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Existence of Positive Solutions for Nonlinear Noncoercive Hemivariational Inequalities

Michael E. Filippakis and Nikolaos S. Papageorgiou

Abstract. In this paper we investigate the existence of positive solutions for nonlinear elliptic problems driven by the p-Laplacian with a nonsmooth potential (hemivariational inequality). Under asymptotic conditions that make the Euler functional indefinite and incorporate in our framework the asymptotically linear problems, using a variational approach based on nonsmooth critical point theory, we obtain positive smooth solutions. Our analysis also leads naturally to multiplicity results.

1 Introduction

The aim of this paper is to study the existence of positive solutions for the following second order nonlinear elliptic differential equation with nonsmooth potential (hemivariational inequality):

$$\left\{ \begin{array}{l} -\operatorname{div}\left(\|Dx(z)\|^{p-2}Dx(z)\right) \in \partial j(z,x(z)) \text{ a.e. on } Z \\ x|_{\partial Z}=0,\ 1<p<\infty. \end{array} \right\} \tag{1.1}$$

Here $Z \subseteq \mathbb{R}^{\mathbb{N}}$ is a bounded domain with a C^2 boundary ∂Z and $j(z,x)$ is a measurable potential function which is locally Lipschitz and in general nonsmooth in $x \in \mathbb{R}$. By $\partial j(z,x)$ we denote the generalized subdifferential of $x \to j(z,x)$ (see Section 2).

Using a variational approach based on the nonsmooth critical point theory (see Gasinski–Papageorgiou [5]), we establish the existence of a positive solution for problem (1.1) under conditions which make the Euler (energy) functional of the problem noncoercive. More precisely, we require that the "slopes" $\left\{\frac{u}{x^{p-1}} : u \in \partial j(z,x)\right\}$ stay above $\lambda_1 > 0$ near $+\infty$. Here $\lambda_1 > 0$ is the principal eigenvalue of the negative p-Laplacian with Dirichlet boundary conditions, *i.e.,* of $(-\triangle_p, W_0^{1,p}(Z))$. Also, near 0^+ we ask that the "slopes" $\left\{\frac{pj(z,x)}{x^p}\right\}$ stay below $\lambda_1 > 0$. Note that this second asymptotic condition is in terms of the potential $j(z,x)$, which is in general less restrictive than an analogous condition involving its subdifferential. When $p=2$, our framework incorporates the so-called "asymptotically linear problems" (see Amann–Zehnder [1], Bartsch–Li [3] and Zhou [11]). This situation was extended recently to problems driven by the p-Laplacian and the question of existence of positive solutions was investigated by several authors. We mention the works of Huang [6],

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Li–Zhou [7] and Fan–Zhao–Huang [4]. In all these papers, the potential function is smooth in $x \in \mathbb{R}$ and $\partial j(z,x) = f(z,x)$ belongs in $C(\overline{Z}\times\mathbb{R})$, is positive on $\overline{Z}\times\mathbb{R}_+$ and also satisfies additional restrictive hypotheses such as that $\frac{f(z,x)}{x}$ is nondecreasing in $x \geq 0$ for almost all $z \in Z$ (see Li–Zhou [7] and Zhou [11]). Our hypotheses here are weaker, permitting nonuniform nonresonance at $+\infty$ and at 0^+; the condition at 0^+ is expressed in terms of the potential and our potential function is nonsmooth in $x \in \mathbb{R}$. Moreover, our work here complements that of Motreanu–Papageorgiou [8], where asymptotically at $+\infty$ and at 0^+, the situation with the "slopes" $\left\{ \frac{u}{x^{p-1}} : u \in \partial j(z,x)\right\}$ and $\left\{ \frac{pj(z,x)}{x^p} \right\}$ is reversed. This makes the Euler functional of Motreanu–Papagergiou [8] coercive, while in our case it is indefinite. Finally we should mention that problems like (1.1) arise in mechanics and engineering when one wants to consider more realistic laws of nonmonotone and multivalued nature, which means that the associated energy functional is nonconvex and nonsmooth. For concrete applications we refer to Naniewicz–Panagiotopoulos [9].

2 Mathematical Background

Let X be a Banach space, X^* its topological dual. By $\langle \cdot, \cdot \rangle$ we denote the duality brackets for the pair (X, X^*). Given a locally Lipschitz function $\varphi\colon X \to \mathbb{R}$, the *generalized directional derivative* of φ at $x \in X$ in the direction $h \in X$, is defined by

$$\varphi^0(x;h) \stackrel{df}{=} \limsup_{\substack{x' \to x \\ \lambda \downarrow 0}} \frac{\varphi(x' + \lambda h) - \varphi(x')}{\lambda}.$$

The function $h \to \varphi^0(x;h)$ is sublinear, continuous and so it is the support function of a nonempty, w^*-compact, convex set $\partial\varphi(x) \subseteq X^*$ defined by

$$\partial\varphi(x) \stackrel{df}{=} \{x^* \in X^* : \langle x^*, h\rangle \leq \varphi^0(x;h) \text{ for all } h \in X\}.$$

The multifunction $x \to \partial\varphi(x)$ is known as the *generalized* (or *Clarke*) *subdifferential* of φ. If φ is continuous convex (hence locally Lipschitz), then the generalized subdifferential and the subdifferential in the sense of convex analysis coincide. Also if $\varphi \in C^1(X)$ (hence it is locally Lipschitz), then $\partial\varphi = \{\varphi'(x)\}$.

A point $x \in X$ is a critical point of the locally Lipschitz function $\varphi\colon X \to \mathbb{R}$, if $0 \in \partial\varphi(x)$. A local extremum of φ is a critical point. In the present nonsmooth setting, the well-known Palais–Smale condition (PS-condition, for short) takes the following form:

> A locally Lipschitz function $\varphi\colon X \to \mathbb{R}$ satisfies the nonsmooth PS-condition if every sequence $\{x_n\}_{n\geq 1} \subseteq X$ satisfying $|\varphi(x_n)| \leq M_1$ for some $M_1 > 0$ and all $n \geq 1$ and also satisfying $m(x_n) = \inf[\|x^*\| : x^* \in \partial\varphi(x_n)] \to 0$ as $n \to \infty$, has a strongly convergent subsequence.

The next theorem is a nonsmooth variant of the "mountain pass theorem".

Theorem 2.1 *If X is a reflexive Banach space, $\varphi\colon X \to \mathbb{R}$ is locally Lipschitz; it satisfies the nonsmooth PS-condition and there exist $x_0, x_1 \in X$ and $r > 0$ such that*

$$\max\{\varphi(x_0), \varphi(x_1)\} \le m_r := \inf\{\varphi(x) : \|x - x_0\| = r\} \text{ and } \|x_1 - x_0\| > r,$$

then there exists $\widehat{x} \in X$ which is a critical point of φ such that $\varphi(\widehat{x}) \ge m_r$.

Consider the following nonlinear eigenvalue problem

$$\left\{ \begin{array}{l} -\operatorname{div}(\|Dx(z)\|^{p-2}Dx(z)) = \lambda|x(z)|^{p-2}x(z) \text{ a.e on } Z \\ x|_{\partial Z} = 0,\ 1 < p < \infty,\ \lambda \in \mathbb{R}. \end{array} \right\} \tag{2.1}$$

The smallest $\lambda \in \mathbb{R}$ for which (2.1) has a notrivial solution, is the first eigenvalue of the negative p-Laplacian with Dirichlet boundary conditions, *i.e.,* $(-\triangle_p, W_0^{1,p}(Z))$. It is positive, isolated and simple, *i.e.,* the corresponding eigenspace is one-dimensional. There is a variational characterization of $\lambda_1 > 0$, namely

$$\lambda_1 = \min\left[\frac{\|Dx\|_p^p}{\|x\|_p^p} : x \in W_0^{1,p}(Z), x \neq 0\right]. \tag{2.2}$$

The minimum is attained at the normalized eigenfunction u_1 corresponding to $\lambda_1 > 0$, *i.e.,* $\|u_1\|_p = 1$. Moreover, $u_1 \in C_0^1(\overline{Z})$ and $u_1(z) > 0$ for all $z \in Z$. For more details on the subjects mentioned in this section, see Gasinski–Papageorgiou [5].

3 Positive Solutions

Our hypotheses on the nonsmooth potential are the following:

$H(j)$ $j\colon Z \times \mathbb{R} \to \mathbb{R}$ is a function such that $j(z, 0) = 0$ a.e. on Z and

- (i) for all $x \in \mathbb{R}$, $z \to j(z, x)$ is measurable;
- (ii) for almost all $z \in Z$, $x \to j(z, x)$ is locally Lipschitz;
- (iii) for almost all $z \in Z$, all $x \in \mathbb{R}$ and all $u \in \partial j(z, x)$, we have $|u| \le \alpha(z) + c|x|^{p-1}$ with $\alpha \in L^\infty(Z)_+$, $c > 0$;
- (iv) there exists $\theta \in L^\infty(Z)_+$, $\theta(z) \ge \lambda_1$ a.e. on Z with strict inequality on a set of positive measure, such that $\liminf_{x\to+\infty} \frac{u}{x^{p-1}} \ge \theta(z)$ uniformly for almost all $z \in Z$ and all $u \in \partial j(z, x)$;
- (v) there exists $\eta \in L^\infty(Z)_+$, $\eta(z) \le \lambda_1$ a.e. on Z with strict inequality on a set of positive measure, such that $\limsup_{x\to 0^+} \frac{p\,j(z,x)}{x^p} \le \eta(z)$ uniformly for almost all $z \in Z$.

We consider the Lipschitz continuous truncation function $\tau\colon \mathbb{R} \to \mathbb{R}$ defined by

$$\tau(x) = \begin{cases} 0 & \text{if } x \le 0, \\ x & \text{if } x > 0. \end{cases}$$

We set $j_1(z, x) = j(z, \tau(x))$ which is still a measurable integrand, locally Lipschitz in $x \in \mathbb{R}$ and introduce the Euler functional $\varphi_1\colon W_0^{1,p}(Z) \to \mathbb{R}$ defined by

$$\varphi_1(x) = \frac{1}{p}\|Dx\|_p^p - \int_Z j_1(z, x(z))\,dz, \quad x \in W_0^{1,p}(Z).$$

We know that φ_1 is locally Lipschitz (see Gasinski–Papageorgiou [5, p. 59]).

Proposition 3.1 *If hypotheses $H(j)$ hold, then φ_1 satisfies the nonsmooth PS-condition.*

Proof Let $\{x_n\}_{n\geq 1} \subseteq W_0^{1,p}(Z)$ be a sequence such that

$$|\varphi_1(x_n)| \leq M_1 \text{ for some } M_1 > 0, \text{ all } n \geq 1 \quad \text{and} \quad m(x_n) \to 0 \text{ as } n \to \infty.$$

Since $\partial\varphi_1(x_n) \subseteq W^{-1,q}(Z)$ ($\frac{1}{p} + \frac{1}{q} = 1$) is weakly compact and the norm functional in a Banach space is weakly lower semicontinuous, by the Weierstrass theorem we can find $x_n^* \in \partial\varphi_1(x_n)$, $n \geq 1$, such that $\|x_n^*\| = m(x_n)$. We have $x_n^* = A(x_n) - u_n$, with $u_n \in L^q(Z)$, $u_n(z) \in \partial j_1(z, x_n(z))$ a.e. on Z and $A\colon W_0^{1,p}(Z) \to W^{-1,q}(Z)$ is defined by

$$\langle A(x), y\rangle = \int_Z \|Dx\|^{p-2}(Dx, Dy)_{\mathbb{R}^N}\, dz \quad \text{for all } x, y \in W_0^{1,p}(Z).$$

Hereafter, we denote the duality brackets for the pair $(W_0^{1,p}(Z), W^{-1,q}(Z))$ by $\langle \cdot, \cdot \rangle$. It is easy to check that A is monotone demicontinuous, hence maximal monotone [5, p. 74].

We claim that $\{x_n\}_{n\geq 1} \subseteq W_0^{1,p}(Z)$ is bounded. Suppose that this is not true. By passing to a suitable subsequence if necessary, we may assume that $\|x_n\| \to \infty$. Set $y_n = \frac{x_n}{\|x_n\|}$. We can say that

$$y_n \xrightarrow{w} y \text{ in } W_0^{1,p}(Z) \quad \text{and} \quad y_n \to y \text{ in } L^p(Z).$$

Because of hypothesis $H(j)$(iii) we have

$$\frac{|u_n(z)|}{\|x_n\|^{p-1}} \leq \frac{\alpha(z)}{\|x_n\|^{p-1}} + c|y_n(z)|^{p-1} \text{ a.e. on } Z \qquad (3.1)$$

$$\Rightarrow \left\{ \frac{u_n(\cdot)}{\|x_n\|^{p-1}} \right\} \subseteq L^q(Z) \text{ is bounded.}$$

So we may assume that $\frac{u_n}{\|x_n\|^{p-1}} \xrightarrow{w} h$ in $L^q(Z)$. Given $\varepsilon > 0$, we consider the sets

$$C_{\varepsilon,n} = \{z \in Z : \; x_n(z) > 0, \frac{u_n(z)}{x_n(z)^{p-1}} \geq \theta(z) - \varepsilon\}, \quad n \geq 1.$$

Note that for almost all $z \in \{y > 0\}$, $x_n(z) \to +\infty$, and so because of hypothesis $H(j)$(iv), we have $\chi_{C_{\varepsilon,n}}(z) \to 1$ a.e. on $\{y > 0\}$. Then

$$\Big\| (1 - \chi_{C_{\varepsilon,n}}) \frac{u_n}{\|x_n\|^{p-1}} \Big\|_{L^q(\{y>0\})} \to 0,$$

and so

$$\chi_{C_{\varepsilon,n}} \frac{u_n}{\|x_n\|^{p-1}} \xrightarrow{w} h \text{ in } L^q(\{y > 0\}) \text{ as } n \to \infty.$$

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On the other hand, from the nonsmooth chain rule [5, p. 55], we have

$$\partial j_1(z,x) \subseteq \begin{cases} \{0\} & \text{if } x < 0, \\ \operatorname{conv}\{\eta \partial j(z,0) : \eta \in [0,1]\} & \text{if } x = 0, \\ \partial j(z,x) & \text{if } x > 0. \end{cases} \tag{3.2}$$

Then from (3.1) and (3.2) it follows that $h(z) = 0$ a.e. on $\{y \le 0\}$. We have

$$\begin{aligned} \chi_{C_{\varepsilon,n}}(z) \frac{u_n(z)}{\|x_n\|^{p-1}} &= \chi_{C_{\varepsilon,n}}(z) \frac{u_n(z)}{x_n(z)^{p-1}} y_n(z)^{p-1} \\ &\ge \chi_{C_{\varepsilon,n}}(z)(\theta(z) - \varepsilon) y_n(z)^{p-1} \quad \text{a.e. on } Z. \end{aligned}$$

Passing to the limit as $n \to \infty$, using Mazur's lemma and letting $\varepsilon \downarrow 0$, we obtain

$$\begin{aligned} & h(z) \ge \theta(z) y^+(z)^{p-1} \text{ a.e. on } Z \\ & \qquad \Rightarrow \ h(z) = g(z) y^+(z)^{p-1} \text{ with } g(z) \ge \theta(z) \text{ a.e. on } Z, \end{aligned}$$

(since $h(z) \le c y^+(z)^{p-1}$ a.e.).

From the choice of the sequence $\{x_n\}_{n \ge 1} \subseteq W_0^{1,p}(Z)$, we have

$$\Big| \langle A(x_n), y_n - y \rangle - \int_Z u_n (y_n - y)\, dz \Big| \le \varepsilon_n \|y_n - y\| \text{ with } \varepsilon_n \downarrow 0.$$

Dividing by $\|x_n\|^{p-1}$, we obtain

$$\Big| \langle A(y_n), y_n - y \rangle - \int_Z \frac{u_n}{\|x_n\|^{p-1}} (y_n - y)\, dz \Big| \le \frac{\varepsilon_n}{\|x_n\|^{p-1}} \|y_n - y\|.$$

Evidently, $\int_Z \frac{u_n}{\|x_n\|^{p-1}} (y_n - y) dz \to 0$. So we obtain $\lim_{n\to\infty} \langle A(y_n), y_n - y \rangle = 0$. But A being maximal monotone, it is generalized pseudomonotone [5, p. 84], and so $\langle A(y_n), y_n \rangle \to \langle A(y), y \rangle$, hence $\|Dy_n\|_p \to \|Dy\|_p$. Since $Dy_n \xrightarrow{w} Dy$ in $L^p(Z, \mathbb{R}^N)$, and $L^p(Z, \mathbb{R}^N)$, being uniformly convex, has the Kadec–Klee property, we infer that $Dy_n \to Dy$ in $L^p(Z, \mathbb{R}^N)$, hence $y_n \to y$ in $W_0^{1,p}(Z)$. Recall that

$$\Big| \langle A(y_n), v \rangle - \int_Z \frac{u_n}{\|x\|^{p-1}} v\, dz \Big| \le \varepsilon_n \|v\| \text{ for all } v \in W_0^{1,p}(Z).$$

In the limit as $n \to \infty$, we obtain

$$\langle A(y), v \rangle = \int_Z hv\, dz = \int_Z g(y^+)^p v\, dz \text{ for all } v \in W_0^{1,p}(Z). \tag{3.3}$$

Recalling that $Dy^+(z) = 0$ a.e on $\{y \le 0\}$, from (3.3) we infer that

$$-\operatorname{div}(\|Dy^+(z)\|^{p-2} Dy^+(z)) = g(z)(y^+(z))^{p-1} \text{ a.e. on } Z,\ y^+|_{\partial Z} = 0. \tag{3.4}$$

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From the strict monotonicity of the principal eigenvalue $\widehat{\lambda}_1(g)$ of a weighted eigenvalue problem on the weight $g \in L^\infty(Z)$, we have $\widehat{\lambda}_1(g) < \widehat{\lambda}_1(\lambda_1) = 1$. So from (3.4) it follows that if $y^+ \neq 0$, then $y^+ \in W_0^{1,p}(Z)$ cannot be the principal eigenfunction of the weighted eigenvalue problem with weight g, and so y^+ must change sign [2], a contradiction. Therefore $y^+ \equiv 0$, and so $y(z) \leq 0$ a.e. on Z. Then since

$$\Big| \langle A(y_n, y_n\rangle - \int_Z \frac{u_n}{\|x_n\|^{p-1}} y_n \, dz \Big| \leq \varepsilon_n' \text{ with } \varepsilon_n' \downarrow 0$$

and

$$\int_Z \frac{u_n}{\|x_n\|^{p-1}} y_n \, dz \to 0 \text{ as } n \to \infty \qquad \text{(see (3.2))}$$

we obtain $\|Dy_n\|_p \to 0$, *i.e.*, $y_n \to 0$ in $W_0^{1,p}(Z)$, a contradiction to the fact that $\|y_n\| = 1$, $n \geq 1$. This implies that $\{x_n\}_{n\geq 1} \subseteq W_0^{1,p}(Z)$ is bounded. So we may assume that $x_n \xrightarrow{w} x$ in $W_0^{1,p}(Z)$ and $x_n \to x$ in $L^p(Z)$. Because $\int_Z u_n(x_n - x)\, dz \to 0$ (see hypothesis H(j)(iii)), we have $\langle A(x_n), x_n - x\rangle \to 0$, and so as before from the generalized pseudomonotonicity of A and the Kadec–Klee property of $L^p(Z, \mathbb{R}^N)$, we conclude that $x_n \to x$ in $W_0^{1,p}(Z)$. ■

In order to show that φ_1 satisfies the mountain pass geometry, we need the following lemma which underlines the significance of the nonuniform nonresonance condition $H(j)(v)$.

Lemma 3.2 *If $\eta \in L^\infty(Z)_+$ satisfies $\eta(z) \leq \lambda_1$ a.e. on Z with strict inequality on a set of positive measure, then there exists $\xi > 0$ such that*

$$\psi(x) = \|Dx\|_p^p - \int_Z \eta(z)|x(z)|^p \, dz \geq \xi \|Dx\|_p^p$$

for all $x \in W_0^{1,p}(Z)$.

Proof From (2.2) we see that $\psi \geq 0$. Suppose that the lemma is not true. Exploiting the p-homogeneity of ψ, we can find $\{x_n\}_{n\geq 1} \subseteq W_0^{1,p}(Z)$ such that $\psi(x_n) \downarrow 0$ and $\|Dx_n\|_p = 1$. Because of Poincaré's inequality, we may assume that $x_n \xrightarrow{w} x$ in $W_0^{1,p}(Z)$ and $x_n \to x$ in $L^p(Z)$. Clearly ψ is w-lower semicontinuous on $W_0^{1,p}(Z)$. So $\psi(x) \leq 0$, hence

(3.5)
$$\|Dx\|_p^p \leq \int_Z \eta(z)|x(z)|^p \, dz \leq \lambda_1 \|x\|_p^p \quad \Rightarrow \quad x = 0 \text{ or } x = \pm u_1 \quad \text{(see (2.2))}.$$

If $x = 0$, then $\|Dx_n\|_p \to 0$, a contradiction to the fact that $\|Dx_n\|_p = 1$. So $x = \pm u_1$. Then $|x(z)| > 0$ for all $z \in Z$. So from the first inequality in (3.5) and the hypothesis on η we infer that $\|Dx\|_p^p < \lambda_1 \|x\|_p^p$, a contradiction to (2.2). ■

Using this lemma, we prove the following proposition.

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Proposition 3.3 *If hypotheses* H(j) *hold, then there exists* $\rho > 0$ *such that* $\varphi_1(x) \geq \beta > 0$ *for all* $x \in W_0^{1,p}(Z)$ *with* $\|x\| = \rho$.

Proof By virtue of hypothesis H(j)(v), given $\varepsilon > 0$, we can find $\delta = \delta(\varepsilon) > 0$ such that

$$(3.6) \qquad j_1(z,x) = j(z,x) \leq \frac{1}{p}(\eta(z) + \varepsilon)x^p \text{ for almost all } z \in Z \text{ and all } x \in (0,\delta].$$

On the other hand, because of hypothesis H(j)(iii) and the mean value problem for locally Lipschitz functions [5, p. 53], we have

$$(3.7) \qquad j_1(z,x) \leq c_1 x^\tau \text{ for almost all } z \in Z, \text{ all } x \geq \delta \text{ and with } c_1 > 0, \tau > p.$$

Since $j(z,0) = 0$ a.e. on Z, from (3.6) and (3.7) we infer that

$$(3.8) \qquad j_1(z,x) \leq \frac{1}{p}(\eta(z) + \varepsilon)|x|^p + c_1|x|^\tau \text{ for almost all } z \in Z \text{ and all } x \in \mathbb{R}.$$

Therefore for all $x \in W_0^{1,p}(Z)$, we have

$$\begin{aligned}\varphi_1(x) &\geq \frac{1}{p}\|Dx\|_p^p - \frac{1}{p}\int_Z \eta(z)|x(z)|^p\,dz - \frac{\varepsilon}{p}\|x\|_p^p - c_1\|x\|^\tau \text{ (see (3.8))}\\ &\geq \frac{1}{p}\Big(\xi - \frac{\varepsilon}{\lambda_1}\Big)\|Dx\|_p^p - c_1\|x\|^\tau \text{ (see Lemma 3.2 and (2.2)).}\end{aligned}$$

Choosing $\varepsilon < \lambda_1\xi$ and using Poincaré's inequality, we obtain

$$\varphi_1(x) \geq c_2\|x\|^p - c_1\|x\|^\tau \text{ for some } c_2 > 0, \text{ all } x \in W_0^{1,p}(Z).$$

Since $\tau > p$, we can find $\rho > 0$ small such that $\varphi_1(x) \geq \beta > 0$ for all $x \in W_0^{1,p}(Z)$ with $\|x\| = \rho$. ■

Proposition 3.4 *If hypotheses* H(j) *hold, then* $\varphi(tu_1) \to -\infty$ *as* $t \to +\infty$.

Proof By virtue of hypotheses H(j)(iii) and (iv), given $\varepsilon > 0$, we can find $c_\varepsilon > 0$ such that

$$(3.9) \qquad j_1(z,x) \geq \frac{1}{p}(\theta(z) - \varepsilon)x^p - c_\varepsilon \text{ for almost all } z \in Z, \text{ all } x \geq 0.$$

Then for $t > 0$ we have

$$\varphi_1(tu_1) \leq \frac{t^p}{p}\int_Z(\lambda_1 - \theta(z))u_1(z)^p\,dz + \frac{\varepsilon t^p}{p}\|u_1\|_p^p + c_\varepsilon|Z|_N$$

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(see (3.9) and recall that $\|Du_1\|_p^p = \lambda_1\|u_1\|_p^p$). Here $|Z|_N$ denotes the Lebesgue measure of $Z \subseteq \mathbb{R}^N$. Note that

$$\int_Z (\lambda_1 - \theta(z))u_1(z)^p \, dz = \gamma < 0 \quad \text{and} \quad \|u_1\|_p = 1.$$

So $\varphi_1(tu_1) \leq \frac{t^p}{p}(\gamma+\varepsilon)+c_\varepsilon|Z|_N$. Choosing $\varepsilon < -\gamma$, we conclude that $\varphi_1(tu_1) \to -\infty$ as $t \to +\infty$. ■

Now we are ready for the theorem on the existence of positive solutions.

Theorem 3.5 *If hypotheses* H(j) *hold, then problem* (1.1) *has a solution* $x \in C_0^1(\overline{Z})$, $x \neq 0$ *and* $x(z) \geq 0$ *a.e. on* Z.

Proof Propositions 3.1, 3.3 and 3.4 permit the application of Theorem 2.1, which gives $x \in W_0^{1,p}(Z)$ such that

$$0 \in \partial\varphi_1(x) \text{ with } \varphi_1(0) = 0 < \varphi_1(x).$$

Evidently $x \neq 0$, and from the inclusion it follows that

$$A(x) = u \text{ with } u \in L^q(Z), u(z) \in \partial j_1(z, x(z)) \text{ a.e. on } Z.$$

Using $-x^- \in W_0^{1,p}(Z)$ as a test function, we obtain $\|Dx^-\|_p^p = 0$ (see (3.2)). Hence $x^- = 0$ and so $x(z) \geq 0$ a.e. on Z. Finally from nonlinear regularity theory [5, pp. 115–116], we have that $x \in C_0^1(\overline{Z})$. So $x \in C_0^1(\overline{Z})$ is a solution of (1.1) (see (3.2)). ■

With a mild additional condition on $j(z,x)$ we can have a strictly positive solution.

Theorem 3.6 *If hypotheses* H(j) *hold and for almost all* $z \in Z$, *all* $x \geq 0$ *and all* $u \in \partial j(z,x)$, $-c_0x^p \leq u$ *with* $c_0 > 0$, *then problem* (1.1) *has a solution* $x \in C_0^1(\overline{Z})$ *with* $x(z) > 0$ *for all* $z \in Z$.

Proof Let $x \in C_0^1(\overline{Z})$, $x \neq 0$, $x \geq 0$ be the solution obtained in Theorem 3.5. Using the nonlinear strict maximum principle of Vazquez [10] (see also [5, p. 117]), we obtain $x(z) > 0$ for all $z \in Z$. ■

Remark 3.7 If $\partial j(z,x) \subseteq \mathbb{R}_+$ for almost all $z \in Z$ and all $x \geq 0$, then the additional condition of Theorem 3.6 it satisfied. Actually this is the situation in the papers mentioned in the introduction.

It is clear from our analysis that if the asymptotic conditions in H(j)(iv) and (v) are symmetric for $\pm\infty$ and for $0^{\pm}$, then we can have a multiplicity result for problem (1.1). So let H(j)′ be the hypotheses H(j) with the limits in H(j)(iv) and in H(j)(v) replaced by

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(iv) $\liminf_{|x|\to\infty} \frac{u}{|x|^{p-2}x} \geq \theta(z)$ uniformly for almost all $z \in Z$ and all $u \in \partial j(z,x)$,

(v) $\limsup_{x\to 0} e \frac{pj(z,x)}{|x|^p} \leq \eta(z)$ uniformly for almost all $z \in Z$,

respectively.

Theorem 3.8 *If hypotheses* H(j)′ *hold, then problem* (1.1) *has at least two nontrivial solutions* $x_0, x_1 \in C_0^1(\overline{Z})$ *such that* $x_0(z) \leq 0 \leq x_1(z)$ *for all* $z \in Z$.

Theorem 3.9 *If hypotheses* H(j)′ *hold and for almost all* $z \in Z$, *all* $x \in \mathbb{R}$ *and all* $u \in \partial j(z,x)$ *we have* $ux \geq 0$, *then problem* (1.1) *has at least two solutions* $x_0, x_1 \in C_0^1(\overline{Z})$ *such that* $x_0(z) < 0 < x_1(z)$ *for all* $z \in Z$.

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Department of Mathematics
National Technical University
Zografou Campus
Athens 15780
Greece
e-mail: mfilip@math.ntua.gr
npapg@math.ntua.gr