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Note on duality of Kelleyspace products

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It has been shown by W.F. LaMartin that the Pontryagin dual of a k-space product of Hausdorff abelian k-groups is the coproduct of their duals. Here we offer a different proof, based partly on LaMartin's, that the dual of a product is the coproduct of the duals in a more general setting.

Introduction

Let $K = (K, 1, \times, \{-, -\}, \ldots)$ denote the cartesian closed category of compactly generated Hausdorff spaces and let $V = (V, I, \otimes, [-, -], \ldots)$ be the category of algebras for a finitary additive commutative algebraic theory on K (see Day [1], Example 4.3). Then V is symmetric monoidal closed, complete, cocomplete, and additive, while $V = V(I, -) : V \to K$ reflects isomorphisms and creates filtered colimits, which summarises the basic properties we require of V .

Duality of products

LEMMA 1. *Let* $\Omega \in V$ *be an algebra with no small subalgebras and let* $f : \prod_{\lambda\in\Lambda} M_\lambda \to \Omega$ *be a homomorphism. Then there exists a finite* $F \subset \Lambda$ *with* $f(M_\lambda) = 0$ *if* $\lambda \notin F$.

Proof. See LaMartin [2], Lemma 8. //

PROPOSITION 2. *With the same hypotheses as in Lemma* 1, $f : \prod_{\lambda\in\Lambda} M_\lambda \to \Omega$ *factors through some projection onto a finite subproduct.*

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Proof. The canonical map $\sum M_\lambda \to \prod M_\lambda$ is dense because, if $(m_\lambda) \in \prod M_\lambda$, then $\prod\{0, m_\lambda\}$ is compact and so has the product topology. Thus it follows that (m_λ) lies in the closure of $\sum M_\lambda$ in the ordinary product $\prod M_\lambda$ since $\sum M_\lambda$ is dense in the ordinary product $\prod M_\lambda$ by additivity of V. From this we see that (*) commutes and the result follows:

$$\begin{array}{ccccc}
\sum\limits_{\lambda\in\Lambda} M_\lambda & \twoheadrightarrow & \prod\limits_{\lambda\in\Lambda} M_\lambda & \xrightarrow[(*)]{f} & \Omega \\
\downarrow & & \downarrow \text{projn.} & \nearrow f_{\text{res.}} & \\
\sum\limits_{\lambda\in F} M_\lambda & \xrightarrow{\cong} & \prod\limits_{\lambda\in F} M_\lambda & &
\end{array} \quad . \qquad //$$

We now write $\prod\limits_{\lambda\in\Lambda} M_\lambda = \lim\limits_{\phi\in\Phi} N_\phi$ where the limit is cofiltered over the finite subsets of Λ. On considering the canonical map:

$$\operatorname*{colim}_{\phi\in\Phi} [N_\phi, \Omega] \to [\lim_{\phi\in\Phi} N_\phi, \Omega]$$

we see that, provided Ω has no small subalgebras, it is a bijection by Proposition 2 and the fact that the colimit is filtered.

THEOREM 3. *If* $\Omega \in V$ *has no small subalgebras then* $\operatorname{colim}[N_\phi, \Omega] \cong [\lim N_\phi, \Omega]$.

Proof. It remains to prove that the spaces $\operatorname{colim}[N_\phi, \Omega]$ and $[\lim N_\phi, \Omega]$ admit the same morphisms from compact Hausdorff spaces. But, given any compact Hausdorff space C, $\{C, \Omega\}$ has no small subalgebras since the sub-basic open neighbourhood $W(C, V) = \{g \in K(C, \Omega); g(C) \subseteq V\}$ is a neighbourhood of $0 \in [C, \Omega]$ which contains no non-trivial subalgebras whenever V is an open neighbourhood of $0 \in \Omega$ containing no non-trivial subalgebras. Thus the result follows from Proposition 2. //

By additivity of V we see that $\operatorname*{colim}\limits_{\phi\in\Phi} [N_\phi, \Omega] \cong \sum\limits_{\lambda\in\Lambda} [M_\lambda, \Omega]$, as required.

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