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Bipositive Isomorphisms Between Beurling Algebras and Between their Second Dual Algebras

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Abstract. Let G be a locally compact group and let ω be a continuous weight on G. We show that for each of the Banach algebras $L^1(G,\omega)$, $M(G,\omega)$, $LUC(G,\omega^{-1})^*$, and $L^1(G,\omega)^{**}$, the order structure combined with the algebra structure determines the weighted group.

1 Introduction and Preliminaries

Let G be a locally compact group with a fixed Haar measure. By the group algebra on G we mean the convolution Banach algebra $L^1(G)$ of Haar-integrable functions on G. A natural question asks to what extent does the algebra structure of $L^1(G)$ determine the topological group structure of G. An account of the progress on this question is contained in W. Rudin's monograph (see [Rud62, Subsection 4.7.7]). It is well known that, in general, the algebra structure of a group algebra does not necessarily determine its underlying topological group structure, even when the groups are finite; see, for example, [Wen51]. So, if only the existence of an algebra isomorphism between group algebras is assumed, then the underlying topological groups are isomorphic only if we impose some constraints, for instance on the norm of the isomorphism, or if we consider some special isomorphism such as a bipositive isomorphism. In this article we focus on bipositive algebra isomorphisms.

Y. Kawada was the first author to study such a question. In [Kaw48], Kawada showed that if we have a bipositive algebra isomorphism between group algebras, then the underlying locally compact groups must be isomorphic. H. Farhadi [Far98] proved similar results for other Banach algebras related to locally compact groups, including the bidual of group algebras. He proved that if G and H are locally compact groups and T is a bipositive algebra isomorphism from $L^1(G)^{**}$ onto $L^1(H)^{**}$, then T is an isometric algebra isomorphism. In [GL88], Ghahramani and Lau proved that the Banach algebra structure of the bidual of the group algebras determine their underlying locally compact groups. It then follows from [GL88] that G and H are isomorphic locally compact groups. In [Far98, Thm. 2.4], Farhadi showed that a similar result holds if we replace the bidual of group algebras with a measure algebra or

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with dual of the space of left uniformly continuous functions equipped with an Arens-type multiplication, by reducing the problem to one covered in the paper of Lau and McKennon [LM80].

In this article we provide an example of a bipositive algebra isomorphism between weighted measure algebras that is not an isometry (see Example 3.9). The same construction can be carried out to provide bipositive algebra isomorphisms between other types of convolution algebras associated with weighted locally compact groups (*e.g.*, weighted group algebras) that are not isometries. This example shows that the same approach as the one in Farhadi's paper [Far98] cannot be used to settle similar problems when the weight is nontrivial. Using techniques different from those used by Farhadi [Far98], we show that the algebra structure together with the order structure of various convolution algebras associated with weighted locally compact groups completely determines the structure of the topological groups together with a constraint on the weights. Moreover, we give a complete description of bipositive algebra isomorphisms between weighted group and between weighted measure algebras in terms of topological group isomorphisms.

Throughout, G denotes a locally compact group with a fixed left Haar measure λ. Integration with respect to the Haar measure λ is denoted by $\int_G \cdots dx$. A *weight* on G is a positive continuous function $\omega\colon G \to \mathbb{R}^+$ such that $\omega(xy) \leq \omega(x)\omega(y)$ for all $x, y \in G$. By a *weighted locally compact group* we mean a pair (G,ω), where G is a locally compact group and ω is a weight function on G.

We recall that given a weight ω on G, $C_0(G,\omega^{-1})$ denotes the Banach space of all (continuous) functions f on G such that $f/\omega \in C_0(G)$, equipped with the norm

$$\|f\|_{\infty,\omega^{-1}} = \sup_{x\in G}\left|\frac{f(x)}{\omega(x)}\right|.$$

We call a continuous function f defined on G *positive* if $f(x) \geq 0$ for $x \in G$.

The set of all regular Borel measures on G such that

$$\int_G \omega(s)\,d|\mu|(s) < \infty,$$

denoted by $M(G,\omega)$, is a Banach space with respect to the norm

$$\|\mu\|_\omega = \int_G \omega(s)\,d|\mu|(s).$$

It can be seen that as a Banach space $M(G,\omega)$ is isometrically isomorphic to $C_0(G,\omega^{-1})^*$, with the duality implemented by the pairing

$$\langle \mu, f\rangle = \int_G f(x)\,d\mu(x) \quad \big(f \in C_0(G,\omega^{-1}), \mu \in M(G,\omega)\big).$$

The Banach space $M(G,\omega)$ is a Banach algebra if it is equipped with the following multiplication $*$ defined by duality:

$$\langle \mu * \nu, f\rangle = \int_G\int_G f(st)\,d\mu(s)\,d\nu(t) \quad \big(\mu,\nu \in M(G,\omega),\ f \in C_0(G,\omega^{-1})\big).$$

The multiplication in $M(G,\omega)$ is separately weak-star continuous.

The measure $\mu \in M(G,\omega)$ is called *positive* if $\langle \mu, f\rangle \geq 0$, for every positive $f \in C_0(G,\omega^{-1})$.

The *Beurling algebra* $L^1(G,\omega)$ is the Banach algebra of all measurable functions ψ satisfying

$$\|\psi\|_{1,\omega} = \int_G |\psi(x)|\omega(x)dx < \infty,$$

together with the convolution product defined by

$$\psi_1 * \psi_2(x) = \int_G \psi_1(y)\psi_2(y^{-1}x)dy, \quad (\psi_1,\psi_2 \in L^1(G,\omega), \lambda \text{ a.e. } x \in G).$$

By [Gha84, Lemma 2.1], the Banach algebra $L^1(G,\omega)$ has a bounded approximate identity. We can identify each $\psi \in L^1(G,\omega)$ with an element $h \mapsto \int_G h(x)\psi(x)dx$ of $C_0(G,\omega^{-1})^*$. It can be seen that with this identification $L^1(G,\omega)$ is a closed ideal of $M(G,\omega)$.

An element $\psi \in L^1(G,\omega)$ is *positive* if $\psi(x) \geq 0$, λ a.e. $x \in G$.

It can readily be seen that $L^1(G,\omega)$ as a Banach space is isometrically isomorphic to $L^1(G)$ via $\psi \mapsto \psi\omega$. This implies that the dual of $L^1(G,\omega)$ is

$$L^\infty(G,\omega^{-1}) := \{f : f/\omega \in L^\infty(G)\},$$

with the norm

$$\|f\|_{\infty,\omega^{-1}} = \left\| \frac{f}{\omega} \right\|_\infty.$$

The above definitions and identifications can be found, for example, in [DL05] or [Gha84].

We recall that a linear operator L on the Banach algebra $\mathcal{A}$ is a left multiplier if

$$L(ab) = L(a)b \quad (a,b \in \mathcal{A}).$$

As shown in [Joh64], if the Banach algebra $\mathcal{A}$ has a bounded approximate identity, then every left multiplier on $\mathcal{A}$ is continuous.

Since $L^1(G,\omega)$ is an ideal in $M(G,\omega)$, for each $\mu \in M(G,\omega)$ we can define the left multiplier $L_\mu: L^1(G,\omega) \to L^1(G,\omega); \psi \mapsto \mu * \psi$. It follows from [Gha84, Lemma 2.3] that every left multiplier on $L^1(G,\omega)$ is an L_μ for a measure $\mu \in M(G,\omega)$.

The space of all continuous left multipliers of a Banach algebra $\mathcal{A}$ equipped with composition of operators as product and with operator norm is a Banach algebra, called the *left multiplier algebra* of $\mathcal{A}$. The reader is referred to [Pal94, Sections 1.2.1–1.2.7] for definitions and basic theorems regarding the left multipliers. Some easy calculations show that the mapping $\mu \mapsto L_\mu$ is a continuous algebra isomorphism from $M(G,\omega)$ onto the left multiplier algebra of $L^1(G,\omega)$ but is not necessarily isometric, unless $\omega(e) = 1$.

As usual, we let $C_b(G)$ be the space of all complex-valued, continuous, and bounded functions on G equipped with the sup-norm, and let $LUC(G)$ be the subspace of $C_b(G)$ consisting of all functions f such that the map $G \to C_b(G); x \mapsto l_x f$ is continuous, where $l_x f$ is the function defined by $l_x f(y) = f(xy)$, for each $y \in G$.

Let $LUC(G,\omega^{-1})$ denote the Banach space of all continuous functions f where $f/\omega \in LUC(G)$, equipped with the norm

$$\|f\|_{\infty,\omega^{-1}} := \sup_{x\in G} \left| \frac{f(x)}{\omega(x)} \right|.$$

We also recall that the second dual space $\mathcal{A}^{**}$ of a Banach algebra $\mathcal{A}$ can be equipped with two Banach algebra products, called *Arens products*, that naturally extend the product of $\mathcal{A}$, as canonically embedded in $\mathcal{A}^{**}$. We will assume that $L^1(G)^{**}$ carries the first (left) product.

Next we recall the definition of a Banach algebra product in $LUC(G,\omega^{-1})^*$. For the definition of a product in $LUC(G,\omega^{-1})^*$, we require that the weight ω also satisfy $\omega(e_G)$=1, where e_G denotes the identity element in the group G.

There is a right action of the Banach algebra $L^1(G,\omega)$ on its dual $L^\infty(G,\omega^{-1})$ given by

$$\langle f\cdot\phi,\psi\rangle=\langle f,\phi*\psi\rangle\quad\big(f\in L^\infty(G,\omega^{-1}),\phi,\psi\in L^1(G,\omega)\big).$$

For the weight ω on G satisfying $\omega(e_G)=1$, we have by [DL05, Proposition 7.15] and [Grø90, Proposition 1.3] that $L^\infty(G,\omega^{-1})\cdot L^1(G,\omega)=LUC(G,\omega^{-1})$. Hence, $LUC(G,\omega^{-1})=LUC(G,\omega^{-1})\cdot L^1(G,\omega)$, since $L^1(G,\omega)$ factors. So we have a left action of $LUC(G,\omega^{-1})^*$ on $LUC(G,\omega^{-1})$, defined by duality as follows:

$$\langle m\cdot f,\psi\rangle=\langle m,f\cdot\psi\rangle,$$

where $m\in LUC(G,\omega^{-1})^*$, $f\in LUC(G,\omega^{-1})$, and $\psi\in L^1(G,\omega)$. We can then define a Banach algebra product $\square$ on $LUC(G,\omega^{-1})^*$ via

$$\langle m\,\square\, n,f\rangle=\langle m,n\cdot f\rangle,$$

where $m,n\in LUC(G,\omega^{-1})^*$, $f\in LUC(G,\omega^{-1})$. We can embed $M(G,\omega)$ isometrically as a Banach algebra into $LUC(G,\omega^{-1})^*$ via the natural embedding $\langle\mu,f\rangle=\int_G f\,d\mu$, $f\in LUC(G,\omega^{-1})$ and $\mu\in M(G,\omega)$.

The above definitions and properties can be found, for example, in [DL05].

Let $\mathcal{A}$ and $\mathcal{B}$ be ordered Banach algebras. Then an operator $T:\mathcal{A}\to\mathcal{B}$ is called *positive* if for each positive element $a\in\mathcal{A}$, $T(a)\geq 0$ in $\mathcal{B}$. In particular, $m\in LUC(G,\omega^{-1})^*$ is positive if $\langle m,f\rangle\geq 0$, for every positive function f in $LUC(G,\omega^{-1})$. The operator T is called *bipositive* if T is a bijection and both T and T^{-1} are positive operators. For a space S of functions, S^+ denotes the subset of all positive elements in S.

We conclude this section with some lemmas that are needed for our work in the subsequent sections. The proof of Lemma 1.1, follows the same lines as [Gre65, Lemma 1.1.3] and is therefore omitted.

We say a net (μ_α) in $M(G,\omega)$ converges to μ in $M(G,\omega)$ in strong operator topology if for each f in $L^1(G,\omega)$,

$$\mu_\alpha*f\xrightarrow{\|\cdot\|_{1,\omega}}\mu*f.$$

Lemma 1.1 *If (G,ω) is a weighted locally compact group, then the strong operator closed convex hull of $\{\gamma\frac{\delta_x}{\omega(x)}: x\in G,\ \gamma\in\mathbb{T}\}$ is the unit ball of $M(G,\omega)$.*

For $\mu\in M(G,\omega)$, we can define a continuous linear functional $\widehat{\mu}$ on $LUC(G,\omega^{-1})$ by the pairing

$$\langle\widehat{\mu},f\rangle=\int_G f(x)\,d\mu(x),\quad\big(f\in LUC(G,\omega^{-1})\big).$$

Since $C_0(G,\omega^{-1})$ is a Banach subspace of $LUC(G,\omega^{-1})$, the mapping $\mu \mapsto \widehat{\mu}$, is an isometric linear isomorphism from $M(G,\omega)$ into $LUC(G,\omega^{-1})^*$. Whenever it is clear from the context, we will just write μ instead of $\widehat{\mu}$ and $M(G,\omega)$ instead of $\widehat{M(G,\omega)}$.

Lemma 1.2 *Let (G,ω) be a weighted locally compact group. Then we have*

(i) $LUC(G,\omega^{-1})^* = M(G,\omega) \oplus_1 C_0(G,\omega^{-1})^\perp$. *In this decomposition, $M(G,\omega)$ is a norm-closed subalgebra and $C_0(G,\omega^{-1})^\perp$ is a w^*-closed ideal.*

(ii) $(LUC(G,\omega^{-1})^*)^+ = M(G,\omega)^+ \oplus_1 (C_0(G,\omega^{-1})^\perp)^+$.

Proof (i) The map $\Phi\colon LUC(G,\omega^{-1}) \to LUC(G)\colon f \mapsto \omega^{-1} f$ is an isometric linear isomorphism mapping $C_0(G,\omega^{-1})$ onto $C_0(G)$. As $\Phi^*\colon \mu \mapsto \omega^{-1}\mu$ maps the copy of $M(G)$ in $LUC(G)^*$ isometrically onto the copy of $M(G,\omega)$ in $LUC(G,\omega^{-1})^*$, and also maps $C_0(G)^\perp$ onto $C_0(G,\omega^{-1})^\perp$ and since by [GLL90, Lemma 1.1], $LUC(G)^* = M(G) \oplus_1 C_0(G)^\perp$, we have $LUC(G,\omega^{-1})^* = M(G,\omega) \oplus_1 C_0(G,\omega^{-1})^\perp$.

To show that $M(G,\omega)$ is a subalgebra of $LUC(G,\omega^{-1})^*$, first we prove that given $\mu \in M(G,\omega)$ and $f \in LUC(G,\omega^{-1})$, $\widehat{\mu} \,\Box\, f = \mu \cdot f$. To this end, let $\phi \in L^1(G,\omega)$. Then

$$\langle \widehat{\mu} \,\Box\, f, \phi\rangle = \langle \widehat{\mu}, f \cdot \phi\rangle = \langle f \cdot \phi, \mu\rangle.$$

Simple calculations using Fubini's theorem show that

$$(f \cdot \phi)(x) = \int_G f(tx)\phi(t)\,dt \quad (x \in G),$$

so that

$$\langle f \cdot \phi, \mu\rangle = \int_G \int_G f(tx)\phi(t)\,dt\,d\mu(x) = \langle f, \phi * \mu\rangle = \langle \mu \cdot f, \phi\rangle.$$

Hence,

$$\widehat{\mu} \,\Box\, f = \mu \cdot f. \tag{1.1}$$

Now given $\mu, \nu \in M(G,\omega)$ and $f \in LUC(G,\omega^{-1})$ we have

$$\langle \widehat{\mu} \,\Box\, \widehat{\nu}, f\rangle = \langle \widehat{\mu}, \widehat{\nu} \,\Box\, f\rangle = \langle \widehat{\mu}, \nu \cdot f\rangle = \langle \nu \cdot f, \mu\rangle.$$

Again simple calculations using Fubini's theorem show that

$$\langle \nu \cdot f, \mu\rangle = \langle f, \mu * \nu\rangle = \left\langle \widehat{(\mu * \nu)}, f\right\rangle$$

and so $\widehat{\mu} \,\Box\, \widehat{\nu} = \widehat{(\mu * \nu)}$, showing that $M(G,\omega)$ is a subalgebra of $LUC(G,\omega^{-1})^*$.

To show that $C_0(G,\omega^{-1})^\perp$ is an ideal in $LUC(G,\omega^{-1})^*$, first we note that if $\mu \in M(G,\omega)$ and $h \in C_0(G,\omega^{-1})$, then from equation (1.1),

$$\mu \,\Box\, h = \mu \cdot h \in C_0(G,\omega^{-1}). \tag{1.2}$$

Since $C_0(G,\omega^{-1})$ is a Banach $M(G,\omega)$-submodule of $LUC(G,\omega^{-1})$, we note that for $\psi \in L^1(G,\omega)$, and $h \in C_0(G,\omega^{-1})$,

$$h \,\Box\, \psi \in C_0(G,\omega^{-1}). \tag{1.3}$$

To see this, first note that because $C_0(G,\omega^{-1})$ is a Banach $M(G,\omega)$-submodule of $LUC(G,\omega^{-1})$, for $\mu \in M(G,\omega) \subseteq LUC(G,\omega^{-1})^*$, $\mu \cdot h, h \cdot \mu \in C_0(G,\omega^{-1})$. Now,

$$\begin{aligned}\langle h \,\Box\, \psi, \phi\rangle_{L^\infty - L^1} &= \langle h, \psi * \phi\rangle_{L^\infty - L^1} = \langle \psi * \phi, h\rangle_{M - C_0} = \langle h \cdot \psi, \phi\rangle_{M^* - M}\\ &= \langle \phi, h \cdot \psi\rangle_{M - C_0} = \langle h \cdot \psi, \phi\rangle_{L^\infty - L^1},\end{aligned}$$

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so $h \square \psi = h \cdot \psi$ in $L^\infty(G, \omega^{-1})$. By continuity, $h \square \psi = h \cdot \psi \in C_0(G, \omega^{-1})$.

Now suppose that $m \in LUC(G, \omega^{-1})^*$, $n \in C_0(G, \omega^{-1})^\perp$, and $h \in C_0(G, \omega^{-1})$. Then for $\psi \in L^1(G, \omega)$, equation (1.3) gives $\langle n \square h, \psi \rangle = \langle n, h \square \psi \rangle = 0$, and hence $\langle m \square n, h \rangle = \langle m, n \square h \rangle = 0$. Thus, $C_0(G, \omega^{-1})^\perp$ is a left ideal in $LUC(G, \omega^{-1})^*$. Writing m as $m = \mu + m_1$, where $\mu \in M(G, \omega)$ and $m_1 \in C_0(G, \omega^{-1})^\perp$, we have $n \square m = n \square \mu + n \square m_1$, with $n \square m_1 \in C_0(G, \omega^{-1})^\perp$ from the above. Also, equation (1.2) gives $\langle n \square \mu, h \rangle = \langle n, \mu \square h \rangle = 0$, so $n \square \mu \in C_0(G, \omega^{-1})^\perp$, as well. Thus, $C_0(G, \omega^{-1})^\perp$ is also a right ideal in $LUC(G, \omega^{-1})^*$.

(ii) Let $p \in (LUC(G, \omega^{-1})^*)^+$ and let $\mu \in M(G, \omega)$ be the measure obtained by restricting p to $C_0(G, \omega^{-1})$. Then μ can be regarded as an element of $LUC(G, \omega^{-1})^*$ through the pairing

$$\langle \mu, f \rangle = \int_G f(t)\, d\mu(t) \quad \big(f \in LUC(G, \omega^{-1}) \big).$$

Obviously, μ is positive. Let $m := p - \mu \in C_0(G, \omega^{-1})^\perp$. We prove that $m \geq 0$. To this end, we take $g \in \big(LUC(G, \omega^{-1}) \big)^+$. Given $\epsilon > 0$, there exists a compact subset $K \subseteq G$ such that

$$\int_{G \setminus K} \omega(t)\, d|\mu|(t) < \epsilon.$$

Let $f \in C_c(G)$ such that $0 \leq f \leq 1$ and $f \equiv 1$ on K. We set $g_1 := fg$; pointwise product. Then $g_1 \in C_c(G) \subseteq LUC(G, \omega^{-1})$, and $g - g_1 \in LUC(G, \omega^{-1})^+$. We have

$$\begin{aligned}
\langle m, g \rangle &= \langle p, g \rangle - \langle \mu, g \rangle = \langle p, g_1 \rangle + \langle p, g - g_1 \rangle - \langle \mu, g_1 \rangle - \langle \mu, g - g_1 \rangle \\
&= \langle p - \mu, g_1 \rangle + \langle p, g - g_1 \rangle - \langle \mu, g - g_1 \rangle = \langle p, g - g_1 \rangle - \langle \mu, g - g_1 \rangle \\
&\geq -\|g - g_1\|_{\infty, \omega^{-1}} \int_{G \setminus K} \omega(t) d|\mu|(t) \\
&\geq -\epsilon \|g - g_1\|_{\infty, \omega^{-1}} \geq -\epsilon \|g\|_{\infty, \omega^{-1}}.
\end{aligned}$$

Hence, $\langle m, g \rangle \geq 0$. ∎

The following proposition is at the heart of this paper. Proposition 1.3(i) generalizes [Kaw48, Thm.2] to weighted measure algebras. Our techniques are mostly different from the ones used by Kawada in the special case $\omega = 1$. Proposition 1.3(ii) is a modification of [GLL90, Corollary 1.2] in the context of ordered Banach algebras.

Proposition 1.3 *Let (G, ω) be a weighted locally compact group.*

(i) *Suppose that $\mu \in (M(G, \omega))^+$ is invertible and μ^{-1} is also positive. Then there exist a positive number γ and an element x in G such that $\mu = \gamma \delta_x$.*

(ii) *Suppose that $m \in (LUC(G, \omega^{-1})^*)^+$ such that m is invertible and m^{-1} is also positive. Then there exist a positive number γ and an element x in G such that $m = \gamma \delta_x$.*

Proof (i) First we observe that there exist positive discrete measures μ_d and ν_d and positive continuous measures μ_c and ν_c such that

$$\mu = \mu_d + \mu_c \quad \text{and} \quad \mu^{-1} = \nu_d + \nu_c.$$

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To see this, we note that since $\omega\mu$ belongs to $M(G)$, by [HR, 19.20 and 19.21], there exist a discrete measure $\omega\mu_d$ and a continuous measure $\omega\mu_c$ in $M(G)$ such that

$$\omega\mu = \omega\mu_d + \omega\mu_c, \quad \text{and} \quad \|\omega\mu\| = \|\omega\mu_d\| + \|\omega\mu_c\|.$$

Now, since $\omega\mu$ is positive, we have

$$\begin{aligned}\|\omega\mu\| &= \langle \omega\mu, 1\rangle = \langle \omega\mu_d, 1\rangle + \langle \omega\mu_c, 1\rangle \\ &= \mathrm{Re}\big(\langle \omega\mu_d, 1\rangle + \langle \omega\mu_c, 1\rangle\big) = \mathrm{Re}\langle \omega\mu_d, 1\rangle + \mathrm{Re}\langle \omega\mu_c, 1\rangle.\end{aligned} \tag{1.4}$$

Obviously, $|\mathrm{Re}\langle \omega\mu_d, 1\rangle| \le \|\omega\mu_d\|$ and $|\mathrm{Re}\langle \omega\mu_c, 1\rangle| \le \|\omega\mu_c\|$. Now if either of these last two inequalities is strict, then from (1.4) we would have

$$\|\omega\mu\| < \|\omega\mu_d\| + \|\omega\mu_c\|,$$

a contradiction. Therefore, we must have $\mathrm{Re}\langle \omega\mu_d, 1\rangle = \|\omega\mu_d\|$ and $\mathrm{Re}\langle \omega\mu_c, 1\rangle = \|\omega\mu_c\|$. Hence, $\omega\mu_d$ and $\omega\mu_c$ are positive measures. It then follows that μ_d and μ_c are positive measures. Similarly, we can show that ν_d and ν_c are positive measures.

Since $\mu * \mu^{-1} = \delta_{e_G}$, we have that

$$(\mu_d + \mu_c) * (\nu_d + \nu_c) = \delta_{e_G},$$

that is

$$\mu_d * \nu_d + \mu_d * \nu_c + \mu_c * \nu_d + \mu_c * \nu_c = \delta_{e_G}.$$

Hence, $\mu_d * \nu_d = \delta_{e_G}$ and $\mu_d * \nu_c + \mu_c * \nu_d + \mu_c * \nu_c = 0$, because $M_d(G,\omega) \cap M_c(G,\omega) = \{0\}$ and $M_c(G,\omega)$ is an ideal in $M(G,\omega)$. Hence $\mu_d * \nu_d = \mu_c * \nu_d = \mu_c * \nu_c = 0$, by positivity of all the measures involved. Reversing roles of ν and μ gives us

$$\nu_d * \mu_d = \delta_e, \quad \nu_d * \mu_c = \nu_c * \mu_d = \nu_c * \mu_c = 0.$$

Hence

$$\nu_c = (\nu_d * \mu_d) * \nu_c = \nu_d * (\mu_d * \nu_c) = 0.$$

Similarly, $\mu_c = 0$. Suppose that

$$\mu_d = \sum_{n=1}^{\infty} a_n \delta_{x_n} \quad \text{and} \quad \nu_d = \sum_{n=1}^{\infty} b_n \delta_{x_n'},$$

where (x_n) and (x_n') are sequences of distinct elements of G and a_n and b_n's are positive numbers such that $\sum_n a_n \omega(x_n) < \infty$ and $\sum_n b_n \omega(x_n') < \infty$. Since $a_n \ge 0$ and $b_n \ge 0$, from $\mu_d * \nu_d = \delta_{e_G}$, it follows that μ_d is concentrated at a single point. The same argument works for ν_d and ν_c.

(ii) By Lemma 1.2, we have $m = \mu_1 + n_1$ and $m^{-1} = \mu_2 + n_2$, with $\mu_i \ge 0$ and $n_i \ge 0$, for all $i = 1, 2$. From

$$\delta_{e_G} = m \,\Box\, m^{-1} = \mu_1 * \mu_2 + \mu_1 \,\Box\, n_2 + n_1 \,\Box\, \mu_2 + n_1 \,\Box\, n_2,$$

it follows that

$$\mu_1 * \mu_2 = \delta_{e_G} \quad \text{and} \quad \mu_1 \,\Box\, n_2 + n_1 \,\Box\, \mu_2 + n_1 \,\Box\, n_2 = 0,$$

since $M(G,\omega) \cap C_0(G,\omega^{-1})^{\perp} = \{0\}$ and $C_0(G,\omega^{-1})^{\perp}$ is an ideal in $LUC(G,\omega^{-1})^*$. Similarly,

$$\mu_2 * \mu_1 = \delta_{e_G} \tag{1.5}$$

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and

$$n_2 \,\square\, \mu_1 + \mu_2 \,\square\, n_1 + n_2 \,\square\, n_1 = 0. \tag{1.6}$$

Hence, by equation (1.5) and part (i), there exist $\gamma > 0$ and an element $x \in G$ such that $\mu_1 = \gamma\delta_x$ and $\mu_2 = \gamma^{-1}\delta_{x^{-1}}$. Since all the terms in equation (1.6) are nonnegative, we have

$$\mu_1 \,\square\, n_2 = 0 \quad \text{and} \quad n_1 \,\square\, \mu_2 = 0,$$

and thus by invertibility of μ_1 and μ_2, we have that $n_1 = n_2 = 0$. ■

Lemma 1.4 *Let (G, ω) be a weighted locally compact group. Then for every $\mu \in M(G, \omega)$, $m \mapsto \mu \,\square\, m; LUC(G, \omega^{-1})^* \to LUC(G, \omega^{-1})^*$ is weak-star continuous.*

Proof It follows from the factorization $LUC(G, \omega^{-1}) = LUC(G, \omega^{-1}) \cdot L^1(G, \omega)$ that $LUC(G, \omega^{-1})$ is a Banach $M(G, \omega)$-submodule of $L^\infty(G, \omega^{-1})$. For $\mu \in M(G, \omega)$ we denote the adjoint of $f \mapsto f \cdot \mu; LUC(G, \omega^{-1}) \to LUC(G, \omega^{-1})$ by $m \mapsto \mu \cdot m; LUC(G, \omega^{-1})^* \to LUC(G, \omega^{-1})^*$. So for weak-star continuity of $m \mapsto \mu \,\square\, m$, it suffices to show that $\mu \,\square\, m = \mu \cdot m$, for all $m \in LUC(G, \omega^{-1})^*$. To this end, first let $\mu = \psi \in L^1(G, \omega)$. Then if $f \in LUC(G, \omega^{-1})$, we have

$$\langle \psi \,\square\, m, f \rangle = \langle \psi, m \cdot f \rangle = \langle m \cdot f, \psi \rangle = \langle m, f \cdot \psi \rangle = \langle \psi \cdot m, f \rangle.$$

Hence, $\psi \,\square\, m = \psi \cdot m$. Now let

$$\mu \in M(G, \omega), \quad m \in LUC(G, \omega^{-1})^*, \quad \text{and } f \in LUC(G, \omega^{-1}).$$

Then $f = g \cdot \psi$, for some $g \in LUC(G, \omega^{-1})$ and $\psi \in L^1(G, \omega)$. Hence,

$$\begin{aligned}
\langle \mu \,\square\, m, f \rangle &= \langle \mu \,\square\, m, g \cdot \psi \rangle = \langle \psi \cdot (\mu \,\square\, m), g \rangle \\
&= \langle \psi \,\square\, (\mu \,\square\, m), g \rangle = \langle (\psi * \mu) \,\square\, m, g \rangle \\
&= \langle (\psi * \mu) \cdot m, g \rangle = \langle \psi \cdot (\mu \cdot m), g \rangle \\
&= \langle \mu \cdot m, g \cdot \psi \rangle = \langle \mu \cdot m, f \rangle.
\end{aligned}$$

Hence, $\mu \,\square\, m = \mu \cdot m$. ■

2 The "Easy" Direction of all the Results

Definition 2.1 Let (G, ω_1) and (H, ω_2) be weighted locally compact groups. A standard isomorphism (γ, ϕ) from (G, ω_1) onto (H, ω_2) consists of the following data: a bicontinuous bijective isomorphism of locally compact groups $\phi\colon G \to H$ and a continuous character $\gamma\colon G \to \mathbb{R}^+$ such that

$$0 < \inf_{x \in G} \gamma(x) \frac{(\omega_2 \circ \phi)(x)}{\omega_1(x)} \quad \text{and} \quad \sup_{x \in G} \gamma(x) \frac{(\omega_2 \circ \phi)(x)}{\omega_1(x)} < \infty.$$

Theorem 2.2 *Let (G, ω_1) and (H, ω_2) be weighted locally compact groups, and suppose that (γ, ϕ) is a standard isomorphism from (G, ω_1) to (H, ω_2). Define*

$$j_{\gamma,\phi}\colon C_0(H, \omega_2^{-1}) \to C_0(G, \omega_1^{-1})$$

by $j_{\gamma,\phi}(f) = \gamma \cdot f \circ \phi$, and $J_{\gamma,\phi}\colon LUC(H, \omega_2^{-1}) \to LUC(G, \omega_1^{-1})$ by

$$J_{\gamma,\phi}(f) = \gamma \cdot f \circ \phi.$$

(i) *$j_{\gamma,\phi}$ is a bounded, bijective, bipositive, bijective linear isomorphism, and*

$$T_{\gamma,\phi} := j^*_{\gamma,\phi}\colon M(G,\omega_1) \to M(H,\omega_2)$$

is a bounded, bipositive, bijective algebra isomorphism.

(ii) *$T_{\gamma,\phi}$ maps $L^1(G,\omega_1)$ bijectively onto $L^1(H,\omega_2)$.*

(iii) *$T^{**}_{\gamma,\phi}\colon L^1(G,\omega_1)^{**} \to L^1(H,\omega_2)^{**}$ is a bounded, bipositive, bijective algebra isomorphism.*

(iv) *$J_{\gamma,\phi}$ is a bounded, bipositive linear isomorphism, mapping $C_0(H,\omega_2^{-1})$ onto $C_0(G,\omega_1^{-1})$, and $J^*_{\gamma,\phi}\colon LUC(G,\omega_1^{-1})^* \to LUC(H,\omega_2^{-1})^*$ is a bounded, bipositive, algebra isomorphism.*

Proof Suppose that (G,ω_1) and (H,ω_2) are weighted locally compact groups, and suppose that (γ,ϕ) is a standard isomorphism from (G,ω_1) to (H,ω_2). By Definition 2.1, $\phi\colon G \to H$ is a bicontinuous isomorphism from G onto H and $\gamma\colon G \to ((0,\infty),\times)$ is a continuous homomorphism. Furthermore, there are positive numbers M and m such that

$$m \le \gamma(x)\frac{\omega_2(\phi(x))}{\omega_1(x)} \le M, \quad (x \in G).$$

(i) We define the mapping

$$j_{\gamma,\phi}\colon C_0(H,\omega_2^{-1}) \to C_0(G,\omega_1^{-1}), \text{ where } j_{\gamma,\phi}(f) := \gamma \cdot f \circ \phi.$$

Then it is straightforward to check that $j_{\gamma,\phi}$ is a bounded bipositive linear isomorphism. Hence, the dual mapping

$$T_{\gamma,\phi} := j^*_{\gamma,\phi}\colon M(G,\omega_1) \to M(H,\omega_2)$$

is also a bounded bipositive linear isomorphism. To show that $T_{\gamma,\phi}$ is multiplicative, we first note that for each $x \in G$ we have

$$T_{\gamma,\phi}(\delta_x) = \gamma(x)\delta_{\phi(x)}.$$

Hence, since γ and ϕ are multiplicative, it can be readily seen that $T_{\gamma,\phi}$ is multiplicative on point masses. Now, to see that $T_{\gamma,\phi}$ is also multiplicative on $M(G,\omega_1)$, we note that the linear span of point masses is weak-star dense in $M(G,\omega_1)$, the convolution product is separately weak-star continuous and $T_{\gamma,\phi} = j^*_{\gamma,\phi}$ is weak-star continuous. Clearly, $T_{\gamma,\phi}$ is invertible with ${T_{\gamma,\phi}}^{-1} = T_{1/\gamma,\phi^{-1}}$. Therefore, $T_{\gamma,\phi}$ is a bipositive algebra isomorphism.

(ii) We observe that $T_{\gamma,\phi}(L^1(G,\omega_1)) = L^1(H,\omega_2)$. Since $\psi' \mapsto \int_G \psi'(\phi(x))dx$ defines a Haar integral on $L^1(H)$, there exists $c > 0$ such that the equation

$$\int_G \psi'(\phi(x))dx = c\int_H \psi'(y)dy \quad (\psi' \in L^1(H))$$

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holds. Let $\psi \in L^1(G,\omega_1)$. Then for each $f \in C_0(H,\omega_2)$, we have

$$\begin{aligned}\langle T_{\gamma,\phi}(\psi), f\rangle &= \langle j^*_{\gamma,\phi}(\psi), f\rangle = \langle \psi, \gamma\cdot f\circ\phi\rangle\\ &= \int_G \psi(x)\gamma(x) f\circ\phi(x)dx\\ &= c\int_H \psi\circ\phi^{-1}(y)\gamma\circ\phi^{-1}(y)f(y)dy\\ &= \langle c\gamma\circ\phi^{-1}\cdot\psi\circ\phi^{-1}, f\rangle.\end{aligned}$$

Therefore, $T_{\gamma,\phi}(\psi) = c\gamma\circ\phi^{-1}\cdot\psi\circ\phi^{-1} \in L^1(H,\omega_2)$, and hence

$$T_{\gamma,\phi}(L^1(G,\omega_1)) \subseteq L^1(H,\omega_2).$$

A similar argument using ${T_{\gamma,\phi}}^{-1} = T_{1/\gamma,\phi^{-1}}$ shows that

$$T_{1/\gamma,\phi^{-1}}(L^1(H,\omega_2)) \subseteq L^1(G,\omega_1). \tag{2.1}$$

Now, by applying $T_{\gamma,\phi}$ to each side of (2.1), we have that $L^1(H,\omega_2) \subseteq T_{\gamma,\phi}(L^1(G,\omega_1))$, and therefore $T_{\gamma,\phi}(L^1(G,\omega_1)) = L^1(H,\omega_2)$.

(iii) Since $T_{\gamma,\phi}: L^1(G,\omega_1) \to L^1(H,\omega_2)$ is a bounded algebra homomorphism, $T^{**}_{\gamma,\phi}: L^1(G,\omega_1)^{**} \to L^1(H,\omega_2)^{**}$ is also a bounded algebra homomorphism (this is a standard fact in the theory of Arens product). It can be readily checked that since $T_{\gamma,\phi}$ is bijective and bipositive $T^{**}_{\gamma,\phi}$ is also bijective and bipositive.

(iv) It is not difficult to see that $J_{\gamma,\phi}$ as defined above is a bipositive (bounded) linear isomorphism mapping $LUC(H,\omega_2^{-1})$ onto $LUC(G,\omega_1^{-1})$. We note that $J_{\gamma,\phi}$ maps $C_0(H,\omega_2)$ onto $C_0(G,\omega_1)$. Now, the dual mapping

$$J^*_{\gamma,\phi}: LUC(G,\omega_1^{-1})^* \to LUC(H,\omega_2^{-1})^*$$

is also a bipositive (bounded) linear isomorphism such that

$$J^*_{\gamma,\phi}(\delta_x) = \gamma(x)\delta_{\phi(x)} \quad (x \in G). \tag{2.2}$$

We observe that $J^*_{\gamma,\phi}$ is also multiplicative. To see this, first note that since γ and ϕ are multiplicative, it can be readily seen from the equation (2.2) that $J^*_{\gamma,\phi}$ is multiplicative on the linear span of point masses. Now, to see that $J^*_{\gamma,\phi}$ is multiplicative on $LUC(G,\omega_1^{-1})^*$, we note that the linear span of point masses is weak-star dense in $LUC(G,\omega_1^{-1})^*$; for each $n \in LUC(G,\omega_1^{-1})^*$; $m \mapsto m\,\square\, n$ is weak-star continuous on $LUC(G,\omega_1^{-1})^*$; for each $\mu \in M(G,\omega_1)$, $n \mapsto \mu\,\square\, n$ is weak-star continuous; and from equation (2.2), $J^*_{\gamma,\phi}$ maps the linear span of point masses in $LUC(G,\omega_1^{-1})^*$ into $M(H,\omega_2)$. So, if $m, n \in LUC(G,\omega_1^{-1})^*$, we can find nets (μ_i) and (ν_j) in the linear span of the point masses such that $w^* - \lim \mu_i = m$ and $w^* - \lim \nu_j = n$, and obtain

$$\begin{aligned}J^*_{\gamma,\phi}(m\,\square\, n) &= J^*_{\gamma,\phi}\Big(w^* - \lim_i\big(w^* - \lim_j(\mu_i\,\square\,\nu_j)\big)\Big)\\ &= w^* - \lim_i\big(w^* - \lim_j J^*_{\gamma,\phi}(\mu_i\,\square\,\nu_j)\big)\\ &= w^* - \lim_i\Big(w^* - \lim_j\big(J^*_{\gamma,\phi}(\mu_i)\,\square\, J^*_{\gamma,\phi}(\nu_j)\big)\Big) = J^*_{\gamma,\phi}(m)\,\square\, J^*_{\gamma,\phi}(n),\end{aligned}$$

as required. In particular, this shows that $LUC(G,\omega_1^{-1})^*$ and $LUC(H,\omega_2^{-1})^*$ are bipositively algebraically isomorphic Banach algebras. ∎

3 The Harder Direction for Weighted Measure Algebras

In this section we show that the existence of a bipositive algebra isomorphism between the weighted measure algebras $M(G,\omega_1)$ and $M(H,\omega_2)$ implies that their underlying locally compact groups must be isomorphic. Corollary 3.7 is a special case of [Far98, Thm.2.4].

Lemma 3.1 *Let (G,ω) be a weighted locally compact group. For each $\psi \in L^1(G,\omega)$, the map $x \mapsto \delta_x * \psi: G \to (L^1(G,\omega), \|\cdot\|_{1,\omega})$ is continuous.*

Proof See [Kan09, Lemma 1.3.6 (ii)]. ∎

Lemma 3.2 *Let (G,ω) be a weighted locally compact group. The left annihilator of $L^1(G,\omega)$ in $M(G,\omega)$ is zero.*

Proof Let $\mu \in M(G,\omega)$ be a left annihilator of $L^1(G,\omega)$ and let (f_i) denote the bounded approximate identity of $L^1(G,\omega)$ given in the proof of [Gha84, Lemma 2.1]. Then by [Gha84, Lemma 2.2], we have that

$$\mu = \mu * \delta_e = w^* - \lim_i(\mu * f_i) = 0. \qquad ∎$$

Lemma 3.3 *Let T be a bounded algebra isomorphism from $M(G,\omega_1)$ to $M(H,\omega_2)$. Then T restricted to bounded subsets is SOT-to-w^* continuous.*

Proof Take a bounded net $(\mu_\beta) \subset M(G,\omega_1)$ that converges in the strong operator topology to $\mu \in M(G,\omega_1)$. We claim that

$$T(\mu_\beta) \xrightarrow{w^*} T(\mu).$$

To see this, let ν be a weak-star limit point of $(T(\mu_\beta))$ in $M(H,\omega_2)$ and let $(\mu_{\beta(i)})$ be a subnet of (μ_β) such that

$$T(\mu_{\beta(i)}) \xrightarrow{w^*} \nu.$$

Observe that it suffices now to show that $\nu = T(\mu)$.

Without loss of generality, we can assume that $T(\mu_\beta) \xrightarrow{w^*} \nu$. Let $\psi \in L^1(G,\omega)$ be fixed. Then

$$\|T(\mu_\beta) * T(\psi) - T(\mu) * T(\psi)\|_{\omega_2} \longrightarrow 0,$$

since T is bounded and (μ_β) tends to μ in strong operator topology. Hence,

$$T(\mu_\beta) * T(\psi) \xrightarrow{w^*} T(\mu) * T(\psi)$$

in $M(H,\omega_2)$. On the other hand, since multiplication in $M(H,\omega_2)$ is separately weak-star continuous we see that

$$T(\mu_\beta) * T(\psi) \xrightarrow{w^*} \nu * T(\psi).$$

Thus,

$$T(\mu * \psi) = T(\mu) * T(\psi) = \nu * T(\psi),$$

and so $\mu * \psi = T^{-1}(\nu) * \psi$. Since $\psi \in L^1(G, \omega_1)$ is arbitrary, we obtain $\mu = T^{-1}(\nu)$ by Lemma 3.2, and so $T(\mu) = \nu$ as required. ∎

Corollary 3.4 *A bounded net in $M(G,\omega)$ that converges in the strong operator topology converges in the weak-star topology.*

Theorem 3.5 *Let T be a bipositive algebra isomorphism from $M(G,\omega_1)$ onto $M(H,\omega_2)$. Then there exists a standard isomorphism (γ,ϕ) from (G,ω_1) to (H,ω_2) such that $T = j^*_{\gamma,\phi}$.*

Proof Suppose that T is a bipositive algebra isomorphism from $M(G,\omega_1)$ onto $M(H,\omega_2)$. By [AB85, Thm. 4.3], T and T^{-1} are bounded operators. Suppose that $x \in G$ is given. Since δ_x is a positive measure and T is a positive operator, $T(\delta_x)$ is a positive measure. Also, since T is an algebra isomorphism and δ_x is an invertible measure, $T(\delta_x)$ is also invertible, with the inverse $T(\delta_{x^{-1}})$. Thus, we have that $T(\delta_x)$ is a positive invertible measure with a positive inverse. It now follows from Proposition 1.3 that there exist an element $\phi(x) \in H$ and a positive number $\gamma(x)$ such that

$$T(\delta_x) = \gamma(x)\delta_{\phi(x)}.$$

Since T is an algebra isomorphism and $\delta_x * \delta_y = \delta_{xy}$ for each $x, y \in G$, we can readily see that both $\gamma : G \to (0,\infty)$ and $\phi\colon G \to H$ are multiplicative. We shall now show that γ and ϕ are continuous. Suppose that (g_α) is a net in G that tends to e_G, the identity element of G. By Lemma 3.1, for every $\psi \in L^1(G,\omega)$,

$$\delta_{g_\alpha} * \psi \xrightarrow{\|\cdot\|_{1,\omega_1}} \psi.$$

Since T is bounded, we have that

$$T(\delta_{g_\alpha} * \psi) \xrightarrow{\|\cdot\|_{1,\omega_2}} T(\psi)$$

in $M(H,\omega_2)$. Hence

$$T(\delta_{g_\alpha}) * T(\psi) \xrightarrow{\|\cdot\|_{1,\omega_2}} T(\psi). \tag{3.1}$$

Let U be a precompact neighbourhood of e_G. Without loss of generality we can assume $g_i \in U$, for all i. Then

$$\|T(\delta_{g_\alpha})\| \le \|T\|\|\delta_{g_\alpha}\| \le \|T\|\omega_1(g_\alpha) \le \|T\| \sup\{\omega_1(t); t \in \overline{U}\}.$$

Hence, the net $(T(\delta_{g_\alpha}))$ is bounded in $M(H,\omega_2)$, and so it has a subnet $(T(\delta_{g_{\alpha(i)}}))$ converging weak-star to some $\mu \in M(H,\omega_2)$. Then by equation (3.1), we have that $\mu * T(\psi) = T(\psi)$. Applying T^{-1} to the two sides of this equation yields $T^{-1}(\mu) * \psi = \psi$. Hence by Lemma 3.2, $T^{-1}(\mu) = \delta_{e_G}$, or equivalently $T(\delta_{e_G}) = \mu$. Hence, $\mu = \gamma(e_G)\delta_{\phi(e_G)} = \gamma(e_G)\delta_{e_H}$. This, in particular, shows that $\gamma(g_i) \to \gamma(e_G)$ and $\phi(g_i) \to \phi(e_G)$. Hence, ϕ and γ are continuous. To prove that ϕ is a bijection, we note that corresponding to T^{-1}, there exist $\beta\colon H \to (0,+\infty)$ and $\psi\colon H \to G$ such that

$$T^{-1}(\delta_y) = \beta(y)\delta_{\psi(y)} \quad (y \in H).$$

It follows from the equations $T(T^{-1}(\delta_y)) = \delta_y$ and $T^{-1}(T(\delta_x)) = \delta_x$ that ψ is a bijection, $\psi = \phi^{-1}$ and $\beta(\phi(x)) = \frac{1}{\gamma(x)}$, for all $x \in G$. By symmetry, ψ is continuous. Therefore, ϕ is an isomorphism of topological groups from G onto H.

Now, we show that for every $x \in G$ we have

$$\|T^{-1}\|^{-1} \leq \gamma(x)\frac{\omega_2(\phi(x))}{\omega_1(x)} \leq \|T\|. \tag{3.2}$$

Since, T is a bounded operator, for all $x \in G$ we have that

$$\gamma(x)\omega_2(\phi(x)) = \gamma(x)\|\delta_{\phi(x)}\| = \|T(\delta_x)\| \leq \|T\|\|\delta_x\| = \|T\|\omega_1(x).$$

A similar argument using T^{-1} shows that

$$\omega_1(x) \leq \|T^{-1}\|\gamma(x)\omega_2(\phi(x)).$$

Thus we have established the inequalities in equation (3.2).

Therefore, by the above argument there exist an isomorphism of locally compact groups ϕ from G onto H, a continuous homomorphism $\gamma : G \to (0, +\infty)$ and positive constants M and m such that

$$T(\delta_x) = \gamma(x)\delta_{\phi(x)} = j^*_{\gamma,\phi}(\delta_x) \quad \text{and} \quad m \leq \gamma(x)\frac{\omega_2(\phi(x))}{\omega_1(x)} \leq M, \tag{3.3}$$

for each $x \in G$.

Given $\mu \in M(G, \omega_1)$, by Lemma 1.1, we can find a bounded net (μ_β) of discrete measures in $M(G, \omega_1)$ such that

$$\lim \|\mu_\beta * \psi - \mu * \psi\|_{\omega_1} = 0 \quad (\psi \in L^1(G, \omega_1)).$$

By equation (3.3),

$$T(\mu_\beta) = j^*_{\gamma,\phi}(\mu_\beta) \tag{3.4}$$

for each β. By Lemma 3.3, $T(\mu_\beta) \xrightarrow{w^*} T(\mu)$. By Corollary 3.4 and weak-star continuity of $j^*_{\gamma,\phi}$, we have that

$$j^*_{\gamma,\phi}(\mu_\beta) \xrightarrow{w^*} j^*_{\gamma,\phi}(\mu).$$

It now follows from equation (3.4) that $T(\mu) = j^*_{\gamma,\phi}(\mu)$. ∎

Proposition 3.6 *Let (G, ω_1) and (H, ω_2) be locally compact weighted groups such that there exists a standard isomorphism (γ, ϕ) from (G, ω_1) to (H, ω_2). Then $\frac{\omega_2(\phi(x))}{\omega_1(x)}$ is bounded below if and only if $\gamma \equiv 1$.*

Proof We show that if $\frac{\omega_2(\phi(x))}{\omega_1(x)}$ is bounded below by a constant, say $C > 0$, then $\gamma \equiv 1$. The converse is obvious. First we note that if there is an element $x \in G$ such that $\gamma(x) < 1$, then $\gamma(x^{-1}) > 1$. Suppose that there exists $x \in G$ such that $\gamma(x) > 1$. Then, since γ is a homomorphism, by using powers of x, we can assume that there is a sequence (y_n) of elements of G such that $\gamma(y_n) > n^3$. Let $\mu := \sum \frac{1}{n^2}\omega_1(y_n)^{-1}\delta_{y_n}$. Then since $T_{\gamma,\phi}$, as introduced in Theorem 2.2, is positive, for each $n \in \mathbb{N}$,

$$T_{\gamma,\phi}(\mu) \geq \frac{1}{n^2}T_{\gamma,\phi}(\omega_1(y_n)^{-1}\delta_{y_n}) = \frac{1}{n^2}\omega_1(y_n)^{-1}\gamma(y_n)\delta_{\phi(y_n)} \geq n\omega_1(y_n)^{-1}\delta_{\phi(y_n)}.$$

This means that for each $n \in \mathbb{N}$,

$$\|T_{\gamma,\phi}(\mu)\| \geq n\omega_1(y_n)^{-1}\omega_2(\phi(y_n))) \geq nC,$$

a contradiction. ∎

The following result is a special case of [Far98, Thm. 2.4].

Corollary 3.7 *Suppose that G and H are locally compact groups. If T is a bipositive algebra isomorphism from $M(G)$ onto $M(H)$, then T is an isometry.*

Proof By Theorem 3.5 and Proposition 3.6, there is an isomorphism of topological groups ϕ from G onto H such that $T = j_\phi^*$. It is readily seen that j_ϕ is an isometry and therefore T is an isometric algebra isomorphism. ∎

In Corollary 3.8 we give a characterization of all bipositive algebra isomorphisms from $M(G,\omega_1)$ onto $M(H,\omega_2)$ that are also isometries.

Corollary 3.8 *Suppose that T is a bipositive algebra isomorphism from $M(G,\omega_1)$ onto $M(H,\omega_2)$. Then T is an isometric isomorphism if there exists an isomorphism of topological groups ϕ from G onto H such that*

$$T(\delta_x) = \frac{\omega_1(x)}{\omega_2(\phi(x))}\delta_{\phi(x)} \quad \textit{and} \quad \frac{\omega_1(x)}{\omega_2(\phi(x))} \textit{ is multiplicative.}$$

Proof Suppose that T is a bipositive isometric isomorphism from $M(G,\omega_1)$ onto $M(H,\omega_2)$. Then by the proof of Theorem 3.5, there exist an isomorphism of topological groups ϕ from G onto H and a continuous homomorphism $\gamma : G \to (0,\infty)$ such that

$$T(\delta_x) = \gamma(x)\delta_{\phi(x)} \quad \text{and} \quad \|T^{-1}\|^{-1} \leq \gamma(x)\frac{\omega_2(\phi(x))}{\omega_1(x)} \leq \|T\|. \tag{3.5}$$

Since T is an isometry, $\|T\| = \|T^{-1}\| = 1$. Equation (3.5) now implies that

$$\gamma(x) = \frac{\omega_1(x)}{\omega_2(\phi(x))} \quad \text{and} \quad T(\delta_x) = \frac{\omega_1(x)}{\omega_2(\phi(x))}\delta_{\phi(x)} \quad (x \in G).$$ ∎

The following example shows that a bipositive algebra isomorphism between weighted measure algebras need not necessarily be an isometry.

Example 3.9 Let $\mathbb{R}$ denote the additive group of real numbers. Define the subadditive function g on $\mathbb{R}$ by

$$g(x) := \begin{cases} 1 & x \leq -1, \\ |x| & -1 \leq x \leq 1, \\ 1 & 1 \leq x. \end{cases}$$

Consider the weight functions $\omega_1 = 1$ and $\omega_2 := e^g$ on $\mathbb{R}$. Let $\phi_0 : \mathbb{R} \to \mathbb{R}; x \mapsto x$ be the identity isomorphism and $\gamma_0 : \mathbb{R} \to (0,\infty); x \mapsto 1$ be the trivial homomorphism on $\mathbb{R}$. Using the notation introduced in Theorem 2.2, we define the operator

$$T_{\gamma_0,\phi_0} : M(\mathbb{R}) \longrightarrow M(\mathbb{R},\omega_2).$$

It can be readily seen that T is a bipositive algebra isomorphism that is not an isometry.

We remark that the above construction can be carried out in the following sections to provide bipositive algebra isomorphisms that are not isometries.

4 The Harder Direction, Beurling Algebras

In this section, among other results, we shall show that every bipositive algebra isomorphism T from $L^1(G,\omega_1)$ onto $L^1(H,\omega_2)$ extends to a bipositive algebra isomorphism $\widetilde{T}$ from $M(G,\omega_1)$ onto $M(H,\omega_2)$. Therefore, by Theorem 3.5, if there exists a bipositive algebra isomorphism from $L^1(G,\omega_1)$ onto $L^1(H,\omega_2)$, then the locally compact groups G and H are isomorphic. This generalizes the result of Kawada [Kaw48] to the context of weighted group algebras.

Lemma 4.1 *Let (G,ω) be a weighted locally compact group. Then a left multiplier $L_\mu\colon\psi\mapsto\mu*\psi$ on $L^1(G,\omega)$ is positive if and only if μ is positive.*

Proof It is obvious that for a positive measure $\mu\in M(G,\omega)$, the left multiplier $L_\mu(\psi)=\mu*\psi$ is positive. For the converse, suppose that L_μ is positive. Then the proof of [Gha84, Lemma 2.3] shows that $\mu=w^*-\lim_i L_\mu(f_i)$, where $f_i:=\chi_{U_i}/\lambda(U_i)$. Since f_i's are positive elements in $L^1(G,\omega)$, we have that $L_\mu(f_i)$'s are also positive and therefore μ is also positive. ■

Let (G,ω) be a weighted locally compact group. Recall that $\mathcal{M}_l(L^1(G,\omega)$ denotes the Banach algebra of all left multipliers on $L^1(G,\omega)$ equipped with the composition of operators and with operator norm. The following theorem shows that there is a bipositive algebra isomorphism from $\mathcal{M}_l(L^1(G,\omega))$ onto $M(G,\omega)$.

Theorem 4.2 *Let (G,ω) be a weighted locally compact group. The left multiplier algebra of $L^1(G,\omega)$ is bipositively and algebraically isomorphic to $M(G,\omega)$.*

Proof Let $\theta\colon M(G,\omega)\to\mathcal{M}_l(L^1(G,\omega))$, $\theta(\mu):=L_\mu$. It follows from [Gha84, Lemma 2.3] and Lemma 3.2 that θ is an algebra isomorphism. Lemma 4.1 now shows that θ is bipositive. ■

Theorem 4.3 *Let G and H be locally compact groups, and let ω_1 and ω_2 be weights on G and H, respectively. Let $T\colon L^1(G,\omega_1)\to L^1(H,\omega_2)$ be a bipositive algebra isomorphism. Then there exists a standard isomorphism (γ,ϕ) from (G,ω_1) to (H,ω_2), such that the map $j^*_{\gamma,\phi}\colon M(G,\omega_1)\to M(H,\omega_2)$, when restricted to $L^1(G,\omega_1)$, agrees with T.*

Proof Suppose that T is a bipositive algebra isomorphism from $L^1(G,\omega_1)$ onto $L^1(H,\omega_2)$. Then it is easy to see that for each $\mu\in M(G,\omega)$,

$$L_\mu\colon L^1(H,\omega_2)\longrightarrow L^1(H,\omega_2);\quad \psi\longmapsto T(\mu*T^{-1}(\psi))$$

is a multiplier. So, by [Gha84, Lemma 2.3], there is a measure $\widetilde{T}(\mu) \in M(H, \omega_2)$ such that

$$L_\mu(\psi) = \widetilde{T}(\mu) * \psi.$$

If $\mu \in M(G, \omega_1)^+$, then by Lemma 4.1, L_μ is also a positive multiplier, and therefore $\widetilde{T}(\mu)$ belongs to $M(H, \omega_2)^+$. An easy calculation shows that $\widetilde{T}: \mu \mapsto \widetilde{T}(\mu)$ from $M(G, \omega_1)$ onto $M(H, \omega_2)$ is a bipositive algebra isomorphism extending T. Now Theorem 3.5 implies the existence of a standard isomorphism (γ, ϕ) from (G, ω_1) to (H, ω_2) such that $j^*_{\gamma,\phi} = \widetilde{T}$. By Theorem 2.2(i), $j^*_{\gamma,\phi}$ maps $L^1(G, \omega_1)$ bijectively onto $L^1(H, \omega_2)$, so T maps $L^1(G, \omega_1)$ onto $L^1(H, \omega_2)$. ∎

By taking $\omega_1 = 1$ and $\omega_2 = 1$ in Theorem 4.3, we obtain the following result, originally proved by Wendel [Wen51, Thm. 2].

Corollary 4.4 *Suppose that G and H are locally compact groups. If T is a bipositive algebra isomorphism from $L^1(G)$ onto $L^1(H)$, then T is an isometry.*

Proof By Theorems 2.2 and 4.3 and Proposition 3.6, there is an isomorphism of topological groups ϕ from G onto H such that for each ψ in $L^1(G)$ we have that $T = c\psi \circ \phi^{-1}$, where c is the measure adjustment constant introduced in Theorem 2.2(ii). It is now readily seen that T is an isometry and therefore an isometric algebra isomorphism. ∎

5 The Harder Direction, for $LUC(G, \omega^{-1})$ and $L^1(G, \omega)^{**}$

In this section, in order for us to be able to define a Banach algebra product on $LUC(G, \omega^{-1})^*$, we require the weight ω additionally to satisfy $\omega(e_G)$=1, where e_G denotes the identity element in the group G. We conclude this section by showing that the order structure combined with algebra structure of the bidual of the weighted group algebra $L^1(G, \omega)^{**}$ determines the locally compact group G. This result generalizes Farhadi [Far98, Thm.2.2] to the context of Beurling algebras. We remark that the same ideas as in the proof of Farhadi [Far98, Thm.2.2] cannot be followed to provide a proof for Theorem 5.2. This is mainly because when $\omega \not\equiv 1$, the function ω is not a multiplicative linear functional on $L^1(G, \omega)$.

Theorem 5.1 *Let (G, ω_1) and (H, ω_2) be a weighted locally compact groups, with $\omega_1(e_G) = 1$ and $\omega_2(e_H) = 1$. Suppose that $T: LUC(G, \omega_1)^* \to LUC(H, \omega_2)^*$ is a bipositive algebra isomorphism. Then there is a standard isomorphism (γ, ϕ) from (G, ω_1) to (H, ω_2).*

Proof Let $T: LUC(G, \omega_1)^* \to LUC(H, \omega_2)^*$ be a bipositive algebra isomorphism. By [AB85, Thm.4.3], T and T^{-1} are bounded operators. Given $x \in G$, since T is a positive operator $T(\delta_x)$ is positive. Since T is an algebra isomorphisms $T(\delta_x)$ is also invertible with a positive inverse $T(\delta_{x^{-1}})$. Therefore, by Proposition 1.3, there exist $\phi(x) \in H$ and a positive number $\gamma(x)$ such that $T(\delta_x) = \gamma(x)\delta_{\phi(x)}$. Since T is an algebra isomorphism, we have that both γ and ϕ are multiplicative.

We shall now show that γ and ϕ are continuous. Suppose that (x_α) is a net in G such that $x_\alpha \to x$ in G. Then by Lemma 3.1, for every ψ in $L^1(G,\omega_1)$, we have that

$$\delta_{x_\alpha} * \psi \xrightarrow{\|\cdot\|_{1,\omega_1}} \delta_x * \psi.$$

Since T is a bounded algebra isomorphism,

$$T(\delta_{x_\alpha}) \Box T(\psi) \xrightarrow{\|\cdot\|_{1,\omega_2}} T(\delta_x) \Box T(\psi).$$

Without loss of generality, we can assume that x_α's are contained in a compact neighbourhood U of x. Since

$$\|T(\delta_{x_\alpha})\| \leq \|T\| \sup\{\omega_1(t) : t \in U\},$$

$(T(\delta_{x_\alpha}))$ is a bounded net in $M(H,\omega_2)$. Thus, there is a subnet $(T(\delta_{x_{\alpha_i}}))$ and an element m in $LUC(H,\omega_2^{-1})^*$ such that $T(\delta_{x_{\alpha_i}}) \to m$ in the weak-star topology of $LUC(H,\omega_2^{-1})^*$. Therefore,

$$T(\delta_{x_{\alpha_i}}) \Box T(\psi) \xrightarrow{w^*} m \Box T(\psi).$$

Hence, for every ψ in $L^1(G,\omega_1)$, $\delta_x \Box \psi = T^{-1}(m) \Box \psi$. Therefore, $T(\delta_x) = m$, since $LUC(G,\omega_1^{-1}) = L^1(G,\omega_1) \cdot LUC(G,\omega_1^{-1})$. A similar argument then shows that every subnet of $(T(\delta_{x_\alpha}))$ has a subnet convergent to $T(\delta_x)$. Hence,

$$\gamma(x_\alpha)\delta_{\phi(x_\alpha)} = T(\delta_{x_\alpha}) \xrightarrow{w^*} T(\delta_x) = \gamma(x)\delta_{\phi(x)}.$$

An argument similar to that in the proof of Theorem 3.5 shows that γ and ϕ are continuous. By considering T^{-1} we can show that ϕ is surjective, with a continuous inverse. ∎

Theorem 5.2 *Suppose that $T: L^1(G,\omega_1)^{**} \to L^1(H,\omega_2)^{**}$ is a bipositive bijective algebra isomorphism. Then there is a standard isomorphism (γ,ϕ) from (G,ω_1) to (H,ω_2).*

Proof Let (f_i) be a bounded approximate identity of $L^1(G,\omega_1)$ with $f_i \geq 0$, for all i (see [Gha84, Lemma 2.1]), and let E be a weak-star cluster point of (f_i). Then, $E \geq 0$. By [BD73, Prop.III.28.7], E is a right identity of $L^1(G,\omega_1)^{**}$. Hence, $T(E)$ is also a positive right identity of $L^1(H,\omega_2)^{**}$. Now we argue as in [GL88].

The maps

$$\tau_E : EL^1(G,\omega_1)^{**} \longrightarrow LUC(G,\omega_1^{-1})^*; \quad En \longmapsto n|_{LUC(G,\omega_1^{-1})^*}$$

and

$$\tau_{T(E)} : T(E)L^1(H,\omega_2)^{**} \longrightarrow LUC(H,\omega_2^{-1})^*; \quad T(E)m \longmapsto m|_{LUC(H,\omega_2^{-1})^*}$$

establish bipositive algebra isomorphisms between each domain and target algebras. Hence, $\tau_E^{-1} \circ T \circ \tau_{T(E)}$ is a bipositive algebra isomorphism from $LUC(G,\omega_1^{-1})^*$ onto $LUC(H,\omega_2^{-1})^*$. The result now follows from Theorem 5.1. ∎

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