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# Polyhedral decompositions of cubic graphs

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To Paul Erdös,
for his five thousand million and sixtieth birthday

A polyhedral decomposition of a finite trivalent graph $G$ is defined as a set of circuits $\underline{\underline{C}} = \{C_1, C_2, \ldots, C_m\}$ with the property that every edge of $G$ occurs exactly twice as an edge of some $C_k$. The decomposition is called even if every $C_k$ is a simple circuit of even length. If $G$ has a Tait colouring by three colours $a, b, c$ then the $(a, b)$, $(b, c)$ and $(c, a)$ circuits obviously form an even polyhedral decomposition. It is shown that the converse is also true: if $G$ has an even polyhedral decomposition then it also has a Tait colouring. This permits an equivalent formulation of the four colour conjecture (and a much stronger conjecture of Branko Grünbaum) in terms of polyhedral decompositions alone.

## 1. Introduction

Grünbaum has conjectured ([2]) that for any triangulation of an orientable surface it is possible to colour the edges by three colours in such a fashion that the edges of each triangle have different colours. In terms of the dual graph the conjecture states that a cubic graph (that is, one in which each vertex is incident with exactly three edges) which is the

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graph of edges and vertices of a polyhedron on an orientable surface admits an edge colouring by three colours. If the orientable surface is the sphere, Grünbaum's conjecture is equivalent to the four colour conjecture ([5], p. 121), but for other surfaces there is no such connection between face and edge colouring of polyhedra, and for instance the well known configuration of seven mutually adjacent countries on the torus admits an edge colouring by three colours.

Superficially it seems that Grünbaum's conjecture cannot be true. For take any trivalent graph $G$ which is not edge colourable by three colours (such as the Petersen graph) and represent it on a suitable orientable surface $S$. This can always be done in such a way ([4], p. 198) that the components of its complement on $S$ are simply connected domains. The dual of $G$ will supply a triangulation of $S$ whose edges are not colourable by three colours in the manner required by Grünbaum.

There are two ways in which this argument can go wrong. First, it may happen that the circuit of edges which forms the boundary of a face is not a simple circuit, that is, it goes through the same edge twice. Secondly, two such circuits may have more than one edge in common. In terms of the dual graph it means that the triangulation has loops and multiple edges which are evidently not allowed in Grünbaum's conjecture. We call a map on a surface *proper* if its dual is a triangulation without loops or multiple edges.

If we admit multiple edges then Petersen's graph yields a counter-example already on the torus. Represent the torus as the Cartesian plane modulo the integral lattice $Z^2$; then the following straight line segments represent a triangulation:

$$[(0,0),(0,\tfrac{1}{2})],\ [(0,0),(\tfrac{1}{2},\tfrac{1}{2})],\ [(0,0),(1,\tfrac{1}{2})],\ [(0,0),(\tfrac{1}{2},0)],$$
$$[(\tfrac{1}{2},0),(1,\tfrac{1}{2})],\ [(\tfrac{1}{2},0),(\tfrac{3}{4},0)],\ [(\tfrac{3}{4},0),(1,\tfrac{1}{2})],\ [(\tfrac{3}{4},0),(1,0)],$$
$$[(0,\tfrac{1}{2}),(0,1)],\ [(0,\tfrac{1}{2}),(\tfrac{1}{2},1)],\ [(0,\tfrac{1}{2}),(\tfrac{1}{2},\tfrac{1}{2})],\ [(\tfrac{1}{2},\tfrac{1}{2}),(\tfrac{1}{2},1)],$$
$$[(\tfrac{1}{2},\tfrac{1}{2}),(\tfrac{3}{4},1)],\ [(\tfrac{1}{2},\tfrac{1}{2}),(1,\tfrac{1}{2})],\ [(\tfrac{3}{4},1),(1,\tfrac{1}{2})]\ .$$

The triangulation has 5 vertices, 15 edges and 10 faces, and it can be verified easily that it does not admit a Grünbaum type edge colouring, either directly by testing all possibilities or by observing that its dual is the Petersen graph, that is the graph of vertices and edges of the regular dodecahedron in which diametrically opposite points have been identified. It is well known that the Petersen graph does not admit edge colouring by three colours. Of course this triangulation has several multiple edges, for instance $[(0, 0), (0, \frac{1}{2})]$ and $[(0, 0), (1, \frac{1}{2})]$ join the same pair of vertices on the torus.

On the other hand the Petersen graph can easily be shown to be the graph of edges and vertices of a proper map on the projective plane (§3). Questions of this kind, whether a trivalent graph is the graph of vertices and edges of a proper map on a surface (orientable or not) can be formulated in purely combinatorial terms, by means of certain circuit decompositions (§2). In §4 we shall give an equivalent formulation of edge colourability by three colours (and hence of the four colour and Grünbaum's conjecture) by means of circuit decompositions alone.

## 2. Polyhedral decompositions

We follow mostly the terminology of Berge [1] and Tutte [7]. A graph $G$ consists of a finite set $X$ of vertices $\{p, q, \ldots, x, y, \ldots\}$ , a finite set $Y$ of arcs (directed edges) $\{\alpha, \beta, \gamma, \ldots\}$ , and an incidence mapping $\Gamma : Y \to X \times X$ which associates with each arc $\gamma$ two vertices, an initial vertex $p$ and a terminal vertex $q$ . Elements of $\Gamma$ are denoted by $(p\gamma q)$ where $(p, q) \in X \times X$ , $\gamma \in Y$ , and we shall interchangeably speak of an arc $\gamma$ or an arc $(p\gamma q)$ . We assume that $\Gamma$ is symmetric, that is, there is an involution $\sigma : Y \to Y$ which associates with every $\gamma \in Y$ a reverse arc $\sigma(\gamma) = \gamma' \neq \gamma$ such that if $(p\gamma q) \in \Gamma$ then $(q\gamma' p) \in \Gamma$ . The couple $\{\gamma, \gamma'\}$ is called an edge of $\Gamma$ and written $[\gamma] = [\gamma']$ ; $p$ and $q$ are called its end vertices. We admit multiple edges, that is we do not assume $\Gamma$ to be injective: we may have $(p\alpha q) \in \Gamma$ , $(p\beta q) \in \Gamma$ with $\alpha \neq \beta$ . We also admit loops, that is, arcs $(p\gamma p)$ with the same initial and terminal vertex; the reverse $(p\gamma' p)$ then is also a loop, distinct from $(p\gamma p)$ .

Unless the contrary is explicitly stated, all graphs will be assumed to be finite and trivalent (or cubic), that is, each vertex is the initial (hence terminal) vertex of exactly three arcs. Thus a graph on two vertices $p, q$ with six arcs $(p\alpha p)$ , $(p\alpha' p)$ , $(p\beta q)$ , $(q\beta' p)$ , $(q\gamma q)$ , $(q\gamma' q)$ is trivalent according to this definition. Generally if

$(p\alpha p)$ is a loop then there is just one more arc $(p\beta q)$ having $p$ as an initial vertex, and then $[\beta]$ is an isthmus, that is an edge whose removal disconnects the component of $G$ in which the edge is situated. Hence a cubic graph without an isthmus contains no loops.

A path $C$ of length $k \geq 1$ is a sequence $(p_0\alpha_1 p_1\alpha_2 \dots \alpha_k p_k)$ such that $(p_{i-1}\alpha_i p_i) \in \Gamma$ for $i = 1, \dots, k$. We say that $\alpha_i$ or $(p_{i-1}\alpha_i p_i)$ is an arc of $C$, $[\alpha_i]$ an edge of $C$, and $p_0$ is the initial vertex, $p_k$ the terminal vertex of $C$. An arc can be regarded as a path of length 1. The reverse $C'$ of $C$ is the path $(p_k\alpha'_k \dots \alpha'_2 p_1\alpha'_1 p_0)$.

$C$ is called *semisimple* if consecutive edges $[\alpha_{i-1}]$, $[\alpha_i]$ are distinct, and *simple* if any two edges $[\alpha_i]$, $[\alpha_j]$, $i \neq j$ are distinct. This definition is at variance with Tutte ([7], pp. 29-30), but it should be noted that in a cubic graph simplicity implies that $p_i \neq p_j$ for $0 < i < j < k$. The path is called reentrant if its initial and terminal vertices coincide. An equivalence class of reentrant paths generated by the relation $(p_k\alpha_1 p_1 \dots \alpha_k p_k) \sim (p_1\alpha_2 p_2 \dots \alpha_k p_k\alpha_1 p_1)$ is called a circuit and denoted $[p_k\alpha_1 p_1 \dots \alpha_k p_k]$; it is semisimple if $[\alpha_i] \neq [\alpha_{i+1}]$ for $i = 1, \dots, k$, where $[\alpha_{k+1}] = [\alpha_1]$. For instance in the previous example the circuit $[p\alpha p\beta q\gamma q\beta' p]$ is semisimple but not simple.

First we show

LEMMA 1. *Given a cubic graph $G$ with incidence mapping $\Gamma$ there is an injection (hence surjection)*

$$L_0 : \Gamma \to \Gamma$$

*such that*

(i) $L_0(p\alpha q) = (s\beta r) \Rightarrow q = s$,

(ii) $L_0(p\alpha q) = (q\beta r) \Rightarrow L_0(r\beta' q) \neq (q\alpha' p)$.

REMARK. Condition *(ii)* implies that $\beta \neq \alpha'$; for $L_0(p\alpha q) = (q\alpha' p)$

contradicts *(ii)* with $\beta = \alpha'$ , $\beta' = \alpha$ , $r = p$ .

Proof. Consider the set $\Lambda$ of all injections

$$L : \Gamma^* \to \Gamma \ , \quad \Gamma^* \subset \Gamma \ ,$$

with properties *(i)* and *(ii)*; in particular $L(p\alpha q) \neq (q\alpha' p)$ for any $(p\alpha q) \in \Gamma^*$ . $\Lambda$ is partially ordered by $L_1 < L_2$ if $L_2$ is an extension of $L_1$ . Let $L_0$ be a maximal element of $\Lambda$ , $\Gamma_0$ its domain; we want to prove that $\Gamma_0 = \Gamma$ .

First we note that if $(p\alpha q) \in \Gamma$ , $(q\beta r) \in \Gamma$ , $\beta \neq \alpha'$ , then either $(q\beta r) = L_0(p\alpha q)$ or $(q\alpha' p) = L_0(r\beta' q)$ . For suppose first that $(p\alpha q) \notin \Gamma_0$ . Then we must have $L_0(r\beta' q) = (q\alpha' p)$ ; for otherwise we could extend $L_0$ to $L \in \Lambda$ with domain $\Gamma^* = \Gamma_0 \cup (p\alpha q)$ by defining $L(p\alpha q) = (q\beta r)$ or $L(p\alpha q) = (q\gamma s)$ , $\gamma \neq \beta$ , $\gamma \neq \alpha'$ , depending on whether or not $L_0(s\gamma' q) = (q\beta r)$ . Indeed if $(q\beta r) \neq L_0(s\gamma' q)$ and $L_0(r\beta' q) \neq (q\alpha' p)$ then $L(p\alpha q) = (q\beta r)$ is compatible with *(ii)* and with injectivity of $L$ ; on the other hand if $L_0(s\gamma' q) = (q\beta r)$ then $(q\gamma s) \neq L_0(r\beta' q)$ hence $L(p\alpha q) = (q\gamma s)$ is compatible with *(ii)* and injectivity. In either case $L_0$ can be properly extended, contrary to the maximality of $L_0$ .

Suppose next that $(p\alpha q) \in \Gamma_0$ , $L_0(p\alpha q) = (q\gamma s)$ where $\gamma$ is as before. Then $L_0(s\gamma' q) \neq (q\alpha' p)$ and $L_0(r\beta' q) \neq (q\gamma s)$ since $L_0$ is injective. Hence again $L_0(r\beta' q) = (q\alpha' p)$ since otherwise we could extend $L_0$ to $L \in \Lambda$ with domain $\Gamma^* = \Gamma_0 \cup (r\beta' q)$ by defining $L(r\beta' q) = (q\alpha' p)$ , which is compatible with *(ii)* and injectivity.

Hence $L_0$ has the property that for every pair of arcs $(p\alpha q)$ , $(q\beta r)$ , $\beta \neq \alpha'$ , either $L_0(p\alpha q) = (q\beta r)$ or $L_0(r\beta' q) = (q\alpha' p)$ . This implies for an arbitrary $(p\alpha q) \in \Gamma$ that if $(q\beta r)$ , $(q\gamma s)$ are the two arcs distinct from $(q\alpha' p)$ with initial vertex $q$ and $(q\beta r) \neq L_0(p\alpha q)$ then $L_0(r\beta' q) = (q\alpha' p)$ hence $(q\alpha' p) \neq L_0(s\gamma' q)$ , by injectivity.

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Therefore $L_0(p\alpha q) = (q\gamma s)$ , $(p\alpha q) \in \Gamma_0$ hence $\Gamma \subset \Gamma_0$ , $\Gamma_0 = \Gamma$ , and the lemma is proved.

From the lemma it follows, by the remark preceding the proof, that the orbits of $L_0$ are semisimple circuits $C_1, C_2, \ldots, C_m$ with the following property:

P₀. every $\gamma \in Y$ (or $(p\gamma q) \in \Gamma$ ) is contained in exactly one $C_k$ .

We call a family of semisimple circuits $\underline{\underline{C}} = \{C_1, C_2, \ldots, C_m\}$ of $G$ a *polyhedral decomposition* if they have the property

P₁. every edge $[\gamma]$ occurs exactly twice in the circuits $C_k$ .

Thus the arc $\gamma$ itself may or may not appear in any $C_i$ ; if it does not appear then $\gamma'$ appears twice. Note that in a polyhedral decomposition any $C_i$ may be replaced by its reverse $C_i'$ .

We shall call the decomposition *coherent* if it has the more stringent property P₀. We have thus proved:

THEOREM 1. *Every cubic graph has at least one coherent polyhedral decomposition.*

With every polyhedral decomposition $\underline{\underline{C}}$ of $G$ there is associated a characteristic

$$\chi(G, \underline{\underline{C}}) = V - E + F$$

where $V$ is the number of vertices, $E$ the number of edges of $G$ , and $F$ the number of distinct circuits in $\underline{\underline{C}}$ . For example in the two-vertex graph described earlier

$$C_1 = [p\alpha p\beta q\gamma q\beta' p] , \quad C_2 = [p\alpha' p] , \quad C_3 = [q\gamma' q]$$

is a coherent polyhedral decomposition with characteristic $2 - 3 + 3 = 2$ . The genus associated with a coherent $\underline{\underline{C}}$ is $g(G, \underline{\underline{C}}) = 1 - \frac{1}{2}\chi(G, \underline{\underline{C}})$ and the genus of $G$ is defined $g(G) = \min_{\underline{\underline{C}}} g(G, \underline{\underline{C}})$ , the minimum taken for all possible coherent polyhedral decompositions of $G$ .

By interpreting the circuits of $\underline{\underline{C}}$ as the boundaries of 2-cells on a topological surface (with the obvious topology at the edges and vertices) we find as a corollary of Theorem 1:

THEOREM 2. *Every connected cubic graph* $G$ *is the graph of vertices and edges of a map on an orientable surface of genus* $g(G)$ .

This of course is a special case of the theorem of Petersen and König [3], but the proof (in Hungarian) is not easily accessible and we preferred to give an independent proof of Theorem 1, because of the importance of polyhedral decompositions for all that follows.

For the purposes of Theorem 2 a map is understood to have simply connected faces (countries) but an edge is not necessarily on the boundary of two distinct countries nor have the boundaries of two countries necessarily only one edge in common. A country may have a single edge $[\gamma]$ for its boundary, namely when $\gamma$ is a loop.

The polyhedral decomposition is called *simple* if all circuits are simple; a simple polyhedral decomposition is called *proper* if two distinct circuits have at most one edge in common. For instance the boundaries of the faces in the dual of a triangulation in Grunbaum's conjecture form a proper coherent polyhedral decomposition.

Clearly the existence of a proper (coherent) polyhedral decomposition of characteristic $\chi$ is equivalent to $G$ being the graph of vertices and edges of a proper map on an (orientable) surface of characteristic $\chi$ .

LEMMA 2. *The number of vertices in a cubic graph* $G$ *is even. The sum of lengths of the circuits in a polyhedral decomposition of* $G$ *is even.*

The first statement is well known and follows from $2E = 3V$ where, as before, $E$ is the number of edges and $V$ the number of vertices of $G$ . The second statement follows from $P_1$ which requires that the sum of lengths of the circuits be $2E$ .

LEMMA 3. *If the cubic graph* $G$ *has an isthmus then it has no simple polyhedral decomposition. If* $G$ *has no isthmus but has a pair of non-adjacent edges whose removal disconnects* $G$ *(that is, is of connectivity* 2 *), or if it has a double edge but not a triple edge, then it has no proper polyhedral decomposition.*

For if $G$ has an isthmus $[\gamma_0]$ then any circuit $C_0$ which contains $(p\gamma_0 q)$ must also contain $(q\gamma_0' p)$ since no arc of $C_0$ following $\gamma_0$ can reach $p$ again before reaching $q$. If $[\gamma_1]$, $[\gamma_2]$ are edges whose removal disconnects $G$ then any simple circuit $C_1$ which contains $(x\gamma_1 y)$ must necessarily pass through the edge $[\gamma_2]$ in order to get back to $x$. If therefore $C_2$ is another simple circuit through $[\gamma_1]$, it must also pass through $[\gamma_2]$, and so $C_1$ and $C_2$ have two edges in common.

Finally if $(x\gamma_1 y)$, $(x\gamma_2 y)$ are distinct arcs of $G$ and $(x\gamma_3 p)$, $(y\gamma_4 q)$ are the other two arcs adjoining $x$ and $y$ where $p \neq y$, $q \neq x$, then any simple circuit containing $(p\gamma_3' x)$ must also contain $(y\gamma_4 q)$ (hence contain either $(p\gamma_3' x\gamma_1 y\gamma_4 q)$ or $(p\gamma_3' x\gamma_2 y\gamma_4 q)$ ). Similarly any simple circuit containing $(x\gamma_3 p)$ must also contain $(q\gamma_4' y)$. Hence every two such circuits have two edges in common.

It is an open question whether every cubic graph without an isthmus has a polyhedral decomposition, or whether every cubic graph with connectivity $> 2$ has a proper polyhedral decomposition. The Petersen graph has both a simple coherent and a proper polyhedral decomposition; this will be verified in §3.

## 3. Graphs with no proper coherent decomposition

Let $P$ be the Petersen graph on the vertex set $\{0, 1, 2, \ldots, 9\}$ with edges (in obvious notation) [01], [12], [23], [34], [40], [56], [67], [78], [89], [95], [05], [17], [29], [36], [48]. Then the following is a simple coherent decomposition.:

$$C_1 = [012340], \quad C_2 = [367843], \quad C_3 = [048950],$$
$$C_4 = [1765921], \quad C_5 = [0563298710].$$

This was the decomposition used in §1 to represent the Petersen graph on the torus; it is of course not proper, $C_5$ has more than one edge in common with the other circuits of the decomposition.

A typical proper polyhedral decomposition is

$$C_1 = [012340] , \quad C_2 = [367843] , \quad C_3 = [295632] ,$$
$$C_4 = [178921] , \quad C_5 = [056710] , \quad C_6 = [048950] .$$

The decomposition is not coherent and its characteristic is 1 ; it represents a polyhedron on the projective plane obtained by identifying opposite points on the regular dodecahedron.

We now show that $P$ has no proper coherent decomposition. We call two non-adjacent edges $[ab]$, $[cd]$ of $P$ *opposite* if the subgraph spanned by $a, b, c, d$ has no other edges in $P$ . For instance [23] and [78] are opposite because none of the edges [27], [28], [37], [38] , exist in $P$ .

THEOREM 3. *Let $H$ be a graph obtained from the Petersen graph $P$ by deleting a pair of opposite edges. Let $G$ be any trivalent graph which contains a subgraph isomorphic to $H$ . Then $G$ has no proper coherent polyhedral decomposition.*

In particular $P$ itself has no proper coherent decomposition. Because of the symmetries of the Petersen graph we may assume without loss of generality that $H$ is obtained from $P$ by omitting the edges [23] and [78] . We also denote by $K$ the graph obtained from $H$ by omitting the set of vertices $S = \{2, 3, 7, 8\}$ and replacing the set of edges

$$A = \{[12], [29], [34], [36], [17], [67], [48], [89]\}$$

by the set $B = \{[16], [19], [46], [49]\}$ . Inspection shows that $K$ is isomorphic to the bipartite Kuratowski $K_{3,3}$ on the vertices 1, 4, 5 and 0, 6, 9 .

Denote by $\Gamma_K$ the set of arcs of $K$ , by $\Gamma_H$ the set of arcs of $H$ , and by $\sigma : \Gamma_H \to \Gamma_K$ an inclusion mapping defined as follows: If $[xy] \in A$ where $y \in S$ then $\sigma(xy) = (xz)$ , $\sigma(yx) = (zx)$ , where $z$ is the unique vertex of $K$ , distinct from $x$ , for which $[yz] \in A$ . If $(xy) \in \Gamma_H$ , $x \notin S$ , $y \notin S$ then we define $\sigma(xy) = (xy)$ . Note that given $[xz] \in B$ there is a unique $y \in S$ such that $[xy] \in A$ , $[yz] \in A$ , and hence $\sigma(xy) = \sigma(yz) = (xz)$ .

Suppose now that $H$ is embedded isomorphically in a trivalent $G$ and that $G$ has a proper coherent decomposition $\underline{\underline{C}} = \{C_1, C_2, \ldots, C_m\}$. By Lemma 3, $G$ has no double edges. We associate with $\underline{\underline{C}}$ a coherent decomposition of $K$ as follows. Suppose first that $(xy) \in \Gamma_K$, $[xy] \notin B$ and let $C_i$ be the (unique) circuit of $\underline{\underline{C}}$ which contains $(xy)$. Then $C_i$ contains a path $(xyz)$ and we define $L_0(xy) = \sigma(yz)$. Suppose next that $[xz] \in B$; then as we have remarked earlier, there is a unique $y \in S$ such that $[xy] \in A$, $[yz] \in A$. Furthermore there is a unique $C_j \in \underline{\underline{C}}$ containing the arc $(yz)$ hence containing a path $(yzt)$, and we define $L_0(xz) = \sigma(zt)$. With this definition of $L_0 : \Gamma_K \to \Gamma_K$ both conditions *(i)* and *(ii)* of Lemma 1 are satisfied; the first one trivially, the second by virtue of property $P_0$ of the circuits $C_i$. The orbits of $L_0$ form a coherent polyhedral decomposition of $K$; this decomposition is not necessarily proper, not even simple.

Let $p, q, r$ denote the vertices 1, 4, 5 in an arbitrary arrangement, $x, y, z$ the vertices 0, 6, 9 in an arbitrary arrangement. Then the only possible coherent decompositions of $K$ are

1. $[pxqyrzp]$ $[pyqzrxp]$ $[pzqxryp]$,

2. $[pxqyp]$ $[pyrzp]$ $[pzqxryqzrxp]$,

3a. $[pxqyrzpyqzrxpzqxryp]$,

3b. $[pxqypzqxryqzrxpyrzp]$.

In Case 1 there are two distinct possibilities:

$$[1056491] \quad [1659401] \quad [1950461]$$

and

$$[1649501] \quad [1046591] \quad [1940561] .$$

By inserting the vertices 2, 3, 7, 8 at the appropriate places we obtain the following circuits in $H$:

1.1 $D_1 = [21056348 92]$, $D_2 = [765984017]$, $D_3 = [2950436712]$,

1.2 $D_1 = [8950176348]$, $D_2 = [210436592]$, $D_3 = [8405671298]$.

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It is sufficient to consider 1.1; for $H$ has the dihedral group of order 8 generated by the permutations (05)(1946)(2837) and (27)(38)(69) for its group of symmetries, and either of these permutations carries 1.1 into 1.2.

Each of the vertices 2, 3, 7, 8 of $S$ appears exactly twice in the circuits $D_i$, and exactly one of these occurrences represents in $G$ a vertex of entry (hence also of departure) of an arc from a vertex of $G - H$ in the original decomposition $\underline{\underline{C}}$ of $G$. We shall refer to such an occurrence of the vertices 2, 3, 7, 8 in the circuits $D_i$ as a "*vertex of entry*" of the circuit. Note that in a $D_i$ there cannot be only one vertex of entry since otherwise the corresponding circuit in $G$ would not be simple. Therefore there are either no vertices of entry in $D_i$ or there are at least two. In the former case $D_i$ must appear as a circuit in $\underline{\underline{C}}$.

Now $D_2$ cannot be a circuit in $\underline{\underline{C}}$ since the subpath (210563) of $D_1$ is part of a circuit in $\underline{\underline{C}}$ and it has more than two vertices in common with $D_2$ which is impossible in a proper decomposition. Hence 7 and 8 are vertices of entry in $D_2$ and therefore they are not vertices of entry in $D_1$ and $D_3$. But neither $D_1$ nor $D_3$ are circuits in $\underline{\underline{C}}$ since they have more than two vertices in common with the subpath (76598) of $D_2$. Therefore 2 and 3 must be vertices of entry both in $D_1$ and $D_3$, which is impossible.

In Case 2 we have 18 distinct possibilities which after inserting the vertices 2, 3, 7, 8 and taking into account the symmetries of $H$, reduce to 5 distinct cases:

| | $D_1$ | $D_2$ | $D_3$ |
|---|---|---|---|
| 2.1 | [1043671] | [1765921] | [129840563489501] |
| 2.2 | [1043671] | [012950] | [176598405634892l] |
| 2.3 | [176348921] | [012950] | [1043659840567l] |
| 2.4 | [9843659] | [012950] | [10489217634056 71] |
| 2.5 | [056340] | [012950] | [104892176598436 71] . |

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Of these, Case 2.1 is ruled out because $D_1$ and $D_2$ both contain only two vertices from $S$ and 7 can only be vertex of entry in one of them, therefore either $D_1$ or $D_2$ is a circuit of $\underline{\underline{C}}$ . But $D_3$ contains the paths (840563) and (895012) which have more than two vertices in common with $D_1$ and $D_2$ .

In Case 2.2, $D_3$ contains the path (8405634892) which is not simple, therefore 3 is a vertex of entry of $D_3$ . This implies that $D_1$ is a circuit of $\underline{\underline{C}}$ (since 3 in $D_1$ is not vertex of entry), which is impossible since it has more than two vertices in common with the subpath (840563) of $D_3$ .

In Case 2.3, 7 is a vertex of entry of $D_3$ , by the same argument as before, and therefore the subpath (21763) of $D_1$ is part of a circuit in $\underline{\underline{C}}$ , which is impossible since it has more than two vertices in common with (71043) in $D_3$ .

In Case 2.4, 3 is a vertex of entry of $D_3$ hence $D_1$ is a circuit of $\underline{\underline{C}}$ , which is impossible since it has more than two vertices in common with the subpath (340567) of $D_3$ .

Finally in Case 2.5, $D_1$ and $D_2$ are circuits of $\underline{\underline{C}}$ since they have only one vertex each from $S$ . Now one of the two occurrences of 7 in $D_3$ is not a vertex of entry and therefore either (3671048) or (2176598) is part of a circuit in $\underline{\underline{C}}$ . But the first of these paths has four vertices in common with $D_1$ , the second has four vertices in common with $D_2$ .

The last remaining cases are 3a and 3b; they are all equivalent under the symmetries of $H$ and we only have to consider

[10436592176348950129840567l] .

Here 2 and 7 are vertices of entry in (89501298) and (8405671043) therefore (365921763) is part of a circuit in $\underline{\underline{C}}$ which is clearly impossible if $\underline{\underline{C}}$ is to be proper. So we have verified Theorem 3 in all cases and the proof is complete.

By a similar argument it can be shown that if $H'$ is obtained from $P$ by deleting any two non-adjacent edges (not necessarily opposite, such as [36] and [48] ) then a trivalent $G$ which contains an isomorphic copy of $H'$ has no proper coherent polyhedral decomposition.

Grünbaum's conjecture suggests a link between Tait colourability and the existence of a proper coherent decomposition. A Tait colouring of a graph is an edge colouring by three colours so that edges meeting at a vertex have distinct colours. Grünbaum's conjecture states that a cubic graph which admits a proper coherent polyhedral decomposition always has a Tait colouring. The result of Theorem 3 shows that the converse is not true, not even if we assume that the graph contains no isthmuses or pairs of edges whose removal disconnects the graph. Indeed $H$ can easily be shown to possess a Tait colouring and by joining two copies of $H$ via edges between the corresponding vertices 2, 3, 7, 8 we obtain a cubic graph with Tait colouring which does not admit a proper coherent decomposition. We shall show somewhat more, namely that although $H$ itself has a Tait colouring, the colouring of the four edges emanating from the vertices 2, 3, 7, 8 when $H$ is embedded in a cubic graph is restricted to two possibilities, each involving only two of the three colours.

LEMMA 4. *Let* $H$ *be as in Theorem* 3, $G$ *the graph obtained from* $H$ *by inserting four new vertices* $p, q, r, s$ *and four new edges* $[\gamma_1] = [2p]$, $[\gamma_2] = [3q]$, $[\gamma_3] = [7r]$, $[\gamma_4] = [8s]$. *Then in any Tait colouring of* $G$, *the edges* $[\gamma_i]$, $i = 1, 2, 3, 4$ *receive two distinct colours. Furthermore, the colours assigned to* $[\gamma_1]$ *and* $[\gamma_2]$ *are distinct and the colours assigned to* $[\gamma_3]$ *and* $[\gamma_4]$ *are distinct.*

Proof. Let $a, b, c$ be the three colours. The circuit [017650] receives, apart from a permutation of colours and cyclic permutation of edges, the successive colours *ababc* . An easy enumeration shows that only four of the five cyclic permutations are feasible; they yield the following Tait colourings of $H$ .

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| [01] | [17] | [76] | [65] | [50] | [63] | [34] | [40] | [12] | [29] | [95] | [98] | [84] |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $c$ | $a$ | $b$ | $a$ | $b$ | $c$ | $b$ | $a$ | $b$ | $a$ | $c$ | $b$ | $c$ |
| $b$ | $c$ | $a$ | $b$ | $a$ | $c$ | $a$ | $c$ | $a$ | $b$ | $c$ | $a$ | $b$ |
| $a$ | $b$ | $c$ | $a$ | $b$ | $b$ | $a$ | $c$ | $c$ | $b$ | $c$ | $a$ | $b$ |
| $b$ | $a$ | $b$ | $c$ | $a$ | $a$ | $b$ | $c$ | $c$ | $a$ | $b$ | $c$ | $a$ |

The corresponding four possibility for $[\gamma_i]$ are

| $[\gamma_1]$ | $[\gamma_2]$ | $[\gamma_3]$ | $[\gamma_4]$ |
|---|---|---|---|
| $c$ | $a$ | $c$ | $a$ |
| $c$ | $b$ | $b$ | $c$ |
| $a$ | $c$ | $a$ | $c$ |
| $b$ | $c$ | $c$ | $b$ |

which proves the lemma. For instance $[\gamma_1] = [2p]$ must have a colour distinct from [12] and [29] and this uniquely determines it in all four cases. Similarly for $[\gamma_2]$, $[\gamma_3]$ and $[\gamma_4]$ .

It can be shown that Lemma 4 is also valid for $H'$ , obtained from $P$ by deleting the non-opposite edges [36] and [48] , with $[\gamma_1] = [3p]$ , $[\gamma_2] = [6q]$ , $[\gamma_3] = [4r]$ , $[\gamma_4] = [8s]$ .

Lemma 4 permits the construction of a cubic graph $Q$ with 50 vertices and 75 edges which does not admit Tait colouring. $Q$ consists of five subgraphs $H_i$ on the disjoint vertex sets $X_i = \{x_{ij};\ 0 \le j \le 9\}$ , $i = 1, 2, 3, 4, 5$ , each isomorphic to $H$ through the vertex assignment $j \to x_{ij}$ , $0 \le j \le 9$ , and the following ten edges between the $H_i$ :

(1) $$[x_{i2}, x_{i+1,3}]\ ,\quad [x_{i7}, x_{i+2,8}]\ ,\quad i = 1, 2, 3, 4, 5$$

where the first subscripts are modulo 5 . There is exactly one edge linking each pair of subgraphs $H_i$ and there is no ambiguity if we write $[i, i+1]$ for $[x_{i2}, x_{i+1,3}]$ , $[i, i+2]$ for $[x_{i7}, x_{i+2,8}]$ .

**THEOREM 4.** *$Q$ has no Tait colouring.*

For suppose that $Q$ has a Tait colouring. Then by Lemma 4, the

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following pairs of edges must receive distinct colours:

(2) $\{[i, i+1], [i, i-1]\}$ , $\{[i, i+2], [i, i-2]\}$ , $i \le i \le 5$ .

Furthermore, the colouring of the four edges $[ij]$ incident with a given $H_i$ must be identical in pairs. We show that these conditions are contradictory.

By condition (2), consecutive edges in the sequence [12], [23], [34], [45], [51] must receive distinct colours. We may assume, by symmetry, that the sequence of colours is *ababc* . This sequence determines uniquely the pair of colours that the four edges $[ij]$ with given $i$ may receive. In particular [13] and [35] may only receive $a, b$ or $b, a$ and similarly [14], [24] may only receive $a, b$ or $b, a$ . But [13] and [14] may only receive $a, c$ or $c, a$ , a contradiction.

## 4. Even decompositions

We consider now cubic graphs with a Tait colouring by three colours $a, b, c$ . Such a graph obviously cannot have a loop. Furthermore, the $(ab)$, $(bc)$, $(ca)$ circuits form a simple polyhedral decomposition in which all circuits are of even length. Let us call such a decomposition *even*; thus every cubic $G$ which has a Tait colouring has an even polyhedral decomposition. The main result of this section is that the converse is also true.

THEOREM 5. *Let the cubic graph* $G$ *have an even polyhedral decomposition* $\underline{\underline{C}} = \{C_1, C_2, \ldots, C_m\}$ *; then* $G$ *has a Tait colouring by three colours* $a, b, c$ *and a mapping* $\tau : \underline{\underline{C}} \to \{a, b, c\}$ *such that no edge of* $C_k \in \underline{\underline{C}}$ *receives the colour* $\tau C_k$ .

We call this the Tait colouring *induced* by $\underline{\underline{C}}$ .

As a corollary we obtain

THEOREM 6. *Necessary and sufficient for* $G$ *to have a Tait colouring is that* $G$ *possesses an even polyhedral decomposition.*

Proof of Theorem 5. We may assume that $G$ is connected. It follows from Lemma 3 that if $G$ has an even decomposition then it has no isthmuses hence no loops. Now the only cubic graph on two vectices and without loops is the triple edge with vertices $p$, $q$ and arcs $(p\alpha q)$, $(p\beta q)$, $(p\gamma q)$ and their reverses. This graph obviously has a Tait colouring. Hence we may assume that $G$ has more than two vertices and that the theorem is true for graphs with fewer vertices than $G$.

We first consider the case when $G$ has a double edge, that is, arcs $(x\alpha y)$, $(x\beta y)$, $\alpha \neq \beta$. Let the other two arcs with initial vertices $x$ and $y$ be $(x\gamma_1 p)$, $(y\gamma_2 q)$. Let $\underline{\underline{C}} = \{C_1, \ldots, C_m\}$ be an even decomposition. One of the circuits, say $C_1$, contains $(p\gamma_1' x)$ (if $C_1$ contains $(x\gamma_1 p)$ then we replace it by its reverse). Since $C_1$ is simple, the only possibility (apart from an interchange of $\alpha$ and $\beta$ ) is that $C_1 = [p\gamma_1' x\alpha y\gamma_2 qUp]$ where $(qUp)$ is a simple path of odd length, avoiding $x$ and $y$. Another circuit, say $C_2$, contains $(q\gamma_2' y)$; it must have the form $C_2 = [q\gamma_2' y\beta' x\gamma_1 pVq]$ where $(pVq)$ is again a simple path of odd length, avoiding $x$ and $y$. Finally there must also be a circuit $C_3 = [x\beta y\alpha' x]$ in $\underline{\underline{C}}$ (again replacing it by its reverse if necessary).

Now let $G^*$ be the cubic graph obtained from $G$ by removing $x$, $y$, $[\alpha]$, $[\beta]$, $[\gamma_1]$, $[\gamma_2]$ and inserting the new arcs $(p\gamma^* q)$, $(q\gamma^{*\prime} p)$ Then $C_1^* = [p\gamma^* qUp]$, $C_2^* = [q\gamma^{*\prime} pVq]$, $C_j^* = C_{j+1}$ for $j \geq 3$ form an even decomposition $\underline{\underline{C}}^* = \{C_1^*, C_2^*, \ldots, C_{m-1}^*\}$ of $G^*$. By the induction hypothesis $\underline{\underline{C}}^*$ induces a Tait colouring in $G^*$. Suppose that $[\gamma^*]$ is a $c$-edge, $C_1^*$ an $(ac)$ circuit, $C_2^*$ a $(bc)$ circuit, then take $[\gamma_1]$ and $[\gamma_2]$ to be $c$-edges, $[\alpha]$ an $a$-edge, $[\beta]$ a $b$-edge, all other edges of $G$ receiving the same colour as in $G^*$. This will clearly yield a Tait colouring for $G$, induced by $\underline{\underline{C}}$.

Next we assume that $G$ has no double edges but contains a triangle with edges $[\alpha_1]$, $[\alpha_2]$, $[\alpha_3]$, corresponding to arcs $(x\alpha_1 y)$, $(y\alpha_2 z)$, $(z\alpha_3 x)$. Let $(x\beta_1 p)$, $(y\beta_2 q)$, $(z\beta_3 r)$ be the other three arcs with initial vertices $x$, $y$, $z$ where $p$, $q$, $r$ are not necessarily distinct. Since $[x\alpha_1 y\alpha_2 z\alpha_3 x]$ is not admissible as a circuit of an even decomposition, the only possibility (apart from trivial renumberings and reversals) for circuits through the vertices $x$, $y$, $z$ is

$$C_1 = [p\beta_1'x\alpha_1 y\alpha_2 z\beta_3 rUp] , \quad C_2 = [q\beta_2'y\alpha_2 z\alpha_3 x\beta_1 pVq] ,$$

$$C_3 = [r\beta_3'z\alpha_3 x\alpha_1 y\beta_2 qWr] ,$$

where $(rUp)$, $(pVq)$, $(qWr)$ are paths of even length (possibly of zero length if the vertices $p$, $q$, $r$ are not distinct).

Let $G^*$ be obtained from $G$ by removing $x$, $y$, $z$ and all edges indicent with them, and inserting a new vertex $t$ and three new edges $[\gamma_1^*]$, $[\gamma_2^*]$, $[\gamma_3^*]$, corresponding to the arcs $(t\gamma_1^*p)$, $(t\gamma_2^*q)$, $(t\gamma_3^*r)$. Then $C_1^* = [qVq\gamma_2^{*\prime}t\gamma_1^*p]$, $C_2^* = [qWr\gamma_3^{*\prime}t\gamma_2^*q]$, $C_3^* = [rUp\gamma_1^{*\prime}t\gamma_3^*r]$, $C_j^* = C_j$ for $j > 3$ form an even decomposition $\underline{\underline{C}}^* = \{C_1^*, C_2^*, \ldots, C_m^*\}$ of $G^*$. By the induction hypothesis, $\underline{\underline{C}}^*$ induces a Tait colouring; let $[\gamma_1^*]$ receive colour $a$, $[\gamma_2^*]$ colour $b$, $[\gamma_3^*]$ colour $c$. Then $C_1^*$ is an $(ab)$ circuit, $V$ an $(ab)$ path starting with a $b$-edge, ending with an $a$-edge. Similarly $C_2^*$ is a $(bc)$ circuit, $W$ a $(bc)$ path starting with a $c$-edge, ending with a $b$-edge, $C_3^*$ is a $(ca)$ circuit, $U$ a $(ca)$ path starting with an $a$-edge, ending with a $c$-edge. Therefore we obtain a Tait colouring for $G$, induced by $\underline{\underline{C}}$, if $[\beta_1]$, $[\alpha_2]$ receive $a$, and $[\beta_3]$, $[\alpha_1]$ receive $c$.

Finally we assume that $G$ has no loops, double edges or triangles. Let $\underline{\underline{C}} = \{C_1, C_2, \ldots, C_m\}$ be an even decomposition of $G$. We call an edge $[\gamma]$ a *canal* of $C_k \in \underline{\underline{C}}$ if $(p\gamma q) \in \Gamma$ and $p$, $q$ are non-consecutive vertices of $C_k$, that is, $p$, $q$ are vertices of $C_k$ but $[\gamma]$ is not an edge of $C_k$. We shall first prove (without making use of the induction hypothesis) that Theorem 5 is true if every edge is a canal of some $C_k$.

(1) Each vertex of $G$ appears in exactly three distinct circuits. For if $p$ is a vertex of $C_k$ then there are exactly two arcs in $C_k$ which have $p$ as initial or terminal vertex (since $C_k$ is simple), and

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there are altogether six such arcs (since $G$ is cubic).

(2) An edge $[\gamma]$ is canal of at most one $C_k$ . For if $(p\gamma q) \in \Gamma$ then $(p\gamma q)$ and $(q\gamma' p)$ appear in two distinct circuits (since all circuits are simple) and $p$ and $q$ may only appear in a single other circuit $C_k$ , by (1).

(3) If all vertices of $C_i$ are endvertices of canals of $C_i$ then $C_i$ must pass through all vertices of $G$ . For the subgraph spanned by the vertices of $C_i$ is trivalent, hence a component of $G$ . It must therefore be the whole of $G$ since $G$ is connected.

(4) If all edges of $G$ are canals of some $C_k$ then $\underline{\underline{C}}$ has just three circuits $C_1$, $C_2$, $C_3$ , each passing through every vertex of $G$ . For let $2\lambda_i$ be the length of $C_i$ and $\mu_i$ the number of canals of $C_i$ , $i = 1, 2, \ldots, m$ . Clearly $\mu_i \le \lambda_i$ . The number of edges of $G$ which are canals is $\mu = \sum_{i=1}^{m} \mu_i$ since each is canal of exactly one $C_i$ , by (2). Hence $\mu \le \sum_{i=1}^{m} \lambda_i$ , equality only if all vertices of $C_i$ are endvertices of canals of $C_i$ for every $i$ . But $\sum \lambda_i$ is the total number of edges in $G$ (since each edge appears in exactly two circuits), hence if all edges are canals then each $C_i$ passes through all vertices of $G$ , by (3). From here it follows, by (1), that there are only three circuits $C_1$, $C_2$, $C_3$ .

We now show that if all edges are canals then $\underline{\underline{C}} = \{C_1, C_2, C_3\}$ induces a Tait colouring. Let $C_1 = [p_0\alpha_1 p_1 \ldots \alpha_{2k} p_{2k}]$ , $p_0 = p_{2k}$ , where by (4), $X = \{p_1, p_2, \ldots, p_{2k}\}$ is the set of vertices of $G$ . Since all edges are canals, $C_1$ must have $k$ canals $[\gamma_1], \ldots, [\gamma_k]$ and each $[\gamma_i]$ is an edge of both $C_2$ and $C_3$ . They are separated by edges of $C_1$ , exactly half of the edges $[\alpha_j]$ appearing in $C_2$ and the other

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half in $C_3$ . Clearly $[\alpha_j]$ and $[\alpha_{j+1}]$ cannot appear simultaneously in $C_2$ ; for they are not consecutive edges, by the basic property of polyhedral decompositions, and the vertex $p_j$ cannot appear more than once since $C_2$ is simple. Hence if say $C_2$ contains $[\alpha_1]$ then it must contain exactly the $[\alpha_j]$ with odd $j$ and $C_3$ contains the $[\alpha_j]$ with even $j$ . Colouring the $[\alpha_j]$ with odd $j$ by $a$ , those with even $j$ by $b$ , and the $[\gamma_i]$ by $c$ , we obtain a Tait colouring induced by $\underline{\underline{C}}$ , and Theorem 3 is proved for this $\underline{\underline{C}}$ .

The last remaining case to be considered is when $G$ has no double edges or triangles, but has an edge $[\gamma]$ which is not a canal of any $C_k$ in the even decomposition $\underline{\underline{C}} = \{C_1, C_2, \ldots, C_m\}$ . Let $(x\gamma y) \in \Gamma$ , and $(x\alpha_1 p)$, $(x\beta_1 q)$, $(y\alpha_2 r)$, $(y\alpha_2 s)$ the arcs in $\Gamma$ distinct from $(x\gamma y)$ or $(y\gamma' x)$ with initial vertices $x$ or $y$ . Since $G$ has no triangles, $p, q, r, s$ are distinct from each other and from $x, y$ . Let $C_1 = [p\alpha_1' x\gamma y\alpha_2 rUp]$ , $C_2 = [s\beta_2' y\gamma' x\beta_1 qVs]$ be the two circuits containing $\gamma$ (where if necessary we replace $C_1$ or $C_2$ by their reverses). Here $(rUp)$, $(qVs)$ are paths of odd length avoiding $\alpha_1, \alpha_2, \beta_1, \beta_2$ .

Since each edge appears twice, there must be a circuit $C_3 = [q\beta_1' x\alpha_1 pWq]$ in $\underline{\underline{C}}$ (distinct from $C_1, C_2$ ) where $(pWq)$ is of even length. Clearly $W$ must avoid $\beta_2$ since otherwise $[\gamma]$ would be a canal of $C_3$ with end vertices $x$ and $y$ . Hence we must have a circuit $C_4 = [r\alpha_2' y\beta_2 sTr]$ , distinct from the other three, in which $(sTr)$ is of even length.

Let $G^*$ be obtained from $G$ be deleting $x, y, [\alpha_1], [\alpha_2], [\beta_1]$, $[\beta_2]$ and inserting the arcs $(p\gamma_1^* r)$, $(q\gamma_2^* s)$ and their reverses. Then $C_1^* = [rUp\gamma_1^* r]$ , $C_2^* = [qVs\gamma_2^{*\prime} q]$ , $C_3^* = [pWq\gamma_2^* sTr\gamma_1^{*\prime} p]$ , $C_i^* = C_{i+1}$ for $4 \le i \le m-1$ form an even decomposition $\underline{\underline{C}}^* = \{C_1^*, C_2^*, \ldots, C_{m-1}^*\}$ of $G^*$ . By the induction hypothesis $\underline{\underline{C}}^*$ induces a Tait colouring; let $C_3^*$ be an

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$(ab)$ circuit. Since $W$, $T$ have even length, $[\gamma_1^*]$ and $[\gamma_2^*]$ obtain distinct colours; we may assign $a$ to $[\gamma_1^*]$, $b$ to $[\gamma_2^*]$. Then $W$, $T$ are $(ab)$ paths, $W$ starting with a $b$-edge and ending with an $a$-edge, $T$ starting with an $a$-edge and ending with a $b$-edge. The colouring of $C_1^*$ and $C_2^*$ is now forced: $U$ is an $(ac)$ path, $V$ is a $(bc)$ path, both of odd length and starting and ending with a $c$-edge.

To obtain a Tait colouring in $G$, retain all colourings form $G^*$ for the common edges and assign $a$ to $[\alpha_1]$, $[\alpha_2]$, $b$ to $[\beta_1]$, $[\beta_2]$, and $c$ to $[\gamma]$. This will obviously produce a Tait colouring induced by $\underline{\underline{C}}$, and Theorem 5 is fully proved.

We mention the following consequence of Theorem 5 which is a reformulation of Theorem 1 in [6].

THEOREM 7. *Let* $\underline{\underline{C}} = \{C_1, \ldots, C_m\}$ *be an even polyhedral decomposition of the trivalent graph* $G$, $\chi$ *the characteristic of the decomposition. Let* $\mu$ *be the number of edges on which the two circuits which contain the edge have the same orientation. Then* $\mu \equiv \chi \pmod 2$.

In terms of maps on surfaces we can formulate the result as follows: Suppose that a map on a (non-orientable) surface of characteristic $\chi$ has the property that

(i) each vertex has degree three, and

(ii) every country has an even number of neighbours.

Provide the boundary of each country with an orientation and let $\mu$ be the number of edges which obtain the same orientation from the two countries adjacent to the edge. Then $\mu \equiv \chi \pmod 2$.

If the surface is orientable and the countries are coherently oriented then $\mu = 0$ and the theorem merely states the well known fact that the characteristic is even.

In view of Theorem 6, the four colour conjecture can be given the following equivalent formulation:

Every trivalent graph which has a proper coherent polyhedral decomposition of genus 0 has an even polyhedral decomposition.

Grünbaum's conjecture can be stated as follows:

Every trivalent graph which has a proper coherent polyhedral decomposition has an even polyhedral decomposition.

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