are stable. Thus, $K_A(0,l,s)$ parametrizes rank 1 torsion-free sheaves on irreducible curves $C \subset A$ where $C \in |L|$, which are generically line bundles. Curves in this linear system have arithmetic genus 4 and by Riemann-Roch, the line bundles parametrized by $K_A(0,l,s)$ have degree $d := s+3$.

We see that $h^0(A,L) = 3$ and $h^1(A,L) = h^2(A,L) = 0$. Thus, there is a map sending elements of $K_A(0,l,s)$ to their supports in the linear system $|L|$:

$$\begin{aligned} f\colon K_A(0,l,s) &\to \mathbb{P}^2 \cong |L| \\ \mathcal{F} &\mapsto \mathrm{supp}(\mathcal{F}) \end{aligned} \tag{6.3}$$

Lemma 6.2. *Let $C \in |L|$ and $h_C\colon C \hookrightarrow A$ be the natural inclusion. The fiber of f over $C \in |L|$ is the fiber over M of the following surjective morphism:*

$$\begin{aligned} \varphi_C\colon \overline{\mathrm{Pic}}^d_C &\to \mathrm{Pic}^m_{\hat{A}} \\ \mathcal{F} &\mapsto \det(\Phi_P(h_{C*}\mathcal{F})). \end{aligned} \tag{6.4}$$

This fiber $f^{-1}(C) = \varphi_C^{-1}(M)$ is a translation of the fiber of the following map over 0_A:

$$\begin{aligned} j_C : \overline{\mathrm{Pic}}^0_C &\to A \\ I_D &\to -\Sigma D, \end{aligned} \tag{6.5}$$

where ΣD is the sum of points in the divisor D using the group law on A.

When C is smooth, j_C is the morphism given by the universal property of the Jacobian, which sends a line bundle (e.g., $\mathcal{O}(p-q)$) to $p-q$.

Remark 6.3. We will analyze the ι^*-invariant portion of $\varphi_C^{-1}(M)$ in later results. This lemma shows that we may reduce to analyzing the ι^*-invariant portion of the fiber of j_C over 0_A, which we call $\ker j_C$, somewhat abusing notation in the singular case.

Proof. Recall that $K_A(0,l,s)$ is the fiber of the Albanese map over (L,M):

$$\mathrm{alb}\colon M_A(0,l,s) \to \mathrm{Pic}^l_A \times \mathrm{Pic}^m_{\hat{A}}. \tag{6.6}$$

We consider the interaction of alb with f. Let $\mathcal{C}$ be the tautological family of curves in $|L|$. We may identify the fiber of (6.6) over $\{L\} \times \mathrm{Pic}^m_{\hat{A}}$ with the relative compactified Jacobian $\overline{\mathrm{Pic}}^d_{\mathcal{C}/\mathbb{P}^2}$, which has a map to supports $g : \overline{\mathrm{Pic}}^d_{\mathcal{C}/\mathbb{P}^2} \to |L|$. Thus, there is an inclusion $K_A(0,l,s) \hookrightarrow \overline{\mathrm{Pic}}^d_{\mathcal{C}/\mathbb{P}^2}$ making the following diagram commute:

$$\begin{tikzcd} K_A(0,l,s) \arrow[r, hook] \arrow[dr, "f"'] & \overline{\mathrm{Pic}}^d_{\mathcal{C}/\mathbb{P}^2} \arrow[d, "g"] \\ & |L| \cong \mathbb{P}^2. \end{tikzcd} \tag{6.7}$$

For any curve $C \in |L|$, the fiber of g over C is $\overline{\mathrm{Pic}}^d_C$, which is isomorphic to Pic^d_C if C is smooth. The morphism φ_C given in the statement of the lemma is the restriction of the Albanese morphism (6.6) on $M_A(0,l,s)$ to $\overline{\mathrm{Pic}}^d_C$. Using (6.7), we see the fiber of f over C is equal to the fiber of φ_C over M.

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Let $\mathcal{L}$ be a line bundle on C and p a point in C. As in [52, §17.2], applying $\det(\Phi_P(h_{C*}-))$ to the short exact sequence

$$0 \to \mathcal{L} \to \mathcal{L} \otimes \mathcal{O}(p) \to k(p) \to 0$$

implies

$$\varphi_C(\mathcal{L} \otimes \mathcal{O}(p)) = \varphi_C(\mathcal{L}) \otimes P_p,$$

where P_p is the line bundle on $\hat{A}$ corresponding to $p \in C \subset A$. Moreover, for any divisor D on C, we have

$$\varphi_C(\mathcal{L} \otimes \mathcal{O}(D)) = \varphi_C(\mathcal{L}) \otimes P_{\Sigma D},$$

where $P_{\Sigma D}$ is the line bundle on $\hat{A}$ corresponding to the point on A that comes from summing D using the group law on A. If C is singular, this argument may be extended to ideal sheaves of generalized divisors D. Thus, φ_C is a translation of the morphism j_C of (6.5) by an element of $\overline{\mathrm{Pic}}^d(C)$.

If C is smooth, the map induced by applying the universal property of the Jacobian to the inclusion $C \hookrightarrow A$ is surjective [10]; thus, φ_C is surjective as well.

Alternately, to prove φ_C is surjective for smooth curves C, we may observe that φ_C is equivariant under the action of $\mathrm{Pic}^0(A)$ and the action of $\mathrm{Pic}^0(A)$ on $\mathrm{Pic}^m \hat{A}$ is transitive. For singular C, if we restrict φ_C to $\mathrm{Pic}^d_C \subset \overline{\mathrm{Pic}}^d_C$, the same argument holds and so φ_C is surjective. □

In [20], Gulbrandsen shows that the map $f\colon K_A(0,l,-1) \to \mathbb{P}^2$ is a Lagrangian fibration. There is a similar Lagrangian fibration of $K_A(0,l,s)$ for any choice of s.

Proposition 6.4. *For any s, the map $f\colon K(0,l,s) \to \mathbb{P}^2$ is a Lagrangian fibration.*

Proof. By [39, Thm. 1], it suffices to prove that f is surjective and its fibers are connected. By Lemma 6.2, the fiber of f over $C \in |L|$ is the fiber of φ_C over M, which is nonempty since φ_C is surjective. Thus, f is surjective. By [10, Lem. 2.6], the fibers of f over smooth curves are connected. By considering the Stein factorization of f, we conclude that f has connected fibers. □

6.4. *The Lagrangian fibration restricted to* $\mathrm{Fix}(\iota^*)$

Since $K(0,l,s)$ is fibered over $|L|$, we begin by analyzing the action of ι^* on $|L|$.

The restriction of the Weil pairing $\langle -, \phi_L(-)\rangle$ on points in A to $A[2]$ yields a quadratic form $q_L\colon A[2] \to \mu_2$. Since $\ker(\phi_L) \cong (\mathbb{Z}/3\mathbb{Z})^2$ (see Claim 3.4), it contains only the trivial element of $A[2]$. Hence, q_L is nondegenerate. Whether q_L is even or odd as a quadratic form (cf. [52, p. 63], [9, §3]) determines several facts about the action of ι^* on $|L|$.

Proposition 6.5. *The action of ι^* on $H^0(A,L)$ decomposes into eigenspaces $H^0(A,L)_+$ and $H^0(A,L)_-$ with eigenvalues ± 1. Furthermore,*

$$\dim(H^0(A,L)_+) = \begin{cases} 2 & \text{if } q_L \text{ is even} \\ 1 & \text{if } q_L \text{ is odd} \end{cases} \qquad \dim(H^0(A,L)_-) = \begin{cases} 1 & \text{if } q_L \text{ is even} \\ 2 & \text{if } q_L \text{ is odd.} \end{cases}$$

We call the 1-dimensional and 2-dimensional eigenspaces, respectively,

$$V_{\mathrm{hyp}} \quad \text{and} \quad V_{\mathrm{ell}}.$$

For generic A, any curve $C \in \mathbb{P}V_{\mathrm{hyp}}$ is smooth and hyperelliptic, and there are 10 points in $A[2]$ though which it passes. If q_L is even, then 0_A is among these 10 points. The remaining 6 points in $A[2]$ are the base locus of $\mathbb{P}V_{\mathrm{ell}}$. If q_L is odd, then 0_A is among these 6 points.

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The space V_{hyp} was named for the fact that the curves in it are hyperelliptic. The name V_{ell} was chosen because, by Riemann–Hurwitz, quotients C/ι of smooth curves $C \in \mathbb{P}V_{\mathrm{ell}}$ are elliptic.

Proof. Calculations on the dimensions of $H^0(A, L)_{\pm}$ and the number of points through which these curves pass have been carried out in [9, §3], [10, §3] and [48]. See [7, Ch. 4] and [52, Ch. 13] for further details. $\square$

By Proposition 4.1, $\mathrm{Fix}(\iota^*)$ consists of a K3 surface and 36 isolated points. Here, we study the geometry of the K3 surface.

Proposition 6.6. *The K3 surface in* $\mathrm{Fix}(\iota^*)$ *is elliptically fibered.*

Proof. By Proposition 6.5,

$$|L|^{\iota^*} = \mathbb{P}V_{\mathrm{ell}} \sqcup \mathbb{P}V_{\mathrm{hyp}} \cong \mathbb{P}^1 \sqcup \mathbb{P}^0,$$

and thus, $\mathrm{Fix}(\iota^*)$ is fibered over $\mathbb{P}^1 \sqcup \mathbb{P}^0$.

By Lemma 6.2, the fiber of $K_A(0, l, s)$ over C is the fiber of φ_C over M. Let $C \in |L|^{\iota^*}$ be smooth. By Remark 6.3, to determine the dimension of the ι^*-invariant parts of this fiber, we examine the eigenvalues of the action of ι^* on the tangent space of $\ker j_C$.

We have a short exact sequence on tangent spaces

$$0 \to T_0 \ker j_C \to T_0 \operatorname{Pic}^0_C \to T_0 A \to 0.$$

The tangent space $T_0 A$ is $H^1(A, \mathcal{O}_A)$, and it is 2-dimensional with ι^* acting as multiplication by -1. The tangent space $T_0 \operatorname{Pic}^0_C$ is $H^1(C, \mathcal{O}_C) \cong H^0(C, \omega_C)^*$, which is 4-dimensional. On the other hand, tensoring the short exact sequence

$$0 \to \mathcal{O}_A(-C) \to \mathcal{O}_A \to \mathcal{O}_C \to 0$$

with $L \cong \mathcal{O}_A(C)$ gives

$$0 \to \mathcal{O}_A \to L \to \mathcal{O}_C(C) \to 0.$$

By adjunction, $\mathcal{O}_C(C) \cong \omega_C$, so we have the following long exact sequence:

$$0 \to H^0(A, \mathcal{O}_A) \to H^0(A, L) \to H^0(C, \omega_C) \to H^1(A, \mathcal{O}_A) \to 0.$$

The map $H^0(A, \mathcal{O}_A) \to H^0(A, L)$ sends the generator of the 1-dimensional space $H^0(A, \mathcal{O}_A)$ to $[C]$. Putting all of this together, the eigenvalues and dimensions of eigenspaces of ι^* acting on the tangent space of $\ker j_C$ are equal to those of ι^* acting on $H^0(A, L)/[C]$.

Suppose $C \in \mathbb{P}V_{\mathrm{hyp}}$. Then, by Proposition 6.5, the eigenvalues of ι^* acting on $H^0(A, L)/[C]$ are both the same: if q_L is even, they are both $+1$, and if q_L is odd, they are both -1. In each case, these eigenvalues are different from the eigenvalue of the action of ι^* on $[C]$. If, instead, $C \in \mathbb{P}V_{\mathrm{ell}}$, then for q_L even or odd, the eigenvalues of ι^* acting on $H^0(A, L)/[C]$ are $+1$ and -1.

The tangent space of the fiber of $\mathrm{Fix}(\iota^*)$ over $C \in \mathbb{P}V_{\mathrm{hyp}} \sqcup \mathbb{P}V_{\mathrm{ell}}$ is isomorphic to the eigenspace of ι^* acting on $T_0 \ker j_C$ with the same eigenvalue as the action of ι^* on $[C]$. Thus, $\mathrm{Fix}(\iota^*)$ has 0-dimensional fibers over $\mathbb{P}V_{\mathrm{hyp}}$ and generically 1-dimensional fibers over $\mathbb{P}V_{\mathrm{ell}} \cong \mathbb{P}^1$. For any $C \in \mathbb{P}V_{\mathrm{ell}}$ that is smooth, the fiber of j_C over 0_A is 2-dimensional and so must be an abelian surface. Since ι^* acts with two different eigenvalues on the tangent space of $\ker j_C$, it must be, up to isogeny, the product of two elliptic curves.

Generically, curves $C \in \mathbb{P}V_{\mathrm{ell}}$ are smooth, and as mentioned above, C/ι is an elliptic curve. Since the quotient map $C \to C/\iota$ is a ramified cyclic double cover mapping between smooth varieties, pullback induces an inclusion $\operatorname{Pic}^0(C/\iota) \hookrightarrow \operatorname{Pic}^0(C)$. We may represent any point in the image as $\mathcal{O}(x + \iota(x))$ for some $x \in C$. Such line bundles are in $\ker j_C$. Similarly, there is an inclusion of $\operatorname{Pic}^d_{C/\iota}$ into $(\varphi_C^{-1}(M))^{\iota^*}$,

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and by the tangent space calculation, we see that generically these elliptic curves $\mathrm{Pic}^d_{C/\iota} \cong C/\iota$ are the 1-dimensional part of the fiber of f over C. □

In the case of $K_A(0,l,-1)$, we are able to give the following refinement by a different argument.

Proposition 6.7. *The fixed locus of ι^* on $K_A(0,l,-1)$ consists of the Kummer K3 surface $K_1(A) \cong K_1(\hat{A})$ and 36 isolated points.*

Proof. Hassett and Tschinkel [21] and Tarí [54] showed that the fixed locus of a symplectic involution on $K_2(A)$ consists of the Kummer K3 surface $K_1(A)$ and 36 additional isolated points.

As discussed in Example 5.4(b), a series of derived equivalences compatible with ι^* gives an isomorphism $K_A(0,l,-1) \cong K_{\hat{A}}(1,0,-3)$. Hence, the K3 surface in the fixed locus of ι^* acting on $K_A(0,l,-1)$ is isomorphic to $K_1(\hat{A})$, which is isomorphic over $\mathbb{C}$ to $K_1(A)$ [24, 53]. □

7. A $(1,3)$-polarized example: Singular fibers of an elliptic K3

The proof of Proposition 6.6 shows that the fibration $K_A(0,l,s) \to |L|$ restricts to a fibration $\mathrm{Fix}(\iota^*) \to \mathbb{P}V_{\mathrm{ell}}$, and when $C \in \mathbb{P}V_{\mathrm{ell}}$ is smooth, the fiber of $\mathrm{Fix}(\iota^*)$ over C is isomorphic to $\mathrm{Pic}^d_{C/\iota}$. It remains to examine the fibers in $\mathrm{Fix}(\iota^*)$ over curves in $\mathbb{P}V_{\mathrm{ell}}$ that are singular. We show below that the singular fibers are the same as the singular fibers of a natural elliptic fibration of the Kummer K3 of A, which we now describe.

In [48], Naruki analyzes an elliptic fibration of Kummer K3 surfaces that are constructed from $(1,3)$-polarized abelian surfaces. He uses the linear system $\mathbb{P}V_{\mathrm{ell}}$ of Proposition 6.5 to induce a linear system we will call W on $K_1(A)$, which yields an elliptic fibration $K_1(A) \to \mathbb{P}^1$ whose fibers are generically C/ι for $C \in \mathbb{P}V_{\mathrm{ell}}$. Since $C \in \mathbb{P}V_{\mathrm{ell}}$ must have arithmetic genus 4 and pass through at least 6 points in $A[2]$, Riemann–Hurwitz shows that if C/ι is a smooth elliptic curve, then C must be smooth as well and pass through exactly 6 points in $A[2]$.

Proposition 7.1 (Naruki [48, §4]). *Under a genericity assumption on A [48, p. 224, (GA)], the linear system W has:*

(i) *Four singular fibers of type I_1.*
(ii) *Ten singular fibers of type I_2. There is one fiber of this type for each point of $A[2]$ that is not in the base locus of $\mathbb{P}V_{\mathrm{ell}}$. The line in $K_1(A)$ that is the blow up of this point is contained in the fiber.*

We show that the same is true for $\mathrm{Fix}(\iota^*)$:

Theorem 7.2. *Let A be an abelian surface satisfying the hypotheses at the beginning of the section such that the singular fibers of W consist of four fibers of type I_1 and ten fibers of type I_2, as in Proposition 7.1. Then, for any s, $\mathrm{Fix}(\iota^*) \subset K_A(0,l,s)$ contains an elliptically fibered K3 whose singular fibers are of the same type.*

Proof. We split the proof into two parts. In Proposition 7.3 below, we show that there are 4 fibers of type I_1. In Proposition 7.4, we show that there are 10 fibers of type I_2.

For topological reasons, this must be all of the one-dimensional locus of $\mathrm{Fix}(\iota^*) \subset K_A(0,l,s)$. Indeed, the 4 singular fibers of type I_1 and 10 singular fibers of type I_2 account for the fact that the topological Euler number of a K3 surface is 24 [27, Rmk. 11.1.12]. □

Proposition 7.3. *Let $C \in \mathbb{P}V_{\mathrm{ell}}$ be a curve inducing a genus 1 singular curve C/ι of type I_1 in W. Then, $\overline{\mathrm{Pic}}^0_{C/\iota} \cong \overline{\mathrm{Pic}}^d_{C/\iota}$ is a singular curve of type I_1 and includes into $(\ker j_C)^{\iota^*}$.*

Proof. By assumption, the curve C/ι is of type I_1 and hence has arithmetic genus 1 with one nodal singularity. Applying the Riemann–Hurwitz formula for singular curves [18, (1.2)] to the double cover $C \to C/\iota$, we see the arithmetic genus 4 curve C has geometric genus 2 with 6 ramification points, so it must have two singular points that are exchanged by ι. We call these points x and ιx and then write $[x, \iota x]$ for the singular point of C/ι.

Consider the induced map on the normalizations of these curves: $C^\nu \to (C/\iota)^\nu$. Since this is a ramified double cover of curves, the pullback map $\mathrm{Pic}^0((C/\iota)^\nu) \to \mathrm{Pic}^0(C^\nu)$ is an inclusion. We have the following map between short exact sequences of groups:

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{C}^* \arrow[r] \arrow[d, hook] & \mathrm{Pic}^0(C/\iota) \arrow[r] \arrow[d] & \mathrm{Pic}^0((C/\iota)^\nu) \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & \mathbb{C}^* \oplus \mathbb{C}^* \arrow[r] & \mathrm{Pic}^0(C) \arrow[r] & \mathrm{Pic}^0(C^\nu) \arrow[r] & 0
\end{tikzcd}$$

The elements of $\mathbb{C}^*$ correspond to all possible choices for identifying the two fibers over a given node. The vertical maps are pullbacks along quotient maps, and $\mathbb{C}^* \to \mathbb{C}^* \oplus \mathbb{C}^*$ is the diagonal map, which corresponds to a choice of gluing a line bundle at the node on C/ι getting mapped to the same choice of gluing at each of the nodes on C. By the five lemma, the map $\mathrm{Pic}^0(C/\iota) \to \mathrm{Pic}^0(C)$ is an injection.

The Abel map (see Section 6.2) shows the points in $\overline{\mathrm{Pic}}^{-1}_{C/\iota}$ correspond to points on the curve C/ι, and all its elements are line bundles except for the sheaf corresponding to the singular point of C/ι.

The pullback map $\overline{\mathrm{Pic}}^{-1}_{C/\iota} \to \overline{\mathrm{Pic}}^{-2}_{C}$ sends $\mathcal{L}([x,\iota x])$ to $\mathcal{L}([x]+[\iota x])$, which is also not a line bundle and maps to 0 under φ_C. Thus, this pullback map is an injection. The sheaf $\mathcal{L}([x,\iota x])$ is fixed by ι^*. We may choose an isomorphism $\overline{\mathrm{Pic}}^{-1}_{C/\iota} \cong \overline{\mathrm{Pic}}^{0}_{C/\iota}$ compatible with ι^* to see that $\overline{\mathrm{Pic}}^{0}_{C/\iota}$ includes into $(\ker j_C)^{\iota^*}$.

By Proposition 6.1, $\overline{\mathrm{Pic}}^{0}_{C/\iota}$ is a singular curve of type I_1. □

Proposition 7.4. *Let $C \in \mathbb{P}V_{\mathrm{ell}}$ be a curve inducing a genus 1 singular curve X of type I_2 in W. Then, $(\ker j_C)^{\iota^*}$ contains a curve of type I_2.*

Proof. By the discussion in [48], the curve X in the linear system W that corresponds to C is the intersection of a line and a conic. The line in X is the blowup of a point $q \in A[2]$ that is one of the 10 such not in the base locus of $\mathbb{P}V_{\mathrm{ell}}$. The curve C thus has a node at q. The normalization $f\colon C^\nu \to C$ inherits an action of ι, and the quotient C^ν/ι is the conic contained in X. Thus, C^ν is hyperelliptic and as a double cover of C^ν/ι, it is ramified at 8 points, consisting of the six points $p_1, \ldots, p_6$ in the base locus of $\mathbb{P}V_{\mathrm{ell}}$ and the two points above q (named q_1, q_2). By Riemann–Hurwitz, C^ν has genus 3. Thus, C has arithmetic genus 4 and geometric genus 3, so the node at p is its unique singularity.

Via Altman and Kleiman's presentation schemes [3], we have the following description of $\overline{\mathrm{Pic}}^0_C$ (cf. [32, §3.3]). Pullback by the normalization map $f : C^\nu \to C$ gives the short exact sequence on Picard groups

$$0 \to \mathbb{C}^* \to \mathrm{Pic}^0(C) \to \mathrm{Pic}^0(C^\nu) \to 0, \tag{7.1}$$

where, again, the elements of $\mathbb{C}^*$ correspond to all possible choices for identifying the two fibers over q. The presentation scheme of f gives a $\mathbb{P}^1$-bundle $\pi\colon P \to \mathrm{Pic}^0_{C^\nu}$, where the fiber over a point $I' \in \mathrm{Pic}^0_{C^\nu}$ is given by presentations of I'; that is, short exact sequences of sheaves on C of the following form:

$$0 \to I \to f_*I' \to k(q) \to 0.$$

It follows that $I \in \overline{\mathrm{Pic}}^0_C$, so there is a natural morphism $\kappa\colon P \to \overline{\mathrm{Pic}}^0_C$, which is an isomorphism when restricted to the preimage of $\mathrm{Pic}^0_C \subset \overline{\mathrm{Pic}}^0_C$. For each $I' \in \mathrm{Pic}^0_{C^\nu}$, there is a $\mathbb{C}^* \subset \mathbb{P}^1 = \pi^{-1}(I')$, exactly the $\mathbb{C}^*$ of (7.1), which gets mapped injectively under κ into Pic^0_C.

Furthermore, there is a closed embedding $\varepsilon'\colon\ \mathrm{Pic}^0_{C^\nu} \times \{q_1, q_2\} \hookrightarrow P$, which sends a pair (I', q_i) to the presentation

$$0 \to f_*I'(-q_i) \to f_*I' \to f_*(I'|_{q_i}) \to 0.$$

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This gives the description of the rest of the $\mathbb{P}^1$-fiber of π over a point $I' \in \mathrm{Pic}^0_{C^\nu}$: these are the two points compactifying the $\mathbb{C}^*$ described above. Thus, to complete the description of $\overline{\mathrm{Pic}}^0_C$, it remains to describe κ restricted to $\varepsilon'(\mathrm{Pic}^0_{C^\nu} \times \{q_1, q_2\})$.

Here, κ is 2-to-1, but does not just trivially glue the two copies of $\mathrm{Pic}^0_{C^\nu}$ together. Rather, they are glued with a twist:

$$\kappa\varepsilon'(I', q_1) = \kappa\varepsilon'(I'(q_1 - q_2), q_2).$$

Since C^ν is hyperelliptic, $2q_1 \sim_{\text{lin}} 2q_2$ and $\mathcal{O}(q_1 - q_2)$ is 2-torsion in $\mathrm{Pic}^0_{C^\nu}$, which further implies that

$$\kappa\varepsilon'(I'(q_1 - q_2), q_1) = \kappa\varepsilon'(I', q_2).$$

With this observation in hand, we now describe the one-dimensional component of the locus of $\overline{\mathrm{Pic}}^0_C$ that is in $(\ker j_C)^{\iota^*}$. While we are working in $\overline{\mathrm{Pic}}^0_C$, we will instead consider the fiber over 0_A of φ_C. By abuse of notation, we will also call this $\ker \varphi_C$.

Claim 7.5. *The locus of* $\overline{\mathrm{Pic}}^0_C$ *that is both fixed by* ι^* *and is in* $\ker \varphi_C$ *contains*

$$\kappa\Big(\pi^{-1}(\mathcal{O}_{C^\nu}) \cup \pi^{-1}(\mathcal{O}_{C^\nu}(q_1 - q_2))\Big),$$

which is two copies of $\mathbb{P}^1$ *intersecting at two points (i.e., a singular curve of type* I_2*).*

Proof. First, we observe that if $I' \in \mathrm{Pic}^0_{C^\nu}$ is fixed by ι^*, and

$$0 \to I \to f_*I' \to k(q) \to 0$$

is a presentation of I', then I is also fixed by ι^*. Indeed, by push-pull, we know that $f_*\iota^*I' \cong \iota^* f_*I'$. We have the short exact sequence

$$0 \to \iota^*I \to \iota^* f_*I' \to \iota^*k(q) \to 0,$$

so if I' is fixed by ι^*, then $I \cong \iota^*I$. Thus, if I' is fixed, then the whole $\mathbb{P}^1$-fiber in P is pointwise fixed as well. Note also that, given a short exact sequence $0 \to I \to f_*I' \to k(q) \to 0$, if $I \in \ker \varphi_C$, then by the discussion in the proof of Lemma 6.2 about the behavior of φ_C in short exact sequences, so are all the other possible I giving presentations of I'.

Since there is a short exact sequence

$$0 \to \mathcal{O}_C \to f_*\mathcal{O}_{C^\nu} \to k(q) \to 0$$

and $\mathcal{O}_C \in \ker \varphi_C$, we know any other kernels of presentations of $f_*\mathcal{O}_{C^\nu}$ will as well. We also have that $\mathcal{O}_{C^\nu}$ is fixed by ι^*, so it follows that $\kappa(\pi^{-1}(\mathcal{O}_{C^\nu}))$ is both fixed by ι^* and in $\ker \varphi_C$.

The same holds for $\kappa(\pi^{-1}(\mathcal{O}_{C^\nu}(q_1 - q_2))$: since q_1 and q_2 are fixed by ι, $\mathcal{O}_{C^\nu}(q_1 - q_2)$ is fixed by ι^*, and we will show that any presentation is sent to 0 by φ_C. For a presentation

$$0 \to I \to f_*\mathcal{O}_{C^\nu}(q_1 - q_2) \to k(q) \to 0,$$

applying the inclusion $h_C : C \hookrightarrow A$ and Φ_P, we have

$$\det \Phi_P(h_{C*}I) \otimes \mathcal{P}_q \cong \det \Phi_P(h_{C*}f_*\mathcal{O}_{C^\nu}(q_1 - q_2)).$$

There is also a presentation

$$0 \to f_*\mathcal{O}_{C^\nu}(-q_1) \to f_*\mathcal{O}_{C^\nu}(q_1 - q_2) \to k(q) \to 0, \tag{7.2}$$

which gives

$$\det \Phi_P(h_{C*} f_* \mathcal{O}_{C^\nu}(-q_1)) \otimes \mathcal{P}_q \cong \det \Phi_P(h_{C*} f_* \mathcal{O}_{C^\nu}(q_1 - q_2)),$$

and hence,

$$\det \Phi_P(h_{C*} I) \cong \det \Phi_P(h_{C*} f_* \mathcal{O}_{C^\nu}(-q_1)).$$

On the other hand, there is a presentation

$$0 \to f_* \mathcal{O}_{C^\nu}(-q_1) \to f_* \mathcal{O}_{C^\nu} \to k(q) \to 0, \tag{7.3}$$

so $f_* \mathcal{O}_{C^\nu}(-q_1) \in \kappa(\pi^{-1}(\mathcal{O}_{C^\nu}))$, which we showed above is in $\ker \varphi_C$. Thus, the same is true for I.

It remains to show that these two $\mathbb{P}^1$'s in $\overline{\mathrm{Pic}}^0_C$ are glued together at two points. But this follows from the description of $\overline{\mathrm{Pic}}^0_C$, since

$$\kappa\varepsilon'(\mathcal{O}_{C^\nu}, q_1) = \kappa\varepsilon'(\mathcal{O}_{C^\nu}(q_1 - q_2), q_2)$$

and

$$\kappa\varepsilon'(\mathcal{O}_{C^\nu}(q_1 - q_2), q_1) = \kappa\varepsilon'(\mathcal{O}_{C^\nu}, q_2). \qquad \square$$

While above we work in degree 0, we can twist by a degree d line bundle on C to get the description in $\overline{\mathrm{Pic}}^d_C$. This completes the proof of Proposition 7.4. $\square$

Remark 7.6. It is interesting to consider $(\ker j_C)^{\iota^*}$ in Propostion 7.4 from the point of view of the Abel map. We consider the case where $d = -4$. The fixed locus $(\ker j_C)^{\iota^*} \subseteq \overline{\mathrm{Pic}}^{-4}_C$ contains all divisors of the form $-(x + \iota x + y + \iota y)$ for $x, y \in C$, but the information from the Abel map alone does not make clear which of these divisors get identified under linear equivalence in $\overline{\mathrm{Pic}}^{-4}_C$. We will show that these divisors are all contained in the curve from Proposition 7.4. We may choose an isomorphism $\overline{\mathrm{Pic}}^0_C \cong \overline{\mathrm{Pic}}^{-4}_C$ by subtracting four copies of a 2-torsion point $p \in C$ not at the node. Let $p' \in C^\nu$ be preimage of p under the normalization map. Since C^ν is hyperelliptic, $\mathcal{O}_{C^\nu}(-4p') \cong \mathcal{O}_{C^\nu}(-x' - \iota x' - y' - \iota y')$, where x' and y' are the preimages of x and y in C^ν. There is a presentation

$$0 \to \mathcal{O}_C(-x - \iota x - y - \iota y) \to (f_* \mathcal{O}_{C^\nu})(-4p) \to k(q) \to 0,$$

so these divisors all lie in the $\mathbb{P}^1$ corresponding to the twist of $\kappa(\pi^{-1}(\mathcal{O}_{C^\nu}))$.

The divisors $-(x + \iota x + y + \iota y)$ correspond to a two-dimensional family in Hilb^4_C, but their image in $\overline{\mathrm{Pic}}^{-4}_C$ is at most 1-dimensional, so they must lie in the exceptional locus of the Abel map. Since C is not hyperelliptic, the canonical morphism gives a closed immersion into $\mathbb{P}^3$ and divisors in the exceptional locus are those that lie on a hyperplane in $\mathbb{P}^3$; we see there must be an interaction of these planes with the action of ι, but the particulars of it are not immediately clear.

It would also be nice to have a description of the elements in $\overline{\mathrm{Pic}}^{-4}_C$ contained in the other copy of $\mathbb{P}^1$ in $(\ker j_C)^{\iota^*}$. Since the canonical bundle ω_C is fixed by ι^* and $f^*\omega_C \cong \omega_{C^\nu} \otimes \mathcal{O}_{C^\nu}(q_1 + q_2) \cong \mathcal{O}_{C^\nu}(4p' + q_1 + q_2)$, line bundles on C which fit into presentations with middle term $f_*\big(f^*\omega_C^{-1} \otimes \mathcal{O}_{C^\nu}(x' + \iota x')\big)$ lie in the $\mathbb{P}^1$ corresponding to the twist of $\kappa(\pi^{-1}(\mathcal{O}_{C^\nu}(q_1 - q_2))$. However, the question of exactly which effective divisors give rise to these line bundles is again dependent on the geometry of the canonical embedding.

Remark 7.7. In this section we have shown that the elliptic K3 surface in $\mathrm{Fix}(\iota^*)$ has the same types of singular fibers as those in the fibration of the Kummer K3 surface studied by Naruki [48]. By Proposition 6.7 for $K_A(0, l, -1)$, the K3 surface in the fixed-point locus is isomorphic to the Kummer

K3 surface. However, it is not apparent that, in general, there is any kind of natural map from the K3 surface studied by Naruki to $\mathrm{Fix}(\iota^*)$, or that these fixed-point loci are Kummer K3 surfaces.

8. A $(1,3)$-polarized example: Isolated points

Finally, we seek a description of the 36 isolated points in $\mathrm{Fix}(\iota^*)$. We will use a combination of the Abel map and information about the geometry of 2-torsion points in a $(1,3)$-polarized abelian surface to finish our description of the fixed loci.

8.1. Geometry of $A[2]$

The description of the isolated points in $\mathrm{Fix}(\iota^*) \subset K_A(0,l,s)$ will require an understanding of line bundles on curves $C \in |L|^{\iota^*}$ corresponding to divisors which sum to 0 in A.

As discussed in [4], the line bundle L^2 on our $(1,3)$-polarized abelian surface gives an embedding of the *desingularized* Kummer K3 surface into $\mathbb{P}^3$. They describe an action of the Heisenberg group on $\mathbb{P}^3$ that connects the geometry of the group action of elements $A[2]$ to the corresponding lines in the Kummer K3 surface.

We use notation from Hudson's analysis of $A[2]$ for principally polarized abelian surfaces [25, Ch. 1,§4], which has the same group structure: We write the group of points of $A[2]$ in multiplicative notation in terms of (a not minimal set of) generators $1, A, B, C, A', B', C'$, where 1 is the identity. The following multiplication tables hold:

	A	B	C
A	1	C	B
B		1	A
C			1

	A'	B'	C'
A'	1	C'	B'
B'		1	A'
C'			1

Following Naruki [48], we see the six points of $A[2]$ that occur in the base locus of $\mathbb{P}V_{\mathrm{ell}}$ must be a set of six in the group that coincides with those that would lie on a plane in Hudson's $(16,6)$ configuration, and so we take the following six points to be in the base locus of $\mathbb{P}V_{\mathrm{ell}}$:

$$AB', AC', BC', BA', CA', CB'. \tag{8.1}$$

Any possible choice of six points will have the same numerical properties described below as they will differ by a translation.

The remaining ten points of $A[2]$ are then:

$$1, A, A', B, B', C, C', AA', BB', CC'. \tag{8.2}$$

We will need the following observations in our identification of the isolated fixed points:

Lemma 8.1.

(a) *The product of any four distinct points in (8.1) cannot be the identity.*
(b) *Given any point in (8.2), there are exactly two ways to then choose three distinct points from those in (8.1) so that the product of the four points is the identity.*
(c) *There are fifteen ways to choose four distinct points from among (8.2) so that their product is the identity.*

Proof. The results may be verified directly.

In part (b), for instance, if we choose 1, we have exactly

$$(1)(AB')(BC')(CA') \quad \text{and} \quad (1)(AC')(BA')(CB').$$

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In part (c), the fifteen possibilities are:

$(1)(A)(A')(AA')$	$(A)(B)(C')(CC')$
$(1)(B)(B')(BB')$	$(A')(B')(C)(CC')$
$(1)(C)(C')(CC')$	$(A)(B')(C)(BB')$
$(1)(AA')(BB')(CC')$	$(A')(B)(C')(BB')$
$(1)(A)(B)(C)$	$(A')(B)(C)(AA')$
$(1)(A')(B')(C')$	$(A)(B')(C')(AA')$
$(A)(A')(BB')(CC')$	$(B)(B')(AA')(CC')$
	$(C)(C')(AA')(BB')$

□

8.2. *The fiber of* $\mathrm{Fix}(\iota^*)$ *over* $\mathbb{P}V_{\mathrm{hyp}}$

Let $C \in \mathbb{P}V_{\mathrm{hyp}} \cong \mathbb{P}^0$. For A a general $(1,3)$-polarized abelian surface, C is smooth by [10, Lem. 3.4], so the kernel of φ_C (6.4) is an abelian surface (see Proposition 6.4). The action $\ker\varphi_C$ inherits from ι^* on $K(0,l,s)$ is the action of $[-1]$ on it as an abelian surface. Thus, there will be exactly 16 isolated fixed points, consisting of the 2-torsion points on $\ker\varphi_C$.

We may also analyze $(\ker\varphi_C)^{\iota^*}$ using the Abel map α from (6.2). Since C is hyperelliptic, the canonical morphism is the degree 2 morphism $\pi\colon C \to \mathbb{P}^1$. The canonical divisors of C are of the form $\pi^{-1}(t_1)+\pi^{-1}(t_2)+\pi^{-1}(t_3)$ for $t_1,t_2,t_3 \in \mathbb{P}^1$. The sets of points in Hilb^4_C which sum to 0 and are fixed by ι^* consist of points of the form $\pi^{-1}(t_1)+\pi^{-1}(t_2)$, which are all linearly equivalent, and of four distinct 2-torsion points that sum to 0.

From this point of view we find that the sixteen isolated points in Pic^{-4}_C that sum to 0 and are fixed by ι^* are (the negative of) the fifteen points given by Lemma 8.1(c) and the one point that is the image under α of all points of the form $\pi^{-1}(t_1)+\pi^{-1}(t_2)$.

This argument may be used to show that the same result holds for Pic^d_C. We can take the isomorphism $\mathrm{Pic}^{-4}_C \cong \mathrm{Pic}^d_C$ to be given by adding $d+4$ copies of a fixed 2-torsion point p, in which case the isomorphism commutes with ι^*. It is not always possible to choose this isomorphism so that it commutes with taking the kernel of the summation map, but we may instead consider the elements in Pic^d_C that sum to $(d+4)\cdot p$, which amounts to simply performing this calculation in a different fiber of the Albanese map (2.2), which is related to our preferred fiber by an isomorphism.

8.3. *The fibers of* $\mathrm{Fix}(\iota^*)$ *over* $\mathbb{P}V_{\mathrm{ell}}$

In the last section, we found 16 of the 36 isolated points in $\mathrm{Fix}(\iota^*)$. To find the rest we examine $(\ker\varphi_C)^{\iota^*}$ as C varies in $\mathbb{P}V_{\mathrm{ell}} \cong \mathbb{P}^1$.

If C is smooth, by our analysis of the tangent space of $\ker\varphi_C$ in the proof of Proposition 6.6, $(\ker\varphi_C)^{\iota^*}$ is isomorphic to the elliptic curve C/ι, and there are no isolated points.

Let $C \in \mathbb{P}V_{\mathrm{ell}}$ be singular of type I_1. Consider the modified Abel map

$$\alpha\colon\ \mathrm{Hilb}^4_C \to \overline{\mathrm{Pic}}^{-4}_C.$$

The curve C passes through exactly six 2-torsion points (8.1) (cf. Proposition 7.3). By Lemma 8.1(a) the only sets of four points on C that sum to 0 and are fixed by ι^* are those of the form $(x,\iota x,y,\iota y)$ for some $x,y \in C$. Points of this form are already contained in the image of the pullback $\mathrm{Pic}^0_{C/\iota} \cong \mathrm{Pic}^{-2}_{C/\iota} \to \mathrm{Pic}^{-4}_C$. Thus, $(\ker\varphi_C)^{\iota^*} \cong C/\iota$, and there are no isolated points.

Now let $C \in \mathbb{P}V_{\mathrm{ell}}$ be singular of type I_2. The curve C passes through seven points in $A[2]$: those in the base locus of $\mathbb{P}V_{\mathrm{ell}}$ (see (8.1)) as well as one additional 2-torsion point. The points in Hilb^4_C that sum to 0 and are fixed by ι^* are those of the form $(x,\iota x,y,\iota y)$ for some $x,y \in C$, as well as any tuple of four

2-torsion points that sum to 0. By Lemma 8.1(a,b), there are exactly two points of the latter form and they are isolated from the points of the former form (cf. Remark 7.6). Thus the fiber of $\mathrm{Fix}(\iota^*)$ over C consists precisely of a singular curve of type I_2 and two isolated fixed points. There are ten such singular curves, and thus all of the 36 isolated points in $\mathrm{Fix}(\iota^*)$ are now accounted for.

Remark 8.2. It would be interesting to use the presentation scheme description of $\overline{\mathrm{Pic}}^d_C$, for $C \in \mathbb{P}V_{\mathrm{ell}}$ singular, to identify the two isolated points in the fiber of $\mathrm{Fix}(\iota^*)$ over C. For example, what line bundles do they pull back to on C^ν?

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